

ON SOME THERMODYNAMIC PROPERTIES OF A MIXTURE OF GAS AND RADIATION

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ABSTRACT. Emden's theorem on the expansion or contraction of a polytropic gas mass with conservation of polytropic index has been extended to the case of a variable polytrope, which under similar conditions is shown to have the distribution of polytropic index unchanged. A relation is worked out between N and n , the polytropic indices for (P, ρ) and (p, ρ) relations, P being the total pressure, and p , the gas pressure. From this, a method has been given for estimating the temperature distribution in a variable polytrope when its maximum and minimum polytropic indices are known.

I N T R O D U C T I O N

In the current astrophysical theories, stellar bodies are studied as configurations of hot gas masses in equilibrium, the radiation pressure playing in general an important role in the interior. Polytropic configurations with constant indices have been investigated very thoroughly as stellar models from this point of view. But every stellar mass may be accurately conceived as made up of shells of matter in polytropic equilibrium with the polytropic index varying continuously within the mass from shell to shell. Some attempts have been made to study such configurations recently. Eddington¹ set the problem of finding those properties of such a variable polytrope, which lie intermediate between the properties of the two polytropes whose indices are constant, and identical with the upper and lower limits of the index of the variable polytrope. In this paper two simple thermodynamic properties of such variable polytropes have been discussed. The first is an extension of Emden's theorem² on the property of four intersecting (p, V) curves, representing two pairs of polytropes of two different indices. It is shown that under similar conditions the distribution of polytropic index within a gas mass remains unchanged. Secondly, the relation between two polytropic indices corresponding to (i) gas pressure, and (ii) the total pressure has been investigated. Writing the gas law as

$$\frac{dp_g}{p_g} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

we define a polytropic index n by

$$\frac{dp_g}{p_g} = \left(1 + \frac{1}{n}\right) \frac{d\rho}{\rho} \quad \dots (1)$$

with
$$n = \frac{d(\log \rho)}{d(\log T)},$$

where n in general is variable. But the actual stellar material is a mixture of gas and radiation, and if P be the total pressure of gas and radiation, a polytropic relation of the type

$$\frac{dP}{P} = \left(1 + \frac{1}{N}\right) \frac{d\rho}{\rho} \quad \dots (2)$$

will subsist, where N is the effective polytropic index for the mixture of gas and radiation. We shall study the relation between N and n from purely thermodynamic considerations. Candler's³ discussion of variable polytropes is based on the (P, ρ) relation, as the total pressure P appears in the equation of hydrostatic equilibrium, and so his results involve N . A relation between N and n enables one to show that *in a sense* the temperature of a variable polytrope is intermediate between the temperatures of two polytropes with constant limiting indices, when the corresponding points are suitably defined.

EXTENSION OF EMDEN'S THEOREM TO VARIABLE POLYTROPIC

Let us take the different parts of a purely gas sphere to be in polytropic conditions characterised by different indices, and let the index n be varying continuously with the radial distance r (fig. 1).

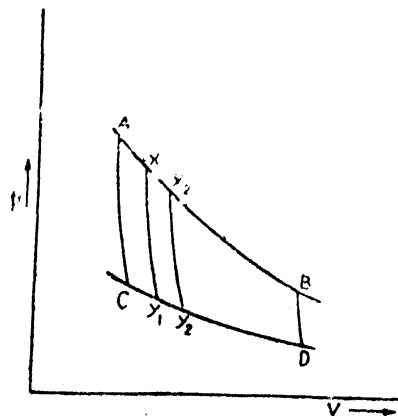


FIGURE 1

Let p be the pressure and V the specific volume at a point, and the (p, V) relation of any part of the gas sphere be represented by the continuous curve AB

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on the (p, V) plane (fig. 1). As the polytropic index is supposed to vary along AB, the heat capacity c will also vary along the curve, since $n = (c_p - c)/(c_p - c_v)$. We now prove the following theorem.

Theorem. If the gas-sphere undergoes some slow change (e.g., general expansion or contraction) such that every gas element undergoes a change along a polytrope of definite index n_0 , and the temperatures (or pressures, or specific volumes) of the corresponding gas elements are in a constant ratio, then the distribution of polytropic index in the gas mass after the change will remain the same as before the change. This is a purely thermodynamic property not connected with the mechanical equilibrium of the gas mass.

