ESSAYS ON MECHANISM DESIGN

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Signed: 

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I would like to dedicate this thesis to my wife Bei Hong, and our baby boy Honghao Li.

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Summary

This thesis consists of three chapters on mechanism design.[1] The first chapter studies the foundations of dominant-strategy mechanisms. The second chapter examines the equivalence of stochastic and deterministic mechanisms. In the third chapter, we focus on the design of efficient mechanisms in dynamic environments with interdependent valuations and evolving private information.

The first chapter. Traditional models in mechanism design make strong assumptions about the agents' hierarchies of beliefs about each other. For example, in models with independent types, agents' beliefs about other agents are common knowledge among the agents and the mechanism designer. This seems peculiar in the context of mechanism design, where the focus is on asymmetric information; imperfect information about others' beliefs seems at least as pervasive as imperfect information about others' preferences. Relaxing these assumptions has been the focus of the literature of robust mechanism design. We consider a revenue-maximizing mechanism designer who has an estimate of the distribution of the agents' payoff-relevant observations, but she does not have any reliable information about the agents' beliefs (including their beliefs about one another's payoff types, their beliefs about these beliefs, etc.).

[1]The first chapter is coauthored with Yi-Chun Chen; the second chapter is coauthored with Yi-Chun Chen, Wei He, and Yeneng Sun; the third chapter is coauthored with Wei He. The third chapter has been published in *Games and Economic Behavior*.

The mechanism designer could use a dominant-strategy mechanism, which does not rely on any assumptions of the agents' beliefs. Alternatively, the mechanism designer could use mechanisms that ask the agents to report their beliefs about one another's payoff types, and to report their beliefs about these beliefs, etc. In the extreme, the mechanism designer could use mechanisms that ask the agents to report everything; that is, their whole infinite hierarchies of beliefs. We examine whether there is any theoretical foundation (in terms of optimality) for the use of dominant-strategy mechanisms.

The second chapter. The mechanism design literature essentially builds on the assumption that a mechanism designer can credibly commit to any outcome. This requirement implies that any outcome of the mechanism must be verifiable before it can be employed. In this vein, a stochastic mechanism demands not only that a randomization device be available to the mechanism designer, but also that the outcome of the randomization device be objectively verified. As noted in Laffont and Martimort (2002, p. 67), "Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced. ... The enforcement of such stochastic mechanisms is thus particularly problematic." In the second chapter, we consider a general social choice environment that has multiple agents, a finite set of alternatives, and independent and dispersed information. We prove that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism. A deterministic mechanism is robust to the availability of the randomization device, and the ability of the mechanism designer to commit to any outcome induced by the randomization device. Our result implies that every mechanism can in fact be deterministically implemented, and thereby irons out the conceptual difficulties associated with stochastic mechanisms.

The third chapter. We focus on the design of efficient mechanisms in dynamic environments with interdependent valuations and evolving private information. Under the assumption that each agent observes her own realized outcome-decision payoff from the previous period, we construct an efficient, incentive-compatible mechanism that is also budget-balanced in every period of the game.

Chapter 1

Revisiting the Foundations of Dominant-Strategy Mechanisms

1.1 Introduction

Suppose that a revenue-maximizing mechanism designer has an estimate of the distribution of the agents' payoff types, but she does not have any reliable information about the agents' beliefs (including their beliefs about one another's payoff types, their beliefs about these beliefs, etc.), as these are arguably never observed. The mechanism designer ranks mechanisms according to their worst-case performance - the minimum expected revenue - where the minimum is taken over all possible agents' beliefs. The use of dominant-strategy mechanisms has a maxmin foundation if the mechanism designer finds it optimal to use a dominant-strategy mechanism.

A closely related notion is the Bayesian foundation. The use of dominant-strategy mechanisms is said to have a Bayesian foundation if there exists a particular assumption about (the distribution of) the agents' beliefs, against which the optimal dominant-strategy mechanism achieves the highest expected revenue among all detail-free mechanisms. Note that if there exists such an assumption, then the worst-case expected revenue of an arbitrary detail-free mechanism obviously cannot exceed its expected revenue against this particular assumption, which in turn cannot exceed the worst-case expected revenue of the optimal dominant-strategy mechanism. Therefore, the Bayesian foundation is a stronger notion than the maxmin foundation.

In the context of a revenue-maximizing auctioneer, Chung and Ely (2007) show that, under a regularity condition on the distribution of the bidders' valuations, the use of dominant-strategy mechanisms has maxmin and Bayesian foundations. What has been missing thus far from the literature on mechanism design is the study of such foundations in general environments. In this paper, we study the maxmin and Bayesian foundations in general social choice environments with quasi-linear preferences and private values. This exposes the underlying logic of the existence of such foundations in the single-unit auction setting, and extends the argument to cases where it was hitherto unknown.

We start with the following contrast between two bilateral trade models (Section 1.3). In the standard bilateral trade model in which traders are ex ante identified buyers or sellers, the use of dominant-strategy mechanisms has maxmin and Bayesian foundations. We then consider a bilateral trade model with ex ante unidentified traders. In this economic environment, we explicitly construct a single Bayesian mechanism that does strictly better than the optimal dominant-strategy mechanism, regardless of the assumption about (the distribution of) the agents' beliefs. In other words, there is neither a Bayesian foundation nor a maxmin foundation. To the best of our knowledge, this is the first example of a revenue maximization setting in which the use of dominant-strategy mechanisms does not have a maxmin foundation.[1]

From this contrast, we abstract the uniform shortest-path tree condition. Our result builds on the recent literature on the network approach to mechanism design, in particular, Rochet and Stole (2003), Heydenreich, Müller, Uetz, and Vohra (2009), Vohra (2011) and Kos and Messner (2013).[2] We formulate the optimal mechanism design question as a network flow problem, and the optimization problem reduces to determining the shortest-path tree (the union of all shortest-paths from the source to all nodes) in this network. We say that there is uniform shortest-path tree if for each agent, the shortest-path tree is the same for all dominant-strategy implementable decision rules and other

[1]Chung and Ely (2007, Proposition 2) construct an example in which a Bayesian foundation does not exist, but their construction is silent about the existence of a maxmin foundation. Bergemann and Morris (2005) study an implementability problem. Börgers (2013) adopts a different notion of optimality.

[2]Also see Rochet (1987), Gui, Müller, and Vohra (2004), and Müller, Perea, and Wolf (2007).

agents' reports.

We show that under an additional regularity condition, the uniform shortest-path tree ensures the maxmin and Bayesian foundations of dominant-strategy mechanisms (Theorem 1.1). The uniform shortest-path tree is largely responsible for the success of mechanism design in numerous applications across various fields. Loosely speaking, the same features that make optimal mechanism design tractable also provide maxmin and Bayesian foundations for the use of dominant-strategy mechanisms. To prove this result, we adopt the linear programming approach to mechanism design, which exposes the underlying logic behind the existence of such foundations.[3] In particular, this gives us a recipe for constructing the assumption about (the distribution of) the agents' beliefs for the Bayesian foundation.

The uniform shortest-path tree condition is of interest because a number of resource allocation problems satisfy this condition. We examine its applicability in prominent environments. First, the uniform shortest-path tree condition is satisfied in environments with linear utilities and one-dimensional types. This fits many classical applications of mechanism design, including single-unit auction (e.g., Myerson (1981)), public good (e.g., Mailath and Postlewaite (1990)), and standard bilateral trade (e.g., Myerson and Satterthwaite (1983)).

[3]We are indebted to Rakesh Vohra for bringing to our attention a closely related paper by Sher and Vohra (2015), as well as for suggestions along this direction.

The uniform shortest-path tree condition also holds in multi-unit auctions with homogeneous or heterogeneous goods, combinatorial auctions and the like, as long as the agents' private values are one-dimensional and utilities are linear. In such a case, the payoff types are linearly ordered via a single path. Second, the uniform shortest-path tree condition can also be satisfied in some multi-dimensional environments. In particular, we consider the multi-unit auction with capacitated bidders (see Malakhov and Vohra (2009)). In this case, the agent's payoff types are located on different paths and are only partially ordered. For both applications, we provide primitive conditions for regularity.

When the uniform shortest-path tree condition is violated, maxmin/ Bayesian foundations might not exist. If the optimal dominant-strategy mechanism exhibits certain properties, we can construct a single Bayesian mechanism that robustly achieves strictly higher expected revenue than the optimal dominant-strategy mechanism, regardless of the agents' beliefs (Theorem 1.2). We stress that as a no-foundation result, this is remarkably strong. In addition to bilateral trade with ex ante unidentified traders, we apply this result to auction with type-dependent outside option.

The remainder of this introduction discusses some related literature. Section 2.2 presents the notations, concepts, and the model. Section 1.3 contrasts two bilateral trade models. Section 1.4 formulates the notion of the uniform shortest-path tree and presents the results. Section 1.5 studies three applications of the results and Section 2.5 concludes with discussions.

1.1.1 Related literature

In a seminal paper, Bergemann and Morris (2005) ask whether a fixed social choice correspondence - mapping payoff type profiles to sets of possible allocations - can or cannot be robustly partially implemented. Thus they focus on a "yes or no" question. In contrast, we consider the objective of revenue maximization for the mechanism designer (under her estimate about the distribution of the agents' payoff types), allowing all possible beliefs and higher-order beliefs of the agents. The best mechanism from the point of view of the mechanism designer will in general not be separable, and thus the results of Bergemann and Morris (2005) do not apply.

This paper joins a growing literature exploring mechanism design with worst case objectives. This includes the seminal work of Bergemann and Morris (2005), Chung and Ely (2007), and more recently, Carroll (2015, 2016), Yamashita (2014, 2016), and Du (2016), among others.

Another recent line of literature studies the equivalence of Bayesian and dominant-strategy mechanisms; see, for example, Manelli and Vincent (2010), Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) and Goeree and Kushnir (2015). Our paper differs from these in that the mechanism designer in our model does not make any assumptions about the agents' beliefs.

1.2 Preliminaries

1.2.1 Notation

There is a finite set $\mathcal{I} = \{1, 2, ..., I\}$ of risk-neutral agents and a finite set $\mathcal{K} = \{1, 2, ..., K\}$ of social alternatives. Agent i's payoff type $v_i \in \mathbb{R}^K$ represents her gross utility under the K alternatives.[4] The set of possible payoff types of agent i is a finite set $V_i \subset \mathbb{R}^K$. The set of possible payoff type profiles is $V = \Pi_{i\in\mathcal{I}} V_i$ with generic payoff type profile $v = (v_1, v_2, ..., v_I)$. We write v_{-i} for a payoff type profile of agent i's opponents, i.e., $v_{-i} \in V_{-i} = \Pi_{j\neq i} V_j$. If Y is a measurable space, then ΔY is the set of all probability measures on Y. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

1.2.2 Types

We follow the standard approach to model agents' information using a type space. A type space, denoted $\Omega = (\Omega_i, f_i, g_i)_{i\in\mathcal{I}}$, is defined by a measurable space of types Ω_i for each agent, and a pair of measurable mappings $f_i : \Omega_i \to V_i$, defining the payoff type of each type, and $g_i : \Omega_i \to \Delta(\Omega_{-i})$, defining each type's belief about the types of the other agents.

[4] We may represent the agent's payoff types in different ways. For instance, when studying one-dimensional payoff types (Section 1.5.1), it is more convenient to represent agent i's payoff type by $v_i \in \mathbb{R}$.

A type space encodes in a parsimonious way the beliefs and all higher-order beliefs of the agents. One simple kind of type space is the naive type space generated by a payoff type distribution $\pi \in \Delta(V)$. In the naive type space, each agent believes that all agents' payoff types are drawn from the distribution π, and this is common knowledge. Formally, a naive type space associated with π is a type space $\Omega^\pi = (\Omega_i, f_i, g_i)_{i \in \mathcal{I}}$ such that $\Omega_i = V_i$, $f_i(v_i) = v_i$, and $g_i(v_i)[v_{-i}] = \pi(v_{-i}|v_i)$ for every v_i and v_{-i}. The naive type space is used almost without exception in auction theory and mechanism design. The cost of this parsimonious model is that it implicitly embeds some strong assumptions about the agents' beliefs, and these assumptions are not innocuous. For example, if the agents' payoff types are independent under π, then in the naive type space, the agents' beliefs are common knowledge. On the other hand, for a generic π, it is common knowledge that there is a one-to-one correspondence between payoff types and beliefs. Myerson (1981) characterizes the optimal auction in the independent case and Crémer and McLean (1988) in the other case. Which of these cases holds makes a big difference for the structure and welfare properties of the optimal auction. The spirit of the Wilson Doctrine is to avoid making such assumptions.

To implement the Wilson Doctrine, the common approach is to maintain the naive type space, but try to diminish its adverse effect by imposing stronger solution concepts. To provide foundations for this methodology, we have to return to the fundamentals. Formally, weaker assumptions about the agents' beliefs are captured by larger type spaces. Indeed, we can remove these assumptions altogether by allowing for every conceivable hierarchy of higher-order beliefs. By the results of Mertens and Zamir (1985), there exists a universal type space, $\Omega^* = (\Omega_i^*, f_i^*, g_i^*)_{i \in \mathcal{I}}$, with the property that, for every payoff type v_i and every infinite hierarchy of beliefs $\hat{h}_i$, there is a type $\omega_i \in \Omega_i^*$ of agent i with payoff type v_i and whose hierarchy is $\hat{h}_i$. Moreover, each Ω_i^* is a compact topological space.[5]

When we start with the universal type space, we remove any implicit assumptions about the agents' beliefs. We can now explicitly model any such assumption as a probability distribution over the agents' universal types. Specifically, an assumption for the mechanism designer is a distribution μ over Ω^*.

1.2.3 Mechanisms

A mechanism consists of a set of messages M_i for each agent i, a decision rule $p : M \to \Delta\mathcal{K}$ and payment functions $t_i : M \to \mathbb{R}$. Each agent i selects a message from M_i. Based on the resulting profile of messages m, the decision rule p specifies the outcome from $\Delta\mathcal{K}$ (lotteries are allowed) and the payment

[5] Also see Heifetz and Neeman (2006).

function t_i specifies the transfer from agent i to the mechanism designer. Agent i obtains utility $p \cdot v_i - t_i$. We write p^k for the probability that alternative k is chosen.

The mechanism defines a game form, which together with the type space constitutes a game of incomplete information. The mechanism design problem is to fix a solution concept and search for the mechanism that delivers the maximum expected revenue for the mechanism designer in some outcome consistent with the solution concept. To implement the Wilson Doctrine and minimize the role of assumptions built into the naive type space, the common approach is to adopt a strong solution concept which does not rely on these assumptions. In practice, the solution concept that is often used for this purpose is dominant-strategy equilibrium. The revelation principle holds, and we can restrict attention to direct mechanisms.

Definition 1.1. *A direct-relevation mechanism Γ for type space Ω is dominant-strategy incentive compatible (dsIC) if for each agent i and type profile $\omega \in \Omega$,*

$$\begin{aligned} p(\omega) \cdot f_i(\omega_i) - t_i(\omega) &\geq 0, \text{ and} \\ p(\omega) \cdot f_i(\omega_i) - t_i(\omega) &\geq p(\omega_i', \omega_{-i}) \cdot f_i(\omega_i) - t_i(\omega_i', \omega_{-i}), \end{aligned}$$

for any alternative type $\omega_i' \in \Omega_i$.

Definition 1.2. *A dominant-strategy mechanism is a dsIC direct-revelation* *mechanism for the naive type space Ω^π. We denote by Φ the class of all dominant-strategy mechanisms.*

To provide a foundation for using dominant-strategy mechanisms, we shall compare it to the route of completely eliminating common knowledge assumptions about beliefs. We maintain the standard solution concept of Bayesian equilibrium, but now we enlarge the type space all the way to the universal type space. By the revelation principle, we restrict attention to direct mechanisms.

Definition 1.3. *A direct-revelation mechanism Γ for type space $\Omega = (\Omega_i, f_i, g_i)$ is Bayesian incentive compatible (BIC) if for each agent i and type $\omega_i \in \Omega_i$,*

$$\begin{aligned}
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega))\, g_i(\omega_i) d\omega_{-i} &\geq 0, \text{ and} \\
\int_{\Omega_{-i}} (p(\omega) \cdot f_i(\omega_i) - t_i(\omega))\, g_i(\omega_i) d\omega_{-i} &\geq \int_{\Omega_{-i}} (p(\omega'_i, \omega_{-i}) \cdot f_i(\omega_i) - t_i(\omega'_i, \omega_{-i}))\, g_i(\omega_i) d\omega_{-i}
\end{aligned}$$

for any alternative type $\omega'_i \in \Omega_i$.

A mechanism, which does not rely on implicit assumptions about higher-order beliefs, should be incentive compatible for all belief hierarchies. In other words, it should be BIC relative to the universal type space.

Definition 1.4. *Let Ψ be the class of all BIC direct-revelation mechanism for the universal type space. We say that such a mechanism is detail free.*

For simplicity of exposition, we add a dummy type v_0 for each agent $i \in \mathcal{I}$ and set $p(v_0, v_{-i}) \cdot v_i = t_i(v_0, v_{-i}) = 0$ for all $v_i \in V_i, v_{-i} \in V_{-i}$.

1.2.4 The mechanism designer as a maxmin decision maker

The mechanism designer has an estimate of the distribution of the agents' payoff types, π. Following Chung and Ely (2007), we assume that π has full support. An assumption μ about the distribution of the payoff types and beliefs of the agents is consistent with this estimate if the induced marginal distribution on V is π. Let $\mathcal{M}(\pi)$ denote the compact subset of such assumptions. For any mechanism Γ, the μ-expected revenue of Γ is

$$R_\mu(\Gamma) = \int_{\Omega^*} \sum_{i \in \mathcal{I}} t_i(\omega) d\mu(\omega).$$

We do not assume that the mechanism designer has confidence in the naive type space as his model of agents' beliefs. Rather he considers other assumptions within the set $\mathcal{M}(\pi)$ as possible as well. The mechanism designer who chooses a mechanism that maximizes the worst-case performance solves the maxmin problem of

$$\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\pi)} R_\mu(\Gamma).$$

If the mechanism designer used a dominant-strategy mechanism, then his maximum revenue would be

$$\Pi^D(\pi) = \sup_{\Gamma \in \Phi} R_\pi(\Gamma),$$

where

$$R_\pi(\Gamma) = \sum_{v \in V} \pi(v) \sum_{i \in \mathcal{I}} t_i(v)$$

for any dominant-strategy mechanism $\Gamma \in \Phi$.

Definition 1.5. *The use of dominant-strategy mechanisms has a maxmin foundation if*

$$\Pi^D(\pi) = \sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\pi)} R_\mu(\Gamma).$$

The use of dominant-strategy mechanisms has a Bayesian foundation if for some belief $\mu^* \in \mathcal{M}(\pi)$*,*

$$\Pi^D(\pi) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma).$$

The Bayesian foundation is a stronger notion than the maxmin foundation. The Bayesian foundation says that there exists an assumption about (the distribution of) agents' beliefs, against which the optimal dominant-strategy mechanism achieves the highest expected revenue among all detail-free mechanisms. It follows that the worse case expected revenue of an arbitrary detail-free mechanism cannot exceed its expected revenue against this particular assumption, which in turn cannot exceed the worst-case expected revenue of the optimal dominant-strategy mechanism. We record this observation as the following proposition.[6]

[6] Also see Chung and Ely (2007, Section 2.5).

