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# THE DISTORTION OF PLANE χ-WAVE AND ITS EFFECT ON ELASTIC SCATTERING IN COULOMB FIELD

By K. C. KAR

(*Received for publication, 20th April, 1944*)

**ABSTRACT**. A new method is discussed for finding the effect of distortion of the incident χ-wave. It is shown that to a first approximation the distortion has no effect on the intensity of elastic scattering and the wavestatistical value of the critical approach.

In the wavestatistical theory of elastic scattering already discussed by the writer (Kar, 1937), it is assumed that the incident electron wave is plane. The wave equation for it is

$$\Delta\chi + k^2\chi = 0; \quad k = \frac{2\pi mv}{h} \qquad \dots \quad (1)$$

On coming within the potential field the wave equation is modified and becomes

$$\Delta\chi + k^2\left(1 - \frac{V}{E}\right)\chi = 0 \qquad \dots \quad (2)$$

At small distance from the scattering centre the potential V becomes appreciable and has the effect of perturbing the wave-function thereby causing scattering. However, at large distance V is small but not negligible as in eq. (1). Its effect is to bring about slight distortion of the plane wave. The distorted plane wave then encounters the large potential field at close distance and is scattered.

The method adopted in the present paper for finding out the distortion is somewhat different from that of Temple (1928) and others.

The well known solution of (1) is

$$\chi = R(r)P_l^m(\cos\theta)e^{im\phi} \qquad \dots \quad (3)$$

where R satisfies the differential equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + k^2R - \frac{l(l+1)}{r^2}R = 0 \qquad \dots \quad (3.1)$$

On putting $\rho = kr$ and $R = \rho^{\frac{1}{2}}R_1$, we have from (3.1) after simple transformations

$$\frac{d^2R_1}{d\rho^2} + \frac{1}{\rho}\frac{dR_1}{d\rho} + \left\{1 - \frac{(l+\frac{1}{2})^2}{\rho^2}\right\}R_1 = 0 \qquad \dots \quad (3.2)$$

which is the well known Bessel's equation whose solution is

$$R_1 = A_l J_{l+\frac{1}{2}}(kr) \qquad \dots \quad (3.3)$$

Thus the solution of the wave equation (1) may be taken in the general form

$$\chi=\frac{1}{\sqrt{kr}}\sum_{l,m} A_{l,m}\, J_{l+\frac{1}{2}}(kr)P_l^m(\cos\theta)e^{im\phi} \quad \cdots \quad (4)$$

where each term of the series is a solution of (1). If, however, it is supposed that the incident electron wave is plane and is proceeding along $x$-axis, the solution of (1) is obviously

$$\chi=Ae^{ikx} \quad \cdots \quad (4.1)$$

As in this case the wave-function $\chi$ has axial symmetry about the $x$-axis, we cannot take the solution in the general form (4) but have to assume $m=0$. Thus (4) becomes

$$\chi=\frac{1}{\sqrt{kr}}\sum_{l} A_l J_{l+\frac{1}{2}}(kr)P_l(\cos\theta) \quad \cdots \quad (4.2)$$

It may be easily seen on expanding $e^{ikx}$ after Rayleigh (*vide* Theory of Sound, Vol. II) that the solutions (4.1) and (4.2) are identical.

Now, in the above treatment we have neglected the Coulomb potential at large distance and obtained the wave-function for a plane wave. If, however, the potential is taken into account, the wave is modified. And we have for the R—equation from (2), taking $V=+(Ze^2/r)$,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+k^2R-\frac{l(l+1)}{r^2}R-\frac{2\alpha' kZ}{r}R=0 \quad \cdots \quad (5)$$

where $\alpha'=\frac{2\pi e^2}{hv}$. On putting $\rho=kr$ and $R=\rho^{-\frac{1}{2}}R_1$ as before, we have from (5)

$$\frac{d^2R_1}{d\rho^2}+\frac{1}{\rho}\frac{dR_1}{d\rho}+\left\{1-\frac{(l+\frac{1}{2})^2}{\rho^2}\right\}R_1-\frac{2\alpha' Z}{\rho}R_1=0 \quad \cdots \quad (5.1)$$

Let us make the substitutions

$$R_1=(\rho-\rho_x)e^{i\alpha' Z}\,.\,R_2,\ \rho=kr,\ \rho_x=kx$$

Hence on differentiating we get

$$\frac{dR_1}{d\rho}=(\rho-\rho_x)^{i\alpha' Z}\left[\frac{dR_2}{d\rho}-\frac{i\alpha' Z}{\rho_x}R_2\right]$$

$$\frac{d^2R_1}{d\rho^2}=(\rho-\rho_x)^{i\alpha' Z}\left[\frac{d^2R_2}{d\rho^2}-\frac{i\alpha' Z}{\rho_x}\frac{dR_2}{d\rho}+\frac{i\alpha' Z}{\rho_x^3}.\rho R_2-\frac{i\alpha' Z}{\rho_x}\left(\frac{dR_2}{d\rho}-\frac{i\alpha' Z}{\rho_x}R_2\right)\right],$$

So we have from (5.1)

$$\frac{d^2R_2}{d\rho^2}+\frac{1}{\rho}\frac{dR_2}{d\rho}+R_2\left\{1-\frac{(l+\frac{1}{2})^2+\alpha'^2Z^2.\frac{\rho^2}{\rho_x^2}}{\rho^2}\right\}-2\alpha' Z\left(\frac{i}{\rho_x}\frac{dR_2}{d\rho}+\frac{R_2}{\rho}\right)$$

$$+\frac{i\alpha' Z}{\rho\rho_x}\left(\frac{\rho^2}{\rho_x^2}-1\right)R_2=0 \quad \cdots \quad (5.2)$$

At large distance $\rho_x \rightarrow \rho$ and so (5.2) may be written in the form

$$\frac{d^2R_2}{d\rho^2}+\frac{1}{\rho}\frac{dR_2}{d\rho}+R_2\left\{1-\frac{(l+\frac{1}{2})^2+\alpha'^2Z^2}{\rho^2}\right\}-\frac{2\alpha'Z}{\rho}\left(i\frac{dR_2}{d\rho}+R_2\right)=0 \qquad (5.3)$$

It may be easily seen that if $\alpha'Z$ is neglected, there would be no distortion and (5.3) would reduce to (3.2). Thus it follows that the distortion is negligible for high speed incident particles and also for scattering by light atoms.