Let the points A, X_1, X_2, \dots, B , of the (p, V) curve move along the curves AC, $X_1Y_1, X_2Y_2, \dots, BD$, all representing polytropes of a definite index n_0 , such that $T_A/T_c = T_{X_1}/T_{Y_1} = T_{X_2}/T_{Y_2} = \dots = T_B/T_b$, then we have to prove that the distribution of polytropic index, i.e., of heat capacity c along CD, will be the same as that along AB.

Take a point X_1 so close to A that along AX_1 we can take the heat capacity c_1 to remain unaltered (or strictly speaking to lie within c_1 and $c_1 + dc_1$). Let X_1 move along a polytrope n_0 , to a position Y_1 such that

$$T_A/T_c = T_{X_1}/T_{Y_1}. \quad \dots (3)$$

Putting $T_{X_1} = T_A + dT_A$, and $T_{Y_1} = T_c + dT_c$, we obtain the relation

$$\frac{dT_A}{T_A} = \frac{dT_c}{T_c}. \quad \dots (3')$$

Now consider an infinitely slow quasistatic cycle round ACY_1X_1 . The closed integral $\int dQ/T$ round this cycle will vanish. The contribution of the side AC is $c_{n_0} \log(T_c/T_A)$, that of CY_1 is $c'_1 dT_c/T_c$, where $(c'_1, c'_1 + dc'_1)$ is the heat capacity for the element CY_1 ; the contributions of Y_1X_1 and X_1A are $c_{n_0} \log(T_{X_1}/T_{Y_1})$, and $-c_1 dT_A/T_A$ respectively. Hence we must have

$$c_{n_0} \log(T_c/T_A) + c'_1 dT_c/T_c + c_{n_0} \log(T_{X_1}/T_{Y_1}) - c_1 dT_A/T_A = 0 \quad \dots (4)$$

But by the above relations (3) the sum of the first and the third terms disappears, and so we have by (3')

$$(c'_1 - c_1) dT_c/T_c = 0,$$

that is

$$c'_1 = c_1. \quad \dots (5)$$

By dividing the curve AB into infinitesimal elements and considering similar circuits, we prove that the distribution of c , i.e., of n along CD, is the same as that along AB.

Since (A,C) and (X_1, Y_1) are on polytropes n_o , we have from the gas law and the polytropic condition

$$\frac{RT_A}{RT_c} = \left(\frac{V_c}{V_A}\right)^{\gamma + \frac{1}{n_o}}, \quad \frac{V_A}{V_c} = \left(\frac{V_c}{V_A}\right)^{\frac{1}{n_o}}, \quad \frac{RT_{x_1}}{RT_{y_1}} = \left(\frac{V_{y_1}}{V_{x_1}}\right)^{\frac{1}{n_o}}$$

whence by (3)

$$V_c/V_A = V_{y_1}/V_{x_1} \quad \dots (6)$$

Similarly

$$\frac{p_c}{p_A} = \frac{p_{y_1}}{p_{x_1}}$$

Hence the condition "temperatures of the corresponding gas elements being in the same ratio" in the above theorem can be replaced by "specific volumes or pressures of the corresponding gas elements being in the same ratio." The proof of the theorem can also be based on properties (5) and (6) of polytropes of constant index. This theorem is a generalisation of the well-known theorem due to Emden in which AB and CD represent two uniform polytropes.

It is known that for uniform expansion or contraction of a gas-sphere every element undergoes a polytropic change belonging to the index 3, and further the temperatures and pressures at corresponding points are proportional. Applying the theorem proved above to uniform expansion or contraction, we obtain the result that when a gas sphere with variable polytropic index expands or contracts uniformly, the distribution of polytropic index along the radius remains unaltered. This indeed may also be proved directly.