Proposition 1.1. *If the use of dominant-strategy mechanisms has a Bayesian foundation, then it has a maxmin foundation.*

1.3 Motivating examples

Before we present the results, it is instructive to contrast two bilateral trade models. In the standard bilateral trade model (see Myerson and Satterthwaite (1983)), whether an agent is the buyer or the seller is exogenously given. Either the seller sells some units to the buyer or no trade occurs. In the bilateral trade model with ex ante unidentified traders (see Cramton, Gibbons, and Klemperer (1987) and Lu and Robert (2001)), each agent may be either the buyer or the seller, depending on the realization of the privately observed information and the choice of the mechanism: the agent's role as the buyer or the seller is endogenously determined by her report and cannot be identified prior to trade. The mechanism designer chooses a mechanism that maximizes the expected profit in both models.

Section 1.3.1 presents the basics shared by both models. Section 1.3.2 studies the standard bilateral trade model. In this case, the use of dominant-strategy mechanisms has maxmin and Bayesian foundations. Section 1.3.3 studies the bilateral trade model with ex ante unidentified traders. We show that there is neither a Bayesian foundation nor a maxmin foundation.

1.3.1 Setup

Consider a broker who chooses trading mechanisms that maximize the expected profit; see for example, Myerson and Satterthwaite (1983, Section 5), Lu and Robert (2001) and Börgers (2015). Each agent is endowed with $\frac{1}{2}$ unit of a good to be traded and has private information about her valuation for the good. Agent 1's valuation for the good could be either 18 or 38. Agent 2's valuation for the good could be either 10 or 30. The broker has the following estimate of the distribution of the agents' valuations:

	$v_1 = 18$	$v_1 = 38$
$v_2 = 10$	$\frac{3}{8}$	$\frac{1}{8}$
$v_2 = 30$	$\frac{1}{8}$	$\frac{3}{8}$

(1.1)

1.3.2 Standard bilateral trade

In the standard bilateral trade model, agent 1 is the buyer and agent 2 is the seller. The trading mechanism is characterized by three outcome functions (p, t_1, t_2), where $p(v_1, v_2)$ is the expected trading amount, $t_1(v_1, v_2)$ is the expected payment from agent 1 to the broker and $t_2(v_1, v_2)$ is the expected payment from agent 2 to the broker, if v_1 and v_2 are the reported valuations of agent 1 and agent 2. Agent 1's utility from purchasing p units of the good and paying a transfer t_1 is $pv_1 - t_1$ and agent 2's utility from selling p unit of the good and paying a transfer t_2 is $-pv_2 - t_2$, where $0 \le p \le \frac{1}{2}$.

Clearly, this model belongs to the class of environments with linear utilities and one-dimensional payoff types; see Section 1.5.1. Following Corollary 1.1, the use of dominant-strategy mechanisms has maxmin and Bayesian foundations.

1.3.3 Bilateral trade with ex ante unidentified traders

In this section, we study the bilateral trade model with ex ante unidentified traders. Each agent may be either the buyer or the seller. The trading mechanism is characterized by three outcomes functions (p, t_B, t_S), where $p(v_1, v_2)$ is the expected trading amount, $t_B(v_1, v_2)$ is the expected payment from *the buyer* to the broker and $t_S(v_1, v_2)$ is the expected payment from *the seller* to the broker, if v_1 and v_2 are the reported valuations of agent 1 and agent 2. The buyer's utility from purchasing p units of the good and paying a transfer t_B is $pv_B - t_B$ and the seller's utility from selling p unit of the good and paying a transfer t_S is $-pv_S - t_S$, where $0 \leq p \leq \frac{1}{2}$.

In the context of this economic environment, this example illustrates that, maxmin/ Bayesian foundations might not exist. Section 1.3.3 calculates the maximum expected revenue that could be achieved by a dominant-strategy mechanism, and Section 1.3.3 explicitly constructs a single Bayesian mechanism that achieves a strictly higher expected revenue, regardless of the assumption about (the distribution of) the agents' beliefs. It should be obvious from the exposition below that this example is robust to small perturbations in the agents' valuations or the broker's estimate of the distribution of the payoff types.

Optimal dominant-strategy mechanism

Using a linear programming solver, we have the optimal dominant-strategy mechanism Γ as follows, where the first number in each cell indicates the amount of good agent 1 buys from agent 2, the second number is the transfer from agent 1 and the third number is the transfer from agent 2. The maximum expected revenue the mechanism designer can generate from a dominant-strategy mechanism is 3.

	$v_1 = 18$	$v_1 = 38$
$v_2 = 10$	$\frac{1}{2}, 9, -5$	$\frac{1}{2}, 9, -15$
$v_2 = 30$	$-\frac{1}{2}, -9, 15$	$\frac{1}{2}, 19, -15$

(1.2)

Neither a Bayesian foundation nor a maxmin foundation

To show that there is no maxmin foundation, it suffices to construct a single Bayesian mechanism and achieve a strictly higher expected revenue than he does using the optimal dominant-strategy mechanism, regardless of the assumption about (the distribution of) the agents' beliefs. Since Bayesian foundation is a stronger notion than maxmin foundation, this further implies that there is no Bayesian foundation.

The construction of the mechanism Γ' follows immediately from Theorem 1.2. We shall save the arguments in Section 1.4. Following Chung and Ely (2007), we use a to denote the first-order belief of a low-valuation type of agent 2 that agent 1 has low valuation. In this mechanism, the mechanism designer elicits agent 2's first-order belief about agent 1's valuation. To see that Γ' is expected revenue improving, note that Γ' achieves revenue of at least 4 everywhere and hence the expected revenue is at least 4, regardless of the agents' beliefs.

	$v_1 = 18$	$v_1 = 38$
$a \in [0, \frac{1}{2})$	$-\frac{1}{2}, -9, 15$	$\frac{1}{2}, 19, -15$
$a \in [\frac{1}{2}, 1]$	$\frac{1}{2}, 9, -5$	$\frac{1}{2}, 9, -5$
$v_2 = 30$	$-\frac{1}{2}, -9, 15$	$\frac{1}{2}, 19, -15$

1.4 Results

We can formulate the optimal dominant-strategy mechanism design problem as follows:

$$\max_{p^k(\cdot)\geq 0, t_i(\cdot)} \sum_{v\in V} \pi(v) \sum_{i\in\mathcal{I}} t_i(v) \qquad (DIC-P)$$

subject to $\forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\}, \forall v_{-i} \in V_{-i}$,

$$p(v_i, v_{-i}) \cdot v_i - t_i(v_i, v_{-i}) \geq p(v_i', v_{-i}) \cdot v_i - t_i(v_i', v_{-i}), \tag{1.3}$$

$$\forall v \in V, \sum_{k\in\mathcal{K}} p^k(v) = 1. \tag{1.4}$$

By compactness arguments, the maximization problem $(DIC - P)$ has a finite optimal value. Denote by V_{DIC-P} the value of the objective function of the program $(DIC - P)$ at an optimum.

Say that a decision rule p is dsIC if there exists transfer scheme t such that the mechanism (p, t) satisfies the incentive constraints (1.3). We omit the proof of the following standard lemma, due to Rochet (1987).

Lemma 1.1. *A necessary and sufficient condition for a decision rule p to be dsIC is the following cyclical monotonicity condition: $\forall i \in \mathcal{I}, \forall v_{-i} \in V_{-i}$ and every sequence of payoff types of agent $i, (v_{i,1}, v_{i,2}, ..., v_{i,k})$ with $v_{i,k} = v_{i,1}$, we have*

$$\sum_{\kappa=1}^{k-1} [p(v_{i,\kappa}, v_{-i}) \cdot v_{i,\kappa+1} - p(v_{i,\kappa}, v_{-i}) \cdot v_{i,\kappa}] \leq 0. \tag{1.5}$$

1.4.1 Uniform shortest-path tree

We first collect some graph-theoretic terminology used in the sequal.

Definition 1.6. *Fix a decision rule p that is dsIC and other agents' reports v_{-i}.*[7] *(1) The set of nodes for agent i is $V_i \cup \{v_0\}$; (2) For any $v_i \in V_i$ and $v_i' \in V_i \backslash \{v_i\} \cup \{v_0\}, v_i' \to v_i$ is a directed edge with length $p(v_i, v_{-i}) \cdot v_i - p(v_i', v_{-i}) \cdot v_i$; and (3) A path from the dummy type v_0 to payoff type $v_{i,k} \in V_i$ is a sequence $P = (v_0, v_{i,1}, v_{i,2}, ..., v_{i,k})$ where (i) $v_{i,j} \in V_i, \forall j = 1, 2, ..., k$; (ii) $v_0 \to v_{i,1}$; (iii)*

[7] In the remainder of this section, whenever we fix a decision rule p, we mean a decision rule p that is dsIC.

$v_{i,j-1} \to v_{i,j}, \forall j = 2, ..., k$ *and (iv)* $j \neq j' \Longrightarrow v_{i,j} \neq v_{i,j'}$.

To understand the maximization problem $(DIC - P)$ and in particular the associated incentive constraints (1.3), it helps to flip to its dual. The dual is a network flow problem that can be described in the following way. Fix a decision rule p and other agents' reports v_{-i}. Introduce one node for each type $v_i \in V_i \cup \{v_0\}$ (the node corresponding to the dummy type v_0 will be the source) and to each directed edge $v_i' \to v_i$, assign a length of $p(v_i, v_{-i}) \cdot v_i - p(v_i', v_{-i}) \cdot v_i$. The optimization problem reduces to determining the shortest-path tree (the union of all shortest-paths from the source to all nodes) in this network. Edges on the shortest-path tree correspond to binding dominant-strategy incentive constraints. Readers unfamiliar with network flows may consult Ahuja, Magnanti, and Orlin (1993) and Vohra (2011).

Definition 1.7. *Fix a decision rule* p *and other agents' reports* v_{-i}*. A shortest-path tree is the union of all shortest-paths from the source to all nodes.*

Note that if v_i' belongs to the shortest-path from the source v_0 to some $v_i \in V_i$, the truncation of the path from v_0 to v_i' defines the shortest-path from v_0 to v_i'.

Definition 1.8. *There is uniform shortest-path tree if for each agent* $i \in \mathcal{I}$*, there is the same shortest-path tree for all decision rules* p *and other agents'* *reports* v_{-i}.

When the uniform shortest-path tree condition is satisfied, we drop the dependence on p, v_{-i}. Uniform shortest-path tree induces an order on the agents' payoff types. For a typical shortest-path $(v_0, v_{i,1}, v_{i,2}, ..., v_{i,k})$ of the shortest-path tree, we write $v_{i,k} \succ_i v_{i,k-1} \succ_i \ldots \succ_i v_{i,1} \succ_i v_0$. It is convenient to represent the uniform shortest-path tree of agent i using $\succ_i$ and its transitive closure by $\succ_i^+$. For notational convenience, write $v_i' \succeq_i^+ v_i$ if $v_i' \succ_i^+ v_i$ or $v_i' = v_i$. If $v_i \succ_i v_i'$, we sometimes write $v_i^- = v_i'$.

With the uniform shortest-path tree, the rent of any payoff type can be easily calculated and all incentive constraints can be replaced by the cyclical monotonicity constraints on the decision rule. We record this as the following proposition.

Proposition 1.2. *With the uniform shortest-path tree $\succ_i$, the maximization problem $(DIC-P)$ is equivalent to*

$$\max_{p(\cdot)} \sum_{i \in \mathcal{I}} \sum_{v_i \in V_i} \sum_{v_{-i} \in V_{-i}} \pi(v_i, v_{-i}) \left[p(v_i, v_{-i}) \cdot v_i - \sum_{v_i' \in V_i : v_i \succeq_i^+ v_i'} p((v_i')^-, v_{-i}) \cdot (v_i' - (v_i')^-) \right], \tag{1.6}$$

subject to $p(\cdot)$ satisfies the cyclical monotonicity constraint (1.5).

Proof. With the uniform shortest-path tree $\succ_i$, for any p and v_{-i}, the rent of payoff type v_i of agent i can be calculated as follows:

$$\begin{aligned} U_i(v_i, v_{-i}) &= p(v_i, v_{-i}) \cdot v_i - t_i(v_i, v_{-i}) \\ &= p(v_i^-, v_{-i}) \cdot v_i - t_i(v_i^-, v_{-i}) \\ &= p(v_i^-, v_{-i}) \cdot v_i^- - t_i(v_i^-, v_{-i}) + p(v_i^-, v_{-i}) \cdot (v_i - v_i^-) \\ &= U_i(v_i^-, v_{-i}) + p(v_i^-, v_{-i}) \cdot (v_i - v_i^-). \end{aligned}$$

By induction,

$$U_i(v_i, v_{-i}) = \sum_{v_i' \in V_i : v_i \succeq_i^+ v_i'} p((v_i')^-, v_{-i}) \cdot (v_i' - (v_i')^-).$$

Therefore,

$$\begin{aligned} t_i(v_i, v_{-i}) &= p(v_i, v_{-i}) \cdot v_i - U_i(v_i, v_{-i}) \\ &= p(v_i, v_{-i}) \cdot v_i - \sum_{v_i' \in V_i : v_i \succeq_i^+ v_i'} p((v_i')^-, v_{-i}) \cdot (v_i' - (v_i')^-). \end{aligned}$$

The maximization problem $(DIC - P)$ is equivalent to

$$\max_{p(\cdot)} \sum_{i \in \mathcal{I}} \sum_{v_i \in V_i} \sum_{v_{-i} \in V_{-i}} \pi(v_i, v_{-i}) \left[p(v_i, v_{-i}) \cdot v_i - \sum_{v_i' \in V_i : v_i \succeq_i^+ v_i'} p((v_i')^-, v_{-i}) \cdot (v_i' - (v_i')^-) \right].$$

By Lemma 1.1, $p(\cdot)$ is subject to the cyclical monotonicity constraint (1.5). □

Definition 1.9. *Say π is regular if the cyclical monotonicity constraint (1.5) is automatically satisfied for p^* that maximizes the reduced objective function (1.6).*

To the best of our knowledge, there is no formal definition of regularity in the general environments. Our definition of regularity captures how it has been used in the literature; see for example, Myerson (1981).[8] In the applications we study in Section 1.5.1 and Section 1.5.2, additional structure is imposed and we provide primitive condition for regularity.

1.4.2 Foundations of dominant-strategy mechanisms

Theorem 1.1. *In environments in which the uniform shortest-path tree condition holds, if π is regular, then the use of dominant-strategy mechanisms has maxmin and Bayesian foundations.*

Proof. The structure of the proof is as follows. Step 1) considers the optimal dominant-strategy mechanism design problem $(DIC - P)$ and derives its dual $(DIC - D)$. Step 2) restricts attention to a subclass of type spaces, formulates the Bayesian mechanism design problem $(BIC - P)$ and derives its dual $(BIC - D)$. Denote by V_{DIC-D} (resp. V_{BIC-P} and V_{BIC-D}) the value of the objective function of the program $(DIC - D)$ (resp. $(BIC - P)$ and $(BIC - D)$) at an optimum. Step 3) then explicitly constructs an assumption about (the distribution of) the agents' beliefs, against which we show in Step

[8] That is, we first ask which decision rule p the mechanism designer would choose if she does not have to make sure that the decision rule p satisfies the cyclical monotonicity constraint. The regularity condition is then imposed to make sure that such optimal decision rule p automatically satisfies the cyclical monotonicity constraint.

4) that, $V_{DIC-D} \geq V_{BIC-D}$. It follows from the duality theorem in linear programming (see for example, Bradley, Hax, and Magnanti (1977, Chapter 4)) that $V_{DIC-P} = V_{DIC-D} \geq V_{BIC-D} \geq V_{BIC-P}$.

Step 1) First consider the optimal dominant-strategy mechanism design problem $(DIC-P)$. We derive its dual $(DIC-D)$, where $\lambda^{DIC}(v_i'; v_i, v_{-i})$ is the multiplier associated with the incentive constraint (1.3) and $\mu^{DIC}(v)$ is the multiplier associated with the feasibility constraint (1.4).

$$\min_{\lambda^{DIC}(v_i';v_i,v_{-i}),\mu^{DIC}(v)} \sum_{v\in V} \mu^{DIC}(v) \qquad (DIC-D)$$

$$\text{subject to } \forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_{-i} \in V_{-i},$$

$$\sum_{v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\}} \lambda^{DIC}(v_i'; v_i, v_{-i}) - \sum_{v_i' \in V_i \backslash \{v_i\}} \lambda^{DIC}(v_i; v_i', v_{-i}) = \pi(v_i, v_{-i}), \quad (1.7)$$

$$\forall v \in V, \forall k \in \mathcal{K},$$

$$\pi(v) \sum_{i\in\mathcal{I}} v_i(k) + \sum_{i\in\mathcal{I}} \sum_{v_i' \in V_i \backslash \{v_i\}} \lambda^{DIC}(v_i; v_i', v_{-i})(v_i(k) - v_i'(k)) \leq \mu^{DIC}(v), \quad (1.8)$$

$$\forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\}, \forall v_{-i} \in V_{-i},$$

$$\lambda^{DIC}(v_i'; v_i, v_{-i}) \geq 0. \quad (1.9)$$

As $\lambda^{DIC}(v_i'; v_i, v_{-i})$ is the multiplier for the incentive constraint (1.3), by the uniform shortest-path tree and regularity, there is a dual optimum satisfying

$$\lambda^{DIC}(v_i'; v_i, v_{-i}) > 0 \text{ only if } v_i \succ_i v_i',$$

and (1.7) simplifies to

$$\lambda^{DIC}(v_i^-; v_i, v_{-i}) - \sum_{v_i': v_i' \succ_i v_i} \lambda^{DIC}(v_i; v_i', v_{-i}) = \pi(v_i, v_{-i}).$$

By induction,

$$\lambda^{DIC}(v_i'; v_i, v_{-i}) = \begin{cases} \sum_{\hat{v}_i: \hat{v}_i \succeq_i^+ v_i} \pi(\hat{v}_i, v_{-i}); & \text{if } v_i \succ_i v_i'; \\ 0; & \text{otherwise.} \end{cases} \tag{1.10}$$

Step 2) Say that a type space is *simple* if for each agent $i \in \mathcal{I}$ and payoff type $v_i \in V_i$, there is a unique type for agent i with valuation v_i. Let the set of types for agent i be equal to the set of possible valuations, i.e. $\Omega_i = V_i$. We take f_i to be the identity, and for notational ease, we will write $\tau_i(\cdot|v_i) = g_i(v_i)$ for the belief of type v_i of agent i about the types of the other agents. From now on, we restrict attention to such type spaces.