The following method may be adopted to solve the differential eq. (5.3). For small distortion the last term of eq. (5.3) is quite small. So we neglect it at first and get the solution in Bessel Function as in (3.2). We have evidently

$$\left.\begin{aligned} R_2 &= A_s J_{s+\frac{1}{2}}(\rho) \\ (s+\tfrac{1}{2})^2 &= (l+\tfrac{1}{2})^2+\alpha'^2Z^2 \end{aligned}\right\} \qquad \ldots \quad (5.4)$$

On using the well known asymptotic value

$$\sqrt{\frac{\pi}{2\rho}}\, J_{s+\frac{1}{2}}(\rho) \backsim i^{-s}\frac{e^{i\rho}}{\rho}$$

it may be easily seen that

$$i\frac{dR_2}{d\rho}=-R_2 \qquad \ldots \quad (6)$$

for large value of $\rho$. Consequently, to this order of approximation, the last term in (5.3) vanishes. Thus (5.4) may be taken as an asymptotic solution of (5.3). Again, as $\alpha'^2Z^2$ is a small quantity of the second order we may neglect it and take $s=l$ in (5.4) and so the asymptotic value of the wave-function for the distorted incident wave becomes

$$\chi=\frac{1}{\sqrt{kr}}\sum_l A_l J_{l+\frac{1}{2}}(kr)\, e^{i\alpha \log k(r-x)} P_l(\cos\theta) \qquad \ldots \quad (7)$$

On summing the series in the usual way we have for the alternative form

$$\chi=Ae^{ikx+i\alpha \log k(r-x)} \qquad \ldots \quad (7.2)$$

in agreement with Temple's formula, his unimportant constant factors being supposed to be included in the unknown constant A.

Let us next find the effect of the above distortion on the first order scattering function. The well known differential equation determining the scattering function without distortion, is

$$\Delta(\lambda_1\chi_1)+k^2.\lambda_1\chi_1=\frac{8\pi^2m}{h^2}V\chi_0 \qquad \ldots \quad (7.2)$$

where $\chi_0$ is the undistorted incident wave-function. If, now, we take account of the effect of distortion, (7.2) should be

$$\Delta(\lambda_1\chi_1)+k^2\left(1-\frac{V}{E}\right)\lambda_1\chi_1=\frac{8\pi^2m}{h^2}V\chi \qquad \ldots \quad (7.3)$$

where $x$ is given by (7.1).

On solving (7.3) we have

$$\lambda_1\chi_1 = -\frac{A}{4\pi}\cdot\frac{8\pi^2 m}{h^2}\int V(r_1)e^{ikx_1 + i\alpha \log k(r_1 - x_1)} \cdot e^{-ikr_1 - i\alpha \log k(r_1 - x_1)}$$

$$\times e^{ikr_2 + i\alpha \log k(r_2 - x_2)} \cdot \frac{d\tau}{r_{12}} \quad \ldots \quad (7.4)$$

$$= -\frac{2\pi m A}{h^2} e^{i\alpha \log k(r_2 x_2)} \int V e^{ikx_1} \frac{e^{ikr_{12}}}{r_{12}} d\tau \quad \ldots \quad (7.5)$$

On integrating in the usual manner as before, we get for the scattaring function

$$\lambda_1\chi_1 = -\frac{A \operatorname{cosec}^2\frac{1}{2}\theta}{2mv^2} \cdot \frac{e^{ikr + i\alpha \log k(r-x)}}{r} F(r_0) \quad \ldots \quad (8)$$

where

$$F(r_0) = k' \int_{r_0}^{\infty} \sin k'r V(r) r dr \quad \ldots \quad (8.1)$$

It is the same as the formula of Temple when the wavestatistical connection for critical approach is neglected. On evaluating $F(r_0)$ for the coulomb repulsive force, we have for the relative intensity of scattering

$$I = \left(\frac{Ze^2}{2mv^2}\right)^2 \operatorname{cosec}^4\tfrac{1}{2}\theta \cos^2 k'r_0 \quad \ldots \quad (8.2)$$

being the same as the formula derived before by Kar (1937) without considering the effect of distortion. Thus, it follows that the distortion of the incident wave has no effect on the relative intensity of scattering. The critical approach $r_0$ is also unaffected and is the same as that given by Kar (1937), *viz.*,

$$r_0 = \rho \cdot \frac{Ze^2}{mv^2}(\operatorname{cosec}\tfrac{1}{2}\theta + 1)\,; \quad \rho = 1.35 \quad \ldots \quad (8.3)$$

In the above treatment I have not considered the effect of relativity. It is, however, obvious that by taking into account the distortion, one would find no change in the relativestic expressions for the intensity of scattering and the critical approach. It may be easily checked if necessary.

PHYSICAL LABORATORY
PRESIDENCY COLLEGE,
CALCUTTA

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