A RELATION BETWEEN N AND n

We shall now obtain a relation between N and n for a mixture of gas and radiation. Let us start with the gas law in the form

$$p_g V = RT \quad \dots (7)$$

where V is the specific volume, so that (2) may be written as

$$\frac{dP}{P} + \left(1 + \frac{1}{N}\right) \frac{dV}{V} = 0. \quad \dots (2')$$

$$\text{Now } dP = d(p_r + p_g) = d\left(\frac{1}{3}aT^4 + \frac{RT}{V}\right) = (4p_r + p_g) \frac{dT}{T} - p_g \frac{dV}{V},$$

whence, putting $p_g/p_r = \beta/(1-\beta)$, we obtain

$$\frac{dP}{P} + \left[1 + (\beta - 1) - (4 - 3\beta) \frac{d(\log T)}{d(\log V)}\right] \frac{dV}{V} = 0. \quad \dots (8)$$

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Comparing (8) with (2') we have

$$\frac{I}{N} = (\beta - 1) - (4 - 3\beta) \frac{d(\log T)}{d(\log V)} = (\beta - 1) + \frac{4 - 3\beta}{n} \quad \dots (9)$$

This equation can also be put as

$$N = \frac{n}{(4 - 3\beta) - n(1 - \beta)} \quad \dots (9')$$

For a fixed β , corresponding to one value of n there is one value of N and *vice versa*. Also as (for fixed n)

$$\frac{dN}{d\beta} = - \frac{n(n-3)}{[(4-3\beta) - n(1-\beta)]^2},$$

with increasing β , N increases if $n < 3$, and decreases if $n > 3$. For $n = 3$, $N = 3$; also as $\beta \rightarrow 1$, $N \rightarrow n$. We cannot calculate N as $\beta \rightarrow 0$ from (9'), as for $\beta = 0$, $p_g = 0$. In case of pure radiation pressure we know $N = 3$ [though (9)' gives $N \rightarrow n/(4-n)$, as $\beta \rightarrow 0$]. We can now plot on the (β, N) plane the curves for fixed values of n .

From (9) and (9)', the following points may be easily seen :

(a) through every point of the (β, N) plane there passes one and only one curve of the family $n = \text{const.}$, so that the curves $n = \text{const.}$ do not intersect (as for given N and β there is only one value of n) ;

(b) a curve $n = \text{const.}$ for $N > 3$, continually approaches the line $N = n$ from above, so that for every value of $\beta \neq 1$, if $N > 3$, then n also is > 3 , but less than N ; at $\beta = 1$, $N = n$; the curves for constant n rapidly rise as the N -axis is approached, and in fact the curve $n = 4$ is asymptotic to the N -axis; for $n > 4$, the line $\beta = (n-4)/(n-3)$ is an asymptote; a smaller value of β would make N negative, and for a time numerically very large ;

(c) any curve $n = \text{const.}$ for $0 < N < 3$ starts from the N -axis, continuously rises and asymptotically approaches the line $N = n$ from below; at every point on this curve (except for $\beta = 1$) $N < n$, so that for every value of $\beta \neq 1$, if N be positive and < 3 , then $n > N$, but less than 3; at $\beta = 1$, $N = n$;

(d) on a line $\beta = \text{const.} \neq 0$, the n values increase continuously upwards with increasing positive values of N ; if on this line $0 < N_1 < N_2$, then $n_1 < n_2$.

(e) the line $N = 3$ is singular in this respect that on this line $N = n = 3$ for every value of β .

The curves drawn in fig. 2 show these properties clearly. In a stellar body, let N lie between two values, say $N_2 (< 3)$ and $N_1 (> 0)$, and in the figure let the lines AB and CD represent $N = N_2$ and N_1 respectively, and let the range of β

values, within which the pressure ratio varies within the stellar body (excluding the extreme outer portion), be marked as in the figure by the lines AC and BD. Draw the curve $n_1 = \text{const.}$ through the point D ; we have proved that $n_1 > N_1$. Similarly if A lies on the curve $n_2 = \text{const.}$, $n_2 > N_2$.