We can formulate the optimal Bayesian mechanism design problem as follows.

$$\max_{p^k(\cdot) \geq 0, t_i(\cdot)} \sum_{v \in V} \pi(v) \sum_{i \in \mathcal{I}} t_i(v) \tag{$BIC-P$}$$

$$\text{subject to } \forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\},$$

$$\sum_{v_{-i} \in V_{-i}} \tau_i(v_{-i}|v_i)(p(v_i, v_{-i}) \cdot v_i - t_i(v_i, v_{-i}))$$

$$\geq \sum_{v_{-i} \in V_{-i}} \tau_i(v_{-i}|v_i)(p(v_i', v_{-i}) \cdot v_i - t_i(v_i', v_{-i})), \tag{1.11}$$

$$\forall v \in V, \sum_{k \in \mathcal{K}} p^k(v) = 1. \tag{1.12}$$

We derive the dual minimization problem $(BIC-D)$, where $\lambda^{BIC}(v_i';v_i)$ is the multiplier for the incentive constraint (1.11) and $\mu^{BIC}(v)$ is the multiplier for the feasibility constraint (1.12).

$$\min_{\lambda^{BIC}(v_i';v_i),\mu^{BIC}(v)} \sum_{v\in V} \mu^{BIC}(v) \qquad (BIC-D)$$

$$\text{subject to } \forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_{-i} \in V_{-i},$$

$$\sum_{v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\}} \lambda^{BIC}(v_i';v_i)\tau_i(v_{-i}|v_i) - \sum_{v_i' \in V_i \backslash \{v_i\}} \lambda^{BIC}(v_i;v_i')\tau_i(v_{-i}|v_i') = \pi(v_i, v_{-i}), \tag{1.13}$$

$$\forall v \in V, \forall k \in \mathcal{K},$$

$$\pi(v)\sum_{i\in\mathcal{I}} v_i(k) + \sum_{i\in\mathcal{I}} \sum_{v_i' \in V_i \backslash \{v_i\}} \lambda^{BIC}(v_i;v_i')\tau_i(v_{-i}|v_i')(v_i(k) - v_i'(k)) \le \mu^{BIC}(v), \tag{1.14}$$

$$\forall i \in \mathcal{I}, \forall v_i \in V_i, \forall v_i' \in \{V_i \backslash \{v_i\}\} \cup \{v_0\},$$

$$\lambda^{BIC}(v_i';v_i) \ge 0. \tag{1.15}$$

Step 3) Now we construct a particular assumption about (the distribution of) agents' beliefs. Given π, for any $v_i \in V_i$, write

$$\pi_i(v_i) = \sum_{v_{-i}\in V_{-i}} \pi(v_i, v_{-i})$$

for the marginal probability of v_i and write

$$G_i(v_i) = \sum_{\hat{v}_i : \hat{v}_i \succeq_i^+ v_i} \pi_i(\hat{v}_i) \tag{1.16}$$

for the associated distribution function. We define agent i's beliefs as follows:

$$\tau_i(v_{-i}|v_i) = \frac{1}{G_i(v_i)} \sum_{\hat{v}_i : \hat{v}_i \succeq_i^+ v_i} \pi(\hat{v}_i, v_{-i}). \tag{1.17}$$

Step 4) Fix any feasible dual variables $\lambda^{DIC}(v_i'; v_i, v_{-i})$ and $\mu^{DIC}(v)$ of the minimization problem $(DIC - D)$ that satisfy (1.10), let

$$\lambda^{BIC}(v_i'; v_i) = \sum_{v_{-i} \in V_{-i}} \lambda^{DIC}(v_i'; v_i, v_{-i})$$
$$\text{and } \mu^{BIC}(v) = \mu^{DIC}(v).$$

If $v_i \succ_i v_i'$, we have

$$\begin{aligned}
\lambda^{BIC}(v_i'; v_i)\tau_i(v_{-i}|v_i) &= \left[\sum_{v_{-i} \in V_{-i}} \lambda^{DIC}(v_i'; v_i, v_{-i})\right] \tau_i(v_{-i}|v_i) \\
&= \left[\sum_{v_{-i} \in V_{-i}} \sum_{\hat{v}_i : \hat{v}_i \succeq_i^+ v_i} \pi(\hat{v}_i, v_{-i})\right] \tau_i(v_{-i}|v_i) \\
&= G_i(v_i)\tau_i(v_{-i}|v_i) \\
&= \sum_{\hat{v}_i : \hat{v}_i \succeq_i^+ v_i} \pi(\hat{v}_i, v_{-i}),
\end{aligned}$$

where the second equality follows from (1.10), the third equality follows from (1.16), and the last equality follows from (1.17). Otherwise,

$$\lambda^{BIC}(v_i'; v_i)\tau_i(v_{-i}|v_i) = 0.$$

In either case, we have

$$\lambda^{BIC}(v_i'; v_i)\tau_i(v_{-i}|v_i) = \lambda^{DIC}(v_i'; v_i, v_{-i}). \tag{1.18}$$

We now show that the dual variables $\lambda^{BIC}(v_i'; v_i)$ and $\mu^{BIC}(v)$ are feasible under the minimization problem $(BIC - D)$. (1.15) are trivially satisfied. It follows from (1.18) that (1.13) reduces to (1.7), and (1.14) reduces to (1.8).

Since $\lambda^{DIC}(v_i'; v_i, v_{-i})$ and $\mu^{DIC}(v)$ are feasible under the problem $(DIC - D)$, $\lambda^{BIC}(v_i'; v_i)$ and $\mu^{BIC}(v)$ are feasible under the minimization problem $(BIC - D)$. Furthermore, the value of the objective function of the minimization problem $(BIC - D)$ is $\sum_{v \in V} \mu^{BIC}(v) = \sum_{v \in V} \mu^{DIC}(v)$. We conclude that $V_{DIC-D} \geq V_{BIC-D}$. □

1.4.3 No foundations of dominant-strategy mechanisms

This subsection considers violations of the uniform shortest-path tree condition. When the uniform shortest-path tree condition is not satisfied, as illustrated in the bilateral trade model with ex ante unidentified traders (Section 1.3.3), maxmin/ Bayesian foundations might not exist. In particular, we explicitly construct a single Bayesian mechanism that does strictly better than the optimal dominant-strategy mechanism, regardless of the assumption about (the distribution of) the agents' beliefs.

In environments where the uniform shortest path is violated, it is difficult to find the optimal dominant-strategy mechanisms, not to mention the construction of the superior Bayesian mechanism. To have a meaningful discussion, we shall take the optimal dominant-strategy mechanisms (the binding structure, and payments of the agents) as primitives. While the conditions of the theorem may be restrictive, the conditions can be verified whenever the optimal dominant-strategy mechanism can be solved (possibly by a linear programming solver). In addition to bilateral trade with ex ante unidentified traders, the result can also be applied to auction with type-dependent outside option (Section 1.5.3).

Theorem 1.2. *In environments with two agents and binary payoff types for each agent, for the optimal dominant-strategy mechanism, if*

	v_1	v_1'
v_2	$p(v_1, v_2), t_1(v_1, v_2), t_2(v_1, v_2)$	$p(v_1, v_2), t_1(v_1', v_2), t_2(v_1', v_2)$
v_2'	$p(v_1, v_2'), t_1(v_1, v_2'), t_2(v_1, v_2')$	$p(v_1, v_2), t_1(v_1', v_2'), t_2(v_1', v_2')$

1) binding structure:

$$p(v_1, v_2') \cdot v_2 - t_2(v_1, v_2') < 0,$$

$$\textit{and } p(v_1', v_2') \cdot v_2 - t_2(v_1', v_2') > 0;$$

2) payment dominance:

$$t_1(v_1, v_2') + t_2(v_1, v_2') \geq t_1(v_1, v_2) + t_2(v_1, v_2),$$

$$\textit{and } t_1(v_1', v_2') + t_2(v_1', v_2') > t_1(v_1', v_2) + t_2(v_1', v_2),$$

then there is neither a Bayesian foundation nor a maxmin foundation.

Remark 1.1. *For ease of exposition, we state Theorem 1.2 in environments with two agents and binary payoff types for each agent. The argument extends* *to environments with multiple agents and each agent has multiple payoff types, as long as there are two agents and two payoff types for each agent, where the structure as stated in Theorem 1.2 exists.*

Proof. Let

$$x = p(v_1, v_2) \cdot v_2 - t_2(v_1, v_2);$$

$$y = p(v_1', v_2) \cdot v_2 - t_2(v_1', v_2);$$

$$z = p(v_1, v_2') \cdot v_2 - t_2(v_1, v_2') < 0;$$

$$w = p(v_1', v_2') \cdot v_2 - t_2(v_1', v_2') > 0.$$

Since the optimal dominant-strategy mechanism necessarily satisfy the incentive constraints, we have $x \geq 0, y \geq w > 0$.[9]

We show that there is no maxmin foundation. That is, the mechanism designer could employ a single Bayesian mechanism and achieve a strictly higher expected revenue than he does using the optimal dominant-strategy mechanism, regardless of the agents' beliefs. To do this, we first explicitly identify one such mechanism and proceed by verifying i) the mechanism is BIC for the universal type space; and ii) this mechanism achieves a strictly higher expected revenue regardless of the agents' beliefs. Since the Bayesian foundation is a

[9] As a matter of fact, it must be that $x = 0$, and $y = w$. Otherwise, the dominant-strategy mechanism would not have been optimal. Note that the uniform shortest-path tree condition is violated.

stronger notion than the maxmin foundation, this further implies that there is no Bayesian foundation.

We use a to denote the first-order belief of payoff type v_2 of agent 2 that agent 1 has payoff type v_1. In this mechanism, the mechanism designer elicits agent 2's first-order belief about agent 1's payoff type. Consider the following Bayesian mechanism Γ':

	v_1	v_1'
$v_2, a \in [0, \frac{w}{w-z})$	$p(v_1, v_2'), t_1(v_1, v_2'), t_2(v_1, v_2')$	$p(v_1', v_2'), t_1(v_1', v_2'), t_2(v_1', v_2')$
$v_2, a \in [\frac{w}{w-z}, 1]$	$p(v_1, v_2), t_1(v_1, v_2), t_2(v_1, v_2) + x$	$p(v_1', v_2), t_1(v_1', v_2), t_2(v_1', v_2) + y$
v_2'	$p(v_1, v_2'), t_1(v_1, v_2'), t_2(v_1, v_2')$	$p(v_1', v_2'), t_1(v_1', v_2'), t_2(v_1', v_2')$

To see that Γ' is BIC for the universal type space, note that

i truth telling continues to be a dominant strategy for agent 1;

ii truth telling continues to be a dominant strategy for payoff type v_2' of agent 2;

iii $a \in [0, \frac{w}{w-z})$ will not announce v_2' as utility is unchanged;

iv $a \in [\frac{w}{w-z}, 1]$ will not announce v_2' as expected utility is lower; and

v between $a \in [0, \frac{w}{w-z})$ and $a \in [\frac{w}{w-z}, 1]$, payoff type v_2 of agent 2 will announce $a \in [\frac{w}{w-z}, 1]$ if and only if $a \in [\frac{w}{w-z}, 1]$.

To see that the mechanism achieves a strictly higher expected revenue than the optimal dominant-strategy mechanism, regardless of the assumption about (the distribution of) the agents' belief, note that

vi $t_1(v_1, v_2') + t_2(v_1, v_2') \geq t_1(v_1, v_2) + t_2(v_1, v_2)$;

vii $t_1(v_1', v_2') + t_2(v_1', v_2') > t_1(v_1', v_2) + t_2(v_1', v_2)$;

viii $x \geq 0$ and $y \geq w > 0$.

□

1.5 Applications

This section is devoted to the applications of the results. The uniform shortest-path tree condition holds in the standard social choice environment with linear utilities and one-dimensional payoff types as well as some multi-dimensional environments. Section 1.5.1 applies our result to environments with linear utilities and one-dimensional types, and Section 1.5.2 considers a multi-dimensional environment. For both applications, we provide primitive conditions for regularity. As we illustrated in Section 1.3, Theorem 1.2 can be applied to bilateral trade with ex ante unidentified traders. Section 1.5.3 applies Theorem 1.2 to another environment, namely, auction with type-dependent outside option.

1.5.1 Linear utilities and one-dimensional payoff types

In this subsection, we consider the standard social choice environment with linear utilities and one-dimensional payoff types.[10] This fits many classical

[10]This set-up covers the environment studied in Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013, Section 2).

applications of mechanism design, including single-unit auction (e.g., Myerson (1981)), public good (e.g., Mailath and Postlewaite (1990)) and standard bilateral trade (e.g., Myerson and Satterthwaite (1983)).

There is a finite set $\mathcal{I} = \{1, 2, ..., I\}$ of risk neutral agents and a finite set $\mathcal{K} = \{1, 2, ..., K\}$ of social alternatives. Agent i's gross utility in alternative k equals $u_i^k(v_i) = a_i^k v_i$, where $v_i \in \mathbb{R}$ is agent i's payoff type, $a_i^k \in \mathbb{R}$ are constants and $a_i^k \geq 0$ for all k. Agent i obtains utility

$$p(v) \cdot A_i v_i - t_i(v)$$

for decision rule $p \in \Delta\mathcal{K}$ and transfer t_i, where $A_i = (a_i^1, a_i^2, ..., a_i^K)$. For notational simplicity, we assume that each agent has M possible payoff types and that the set V_i is the same for each agent: $V_i = \{v^1, v^2, ..., v^M\}$, where $v^m - v^{m-1} = \gamma$ for each m and some $\gamma > 0$.

We can formulate the optimal dominant-strategy mechanism design problem as follows:

$$\max_{p(\cdot),t(\cdot)} \sum_{v \in V} \pi(v) \sum_{i \in \mathcal{I}} t_i(v)$$

subject to $\forall i \in \mathcal{I}, \forall m, l = 1, 2, ..., M, \forall v_{-i} \in V_{-i}$,

$$p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq 0, \tag{1.19}$$

$$p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) \geq p(v^l, v_{-i}) \cdot A_i v^m - t_i(v^l, v_{-i}). \tag{1.20}$$

In the environment with linear utilities and one-dimensional payoff types, we say that a decision rule p is dsIC if there exists transfer scheme t such that the mechanism (p, t) satisfies the constraints (1.19) and (1.20).

Uniform shortest-path tree condition is naturally satisfied in such settings. In particular, for any agent $i \in \mathcal{I}$, the payoff types are completely ordered via a single path. We omit the proof of the following standard lemma.

Lemma 1.2. *Fix any decision rule p that is dsIC, the shortest path from the source v_0 to any payoff type $v^m \in V_i$ is $P = (v_0, v^1, v^2, ..., v^m)$ and*

$$t_i(v^m, v_{-i}) = p(v^m, v_{-i}) \cdot A_i v^m - \gamma \sum_{m'=1}^{m-1} p(v^{m'}, v_{-i}) \cdot A_i.$$

Next, we present the primitive condition for regularity. It is well known that an equivalent formulation of the problem is

$$\max_{p(\cdot),t(\cdot)} \sum_{v \in V} \pi(v) \sum_{i \in \mathcal{I}} t_i(v)$$
$$\text{subject to } \forall i \in \mathcal{I}, \forall m, l = 1, 2, ..., M, \forall v_{-i} \in V_{-i},$$
$$p(v^1, v_{-i}) \cdot A_i v^1 - t_i(v^1, v_{-i}) = 0,$$
$$p(v^m, v_{-i}) \cdot A_i v^m - t_i(v^m, v_{-i}) = p(v^{m-1}, v_{-i}) \cdot A_i v^m - t_i(v^{m-1}, v_{-i}),$$
$$p(v^m, v_{-i}) \cdot A_i \geq p(v^l, v_{-i}) \cdot A_i, \text{ for } m \geq l.$$

Let $F_i(v_i, v_{-i}) = \sum_{\hat{v}_i \leq v_i} \pi(\hat{v}_i, v_{-i})$ denote the cumulative distribution function of i's valuation conditional on the other agents having payoff type profile

v_{-i}. Define the virtual valuation of agent i as

$$r_i(v) = v_i - \gamma \frac{1 - F_i(v)}{\pi(v)},$$

and rewrite the objective function as

$$\begin{aligned} \sum_{v \in V} \pi(v) \sum_{i \in \mathcal{I}} t_i(v) &= \sum_{v \in V} \sum_{i \in \mathcal{I}} \pi(v) p(v) \cdot A_i r_i(v) \\ &= \sum_{v \in V} \pi(v) p(v) \cdot \sum_{i \in \mathcal{I}} A_i r_i(v). \end{aligned} \tag{1.21}$$

For each alternative k, let $K_i^{k,\inf} = \{k' \in \mathcal{K} : a_i^{k'} < a_i^k\}$. That is, $K_i^{k,\inf}$ is the collection of alternatives that agent i considers inferior to alternative k.

Definition 1.10. *We say that π is regular if the virtual valuations satisfy the following condition: for each $v \in V, j \in \mathcal{I}$,*

$$k \in \arg\max_k \sum_{i \in \mathcal{I}} a_i^k r_i(v) \Rightarrow K_j^{k,\inf} \cap \arg\max_k \sum_{i \in \mathcal{I}} a_i^k r_i(\hat{v}_j, v_{-j}) = \emptyset \tag{1.22}$$

for every $\hat{v}_j > v_j$.

We establish the foundations of dominant-strategy mechanisms in Corollary 1.1.

Corollary 1.1. *If π satisfies the regularity condition (1.22), the use of dominant-strategy mechanisms has a Bayesian/ maxmin foundation.*

Proof. By (1.21), the objective function becomes

$$\sum_{v \in V} \pi(v) p(v) \cdot \sum_{i \in \mathcal{I}} A_i r_i(v).$$

Regularity condition (1.22) ensures that for any alternative k chosen with positive probability for payoff type profile (v^l, v_{-i}), when agent i's payoff type increases say from v^l to v^m, alternatives that are inferior than alternative k from agent i's point of view will not be chosen. It must be that $p(v^m, v_{-i}) \cdot A_i \geq p(v^l, v_{-i}) \cdot A_i$, for $m \geq l$. It is well known that this is equivalent to cyclical monotonicity in environments with linear utilities and one-dimensional payoff types. The uniform shortest-path tree condition follows from Lemma 1.2. The result then follows from Theorem 1.1. □

1.5.2 Multi-unit auction with capacity-constrained bidders

In addition to environments with linear utilities and one-dimensional payoff types, the uniform shortest-path tree condition is also satisfied in some multi-dimensional environments. Solving for the optimal mechanism in a multi-dimensional environment is in general a daunting task. In this section, we examine a specific case where the multi-dimensional analysis can be simplified.

Consider the problem of finding the revenue maximizing auction when bidders have constant marginal valuations as well as capacity constraints.[11] Both the marginal values and capacity constraints are private information to the bidders. Bidder i's payoff type is represented by $v_i = (a, b)$, where a is

[11] Malakhov and Vohra (2009) studies the optimal Bayesian mechanism in such an environment, assuming independent types.

the maximum amount she is willing to pay for each unit and b is the largest number of units she seeks. Units beyond the b^{th} unit are worthless. Let the range of a be $\mathcal{A} = \{1, 2, ..., A\}$ and the range of b be $\mathcal{B} = \{1, 2, ..., B\}$. The seller has Q units to sell.

A crucial assumption is that bidders cannot inflate the capacity but can shade it down. In other words, the auctioneer can verify, partially, the claims made by a bidder. Although this assumption seems odd in the selling context, it is natural in a procurement setting. Consider a procurement auction where the auctioneer wishes to procure Q units from bidders with constant marginal costs and limited capacity. No bidder will inflate his capacity when bidding because of the huge penalties associated with not being able to fulfill the order. Equivalently, we may suppose that the mechanism designer can verify that claims that exceed capacity are false.

Lemma 1.3. *Fix any decision rule p that is dsIC, the shortest-path from the source v_0 to any payoff type (a, b) is*

$$(a, b) \succ_i (a - 1, b) \succ_i ... \succ_i (1, b) \succ_i (1, b - 1) \succ_i ... \succ_i (1, 1) \succ_i v_0.$$

Let $F_{b,v_{-i}}(a) = \sum_{x=1}^{a} \pi((x, b), v_{-i})$.

Corollary 1.2. *If π satisfies the following regularity condition: $\forall v_{-i}, \forall (a, b) \geq$*

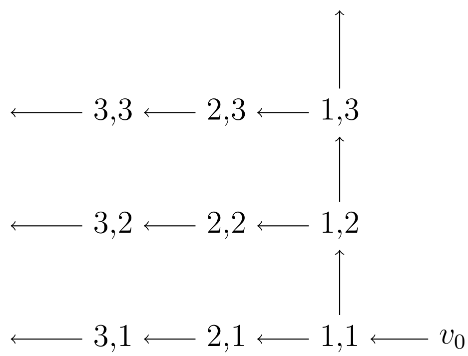

Figure 1.1: $(a, b) \succ_i (a-1, b) \succ_i \ldots \succ_i (1, b) \succ_i (1, b-1) \succ_i \ldots \succ_i (1, 1)$.

(a', b'),

$$a - \frac{1 - F_{b,v_{-i}}(a)}{\pi((a,b), v_{-i})} \geq a' - \frac{1 - F_{b',v_{-i}}(a')}{\pi((a',b'), v_{-i})}, \tag{1.23}$$

then the use of dominant-strategy mechanisms has maxmin and Bayesian foundations.

The proof of Lemma 1.3 and the derivation of the regularity condition (1.23) is a straightforward extension of Malakhov and Vohra (2009) and omitted. When π is independent, the regularity condition (1.23) reduces to the regularity condition in Malakhov and Vohra (2009). Corollary 1.2 then follows from Theorem 1.1.

1.5.3 Auction with type-dependent outside option

Besides the bilateral trade model with ex ante unidentified traders (Section 1.3), we present here another environment to illustrate the usefulness of Theorem 1.2. A single unit of an indivisible object is up for sale. There are two risk-neutral bidders. Each bidder's payoff type is represented by $(a, b) \in \mathbb{R}_+^2$ where a is the maximum amount she is willing to pay and b is the value of her outside option. Bidder 1's private information could be either $(20, 0)$ or $(40, 5)$. Bidder 2's private information could be either $(10, 0)$ or $(30, 5)$. The auctioneer has the following estimate of the distribution of the agents' valuations:

	$v_1 = (20, 0)$	$v_1 = (40, 5)$
$v_2 = (10, 0)$	$\frac{3}{8}$	$\frac{1}{8}$
$v_2 = (30, 5)$	$\frac{1}{8}$	$\frac{3}{8}$

(1.24)

The optimal dominant-strategy mechanism Γ is as follows, where the first number in each cell indicates the probability that agent 1 gets the object, the second number is the probability that agent 2 gets the object, the third number is the transfer from agent 1 to the auctioneer and the fourth number is the transfer from agent 2 to the auctioneer. Following Theorem 1.2, there is neither a Bayesian foundation nor a maxmin foundation.

	$v_1 = (20, 0)$	$v_1 = (40, 5)$
$v_2 = (10, 0)$	$1, 0, 20, 0$	$1, 0, 20, -5$
$v_2 = (30, 5)$	$0, 1, 0, 25$	$1, 0, 35, -5$

(1.25)

1.6 Discussion

1.6.1 Foundations of ex post incentive-compatible mechanisms

Our paper focuses on the private-value setting. The uniform shortest-path tree condition has a natural counterpart in the interdependent-value setting that, under an additional regularity condition, ensures the maxmin and Bayesian foundations of ex post incentive-compatible mechanisms. Indeed, in an independent and contemporaneous work, Yamashita and Zhu (2014) study the so-called "digital-goods" auctions in the interdependent-value setting. They show that under "ordinal invariability" (which entails that each agent has a stable preference ordering over all her payoff types, regardless of what payoff type profile the other agents have) and additional assumptions, the use of ex post incentive-compatible mechanisms has maxmin and Bayesian foundations.

1.6.2 On the notion of the maxmin foundation

Börgers (2013) argues that the maxmin foundation requires too little of an optimal mechanism. For every dominant-strategy mechanism, Börgers constructs another mechanism which never yields lower revenue and sometimes yields strictly higher revenue. The construction builds on the possibility of side bets among agents, and the mechanism designer charges a small fee for each bet. As Börgers points out, the argument would not be valid i) if agents could arrange side bets without requiring the mechanism designer as an intermediary; and ii) if the mechanism designer restricts her attention to the type spaces characterized by Morris (1994), which do not allow speculative trade.

We view the maxmin foundation as *the minimum requirement* that the optimal mechanism needs to satisfy. Indeed, if the use of dominant-strategy mechanisms does not have a maxmin foundation, then by definition, there exists a single Bayesian mechanism that achieves strictly higher expected revenue for every assumption about the agents' beliefs. Consequently, it becomes problematic to rationalize the use of dominant-strategy mechanisms. In settings in which the uniform shortest-path tree condition is violated, dominant-strategy mechanisms may not even satisfy *the minimum requirement* of the maxmin foundation.

Chapter 2

Equivalence of Stochastic and Deterministic Mechanisms

2.1 Introduction

Myerson (1981) provides the framework that has become the paradigm for the study of optimal auction design. Under a "regularity" condition, the optimal auction allocates the object to the bidder with the highest "virtual value", provided that this virtual value is above the seller's opportunity cost. In other words, the optimal auction in Myerson's setting is deterministic.[1]

A natural conjecture is that the optimality of deterministic mechanisms generalizes beyond Myerson's setting. McAfee and McMillan (1988, Section 4) claim that under a general regularity condition on consumers' demand, stochastic delivery was not optimal for a multi-product monopolist. However,

[1] Also see Riley and Zeckhauser (1983) who consider a one-good monopolist selling to a population of consumers with unit demand and show that lotteries do not help the one-good monopolist.

this result has been proven to be incorrect with a single agent. Several papers have shown that a multi-product monopolist may find it beneficial to include lotteries as part of the selling mechanism; see for example, Thanassoulis (2004), Manelli and Vincent (2006, 2007), Pycia (2006), Pavlov (2011), and more recently, Hart and Reny (2015), and Rochet and Thanassoulis (2015).[2] In this paper, we restore the optimality of deterministic mechanisms in remarkably general environments with multiple agents.

We consider a general social choice environment that has multiple agents, a finite set of alternatives, and independent and dispersed information.[3] We show that for any Bayesian incentive compatible mechanism, there exists an equivalent deterministic mechanism that i) is Bayesian incentive compatible; ii) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and iii) delivers the same ex ante expected social surplus. In addition to the standard social choice environments with linear utilities and one-dimensional, private types, our result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and non-transferable utilities.

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[2] In environments in which different types are associated with different risk attitudes, it is known that stochastic mechanisms may perform better; see for example, Laffont and Martimort (2002, p. 67) and Strausz (2006).

[3] Throughout this paper, we say that an agent has "dispersed information" if her type distribution is atomless.

Our result implies that any mechanism, including the optimal mechanisms (whether in terms of revenue or efficiency), can be implemented using a deterministic mechanism and nothing can be gained from designing more intricate mechanisms with possibly more complex randomization. As pointed out in Hart and Reny (2015, p. 912), Aumann commented that it is surprising that randomization can not increase revenue when there is only one good. Indeed, aforementioned papers in the screening literature establish that randomization helps when there are multiple goods. Nevertheless, we show that in general social choice environment with multiple agents, the revenue maximizing mechanism can always be deterministically implemented. This is in sharp contrast with the results in the screening literature.

Our result also has important implications beyond the revenue contrast. The mechanism design literature essentially builds on the assumption that a mechanism designer can credibly commit to any outcome of a mechanism. This requirement implies that any outcome of the mechanism must be verifiable before it can be employed. In this vein, a stochastic mechanism demands not only that a randomization device be available to the mechanism designer, but also that the outcome of the randomization device be objectively verified. As noted in Laffont and Martimort (2002, p. 67), “Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once sufficiently many realizations of the contracts have been observed. ... The enforcement of such stochastic mechanisms is thus particularly problematic. This has led scholars to give up those random mechanisms or, at least, to focus on economic settings where they are not optimal."[4] Our result implies that every mechanism can in fact be deterministically implemented, and thereby irons out the conceptual difficulties associated with stochastic mechanisms.

This paper joins the strand of literature that studies mechanism equivalence. Though motivations vary, these results show that it is without loss of generality to consider the various subclasses of mechanisms. As in the case of dominant-strategy mechanisms (see Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013)) and symmetric auctions (see Deb and Pai (2015)), our findings imply that the requirement of deterministic mechanisms is not restrictive in itself.[5] In this sense, our result provides a foundation for the use of deterministic allocations in mechanism design settings such as auctions,

[4] Also see Bester and Strausz (2001) and Strausz (2003).

[5] Manelli and Vincent (2010) show that for any Bayesian incentive compatible auction, there exists an equivalent dominant-strategy incentive compatible auction that yields the same interim expected utilities for all agents. Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) extend this equivalence result to social choice environments with linear utilities and independent, one-dimensional, private types; also see Footnote 12 for related discussion. Deb and Pai (2015) show that restricting the seller to a using symmetric auction imposes virtually no restriction on her ability to achieve discriminatory outcomes.

bilateral trades, and so on.

In order to prove the existence of an equivalent deterministic mechanism, we develop a new methodology of "mutual purification", and establish its link with the literature of mechanism design.[6] The notion of mutual purification is both conceptually and technically different from the usual purification principle in the literature related to Bayesian games. We shall clarify these two different notions of purification in the next three paragraphs.

It follows from the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) that any behavioral-strategy Nash equilibrium in a finite-action Bayesian game with independent and dispersed information corresponds to some pure-strategy Bayesian Nash equilibrium with the same payoff.[7] In particular, independent and dispersed information allows the agents to replace their behavioral strategies by some equivalent pure strategies one-by-one.[8] The point is that under the independent information assumption, any agent who has dispersed information could purify her own behavioral strategy regardless whether other agents have dispersed information. Example 2.2 illustrates this

[6]Some of our technical results extend the corresponding mathematical results in Arkin and Levin (1972); see the Appendices for more detailed discussion.

[7]See Radner and Rosenthal (1982), Milgrom and Weber (1985) and Khan, Rath, and Sun (2006). Furthermore, by applying the purification idea to a sequence of Bayesian games, Harsanyi (1973) provided an interpretation of mixed-strategy equilibrium in complete information games; see Govindan, Reny, and Robson (2003) and Morris (2008) for more discussion.

[8]See the proof of Theorem 1 in Khan, Rath, and Sun (2006).

idea of "self purification". Given a behavioral-strategy Nash equilibrium in a 2-agent Bayesian game with independent information, there is an equivalent pure strategy for the agent with dispersed information, while the other agent with an atom in her type space could not purify her behavioral strategy.

In contrast, the purification result of this paper is based on the dispersed information associated with the other agents. Example 2.3 partially illustrates this idea of "mutual purification". For a given randomized mechanism in a 2-agent setting with independent information, the agent with an atom in her type space can achieve the same interim payoff by some deterministic mechanism, while there does not exist such a deterministic mechanism for the other agent with dispersed information. In other words, our result becomes possible because each agent relies on the dispersed information of the other agents rather than her own. This also explains why a similar result does not hold in the one-agent setting since there is no dispersed information from other agents for such a single agent to purify the relevant randomized mechanism. In addition, we emphasize that in the multiple-agent setting, the notion of "mutual purification" requires not only that each agent obtain the same interim payoff under some deterministic mechanism, but also that a *single* deterministic mechanism deliver the same interim payoffs for all the agents *simultaneously*.

From a methodological point of view, the general purification principle in

Dvoretzky, Wald, and Wolfowitz (1950) is simply a version of the classical Lyapunov Theorem about the convex range of an atomless finite-dimensional vector measure. Our purification result is technically different. First, the problem we consider is infinite-dimensional because we require the same expected allocation probabilities/ utilities for the equivalent mechanism at the interim level with a continuum of types. Note that Lyapunov's Theorem fails in an infinite-dimensional setting.[9] Second, it is clearly impossible to obtain a purified deterministic mechanism that delivers the same expected allocation probabilities as the original stochastic mechanism, conditioned on the *joint* types of all the agents.[10] However, our result on mutual purification shows that such an equivalence becomes possible when the conditioning operation is imposed on the *individual* types of every agent simultaneously, although the combination of the individual types of every agent is the joint types of all the agents. To the best of our knowledge, this paper is the first to consider the purification of a randomized decision rule that retains the same expected payoffs conditioned on the individual types of every agent in an economic model.

Our paper contributes to the Bayesian mechanism design literature in relying

[9]See, for example, Diestel and Uhl (1977, p. 261).

[10]Since the joint types of all the agents carry the full information, the expected allocation probability of a stochastic mechanism conditioned on the joint types is simply the stochastic mechanism itself.

on specific aspects of agents' private information. These information aspects are often crucial in pinning down different properties of the optimal mechanism. For instance, agents with independent types retain information rents (see Myerson (1981)), whereas the mechanism designer can fully extract the surplus when the agents' types are correlated (see Crémer and McLean (1988)). Our result builds on the assumption that the agents' private information is independent and dispersed. This assumption facilitates the development of the novel methodology of "mutual purification", which lies at the core of our arguments.

The rest of the paper is organized as follows. Section 2.2 introduces the basics. Section 2.3 illustrates our equivalence notion and the idea of "mutual purification" through examples. Section 2.4 presents the equivalence result. Section 2.5 discusses the benefit of randomness, an implementation perspective of our result, and various assumptions of our result. Section 2.6 concludes. The appendix contains proofs omitted from the main body of the paper.

2.2 Preliminaries

2.2.1 Notation

There is a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk neutral agents with $I \geq 2$ and a finite set $\mathcal{K} = \{1, 2, ..., K\}$ of social alternatives. The set of possible types V_i of agent i is a closed subset of finite dimensional Euclidean space $\mathbb{R}^l$ with generic element v_i. The set of possible type profiles is $V \equiv V_1 \times V_2 \times \cdots \times V_I$ with generic element $v = (v_1, v_2, \ldots, v_I)$. Write v_{-i} for a type profile of agent i's opponents; that is, $v_{-i} \in V_{-i} = \Pi_{j \neq i} V_j$. Denote by λ the common prior distribution on V. For each $i \in \mathcal{I}$, λ_i is the marginal distribution of λ on V_i and is assumed to be atomless. Throughout this paper, types are assumed to be independent.[11] If $(Y, \mathcal{Y})$ is a measurable space, then ΔY is the set of all probability measures on $(Y, \mathcal{Y})$. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2.2 Mechanism

The revelation principle applies, and we restrict attention to direct mechanisms characterized by $K + I$ functions, $\{q^k(v)\}_{k \in \mathcal{K}}$ and $\{t_i(v)\}_{i \in \mathcal{I}}$, where v is the profile of reports, $q^k(v) \geq 0$ is the probability that alternative k is implemented with $\sum_{k \in \mathcal{K}} q^k(v) = 1$, and $t_i(v)$ is the monetary transfer that agent i makes to the mechanism designer. Agent i's gross utility in alternative k is $u_i^k(v_i, v_{-i})$.

For simplicity of exposition, we denote

$$Q_i^k(v_i) = \int_{V_{-i}} q^k(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i})$$

for the interim expected allocation probability (from agent i's perspective) that

[11] Note that we do not make any assumption regarding the correlation of the different coordinates of type v_i for any $i \in \mathcal{I}$.

alternative k is implemented. Also write

$$T_i(v_i) = \int_{V_{-i}} t_i(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i})$$

for the interim expected transfer from agent i to the mechanism designer. Agent i's interim expected utility is

$$\begin{aligned} U_i(v_i) &= \int_{V_{-i}} \left[\sum_{1 \le k \le K} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{V_{-i}} \left[\sum_{1 \le k \le K} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i). \end{aligned}$$

A mechanism is Bayesian incentive compatible (BIC) if for each agent $i \in \mathcal{I}$ and each type $v_i \in V_i$,

$$\begin{aligned} &U_i(v_i) \ge 0, \text{ and} \\ &U_i(v_i) \ge \int_{V_{-i}} \left[\sum_{1 \le k \le K} u_i^k(v_i, v_{-i}) q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \end{aligned}$$

for any alternative type $v_i' \in V_i$.

A mechanism (q, t) is said to be "deterministic" if for almost all type profiles, the mechanism implements some alternative k for sure. That is, for λ-almost all $v \in V, q^k(v) = 1$ for some $1 \le k \le K$.

2.2.3 Mechanism equivalence

We shall employ the following notion of mechanism equivalence.

Definition 2.1. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they deliver the same interim expected allocation probabilities and the same* *interim expected utilities for all agents, and the same ex ante expected social surplus.*

Remark 2.1. *Our equivalence is stronger than the prevailing mechanism equivalence notion. For example, Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) define two mechanisms to be equivalent if they deliver the same interim expected utilities for all agents and the same ex ante expected social surplus.*[12]

Remark 2.2. *The equivalent deterministic mechanism also guarantees the same ex post monetary transfers, and hence the same expected revenue; see Theorem 2.1.*

2.3 Examples

2.3.1 An illustration of equivalent deterministic mechanism

In the first example, we illustrate our mechanism equivalence notion in a single-unit auction environment.[13] The example is kept deliberately simple and its only purpose is to illustrate what we mean by equivalent deterministic

[12]Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013, Section 4.1) show that the BIC-DIC equivalence breaks down when requiring the same interim expected allocation probability. They also note that "this notion (of interim expected allocation probabilities) becomes relevant when, for instance, the designer is not utilitarian or when preferences of agents outside the mechanism play a role".

[13]With slight adjustments, this example applies to the irregular case in Myerson's setting where the agents' ironed virtual values are the same in some interval.

mechanism. Our main result is far more general and the proof is much more complex.

Example 2.1. *There are two bidders, whose valuations are uniformly distributed in* $[0,1]$. *Consider the following mechanism. Types are divided into intervals of equal probability and types in the same interval are treated equally. If agents' types belong to the same interval, each agent receives the object with probability* $\frac{1}{2}$ *and if agents' types belong to different intervals, the agent whose type belongs to* $[\frac{1}{2},1]$ *gets the object. In each cell, the first number is the probability that agent 1 gets the object and the second number is the probability that agent 2 gets the object.*

	$[\frac{1}{2},1]$	$[0,\frac{1}{2})$
$[\frac{1}{2},1]$	$\frac{1}{2},\frac{1}{2}$	$1,0$
$[0,\frac{1}{2})$	$0,1$	$\frac{1}{2},\frac{1}{2}$

It is immediate that, the following deterministic mechanism is equivalent in terms of interim expected allocation probabilities. Keeping the transfers unchanged, it is also easy to see the deterministic mechanism is equivalent in terms of interim expected utilities and ex ante social welfare.

	$[\frac{3}{4},1]$	$[\frac{2}{4},\frac{3}{4})$	$[\frac{1}{4},\frac{2}{4})$	$[0,\frac{1}{4})$
$[\frac{3}{4},1]$	$1,0$	$0,1$	$1,0$	$1,0$
$[\frac{2}{4},\frac{3}{4})$	$0,1$	$1,0$	$1,0$	$1,0$
$[\frac{1}{4},\frac{2}{4})$	$0,1$	$0,1$	$1,0$	$0,1$
$[0,\frac{1}{4})$	$0,1$	$0,1$	$0,1$	$1,0$

In Section 2.4, we show that for whatever randomized mechanism that the mechanism designer may choose to use, however complicated, there exists an equivalent mechanism that is deterministic. In other words, going from mechanisms that are deterministic to randomized mechanisms in general does not enlarge the set of obtainable outcomes.

2.3.2 Self purification and mutual purification

In this section, we provide two examples to demonstrate the conceptual difference between the existing approach of "self purification" and our approach of "mutual purification".

The first example is motivated by the game of matching pennies, while the second example is a single unit auction. Both games have two agents, and share the same information structure as follows.

1. Agent 1's type is uniformly distributed on $(0,1]$ with the total probability $1-\lambda_1(0)$, and has an atom at the point 0 with $\lambda_1(0)>0$.

2. Agent 2's type is uniformly distributed on $[0,1]$.

3. Agents' types are independently distributed.

Example 2.2 below illustrates the idea of "self purification". The behavioral strategy of agent 2 can be purified since the distribution of agent 2's type is atomless, while the behavioral strategy of agent 1 cannot be purified since agent 1's type has an atom.