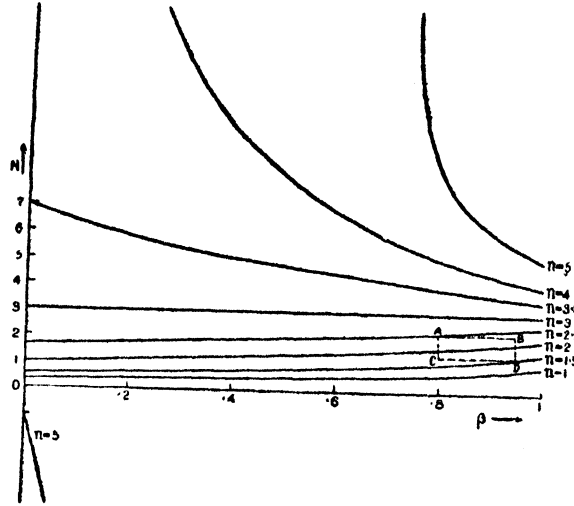


FIGURE 2

Then for any point within the stellar body the index n shall lie between n_1 and n_2 , so that

$$n_1 < \frac{d(\log \rho)}{d(\log T)} < n_2, \quad \dots (10)$$

whence

$$\left(\frac{T}{T_0} \right)^{n_1} < \left(\frac{\rho}{\rho_0} \right) < \left(\frac{T}{T_0} \right)^{n_2} \quad \dots (10')$$

T_0, ρ_0 being the temperature and density at the centre. Hence the density increases at a rate intermediate between the n_1 - and n_2 -th power of T . Relations (10) and (10') evidently subsist also for values of $N > 3$.

LIMITS OF TEMPERATURE IN A VARIABLE POLYTROPE

We can also use the above considerations to find a pair of limits within which the temperature of a gas configuration will lie. For this purpose we shall call points within different gas masses with the same value of β (otherwise the ratio of radiation to gas pressure) *corresponding points*. Any two such masses need not have corresponding points, but for the purpose of the arguments which follow, the different gas masses will be supposed to have corresponding points.

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Suppose we have a stellar body whose polytropic index N varies from N_1 to N_2 , and first suppose $0 < N_1 < N_2 < 3$. If the range of β values within the gas mass be known, we can find in the manner of the previous article two index-numbers n_1 and n_2 ($n_1 < n_2$) such that all n values of the configuration lie between n_1 and n_2 . When both N_1 and N_2 are less than 3, we have seen that $n_1 > N_1$, $n_2 > N_2$, but n_1, n_2 are also both less than 3. We now compare the given gas configuration with two polytropes of constant indices n_1 and n_2 . Such a polytrope will have two disposable parameters, one, say, the central temperature, and the second the factor of proportionality between the pressure and, say, the $\left(1 + \frac{1}{n_1}\right)$ th power of the density (for the first polytrope) in the pressure-density relation. Thus we can choose the first polytrope n_1 as one whose central values of β will be the same as that of the given gas configuration, and also the central temperature of the two identical. It can easily be shown that these two conditions determine the polytrope n_1 completely. Similarly, with the same conditions we determine the polytrope n_2 . We may, for the sake of generality, drop the condition of the equality of central temperatures, and have a single infinity of polytropic configurations for n_1 or n_2 . The following arguments will apply to this general case. With these adjustments we shall have a series of triplets of corresponding points starting from the centre for the given configuration and the two polytropes n_1, n_2 , since in the polytropes $(n_1, n_2) < 3$, β will increase outwards and tend towards 1 near the surface and in the given gas configuration ($n < 3$), β will also do so. Consider a triplet of corresponding points on the configurations characterised by n_1, n and n_2 . At these points we have

$$\frac{\rho_1}{\mu_1 T_1^3} = \frac{\rho}{\mu T^3} = \frac{\rho_2}{\mu_2 T_2^3}$$

so that (μ being supposed uniform throughout any one configuration and for simplicity may be taken to be the same in all of them)

$$\frac{d\rho_1}{\rho_1} - 3 \frac{dT_1}{T_1} = \frac{d\rho}{\rho} - 3 \frac{dT}{T} = \frac{d\rho_2}{\rho_2} - 3 \frac{dT_2}{T_2} \quad \dots \quad (11)$$

We have further

$$\frac{d\rho_1}{\rho_1} = n_1 \frac{dT_1}{T_1}, \quad \frac{d\rho}{\rho} = n \frac{dT}{T}, \quad \frac{d\rho_2}{\rho_2} = n_2 \frac{dT_2}{T_2} \quad \dots \quad (12)$$