Example 2.2. *Consider an $m \times m$ zero-sum generalized "matching pennies" game with incomplete information, where the positive integer m is sufficiently large such that $\frac{1}{m} < \lambda_1(0)$. The information structure is described in the beginning of this subsection. The action space for both agents is $A_1 = A_2 = \{a_1, a_2, \ldots, a_m\}$. The payoff matrix for agent 1 is given below. Notice that the payoffs of both agents do not depend on the type profile.*

		Agent 2				
		a_1	a_2	a_3	$\cdots$	a_m
	a_1	1	-1	0	$\cdots$	0
	a_2	0	1	-1	$\cdots$	0
Agent 1	a_3	0	0	1	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a_m	-1	0	$\cdots$	0	1

Suppose that both agents adopt the behavioral strategy $f_1(v) = f_2(v) = \frac{1}{m}\sum_{1 \le s \le m} \delta_{a_s}$, where δ_{a_s} is the Dirac measure at the point a_s. It is easy to see that (f_1, f_2) is a Bayesian Nash equilibrium and the expected payoffs of both agents are 0.

Claim 2.1. *Agent 2 has a pure strategy f_2' such that (f_1, f_2') is still a behavioral-strategy equilibrium and provides both agents the same expected payoffs, while agent 1 does not have such a pure strategy.*

Proof. It is easy to see that the following pure strategy f_2' gives agent 2 the same expected payoff and (f_1, f_2') is still a Bayesian Nash equilibrium, where

$$f_2'(v) = \begin{cases} a_s, & v \in [\frac{s-1}{m}, \frac{s}{m}), 1 \le s \le m-1; \\ a_m, & v \in [\frac{m-1}{m}, 1]. \end{cases}$$

We next show that there does not exist a pure strategy g_1 of agent 1 such that g_1 is a component of a Bayesian Nash equilibrium with each agent's expected payoff being 0. Suppose that (g_1, g_2) is a Bayesian Nash equilibrium such that g_1 is a pure strategy of agent 1. Let $D_s = \{v_1 \in V_1 \colon g_1(v_1) = a_s\}$ for $1 \le s \le m$. Without loss of generality, we assume that $0 \in D_1$. Let $S = \arg\max_{1 \le s \le m} \lambda_1(D_s)$. Since $\lambda_1(D_s) \ge \lambda_1(D_1) \ge \lambda_1(0) > \frac{1}{m}$ for each $s \in S$, S must be a strict subset of $\{1, \ldots, m\}$. Without loss of generality, we assume that $s^* \in S$ and $s^* + 1 \notin S$. Given agent 1's strategy g_1, agent 2 can adopt the pure strategy $g_2'(v_2) = a_{s^*+1}$ for any $v_2 \in V_2$. Then the expected payoff of agent 2 is $\lambda_1(D_{s^*}) - \lambda_1(D_{s^*+1}) > 0$ with the strategy profile (g_1, g_2'). Since (g_1, g_2) is a Bayesian Nash equilibrium, the expected payoff of agent 2 must be at least $\lambda_1(D_{s^*}) - \lambda_1(D_{s^*+1})$ with the strategy profile (g_1, g_2), which is strictly positive. This is a contradiction. □

Example 2.3 below shows how a purification for an agent relies on the dispersed information of the other agent, which partially illustrates the idea of "mutual purification". In particular, for some given randomized mechanism in the 2-agent setting with independent information as specified above, agent 1 who has an atom in her type space can achieve the same interim expected payoff by some deterministic mechanism,[14] while there does not exist such a deterministic mechanism for agent 2 who has dispersed information.

Example 2.3. *Consider a single unit auction with two agents. The information structure is described as above. The payoff function of agent i is $\epsilon v_i + (1 - v_j)^m$ for $i, j = 1, 2$ and $i \neq j$, where m is sufficiently large and ϵ is sufficiently small such that*

$$\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}.$$

The allocation rule q is defined as follows. Let $q^i(v)$ be the probability that agent i gets the object, and $q^1(v_1, v_2) = q^2(v_1, v_2) = \frac{1}{2}$ for any (v_1, v_2). The interim expected payoff of agent 1 with value v_1 is

$$\int_{V_2} (\epsilon v_1 + (1 - v_2)^m) q^1(v_1, v_2) \lambda_2(\mathrm{d}v_2) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}.$$

The interim expected payoff of agent 2 with value v_2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m) q^2(v_1, v_2) \lambda_1(\mathrm{d}v_1) = \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0)) \frac{1}{2(m+1)}.$$

[14] For simplicity, we only consider such an equivalence in terms of interim expected payoffs.

Claim 2.2. *There exists a deterministic mechanism which gives agent 1 the same interim expected payoff; but there does not exist such a deterministic mechanism for agent 2.*

Proof. We first construct a deterministic mechanism which gives agent 1 the same interim expected payoff. Define a function G on $V_1 \times V_2 = [0,1]^2$ by letting

$$G(v_1, v_2) = \int_0^{v_2} [\epsilon v_1 + (1 - v_2')^m] \, \lambda_2(\mathrm{d}v_2') - \left[\frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}\right],$$

for any $(v_1, v_2) \in V_1 \times V_2$. It is clear that for any $v_1 \in [0,1]$, $G(v_1, 0) < 0 < G(v_1, 1) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}$. One can also check that $\frac{\partial G}{\partial v_2} = \epsilon v_1 + (1 - v_2)^m > 0$ for any $v_1 \in [0,1]$ and $v_2 \in [0,1)$. Hence, for each $v_1 \in [0,1]$, there exists a unique number $g(v_1) \in (0,1)$ such that $G(v_1, g(v_1)) = 0$. By the usual implicit function theorem, g must be differentiable, and hence measurable. Let $\hat{q}^1(v_1, v_2) = 1$ if $0 \leq v_2 \leq g(v_1)$ and 0 otherwise, and $\hat{q}^2(v_1, v_2) = 1 - \hat{q}^1(v_1, v_2)$. Then the mechanism $\hat{q}$ gives agent 1 the same interim expected payoff.

We next show that there does not exist any deterministic mechanism that gives agent 2 the same interim expected payoff. Suppose that there exists a deterministic mechanism $\tilde{q}$ that gives agent 2 the same interim expected payoff. Fix value $v_2 \in V_2 = [0,1]$.

Suppose that $\tilde{q}^2(0, v_2) = 1$. Then the interim expected payoff of agent 2 with value v_2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m)\tilde{q}^2(v_1, v_2)\lambda_1(\mathrm{d}v_1) \geq (\epsilon v_2 + 1)\lambda_1(0).$$

Recall that $\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}$. Hence we have

$$(\epsilon v_2+1)\lambda_1(0) \geq \lambda_1(0) > \frac{\lambda_1(0)}{2}+\epsilon+\frac{1}{m+1} > \frac{\epsilon v_2}{2}+\frac{\lambda_1(0)}{2}+(1-\lambda_1(0))\frac{1}{2(m+1)}.$$

Thus, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly greater than the interim expected payoff of agent 2 under the mechanism q. This is a contradiction. Therefore, we must have $\tilde{q}^2(0, v_2) = 0$ since $\tilde{q}$ is a deterministic mechanism.

Next, since $\tilde{q}^2(0, v_2) = 0$, the interim expected payoff of agent 2 is

$$\begin{aligned}
&\int_{V_1} (\epsilon v_2 + (1 - v_1)^m)\tilde{q}^2(v_1, v_2)\lambda_1(\mathrm{d}v_1) = \int_{(0,1]} (\epsilon v_2 + (1 - v_1)^m)\tilde{q}^2(v_1, v_2)\lambda_1(\mathrm{d}v_1) \\
&\leq (1 - \lambda_1(0)) \int_0^1 (\epsilon v_2 + (1 - v_1)^m) \,\mathrm{d}v_1 = (1 - \lambda_1(0))\epsilon v_2 + \frac{1 - \lambda_1(0)}{m + 1} \\
&< \epsilon + \frac{1}{m + 1} < \frac{\lambda_1(0)}{2} \\
&< \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0))\frac{1}{2(m + 1)}.
\end{aligned}$$

That is, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly less than the interim expected payoff of agent 2 under the mechanism q. This is also a contradiction. Therefore, there does not exist any deterministic mechanism that gives agent 2 the same interim expected payoff. □

We hasten to emphasize the key difference between our approach and the purification method used in the literature. With the classical purification method, each agent uses her own dispersed information to purify her behavioral strategy, which we call “self purification”. In contrast, the purification approach we adopt to achieve our main result is to purify the randomized mechanism via other agents’ dispersed information while keeping each agent’s interim expected allocation probability and interim expected payoff unchanged *simultaneously*, which we call “mutual purification”.

2.4 Results

This section establishes the main result of this paper. We consider a general environment in which agents could have nonlinear and interdependent payoffs. In particular, we assume that all agents have “separable payoffs” in the following sense.

Definition 2.1. *For each $i \in \mathcal{I}$, agent i is said to have separable payoff if for any outcome $k \in \mathcal{K}$ and type profile $v = (v_1, v_2, \ldots, v_I) \in V$, her payoff function can be written as follows:*

$$u_i^k(v_1, \ldots, v_I) = \sum_{1 \leq m \leq M} w_{im}^k(v_i) r_{im}^k(v_{-i}),$$

where M is a positive integer, and w_{im}^k (resp. r_{im}^k) is λ_i-integrable (resp. λ_{-i}-integrable) on V_i (resp. on V_{-i}) for $1 \leq m \leq M$.

That is, the payoff of each agent i is a summation of finite terms, where each term is a product of two components: the first component only depends on agent i's own type, while the second component depends on other agents' types. This setup is sufficiently general to cover most applications. In particular, it includes the interdependent payoff function as in Jehiel and Moldovanu (2001), and obviously covers the widely adopted private value payoffs as a special case.

Theorem 2.1. *Suppose that for each agent $i \in \mathcal{I}$, his payoff function is separable. Then for any mechanism (q, t), there exists a deterministic allocation rule $\tilde{q}$ such that*

1. *q and $\tilde{q}$ induce the same interim expected allocation probability;*

2. *$(\tilde{q}, t)$ delivers the same interim expected utility with (q, t) for each agent $i \in \mathcal{I}$.*

Thus, if (q, t) is BIC, then $(\tilde{q}, t)$ is also BIC.

Remark 2.3. *We prove a stronger result. First, it is clear from the proof of Theorem 2.1 that the equivalent deterministic mechanism $(\tilde{q}, t)$ also guarantees the same ex post monetary transfers. Therefore, our deterministic mechanism equivalence result does not require transferable utility. Second, the equivalence result is immune against coalitions; that is, when there is sharing of information between the coalition members (except for the grand coalition).*[15] *The second*

[15] Jackson and Sonnenschein (2007) also consider the issue of coalitional incentive compatibility. They show that the "linking mechanisms" are immune to manipulations by coalitions.

point is proved explicitly.

2.5 Discussions

2.5.1 Benefit of randomness revisited

Chawla, Malec, and Sivan (2015) consider *multi-agent* setting and focus on the case where the agents' values are independent both across different agents' types and different coordinates of an agent's type. In particular, Chawla, Malec, and Sivan (2015, Theorem13) establish a constant factor upper bound for the benefit of randomness when the agents' values are independent. In the special case of multi-unit multi-item auctions, they show that the revenue of any Bayesian incentive compatible, individually rational randomized mechanism is at most 33.75 times the revenue of the optimal deterministic mechanism. In this paper, we push this result to the extreme and show that the revenue maximizing auction can be deterministically implemented.[16]

2.5.2 An implementation perspective

We have motivated our result broadly, in terms of revenue, social surplus, interim expected allocation probabilities, interim expected utilities and even ex post payments. Alternatively, we may take an implementation perspective to formulate our result. Beyond the equivalence notion discussed throughout the

[16]Chawla, Malec, and Sivan (2015, p. 316) remarked that "our bounds on the benefit of randomness are in some cases quite large and we believe they can be improved".

paper, the deterministic allocation rule can also be required to pick some allocation in the support of the randomized allocation in the stochastic mechanism for each type profile v. Therefore, when a stochastic mechanism implements some social goal (i.e., at every type profile v, every realized allocation is consistent with the social goal), our equivalent deterministic mechanism also has the same property. We shall explain this point in the following paragraph.

Suppose that q is a random allocation rule. Given the K alternatives, the set of all nonempty subsets of $\{1, \ldots, K\}$ can have at most $2^K - 1$ elements $\{C_j\}_{1 \leq j \leq 2^K - 1}$. As a result, the set of type profiles V can be divided into $2^K - 1$ disjoint subsets $\{D_j\}_{1 \leq j \leq 2^K - 1}$ such that

1. the support of $q(v)$ is C_j for all $v \in D_j$;

2. $\lambda(\cup_{1 \leq j \leq 2^K - 1} D_j) = 1$.

We define $2^K - 1$ functions $\{\beta_j\}_{1 \leq j \leq 2^K - 1}$ such that $\beta_j = 1 + \mathbf{1}_{D_j}$ for each j; that is, β_j is the summation of 1 and the indicator function of the set D_j. Instead of working with the function h, we can work with the new function $h' = (h, \beta_1, \ldots, \beta_{2^K - 1})$. Lemma A.4 and Proposition A.1 (in the Appendix) still hold, and we can obtain a deterministic mechanism $\tilde{q}$ such that

$$\int_V q \beta_j \, \mathrm{d}\lambda = \int_V \tilde{q} \beta_j \, \mathrm{d}\lambda$$

for each j, and

$$\int_V q\,\mathrm{d}\lambda = \int_V \tilde{q}\,\mathrm{d}\lambda.$$

That is, $\int_{D_j} q\,\mathrm{d}\lambda = \int_{D_j} \tilde{q}\,\mathrm{d}\lambda$ for each j. Since $\sum_{k \in C_j} q^k(v) = 1$ for λ-almost all $v \in D_j$, $\int_{D_j} \sum_{k \in C_j} q^k(v)\lambda(\mathrm{d}v) = \lambda(D_j)$, which implies that $\int_{D_j} \sum_{k \in C_j} \tilde{q}^k(v)\lambda(\mathrm{d}v) = \lambda(D_j)$. As a result, for λ-almost all $v \in D_j$, $\tilde{q}^k = 1$ for some $k \in C_j$. This proves our claim that the deterministic allocation rule lies in the support of the random allocation rule.

2.5.3 Assumptions

This subsection discusses the assumptions behind our equivalence result. The requirement of multiple agents needs no further explanation. Atomless distribution is an indispensable requirement for almost all purification results. See Example 2.3 where we cannot purify the allocation for agent 2 while keeping her interim expected utility unchanged because agent 1's type distribution has an atom, let alone the stronger requirement that the deterministic mechanism requires such purification for all agents simultaneously. While our result requires independence, it is worth mentioning that we only require independence across agents and we do not make any assumption regarding the correlation of the different coordinates of type v_i for any agent $i \in \mathcal{I}$. Though separable payoff is a restriction, this setup is sufficiently general to cover most applications; see

Section 2.4 for details.

2.6 Conclusion

We prove the following mechanism equivalence result: in a general social choice environment with multiple agents, for any stochastic mechanism, there exists an equivalent deterministic mechanism. On the one hand, our result implies that it is without loss of generality to work with stochastic mechanisms, even if the mechanism designer does not have access to a randomization device, or cannot fully commit to the outcomes induced by a randomization device. On the other hand, our result implies that the requirement of deterministic mechanisms is not restrictive in itself. Even if one is constrained to employ only deterministic mechanisms, there is no loss of revenue or social welfare. Therefore, our result provides a foundation for the use of deterministic mechanisms in mechanism design settings, such as auctions, bilateral trades, etc.

Chapter 3

Efficient Dynamic Mechanisms with Interdependent Valuations

3.1 Introduction

An important strand of mechanism design theory is concerned with the design of efficient mechanisms. The mechanism designer would like to allocate the good to the bidder with the highest valuation,[1] provide the public good if and only if the sum of the agents' valuations is greater than the cost, and facilitate trading if and only if the buyer's valuation is higher than the seller's valuation, etc.[2]

The renowned Vickrey-Clarke-Groves (VCG) mechanism established the

[1]A leading rationale for the widespread privatization of state-owned assets is to enhance efficiency; see Dasgupta and Maskin (2000). For example, the U. S. Congress explicitly mandated the Federal Communications Commission to promote efficiency in its auctions of frequency bands for telecommunications.

[2]The problem of implementing socially efficient outcomes has also been extensively studied in the dynamic setting; see, for example, Bergemann and Välimäki (2010), Athey and Segal (2013), and Guo and Hörner (2015). Pavan, Segal, and Toikka (2014) provide a general treatment of the dynamic mechanism design problem in the independent private-value setting (see also references therein).

existence of an efficient, incentive-compatible mechanism for a general class of static mechanism design problems with private values and quasilinear preferences; see Clarke (1971), Groves (1973) and Vickrey (1961). Subsequently, a pair of classic papers, Arrow (1979) and d'Aspremont and Gérard-Varet (1979) (AGV), constructed an efficient, incentive-compatible mechanism in which the transfers were also budget-balanced, using the solution concept of Bayesian-Nash equilibrium, under the additional assumption that private information is independent across agents.

In dynamic mechanism design problems with private values, Bergemann and Välimäki (2010) and Athey and Segal (2013) have successfully addressed this question, by means of dynamic extensions of the VCG and AGV mechanisms. However, it is well known that VCG and AGV mechanisms no longer work in settings with interdependent valuations. Indeed, Maskin (1992), Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) have demonstrated, in increasing generality, that if information signals are statistically independent, multidimensional (or, if they are single dimensional, but a single crossing condition is violated), and interdependent, then the implementation of efficient mechanisms is generically impossible.

In this paper, we study efficient mechanism design in a dynamic environment with interdependent valuations and evolving private information. Our aim is to construct an efficient, incentive-compatible dynamic mechanism that is also budget-balanced in every period of the game. As in the AGV mechanism and Athey and Segal (2013), we place emphasis on budget balance.

As discussed above, implementation of efficient mechanisms with interdependent valuations runs into difficulties even in the static setting. To overcome such difficulties, we extend the following insight from Mezzetti (2004) to the dynamic setting. In a static mechanism design problem, Mezzetti (2004) constructs a novel and elegant "generalized (or two-stage) Groves mechanism" that bypasses the above difficulties, with the assumption that each agent observes her own realized outcome-decision payoff after the final outcome decision, but before final transfers, are made.[3] While Mezzetti (2004) resolves incentive compatibility, requiring agents to be able to observe their own payoffs before the mechanism ends is a strong assumption in the static setting from an applied perspective. In the dynamic setting, it may seem natural to assume that in each period, each agent could observe her own realized outcome-decision payoff from the previous period.

This assumption is related to the literature on contingent payments; see Hansen (1985), Crémer (1987), Samuelson (1987) and more recently, DeMarzo,

[3]Two-stage mechanisms can also be used to achieve goals other than efficiency (e.g., surplus extraction); see Mezzetti (2007).

Kremer, and Skrzypacz (2005) and Che and Kim (2010) among others.[4] In this paper, we do not require that the realized outcome-decision payoffs are observable to the mechanism designer, but we rely instead on the agents' reports of their own realized payoffs.

This paper places emphasis on budget balance in every period of the game. Indeed, the construction of an efficient, incentive-compatible mechanism is straightforward. In each period, the mechanism designer makes a transfer to each agent that is an adjusted amount of the sum of the other agents' outcome-decision payoffs from the previous period. This suffices to make each agent the residual claimant of the social surplus and provide the agents with the incentive to be truthful as long as the mechanism prescribes an efficient decision rule. Under the assumption of independent types, we show that dynamic efficiency can be achieved with balanced budget. As in the AGV mechanism and Athey and Segal (2013), our construction of the budget-balanced mechanism requires all the other agents to pitch in to pay each agent's incentive term. This ensures that the budget is balanced in every period of the game. The key difference between our mechanism and the "balanced team mechanism" in Athey and Segal (2013) is as follows. In their paper, only the transfers of the most recent two periods are relevant for each agent's incentive in the current period, since

[4]Lehrer (1992) and Tomala (1999) have also adopted similar assumptions of observable payoff in the environment of repeated games.

the expectation of the transfers afterwards is zero.[5] However, in our mechanism, all the future transfers could influence the incentive of the current period.

Another approach that studies efficient mechanism design exploits the correlation of private information; see the seminal contribution of Crémer and McLean (1988) in the static setting. More recently, Liu (2014) and Noda (2015) extend the insight of Crémer and McLean (1988) to the dynamic setting and construct efficient and incentive-compatible mechanisms respectively. These results leverage on the inter-temporal correlation of private information and do not apply in our setting. Hörner, Takahashi, and Vieille (2015) apply a similar technique to dynamic Bayesian games.

The rest of the paper is organized as follows. Section 3.2 introduces the model. Section 3.3 constructs the efficient, incentive-compatible and budget-balanced mechanism and Section 3.4 concludes.

3.2 Model

3.2.1 Setup

Notation. We consider a dynamic mechanism design environment with interdependent valuations in a discrete-time, infinite-horizon model. There is a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk neutral agents. Time is discrete, indexed by $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The state of the world θ_t^i for agent i is a general

[5] See the proof of Proposition 2 in Athey and Segal (2013).

Markov process on the state space Θ^i. The aggregate state is given by the vector $\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^I)$ with $\Theta = \prod_{i\in\mathcal{I}} \Theta^i$. Write $\theta_t^{-i} \in \Theta^{-i} = \prod_{j\neq i} \Theta^j$ for the state of all agents except agent i. The outcome space is a measurable set X endowed with the σ-algebra $\mathcal{X}$. The initial state $\theta_0 \in \Theta$ is assumed to be publicly known. The current state $\theta_t \in \Theta$ and current decision $x_t \in X$ define a probability distribution for state variable θ_{t+1} on Θ by the law of motion $Q(\cdot|x_t, \theta_t)$.

Timing. We consider mechanisms in which, following a publicly observed initial state $\theta_0 \in \Theta$, a decision $x_0 \in X$ is made. Then in each period $t \geq 1$, each agent privately observes her type $\theta_t^i \in \Theta^i$. Agents make reports simultaneously and a public decision $x_t \in X$ is made at the end of each period. Each agent i also receives a transfer $y_t^i \in \mathbb{R}$. We assume that the past reports of each agent and the public decision are observable to all agents. All agents discount the future with a common discount factor $\delta \in (0, 1)$.

Interdependent valuations. We allow agents to have interdependent valuations in the sense that agent i's payoff could depend on the signals of all the other agents for each $i \in \mathcal{I}$. If a sequence of types $\{\theta_t\}_{t\geq 0}$ is realized, a sequence of public decisions $\{x_t\}_{t\geq 0}$ and transfers $\{y_t\}_{t\geq 0}$ are determined, then the discounted payoff of agent i is

$$\sum_{t\geq 0} \delta^t [u_i(x_t, \theta_t) + y_t^i],$$

where $u_i\colon X \times \Theta \to \mathbb{R}$ is assumed to be measurable and bounded. We will refer to u_i as the outcome-decision payoff of agent i.

Independent types. Throughout this paper, we shall assume independent types. That is, conditional on decisions, the private information of agent i does not have any direct effect on the distribution of the current and future types of other agents (we still allow one agent's reports to affect the future types of other agents through the implemented decisions). More formally,

Definition 3.1. *Agents have independent types if given any $x_t \in X$ and $\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^I) \in \Theta$, the transition probability $Q(\cdot|x_t, \theta_t) = \otimes_{i \in \mathcal{I}} Q_i(\cdot|x_t, \theta_t^i)$, where $Q_i(\cdot|x_t, \theta_t^i)$ is a transition probability from $X \times \Theta^i$ to $\triangle(\Theta^i)$.*

Equilibrium notion. The truthtelling strategy of agent i always reports her state θ_t^i in every period $t \geq 1$ truthfully, regardless of the observed past (in particular, regardless of whether she has lied in the past). We will consider perfect Bayesian equilibrium (PBE) in truthtelling strategies, with beliefs that assign probability 1 to the other agents' latest reports being truthful.

3.2.2 Efficiency

A social policy is a measurable function $\chi\colon \Theta \to X$, where $\chi(\theta)$ represents the decision made when the realized state in this period is θ. Starting from an initial type $\theta_0 \in \Theta$, a social policy χ together with the transition probability Q uniquely determine a probability measure over the sequence of states $(\theta_t)_{t\geq 0} \in \Theta^{\mathbb{N}}$.

In period t, efficiency can be obtained at type θ_t by maximizing the discounted expected surplus:

$$\sup_{\{x_s\}_{s\geq t}} \mathbb{E}\left[\sum_{s\geq t} \delta^{s-t} \sum_{i\in\mathcal{I}} u_i(x_s, \theta_s)\right].$$

We characterize the efficient social policy $\chi^*\colon \Theta \to X$ and the associated social value function $V\colon \Theta \to \mathbb{R}$ by the following recursion using the principle of dynamic programming:

$$\begin{aligned} V(\theta) &= v(\theta) + \delta \int_\Theta V(\tilde{\theta}) Q(d\tilde{\theta}|\chi^*(\theta), \theta) \\ &= \sup_{x\in X}\left[\sum_{i\in\mathcal{I}} u_i(x,\theta) + \delta \int_\Theta V(\tilde{\theta}) Q(d\tilde{\theta}|x,\theta)\right], \end{aligned}$$

where $v(\theta) = \sum_{i\in\mathcal{I}} u_i(\chi^*(\theta), \theta)$.[6]

3.3 Mechanism

In this section, we construct an efficient and budget-balanced dynamic mechanism such that truthtelling strategies form a perfect Bayesian equilibrium. As discussed in the introduction, we assume that in each period, each agent observes her own realized outcome-decision payoff from the previous period.

Assumption 3.1. *In each period $t+1$ ($t \geq 1$) and for any $x_t \in X$, each agent i observes her own realized outcome-decision payoff from period t.*

[6] Throughout this paper, we assume that the efficient social policy exists.

For simplicity of exposition, in what follows, we go a step further and assume that in each period, the realized outcome-decision payoffs from the previous period are observable to the mechanism designer. The mechanism we construct still works under the original assumption. Indeed, in each period $t+1$, the mechanism designer could require the agents to report their realized outcome-decision payoffs from period t. Since for each agent i, the report of her own outcome-decision payoff does not affect her utility, we can assume that agent i truthfully reports her outcome-decision payoff from the previous period; see Mezzetti (2004) for further discussions.

In period $t \geq 1$, given reported types r_{t-1} and r_t in periods $t-1$ and t respectively,[7] let

$$\Phi_i(r_{t-1}, r_t^i) = \int_{\Theta_{-i}} V(r_t^i, \tilde{\theta}_t^{-i}) Q_{-i}(d\tilde{\theta}_t^{-i} | \chi^*(r_{t-1}), r_{t-1}^{-i}) - \int_{\Theta} V(\tilde{\theta}_t) Q(d\tilde{\theta}_t | \chi^*(r_{t-1}), r_{t-1}) \text{ and}$$

$$\Psi_i(r_{t-1}, r_t) = \Phi_i(r_{t-1}, r_t^i) - \frac{1}{I-1} \sum_{j \neq i} \Phi_j(r_{t-1}, r_t^j)$$

for each $i \in \mathcal{I}$.

From the mechanism designer's perspective, $\Phi_i(r_{t-1}, r_t^i)$ characterizes the change in the expected social value if agent i reports r_t^i in period t, given the report r_{t-1} in period $t-1$.

Construct the following mechanism (χ^*, y):

1. The socially efficient policy χ^* is implemented in every period; that is, in

[7]Since the initial state $\theta_0 \in \Theta$ is publicly known, we assume $r_0 \equiv \theta_0$.

period t, the allocation is $\chi^*(r_t)$ based on the reports r_t.

2. For each $i \in \mathcal{I}$, the transfer to agent i in period $t+1$ (for $t \geq 1$) is

$$y_{t+1}^i = \frac{1}{\delta} \sum_{j \neq i} w_t^j - \frac{I-1}{I\delta} \left[v(r_t) - \Psi_i(r_{t-1}, r_t) \right],$$

where w_t^i is the realized outcome-decision payoff of agent $i \in \mathcal{I}$ in period t.[8]

As in the AGV mechanism and Athey and Segal (2013), our mechanism requires all the other agents to pitch in to pay each agent's incentive term, which ensures that the budget is balanced on the equilibrium path.

Theorem 3.1. *Truthtelling strategies form a perfect Bayesian equilibrium in the mechanism* (χ^*, y)*. Furthermore, on the equilibrium path, the mechanism* (χ^*, y) *is budget-balanced in every period of the game.*

Proof. The logic of the proof is summarized as follows. Step 1 begins by considering a simpler mechanism (χ^*, z) where the transfer z_t^i to agent i is an adjusted amount of the sum of the realized outcome-decision payoffs of all the other agents in period $t-1$. We show that truthtelling strategies form a PBE in this mechanism. The idea, as in the standard VCG mechanism, is to make each agent the residual claimant of the full surplus. Step 2 proves that

[8] We let $y_0^i = y_1^i \equiv 0$ for each agent $i \in \mathcal{I}$.

the expected present value of agent i's gain from deviating in the mechanism (χ^*, y) is the same as in the simple mechanism (χ^*, z). Therefore, truthtelling strategies still form a perfect Bayesian equilibrium in the mechanism (χ^*, y). Lastly, Step 3 verifies that on the equilibrium path, the mechanism (χ^*, y) is budget-balanced in every period of the game.

Step 1. We consider a simpler mechanism (χ^*, z) where the allocation rule is still the efficient social policy χ^*, but the transfer agent i receives in period $t \geq 2$ is $z_t^i = \frac{1}{\delta} \sum_{j \neq i} w_{t-1}^j$, where $(w_{t-1}^1, w_{t-1}^2, \ldots, w_{t-1}^I)$ are the realized outcome-decision payoffs in period $t-1$.[9] By the one-stage deviation principle, to verify PBE it suffices to show that a one-stage deviation of any agent $i \in \mathcal{I}$ to reporting any $r_t^i \in \Theta^i$ instead of her true type $\theta_t^i \in \Theta^i$ in period t is unprofitable. If all agents choose the truthtelling strategy, then the expected discounted payoff of agent i in period t is

$$
\begin{aligned}
& u_i(\chi^*(\theta_t), \theta_t) + z_t^i + \mathbb{E}\left[\sum_{k \geq 1} \delta^k (u_i(\chi^*(\theta_{t+k}), \theta_{t+k}) + z_{t+k}^i) | \chi^*(\theta_t), \theta_t\right] \\
&= u_i(\chi^*(\theta_t), \theta_t) + z_t^i + \mathbb{E}\left[\sum_{k \geq 1} \delta^k (u_i(\chi^*(\theta_{t+k}), \theta_{t+k}) + \frac{1}{\delta} \sum_{j \neq i} u_j(\chi^*(\theta_{t+k-1}), \theta_{t+k-1})) | \chi^*(\theta_t), \theta_t\right] \\
&= \sum_{j \in \mathcal{I}} u_j(\chi^*(\theta_t), \theta_t) + z_t^i + \mathbb{E}\left[\sum_{k \geq 1} \delta^k \sum_{j \in \mathcal{I}} u_j(\chi^*(\theta_{t+k}), \theta_{t+k}) | \chi^*(\theta_t), \theta_t\right] \\
&= V(\theta_t) + z_t^i.
\end{aligned}
$$

[9] We let $z_0^i = z_1^i \equiv 0$ for each agent $i \in \mathcal{I}$.

Suppose that agent i reports r_t^i instead. Let $x = \chi^*(r_t^i, \theta_t^{-i})$. Then her expected discounted payoff in period t is

$$
\begin{aligned}
& u_i(x,\theta_t) + z_t^i + \mathbb{E}\left[\sum_{k\geq 1} \delta^k (u_i(\chi^*(\theta_{t+k}),\theta_{t+k}) + z_{t+k}^i)|x,\theta_t\right] \\
&= u_i(x,\theta_t) + z_t^i + \mathbb{E}\Bigg[\sum_{k\geq 2} \delta^k (u_i(\chi^*(\theta_{t+k}),\theta_{t+k}) + \frac{1}{\delta}\sum_{j\neq i} u_j(\chi^*(\theta_{t+k-1}),\theta_{t+k-1})) \\
&+ \delta(u_i(\chi^*(\theta_{t+1}),\theta_{t+1}) + \frac{1}{\delta}\sum_{j\neq i} u_j(x,\theta_t))|x,\theta_t\Bigg] \\
&= \sum_{j\in\mathcal{I}} u_j(x,\theta_t) + z_t^i + \mathbb{E}\left[\sum_{k\geq 1}\delta^k \sum_{j\in\mathcal{I}} u_j(\chi^*(\theta_{t+k}),\theta_{t+k})|x,\theta_t\right] \\
&= \sum_{j\in\mathcal{I}} u_j(x,\theta_t) + z_t^i + \delta\int_\Theta V(\tilde{\theta})Q(\mathrm{d}\tilde{\theta}|x,\theta_t).
\end{aligned}
$$

Since V is the social value function when the decision policy is χ^*, we have

$$
\begin{aligned}
V(\theta_t) &= \sum_{j\in\mathcal{I}} u_j(\chi^*(\theta_t),\theta_t) + \delta\int_\Theta V(\tilde{\theta})Q(d\tilde{\theta}|\chi^*(\theta_t),\theta_t) \\
&\geq \sum_{j\in\mathcal{I}} u_j(x,\theta_t) + \delta\int_\Theta V(\tilde{\theta})Q(d\tilde{\theta}|x,\theta_t).
\end{aligned}
$$

Thus, a one-stage deviation of any agent $i \in \mathcal{I}$ to reporting any $r_t^i \in \Theta^i$ instead of her true type $\theta_t^i \in \Theta^i$ in period t is unprofitable. Truthtelling strategies form a perfect Bayesian equilibrium.

Step 2. We prove that the expected present value of agent i's gain from deviating in the mechanism (χ^*, y) is the same as in the simple mechanism (χ^*, z).

In period $t-1$, consider the case where the true type profile is θ_{t-1} and the reported type profile is $(r_{t-1}^{-j}, \theta_{t-1}^{j})$ for some $j \in \mathcal{I}$. That is, agent j truthfully reports $\theta_{t-1}^{j} \in \Theta^{j}$ while the other agents arbitrarily report $r_{t-1}^{-j} \in \Theta^{-j}$. We have

$$
\begin{aligned}
&\int_{\Theta_j} \Phi_j(r_{t-1}^{-j}, \theta_{t-1}^{j}, \tilde{\theta}_t^{j}) Q_j(d\tilde{\theta}_t^{j} | \chi^*(r_{t-1}^{-j}, \theta_{t-1}^{j}), \theta_{t-1}^{j}) \\
&= \int_{\Theta_j} \int_{\Theta_{-j}} V(\tilde{\theta}_t^{j}, \tilde{\theta}_t^{-j}) Q_{-j}(d\tilde{\theta}_t^{-j} | \chi^*(r_{t-1}^{-j}, \theta_{t-1}^{j}), r_{t-1}^{-j}) Q_j(d\tilde{\theta}_t^{j} | \chi^*(r_{t-1}^{-j}, \theta_{t-1}^{j}), \theta_{t-1}^{j}) \\
&- \int_{\Theta} V(\tilde{\theta}_t) Q(d\tilde{\theta}_t | \chi^*(r_{t-1}^{-j}, \theta_{t-1}^{j}), r_{t-1}^{-j}, \theta_{t-1}^{j}) \\
&= 0,
\end{aligned}
$$

where the first equality follows from the definition of Φ_i.

Thus, for each agent $i \in \mathcal{I}$, if all the other agents truthfully report their types, then the expectation of the term $\sum_{j \neq i} \Phi_j(r_{t-1}, r_t^{j})$ in $\Psi_i(r_{t-1}, r_t)$ is 0 (from agent i's perspective) regardless of her own report. In other words, if agent i assigns probability 1 to the event that all the other agents truthfully report their types, then the term $\sum_{j \neq i} \Phi_j(r_{t-1}, r_t^{j})$ in the transfer y_{t+1}^{i} cannot distort her incentive.

Next we consider other terms $v(r_t) - \Phi_i(r_{t-1}, r_t^{i})$ in the transfer y_{t+1}^{i} that could potentially distort agent i's incentives. Suppose that all the other agents adopt the truthtelling strategy; that is, $r_{t-1}^{-i} = \theta_{t-1}^{-i}$ in period $t-1$. As for agent i, her past types are payoff-irrelevant since (1) the past types do not enter into her future outcome-decision payoff functions and transfers; and (2) her belief about the opponents' current types depends on her report, but not the true type, in the previous period. As a result, we can assume that agent i truthfully reports in period $t-1$. We focus on the case that the true type of agent i is θ_t^i but she reports r_t^i in period t.

In what follows, we consider the summation of the expectation of $v(r_{t+k}) - \Phi_i(r_{t+k-1}, r_{t+k}^i)$ for $k \geq 0$ (from agent i's perspective). If agent i deviates from

θ_t^i to r_t^i, we have

$$\int_{\Theta_{-i}} v(r_t^i,\tilde{\theta}_t^{-i})Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) - \Phi_i(\theta_{t-1},r_t^i)$$
$$+\sum_{k\geq 1}\delta^k\int_{\Theta_{-i}}\mathbb{E}\left(v(\tilde{\theta}_{t+k})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),\theta_t^i,\tilde{\theta}_t^{-i}\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i})$$
$$-\int_{\Theta_{-i}}\mathbb{E}\left(\Phi_i(r_t^i,\tilde{\theta}_t^{-i},\tilde{\theta}_{t+1}^i)|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),r_t^i,\tilde{\theta}_t^{-i})\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i})$$
$$-\sum_{k\geq 2}\delta^k\int_{\Theta_{-i}}\mathbb{E}\left(\Phi_i(\tilde{\theta}_{t+k},\tilde{\theta}_{t+k}^i)|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),r_t^i,\tilde{\theta}_t^{-i})\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i})$$
$$=\int_{\Theta_{-i}} v(r_t^i,\tilde{\theta}_t^{-i})Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.1}$$
$$-\int_{\Theta_{-i}} V(r_t^i,\tilde{\theta}_t^{-i})Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.2}$$
$$+\int_{\Theta} V(\tilde{\theta}_t)Q(d\tilde{\theta}_t|\chi^*(\theta_{t-1}),\theta_{t-1})$$
$$+\sum_{k\geq 1}\delta^k\int_{\Theta_{-i}}\mathbb{E}\left(v(\tilde{\theta}_{t+k})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),\theta_t^i,\tilde{\theta}_t^{-i}\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.3}$$
$$-\sum_{k\geq 1}\delta^k\int_{\Theta_{-i}}\mathbb{E}\left(V(\tilde{\theta}_{t+k})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),\theta_t^i,\tilde{\theta}_t^{-i})\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.4}$$
$$+\delta\int_{\Theta_{-i}}\mathbb{E}\left(V(\tilde{\theta}_{t+1})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),r_t^i,\tilde{\theta}_t^{-i})\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.5}$$
$$+\sum_{k\geq 2}\delta^k\int_{\Theta_{-i}}\mathbb{E}\left(V(\tilde{\theta}_{t+k})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),\theta_t^i,\tilde{\theta}_t^{-i})\right)Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.6}$$
$$=\int_{\Theta_{-i}}\Bigg[v(r_t^i,\tilde{\theta}_t^{-i}) - V(r_t^i,\tilde{\theta}_t^{-i})$$
$$+\delta\mathbb{E}\left(V(\tilde{\theta}_{t+1})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),r_t^i,\tilde{\theta}_t^{-i})\right)\Bigg]Q_{-i}(d\tilde{\theta}_t^{-i}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.7}$$
$$+\int_{\Theta} V(\hat{\theta}_t)Q(d\tilde{\theta}_t|\chi^*(\theta_{t-1}),\theta_{t-1})$$
$$+\sum_{k\geq 1}\delta^k\int_{\Theta_{-i}}\mathbb{E}\Bigg[v(\tilde{\theta}_{t+k}) - V(\tilde{\theta}_{t+k})$$
$$+\delta V(\tilde{\theta}_{t+k+1})|\chi^*(r_t^i,\tilde{\theta}_t^{-i}),\theta_t^i,\tilde{\theta}_t^{-i})\Bigg]Q_{-i}(d\hat{\theta}_{t(-i)}|\chi^*(\theta_{t-1}),\theta_{t-1}^{-i}) \tag{3.8}$$
$$=\int_{\Theta} V(\tilde{\theta}_t)Q(d\tilde{\theta}_t|\chi^*(\theta_{t-1}),\theta_{t-1}). \tag{3.9}$$

The first equality follows from the definition of Φ_i. Terms (3.1), (3.2), (3.5) aggregate to term (3.7) and terms (3.3), (3.4), (3.6) aggregate to term (3.8) respectively. It is easy to see that both terms (3.7) and (3.8) are equal to zero. Finally, (3.9) does not depend on agent i's report.