From these two sets of equations follows

$$(3 - n_1) \frac{dT_1}{T_1} = (3 - n) \frac{dT}{T} = (3 - n_2) \frac{dT_2}{T_2} \quad \dots \quad (13)$$

as a relation between the temperature increments at corresponding points. Since $0 < n_1 < n < n_2 < 3$, for positive increments of temperature we must have

$$\frac{dT_2}{T_2} > \frac{dT}{T} > \frac{dT_1}{T_1},$$

whence integrating we obtain

$$\frac{T_2}{T_2'} < \frac{T}{T'} < \frac{T_1}{T_1'} \quad \dots \quad (14)$$

where (T_1', T', T_2') are the temperatures at a triplet of corresponding points on the (n_1, n, n_2) configurations, which are respectively *interior* to the corresponding points where the temperatures are T_1, T, T_2 . (14) shows that the relative temperature decrease in the given configuration is intermediate between the relative decreases in the two polytropes n_1 and n_2 at corresponding points. This relation will apply as long as triplets of corresponding points are available in the three gas masses, and so for the entire given gas configuration.

If we take $T_2' = T' = T_1'$ = the central temperature of the given configuration, then $T_2 < T < T_1$. The three masses have then also the same central densities and temperatures. Hence if $N < 3$, the temperature of the given configuration at any point is intermediate between the temperatures at corresponding points of the polytropes n_1 and n_2 with the same central density and temperature as the given gas mass.

In a similar manner, it can be shown that if N lies between N_1 and N_2 both of which exceed 3, then in the same manner as before two indices $n_1 (< N_1)$ and $n_2 (> N_2)$ both > 3 can be obtained such that any n corresponding to N lies between n_1 and n_2 . In this case we obtain by comparing with polytropes n_1 and n_2 as above the inequality (14) reversed thus

$$\frac{T_1}{T_1'} < \frac{T}{T'} < \frac{T_2}{T_2'}; \quad \dots \quad (14a)$$

and for polytropes of the same central density and temperature, $T_1 < T < T_2$.

The important case, however, is when n_1 is less than 3, but n_2 greater than 3. In such a case β should increase outwards for some parts of the gas mass (so long as $N < 3$), and decrease for others (when $N > 3$). An estimate like (14) or (14a) is then possible only if the β value corresponding to the point where N attains the value 3 (then n also is 3) is known. A rough guess can sometimes be made for known stellar mass when it is considered to conform to an Eddington model for which β is connected with the mass. We can then find $n_1 (< 3)$ and $n_2 (> 3)$ such that for values of N from N_1 to 3 (β increasing outward) the n values corresponding to N are greater than n_1 , and between the values of N from 3 to N_2 the corresponding n values are less than n_2 . n_1 and n_2 are indeed the n values on the lines $N_1 = \text{const.}$ and $N_2 = \text{const.}$ respectively of the points where they are cut by the line $\beta_{(3)} = \text{const.}$, this $\beta_{(3)}$ being the point where the N

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curve of the given gas mass cuts the line $N = 3$. Knowing n_1 and n_2 , equations like (14) and (14a) can be obtained easily for the two parts of the gas mass.

All these results depend only on the thermodynamic properties of the gas mass and not on its mechanical property of equilibrium. They may be useful in guessing rough values in the construction of stellar models.

BEHAVIOURS OF N AND n FOR LARGE AND SMALL VALUES

As the n value approaches and exceeds 4, the thermodynamically allowable values of N may increase to any positive value up to ∞ , and may even become negative. The infinite value of N occurs as we have seen for $\beta = (n-4)/(n-3)$, i.e., for $\beta/(1-\beta) = n-4$. Thus, if n be near to, say, 4.5, large values of N will occur when the radiation pressure is nearly double the gas pressure; on the other hand, if n be near 5, large values of N will occur if these two pressures be nearly equal. For smaller values of $\beta/(1-\beta)$, even negative values of N are thermodynamically permissible. On the other hand n may be negative, if N be negative and $|N|$ less than $(1-\beta)^{-1}$.

It may be noted that N and n vanish together in the ratio of 1 to $4-3\beta$. This fact will be of importance in a later investigation. Moreover as $N \rightarrow \infty$, i.e., the gas mass is approximately isothermal, n will tend to the value $3 - (1-\beta)^{-1}$. Lastly, as $n \rightarrow \infty$, $N \rightarrow -(1-\beta)^{-1}$.

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