Therefore, the transfer scheme y together with χ^* provides each agent the same expected gain from deviating as the simple mechanism (χ^*, z). Since truthtelling strategies form a PBE in the latter mechanism, truthtelling strategies also form a PBE if the mechanism (χ^*, y) is adopted.

Step 3. We show that in the mechanism (χ^*, y), the transfers y^i_{t+1} balance the budget on the equilibrium path; that is, $\sum_{i \in \mathcal{I}} y^i_{t+1} = 0$. On the equilibrium path, agents truthfully report their types, $r_t = \theta_t$ and $w^i_t = u_i(\chi^*(\theta_t), \theta_t)$. We have

$$
\begin{aligned}
&\sum_{i\in\mathcal{I}} y_{t+1}^i \\
&= \sum_{i\in\mathcal{I}} \left[\frac{1}{\delta} \sum_{j\neq i} w_t^j - \frac{I-1}{I\delta} \left[v(\theta_t) - \Psi_i(\theta_{t-1}, \theta_t) \right] \right] \\
&= \frac{1}{\delta} \left[\sum_{i\in\mathcal{I}} \sum_{j\neq i} w_t^j - \sum_{i\in\mathcal{I}} \frac{I-1}{I} \left[v(\theta_t) - \Psi_i(\theta_{t-1}, \theta_t) \right] \right] \\
&= \frac{1}{\delta} \left[\sum_{i\in\mathcal{I}} \sum_{j\neq i} w_t^j - \sum_{i\in\mathcal{I}} \frac{I-1}{I} v(\theta_t) \right] \\
&= \frac{1}{\delta} \left[\sum_{i\in\mathcal{I}} \sum_{j\neq i} u_j(\chi^*(\theta_t), \theta_t) - (I-1) \sum_{i\in\mathcal{I}} u_i(\chi^*(\theta_t), \theta_t) \right] \\
&= 0,
\end{aligned}
$$

where the third equality is due to the following:

$$
\sum_{i\in\mathcal{I}} \Psi_i(\theta_{t-1}, \theta_t) = \sum_{i\in\mathcal{I}} \Phi_i(\theta_{t-1}, \theta_t^i) - \frac{1}{I-1} \sum_{i\in\mathcal{I}} \sum_{j\neq i} \Phi_j(\theta_{t-1}, \theta_t^j) = 0.
$$

□

3.4 Conclusion

In a dynamic environment with interdependent valuations and evolving private information, we construct an efficient, incentive-compatible dynamic mechanism that is also budget-balanced in every period of the game. To overcome the difficulties with interdependent valuations, we assume that in each period, each agent observes her own realized outcome-decision payoffs from the previous period. This extends the insight of Mezzetti (2004) to the dynamic setting.

We conclude with several observations. Firstly, our result can be generalized to the case where each agent only observes her own realized outcome-decision payoff after any finite number of periods. Secondly, we see no difficulties in extending our result to the case of time-dependent payoffs. This allows us to cover finite-horizon environments and in particular, Mezzetti (2004). Finally, in dynamic mechanism design problems with private values, the assumption that each agent observes her own realized outcome-decision payoff is trivially satisfied. Therefore, our result can also be viewed as a construction of an efficient, incentive-compatible and budget-balanced mechanism in this setting.

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Appendix A

Proofs of Chapter Two

A.1 Technical lemmas

In the following, we present several lemmas as the technical preparation for the proof of Proposition 1.[1]

If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measurable spaces, then a measurable rectangle is a subset $A \times B$ of $X \times Y$, where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ are measurable subsets of X and Y, respectively. The "sides" A, B of the measurable rectangle $A \times B$ can be arbitrary measurable sets; they are not required to be intervals. A discrete rectangle is a measurable rectangle such that each of its sides is a finite set.

[1] These lemmas extend the corresponding mathematical results in Arkin and Levin (1972) from the special case with $I = 2$ and λ the uniform distribution on $[0,1] \times [0,1]$ to the general setting in this paper. The corresponding mathematical results in Arkin and Levin (1972) were used to show the following result (see Theorem 2.3 therein): "Suppose that $f_1 \in L_1^\eta(X \times Y, \mathbb{R}^{l_1})$, $f_2 \in L_1^\eta(X \times Y, \mathbb{R}^{l_2})$ and $f_3 \in L_1^\eta(X \times Y, \mathbb{R}^{l_3})$, where $X = Y = [0,1]$ and η is the uniform distribution on $[0,1] \times [0,1]$. Let A be the simplex $\{a = (a_1, \ldots, a_K) \colon \sum_{1 \le k \le K} a_k = 1, a_k \ge 0\}$. Given any measurable function α from $X \times Y$ to A, there exists another measurable function $\overline{\alpha}$ from $X \times Y$ to the vertices of the simplex A such that $\int_{[0,1]} f_1(x,y)\alpha(x,y)\,\mathrm{d}y = \int_{[0,1]} f_1(x,y)\overline{\alpha}(x,y)\,\mathrm{d}y$, $\int_{[0,1]} f_2(x,y)\alpha(x,y)\,\mathrm{d}x = \int_{[0,1]} f_2(x,y)\overline{\alpha}(x,y)\,\mathrm{d}x$ and $\int_{[0,1]}\int_{[0,1]} f_3(x,y)\alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,1]}\int_{[0,1]} f_3(x,y)\overline{\alpha}(x,y)\,\mathrm{d}x\,\mathrm{d}y$."

Lemma A.1. *Let D be a Borel measurable subset of V, and $F \subseteq V$ a measurable rectangle with sides $Y_i \subseteq V_i$ of measure l_i, $i \in \mathcal{I}$. Assume that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$ for some $0 < \epsilon < 1$. Then for any i,*

$$\lambda_i\{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\} \geq (1-\sqrt{\epsilon})l_i.$$

Proof. Denote

$$\Gamma_i = \{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\}.$$

Let Γ_i^C be the complement of Γ_i in V_i. Then

$$\begin{aligned}
(1-\epsilon)\Pi_{1\leq j\leq I} l_j &= (1-\epsilon)\lambda(F) \\
&\leq \lambda(D \cap F) \\
&= (\int_{\Gamma_i} + \int_{\Gamma_i^C})\lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \int_{\Gamma_i} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) + \int_{\Gamma_i^C} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&\leq \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{\Gamma_i^C} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{Y_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\lambda_i(\Gamma_i)\cdot\Pi_{j\neq i} l_j + (1-\sqrt{\epsilon})\Pi_{1\leq j\leq I} l_j.
\end{aligned}$$

The first inequality holds due to the condition that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$. The second inequality is true since $\lambda_{-i}(D_{v_i} \cap F_{v_i}) \leq (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})$ for $v_i \in \Gamma_i^C$. All the equalities are just simple algebras. Rearranging the terms, we have

$$\lambda_i(\Gamma_i) \geq (1-\sqrt{\epsilon})l_i.$$

This completes the proof. $\square$

Lemma A.2. *Let D be a Borel measurable subset of V with $\lambda(D) > 0$, $\tilde{i}_1, \ldots, \tilde{i}_I$ be positive natural numbers, and $0 < \epsilon < 1$ be sufficiently small such that $\epsilon' = \Pi_{1\le j\le I}\tilde{i}_j \cdot \epsilon < 1$ and $\Pi_{1\le j\le I}\tilde{i}_j \cdot \epsilon'^{\frac{1}{2^I}} < 1$.*

Consider the system of measurable rectangles $F^{i_1,\ldots,i_I} = \prod_{1\le j\le I} Y_j^{i_j}$, where $1 \le i_j \le \tilde{i}_j$ and $Y_j^1, \ldots, Y_j^{\tilde{i}_j}$ are pairwise disjoint subsets on V_j for $1 \le j \le I$ such that $\lambda(F^{i_1,\ldots,i_I} \cap D) \ge (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$. Then there exists a discrete rectangle $\{v_1^{i_1}, \ldots, v_I^{i_I}\}_{\{1\le i_j\le \tilde{i}_j, 1\le j\le I\}}$ such that

1. *$(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D$ for $1 \le i_j \le \tilde{i}_j$ and $1 \le j \le I$;*

2. *for each $1 \le j \le I$, $\{v_j^{i_j}\}$ are different points for $1 \le i_j \le \tilde{i}_j$.*

Proof. First, we consider the set

$$\Gamma_1^{i_1,\ldots,i_I} = \{v_1 \in Y_1^{i_1} : \lambda_{-1}(D_{v_1} \cap F_{v_1}^{i_1,\ldots,i_I}) > (1-\sqrt{\epsilon'})\lambda_{-1}(F_{v_1}^{i_1,\ldots,i_I})\}.$$

Denote $\Gamma_1^{i_1} = \cap_{1\le i_k\le \tilde{i}_k, 2\le k\le I}\Gamma_1^{i_1,\ldots,i_I}$. We have

$$\begin{aligned}
\lambda_1(\Gamma_1^{i_1}) &= \lambda_1(Y_1^{i_1}) - \lambda_1\left(\cup_{1\le i_k\le\tilde{i}_k,2\le k\le I}(Y_1^{i_1} \setminus \Gamma_1^{i_1,\ldots,i_I})\right) \\
&\ge \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le\tilde{i}_k,2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - \lambda_1(\Gamma_1^{i_1,\ldots,i_I})\right) \\
&\ge \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le\tilde{i}_k,2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - (1-\sqrt{\epsilon'})\lambda_1(Y_1^{i_1})\right) \\
&= \left(1 - \Pi_{2\le k\le I}\tilde{i}_k \cdot \sqrt{\epsilon'}\right)\lambda_1(Y_1^{i_1}) \\
&> 0.
\end{aligned}$$

The second inequality holds due to Lemma A.1. We fix points $y_1^{i_1} \in \Gamma_1^{i_1}$ arbitrarily, as long as they are all distinct.

Second, let

$$\Gamma_2^{i_1,\ldots,i_I} = \{v_2 \in Y_2^{i_2} \colon (\bigotimes_{3\le k\le I} \lambda_k)(D_{(y_1^{i_1},v_2)} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)}) > (1-\epsilon'^{\frac{1}{4}})(\bigotimes_{3\le k\le I} \lambda_k)(F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)})\}.$$

Since $y_1^{i_1} \in \Gamma_1^{i_1}$ for any i_1, we have $y_1^{i_1} \in \Gamma_1^{i_1,\ldots,i_I}$ and

$$(\bigotimes_{2\le k\le I} \lambda_k)(D_{y_1^{i_1}} \cap F^{i_1,\ldots,i_I}_{y_1^{i_1}}) > (1-\sqrt{\epsilon'})(\bigotimes_{2\le k\le I} \lambda_k)(F^{i_1,\ldots,i_I}_{y_1^{i_1}}).$$

By Lemma A.1, we have

$$\lambda_2(\Gamma_2^{i_1,\ldots,i_I}) \ge (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2}).$$

Denote $\Gamma_2^{i_2} = \cap_{1\le i_j\le \tilde{i}_j, j\ne 2}\Gamma_2^{i_1,\ldots,i_I}$. We have

$$\begin{aligned}
\lambda_2(\Gamma_2^{i_2}) &= \lambda_2(Y_2^{i_2}) - \lambda_2\left(\cup_{1\le i_k\le \tilde{i}_k, k\ne 2}(Y_2^{i_2}\setminus \Gamma_2^{i_1,\ldots,i_I})\right)\\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le \tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - \lambda_2(\Gamma_2^{i_1,\ldots,i_I})\right)\\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le \tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2})\right)\\
&= \left(1 - \Pi_{1\le k\le I, k\ne 2}\tilde{i}_k \cdot \epsilon'^{\frac{1}{4}}\right)\lambda_2(Y_2^{i_2})\\
&> 0.
\end{aligned}$$

We fix points $y_2^{i_2} \in \Gamma_2^{i_2}$ arbitrarily, as long as they are all distinct, and are also different from $\{y_1^{i_1}\}$.

Repeating this procedure until $I-1$, we can find $y_k^{i_k} \in \Gamma_k^{i_k}$ for $1 \le i_k \le \tilde{i}_k$ and $1 \le k \le I-1$, where $\Gamma_k^{i_k} = \cap_{1\le i_j\le \tilde{i}_j, j\ne k}\Gamma_k^{i_1,\ldots,i_I}$ and $\lambda_k(\Gamma_k^{i_k}) > 0$. In particular,

$$\begin{aligned}\Gamma_{I-1}^{i_1,\ldots,i_I} =& \Big\{ v_{I-1} \in Y_{I-1}^{i_{I-1}} : \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})}) \\ & > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})})\Big\}.\end{aligned}$$

Finally, consider the set

$$E^{i_I} = \cap_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} \left(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I} \right).$$

Notice that $F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} = Y_I^{i_I}$ for any $i_1,\ldots,i_I$. Then

$$\begin{aligned}\lambda_I(E^{i_I}) &= \lambda_I\left(\cap_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} (D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \lambda_I\left(\cup_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} (Y_I^{i_I} \setminus D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &\ge \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} \left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} \left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &> \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} \left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k \le \tilde{i}_k, 1\le k\le I-1} \left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(Y_I^{i_I}) \right) \\ &= \left(1 - \Pi_{1\le k\le I-1} \tilde{i}_k \cdot \epsilon'^{\frac{1}{2^{I-1}}} \right) \lambda_I(Y_I^{i_I}) \\ &> 0.\end{aligned}$$

The second inequality holds since $y_{I-1}^{i_{I-1}} \in \Gamma_{I-1}^{i_{I-1}} \subseteq \Gamma_{I-1}^{i_1,\ldots,i_I}$, and hence

$$\lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}).$$

Fix points $y_I^{i_I} \in E^{i_I}$ arbitrarily, as long as they are all different, and are different from $\{y_j^{i_j}\}_{1\le j\le I-1, 1\le i_j\le \tilde{i}_j}$. By the choice of E^{i_I}, $(y_1^{i_1}, \ldots, y_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D$ for any $1 \le i_j \le \tilde{i}_j$ and $1 \le j \le I$. This completes the proof. □

Now we prove the last lemma.

Lemma A.3. *$\mathscr{E}$ is not dense in $L_1^\lambda(D,\mathbb{R})$.*[2] *In particular, there is a measurable function $d(v)$ with a finite set of values, which cannot be approximated in measure on $(D, \mathcal{B}(D), \lambda)$ by functions in $\mathscr{E}$.*

Proof. Let $g = \mathbf{1}_D$ be the indicator function of the set D, and $g_\delta(v) = \frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} g \, d\lambda$. By Lemma 4.1.2 in Ledrappier and Young (1985), $g_\delta \to g$ for λ-almost all $v \in \mathbb{R}^{lI}$ as $\delta \to 0$. Without loss of generality, we assume that this convergence result holds for each point of D and the function h is continuous on D.

Fix natural numbers $\tilde{i}_j$ satisfying the condition that $l \cdot \sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j) < \Pi_{1\le j\le I}\tilde{i}_j$. For any discrete rectangle $L = \{(v_1^{i_1}, \ldots, v_I^{i_I}) \in D\colon 1 \le i_j \le \tilde{i}_j, 1 \le j \le I\}$, we associate a linear mapping T_L from $\mathbb{R}^{\Pi_{1\le j\le I}\tilde{i}_j}$ to $\mathbb{R}^{l_0\cdot\sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j)}$:

$$T_L(w) = \{\sum_{j\notin J, 1\le i_j\le \tilde{i}_j} h(v_1^{i_1}, \ldots, v_I^{i_I}) \cdot w^{i_1,\ldots,i_I}\}_{1\le i_j\le\tilde{i}_j, j\in J, J\in\mathcal{J}},$$

where $l_0 = IKM + 1$, w is a vector with dimensions $\tilde{i}_1, \ldots, \tilde{i}_I$ and $w^{i_1,\ldots,i_I}$ is the corresponding component.

[2]Recall that $\mathscr{E}$ is defined in the proof of Proposition 1.

Fix a discrete rectangle $\overline{L} \subseteq D$ such that

- $\overline{L} = \{(\overline{v}_1^{i_1}, \ldots, \overline{v}_I^{i_I}) \in D \colon 1 \le i_j \le \tilde{i}_j, 1 \le j \le I\}$;

- the rank of the mapping $T_{\overline{L}}$ is maximal, say r.

Consider the system of $\sum_{J \in \mathcal{J}} (\Pi_{j \in J} \tilde{i}_j)$ homogeneous linear equations with $\Pi_{1 \le j \le I} \tilde{i}_j$ unknowns:

$$T_{\overline{L}}(w) = 0.$$

We take r equations and r unknowns for which the corresponding determinant is nonzero. Without loss of generality, we focus on this $r \times r$ matrix and denote it as $\overline{L}_s$, then $\det(\overline{L}_s) \neq 0$. For any discrete rectangle L, denote L_s as the restriction of the vector generated by the operator T_L onto the same matrix. Since h is continuous, $\det(L_s) \neq 0$ for any discrete rectangle L in a small open neighborhood of $\overline{L}$.

Let $w_{\overline{L}}$ be a nontrivial solution of the system corresponding to the discrete rectangle $\overline{L}$ in the sense that $T_{\overline{L}}(w_{\overline{L}}) = 0$. For any discrete rectangle $L \subseteq D$ such that $\det(L_s) \neq 0$, we provide a solution w_L below such that $T_L(w_L) = 0$.

- Since $\det(L_s) \neq 0$, the rank of the system corresponding to the operator T_L is at least r. Due to the choice of $\overline{L}$, the rank of the system corresponding to the operator T_L is at most r, and hence is r. As a result, the equations that do not occur in the determinant $\det(L_s)$ are linear combinations of the r equations that do.

- We focus on the r equations that occur in the determinant $\det(L_s)$, and let $w_L^{i_1,\dots,i_I} = w_{\underline{L}}^{i_1,\dots,i_I}$ if the column corresponding to the unknown $w_L^{i_1,\dots,i_I}$ does not occur in the determinant $\det(L_s)$.

- The remaining r unknowns of $w_L^{i_1,\dots,i_I}$, corresponding to the columns that occur in the determinant $\det(L_s)$, can be obtained by Cramer's rule.

By the above construction, it is obvious that w_L depends continuously on the r nodes of the discrete rectangle L corresponding to the columns of $\det(L_s)$.

Pick numbers $d^{i_1,\dots,i_I}$ subject to $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\dots,i_I}\cdot w_{\underline{L}}^{i_1,\dots,i_I} = 1$. Consider the measurable rectangles

$$G^{i_1,\dots,i_I} = \{v = (v_1,\dots,v_I)\in\mathbb{R}^{lI} \colon |v_j - \overline{v}_j^{i_j}| \le \delta,\ 1\le j\le I\},$$

and

$$F^{i_1,\dots,i_I} = \{v = (v_1,\dots,v_I)\in V \colon |v_j - \overline{v}_j^{i_j}| \le \delta,\ 1\le j\le I\}.$$

Then for sufficiently small δ, $\{G^{i_1,\dots,i_I}\}$ are pairwise disjoint, and $\{F^{i_1,\dots,i_I}\}$ are also pairwise disjoint.

By the first paragraph of this proof, $\frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} \mathbf{1}_D \,\mathrm{d}\lambda \to \mathbf{1}_D(v)$ for each $v\in D$. Since $(\overline{v}_1^{i_1},\dots,\overline{v}_I^{i_I})\in D$, $\lambda(G^{i_1,\dots,i_I}\cap D)\ge(1-\epsilon)\lambda(G^{i_1,\dots,i_I})$ for sufficiently small δ, where ϵ is given in the proof of Lemma A.2. Since D is a subset of V, we have

$$\lambda(F^{i_1,\ldots,i_I} \cap D) = \lambda(G^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(G^{i_1,\ldots,i_I}) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I}).$$

In addition, since $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I}$ is continuous in the discrete rectangle, for sufficiently small δ, $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \geq \frac{1}{2}$ for

$$L = \{(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}.$$

To summarize, we pick $\delta > 0$ sufficiently small such that

1. $\lambda(F^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$; and

2. $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \geq \frac{1}{2}$ for any discrete rectangle

$$L = \{(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}.$$

Let

$$d(v) = \begin{cases} d^{i_1,\ldots,i_I}, & \text{if } v \in F^{i_1,\ldots,i_I} \cap D; \\ 0, & \text{otherwise.} \end{cases}$$

If it could be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure, then there is a sequence $d_n(v) = h(v) \cdot \sum_{J\in\mathcal{J}} \psi_J^n(v_J)$ which converges to d on some Borel measurable subset C such that $\lambda(C) = \lambda(D)$.

By condition (1) above and Lemma A.2, there exists a discrete rectangle $L = \{(v_1^{i_1}, \ldots, v_I^{i_I})\}_{\{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I\}}$ such that $(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap C$ for

$1 \leq i_j \leq \tilde{i}_j$ and $1 \leq j \leq I$. Since $\sum_{1\leq i_j \leq \tilde{i}_j, j\notin J} w_L^{i_1,\ldots,i_I} h(v_1^{i_1},\ldots,v_I^{i_I}) = 0$ for any $J \in \mathcal{J}$, we have

$$
\begin{aligned}
&\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d_n(v_1^{i_1},\ldots,v_I^{i_I}) \cdot w_L^{i_1,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} \left(h(v_1^{i_1},\ldots,v_I^{i_I}) \cdot \sum_{J\in\mathcal{J}} \psi_J^n(v_J^{i_J}) \right) w_L^{i_1,\ldots,i_I} \\
&= \lim_{n\to\infty} \sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} \left\{ \left(w_L^{i_1,\ldots,i_I} h(v_1^{i_1},\ldots,v_I^{i_I}) \right) \cdot \sum_{J\in\mathcal{J}} \psi_J^n(v_J^{i_J}) \right\} \\
&= \lim_{n\to\infty} \sum_{J\in\mathcal{J}} \sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} \left\{ \left(w_L^{i_1,\ldots,i_I} h(v_1^{i_1},\ldots,v_I^{i_I}) \right) \cdot \psi_J^n(v_J^{i_J}) \right\} \\
&= \lim_{n\to\infty} \sum_{J\in\mathcal{J}} \sum_{1\leq i_j\leq \tilde{i}_j, j\in J} \left\{ \sum_{1\leq i_j\leq \tilde{i}_j, j\notin J} w_L^{i_1,\ldots,i_I} h(v_1^{i_1},\ldots,v_I^{i_I}) \right\} \cdot \psi_J^n(v_J^{i_J}) \\
&= 0,
\end{aligned}
$$

where $v_J^{i_J}$ denotes the vector $(v_j^{i_j})_{j\in J}$. However, $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \geq \frac{1}{2}$ by condition (2) above, which is a contradiction. As a result, the function d cannot be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. This completes the proof. □

A.2 Proof of Theorem 2.1

Let h be a function from V to $\mathbb{R}_{++}^{IKM+1}$ such that $h_0(v) \equiv 1$, and $h_{ikm}(v) = r_{im}^k(v_{-i})$[3] for each $i \in \mathcal{I}$, $1 \le k \le K$ and $1 \le m \le M$.[4] Let $\mathcal{J}$ be the set of all nonempty proper subsets of $\mathcal{I}$, and Υ be the set of all allocation rules. That is, given any $\tilde{q} \in \Upsilon$, $\tilde{q}$ is a measurable function and $\sum_{k \in \mathcal{K}} \tilde{q}^k(v) = 1$ for λ-almost all $v \in V$. For any coalition $J \subseteq \mathcal{I}$, denote $\lambda_J = \bigotimes_{j \in J} \lambda_j$.

Fix a Bayesian incentive compatible mechanism (q, t). We consider the allocation rule $\tilde{q} \in \Upsilon$ such that for any $J \in \mathcal{J}$ and λ_J-almost all $v_J \in V_J$,

$$\mathbb{E}\left(\tilde{q} h_j | v_J\right) = \mathbb{E}\left(q h_j | v_J\right) \tag{A.1}$$

for $j = 0$ or $j = ikm$, $i \in \mathcal{I}$, $1 \le k \le K$ and $1 \le m \le M$.

Definition A.1. *We define the following set Υ_q:*

$$\Upsilon_q = \{\tilde{q} \in \Upsilon \colon \tilde{q} \text{ satisfies Equation (A.1)}\}.$$

In what follows, we first provide the following characterization result for the set Υ_q: Υ_q is a nonempty, convex and weakly compact set in some Banach space.

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[3]Throughout this paper, IKM is the product of the integers I, K and M. However, the subscript ikm is not the product of the numbers i, k and m, but refers to the vector (i, k, m) identifying the function r_{im}^k.

[4]Denote $\mathbb{R}_{++}$ as the strictly positive real line. We assume that h is strictly positive without loss of generality. Indeed, we can work with the function h' from V to $\mathbb{R}_{+}^{2IKM+1}$ such that $h'_0(v) \equiv 1$, $h'^1_{ikm}(v) = |r_{im}^k(v_{-i})| + 1$, and $h'^2_{ikm}(v) = r_{im}^k(v_{-i}) + |r_{im}^k(v_{-i})| + 1$ for each $i \in \mathcal{I}$, $1 \le k \le K$ and $1 \le m \le M$. The function h' is strictly positive and suffices for our purpose.

Therefore, the classical Krein-Milman Theorem (see Royden and Fitzpatrick (2010, p. 296)) implies that Υ_q admits extreme points. We proceed by showing that all extreme points of the set Υ_q are deterministic mechanisms.[5] The existence of a deterministic mechanism that is equivalent in terms of interim expected allocation probabilities immediately follows. The equivalence in terms of interim expected utilities and ex ante expected social surplus follows from Equation (A.4) and the separable payoff assumption. The incentive compatibility of the deterministic mechanism follows from Equation (A.4) and the assumption that types are independent.

The following lemma characterizes the set Υ_q.

Lemma A.4. Υ_q *is a nonempty, convex and weakly compact subset.*

Proof of Lemma A.4. Clearly, the set Υ_q is nonempty and convex. We first show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$, where $L_1^\lambda(V, \mathbb{R}^K)$ is the L_1 space of all measurable mappings from V to $\mathbb{R}^K$ under the probability measure λ.

Suppose that the sequence $\{q_m\} \subseteq \Upsilon_q$ and $q_m \to q_0$ in $L_1^\lambda(V, \mathbb{R}^K)$. Then by the Riesz-Fischer Theorem (see Royden and Fitzpatrick (2010, p. 398)), there

[5] Manelli and Vincent (2007) use a related technique in the screening literature. Manelli and Vincent (2007) consider revenue maximizing multi-product monopolist and study the extreme points of the set of feasible mechanisms. They show that, with multiple goods, extreme points could be stochastic mechanisms. In contrast, we work with the mechanism design setting, study a particular set of interest Υ_q and show that all extreme points are deterministic. Apart from this general approach, the technical parts of the proofs are dramatically different.

exists a subsequence $\{q_{m_s}\}$ of $\{q_m\}$, which converges to q_0 λ-almost everywhere. Since $\sum_{k\in\mathcal{K}} q^k_{m_s}(v) = 1$ for λ-almost all v, $\sum_{k\in\mathcal{K}} q^k_0(v) = 1$ for λ-almost all v. As a result, $q_0 \in \Upsilon$.

For any $k \in \mathcal{K}$, $J \in \mathcal{J}$, and $\mathcal{B}(V_J) \otimes \left(\bigotimes_{1\le j\le I, j\notin J}\{V_j, \emptyset\}\right)$-measurable bounded mapping $p\colon V \to \mathbb{R}^K$,

$$\int_V (q^k_0 h_j) p\lambda(\mathrm{d}v) = \lim_{s\to\infty} \int_V (q^k_{m_s} h_j) p\lambda(\mathrm{d}v) = \int_V (q^k h_j) p\lambda(\mathrm{d}v)$$

for $j = 0$ or $j = ikm$. The first equality is due to the dominated convergence theorem, and the second equality holds since $\{q_{m_s}\} \subseteq \Upsilon_q$. Thus, $q_0 \in \Upsilon_q$, which implies that Υ_q is norm closed in $L^\lambda_1(V, \mathbb{R}^K)$.

Since Υ_q is convex, Υ_q is also weakly closed in $L^\lambda_1(V, \mathbb{R}^K)$ by Mazur's Theorem (see Royden and Fitzpatrick (2010, p. 292)). As Υ is weakly compact in $L^\lambda_1(V, \mathbb{R}^K)$, we have that Υ_q is weakly compact in $L^\lambda_1(V, \mathbb{R}^K)$, and hence has extreme points. □

Since Υ_q is a nonempty, convex and weakly compact set, Υ_q has extreme points. The following result shows that all extreme points of Υ_q are deterministic allocations.

Proposition A.1. *All extreme points of Υ_q are deterministic allocations.*

Proof of Proposition A.1. Pick an allocation rule $\tilde{q} \in \Upsilon_q$ which is not deterministic, we shall show that $\tilde{q}$ is not an extreme point of Υ_q.

Since $\tilde{q}$ is not deterministic, there is a positive number $0 < \delta < 1$, a Borel measurable set $D \subseteq V$ such that $\lambda(D) > 0$, and indices j_1, j_2 such that $\delta \leq \tilde{q}^{j_1}(v), \tilde{q}^{j_2}(v) \leq 1-\delta$ for any $v \in D$. For any $J \in \mathcal{J}$, let D_J be the projection of D on $\prod_{j \in J} V_j$. For any $v_J \in D_J$, let $D_{-J}(v_J) = \{v_{-J} \colon (v_J, v_{-J}) \in D\}$ (abbreviated as D_{v_J}).

Consider the following problem on $\alpha \in L^{\lambda}_{\infty}(D, \mathbb{R})$: for any $J \in \mathcal{J}$ and $v_J \in D_J$,

$$\int_{D_{-J}(v_J)} \alpha(v_J, v_{-J}) h(v_J, v_{-J}) \lambda_{-J}(\mathrm{d}v_{-J}) = 0. \tag{A.2}$$

Recall that h is a function taking values in $\mathbb{R}^{IKM+1}$. For simplicity, denote $l_0 = IKM + 1$. Define the set $\mathscr{E}$ as

$$\mathscr{E} = \{h(v) \cdot \sum_{J \in \mathcal{J}} \psi_J(v_J) \colon \psi_J \in L^{\lambda}_{\infty}(D_J, \mathbb{R}^{l_0}), \forall J \in \mathcal{J}\}.$$

Then a bounded measurable function α in $L^{\lambda}_{\infty}(D, \mathbb{R})$ is a solution to Problem (A.2) if and only if $\int_D \alpha \varphi d\lambda = 0$ for any $\varphi \in \mathscr{E}$. Lemma A.3 shows that $\mathscr{E}$ is not dense in $L^{\lambda}_{1}(D, \mathbb{R})$. By Corollary 5.108 in Aliprantis and Border (2006), Problem (A.2) has a nontrivial bounded solution α.

Without loss of generality, we assume that $|\alpha| \leq \delta$. We extend the domain of α to V by letting $\alpha(v) = 0$ when $v \notin D$. For every $v \in V$, define

$$\hat{q}(v) = \tilde{q}(v) + \alpha(v) \left(e_{j_1} - e_{j_2}\right);$$

$$\overline{q}(v) = \tilde{q}(v) + \alpha(v) \left(e_{j_2} - e_{j_1}\right).$$

Then $\sum_{k\in\mathcal{K}} \hat{q}^k(v) = \sum_{k\in\mathcal{K}} \overline{q}^k(v) = \sum_{k\in\mathcal{K}} \tilde{q}^k(v) = 1$. If $v \in D$, then $0 \le \hat{q}^{j_1}(v), \overline{q}^{j_2}(v) \le 1$ as $\delta \le \tilde{q}^{j_1}(v), \tilde{q}^{j_2}(v) \le 1-\delta$, and $\hat{q}^j(v) = \overline{q}^j(v) = \tilde{q}^j(v)$ for $j \neq j_1, j_2$. If $v \notin D$, then $\hat{q}(v) = \overline{q}(v) = \tilde{q}(v)$ as $\alpha(v) = 0$. Thus, $\hat{q}$, $\overline{q} \in \Upsilon$.

For any $J \in \mathcal{J}$ and $\mathcal{B}(V_J) \otimes \left(\bigotimes_{1\le j\le I, j\notin J}\{V_j, \emptyset\}\right)$-bounded measurable mapping $p \in L_\infty^\lambda(V, \mathbb{R}^K)$,

$$\int_V (\hat{q}\cdot p)h\lambda(\mathrm{d}v) = \int_V (\tilde{q}\cdot p)h\lambda(\mathrm{d}v) + \int_V \alpha(v)\left((e_{j_1} - e_{j_2})\cdot p(v)\right)h(v)\lambda(\mathrm{d}v).$$

Since

$$\begin{aligned}
&\int_V \alpha(v)\left((e_{j_1} - e_{j_2})\cdot p(v)\right)h(v)\lambda(\mathrm{d}v) \\
&= \int_{V_J}\int_{V_{-J}} \alpha(v)\left((e_{j_1} - e_{j_2})\cdot p(v)\right)h(v)\lambda_{-J}(\mathrm{d}v_{-J})\lambda_J(\mathrm{d}v_J) \\
&= \int_{V_J} (e_{j_1} - e_{j_2})\cdot p(v)\int_{V_{-J}} \alpha(v)h(v)\lambda_{-J}(\mathrm{d}v_{-J})\lambda_J(\mathrm{d}v_J) \\
&= 0,
\end{aligned}$$

we have that

$$\int_V (\hat{q}\cdot p)h\lambda(\mathrm{d}v) = \int_V (\tilde{q}\cdot p)h\lambda(\mathrm{d}v),$$

which implies that $\hat{q} \in \Upsilon_q$. Similarly, one can show that $\overline{q} \in \Upsilon_q$. Since $\hat{q}$ and $\overline{q}$ are distinct and $\tilde{q} = \frac{1}{2}(\hat{q} + \overline{q})$, $\tilde{q}$ is not an extreme point of Υ_q. □

Now we are ready to prove our main result.

Proof of Theorem 2.1. Fix a mechanism (q, t). The proof is then divided into two steps. In the first step, we obtain a deterministic allocation rule $\tilde{q}$

which has the same interim expected allocation probability with q. In the second step, we verify that $(\tilde{q}, t)$ and (q, t) deliver the same interim expected utility for each agent.

By Proposition A.1, every extreme point of Υ_q is a deterministic allocation rule. Therefore, we can fix a measurable allocation rule $\tilde{q}$ such that

1. $\tilde{q}^k(v) = 0$ or 1 for λ-almost all $v \in V$ and $1 \leq k \leq K$;

2. for any agent i and λ_i-almost all $v_i \in V_i$,

$$\int_{V_{-i}} \tilde{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}), \tag{A.3}$$

   and

$$\int_{V_{-i}} \tilde{q}(v_i, v_{-i})h_{jkm}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})h_{jkm}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) \tag{A.4}$$

   for any $j \in \mathcal{I}$, $1 \leq k \leq K$ and $1 \leq m \leq M$.

Let D_i be the subset of V_i such that Equation (A.3) or (A.4) does not hold. Then $\lambda_i(D_i) = 0$. Define a new allocation rule $\hat{q}$ such that

$$\hat{q}(v) = \begin{cases} q(v), & \text{if } v_i \in D_i \text{ for some } i \in \mathcal{I}; \\ \tilde{q}(v), & \text{otherwise.} \end{cases}$$

Then $\hat{q}^k(v) = 0$ or 1 for λ-almost all $v \in V$ and $1 \leq k \leq K$.

Fix agent i and $v_i \in V_i$. If $v_i \in D_i$, then $\hat{q}(v_i, v_{-i}) = q(v_i, v_{-i})$ and $\int_{V_{-i}} \hat{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i})$. If $v_i \notin D_i$, then

$$\begin{aligned}
\int_{V_{-i}} \hat{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{D_{-i}} \hat{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i}\setminus D_{-i}} \hat{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \int_{D_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i}\setminus D_{-i}} \tilde{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) \\
&= 0 + \int_{V_{-i}} \tilde{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}),
\end{aligned}$$

where $D_{-i} = \cup_{j\in\mathcal{I}, j\neq i}\left(D_j \times \prod_{s\in\mathcal{I}, s\neq i,j} V_s\right)$. The first equality holds by dividing V_{-i} as D_{-i} and $V_{-i} \setminus D_{-i}$. The second equality is due to the definition of $\hat{q}$. The third equality holds since $\lambda_{-i}(D_{-i}) = 0$. The last equality is due to the condition that $v_i \notin D_i$. As a result, Equation (A.3) holds for $\hat{q}$ and every $v_i \in V_i$. Similarly, one can check that Equation (A.4) also holds for $\hat{q}$ and every $v_i \in V_i$.

Suppose that the mechanism $(\hat{q}, t)$ is adopted. By Equation (A.3), the allocation rules q and $\hat{q}$ induce the same interim expected allocation. We need to check that they induce the same interim expected utility. If agent i observes the state v_i but reports v_i', then his payoff is

$$\begin{aligned}
&\int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})\hat{q}^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \\
&= \sum_{1\le k\le K} \sum_{1\le m\le M} \int_{V_{-i}} w_{im}^k(v_i) r_{im}^k(v_{-i})\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i') \\
&= \sum_{1\le k\le K} \sum_{1\le m\le M} w_{im}^k(v_i) \int_{V_{-i}} r_{im}^k(v_{-i})\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i') \\
&= \sum_{1\le k\le K} \sum_{1\le m\le M} w_{im}^k(v_i) \int_{V_{-i}} h_{ikm}(v)\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i') \\
&= \sum_{1\le k\le K} \sum_{1\le m\le M} w_{im}^k(v_i) \int_{V_{-i}} h_{ikm}(v)q^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i') \\
&= \sum_{1\le k\le K} \sum_{1\le m\le M} w_{im}^k(v_i) \int_{V_{-i}} r_{im}^k(v_{-i})q^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i') \\
&= \int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}).
\end{aligned}$$

The first and second equalities follow from the separable payoff assumption. The fourth equality follows from Equation (A.4) and also the assumption that types are independent. All other equalities are simple algebras. Thus, these two mechanisms (q, t) and $(\hat{q}, t)$ deliver the same interim expected utility for every agent. If (q, t) is Bayesian incentive compatible, then $(\hat{q}, t)$ is clearly Bayesian incentive compatible. This completes the proof. □