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ON A THEORETICAL ESTIMATE OF AN UPPER LIMIT OF STELLAR DIAMETERS

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ABSTRACT. Considering the stellar body as a polytrope whose index varies from shell to shell, an upper bound has been obtained, from the conditions of mechanical equilibrium, for the product rT (radial distance × temperature), when the minimum value of the polytropic index and the maximum value of the ratio of radiation to gas pressure within the gas mass are known. Taking the minimum value of the index as 1.5 and 3 approximately for small and large stars respectively, and defining the "radius" as the distance where the temperature falls to about a million degrees, the value of this radius has been calculated in terms of the mass of the configuration, and also of the maximum value of the ratio of the pressures. For stars of small masses these calculated rough upper bounds are not unsatisfactory, but for large masses they are rather too high.

INTRODUCTION

Though stellar bodies generally show a small range of variation in their masses, their radii vary within wide limits. Several inequalities are known giving tolerably good estimates of some of the physical characteristics, such as the central pressure, mean temperature, etc., of stars of known masses and radii, but no purely theoretical formula has been given for an estimation of the radius. In the present note is attempted an estimate of an upper limit to the size of a stellar body, primarily in terms of the ratio of the pressures, and finally in terms of the mass. The limits indeed are quite rough, but considering the fact that they do not involve the opacity factor, the law of energy generation, etc., and depend only on the condition of mechanical equilibrium, these rough values may be of some interest as setting some limit to the arbitrariness of the dimensions of stars purely from conditions of mechanical equilibrium. In the deduction of the relation it is necessary to assume some compressibility condition, a relation between pressure and density. We have taken quite a general type of such a relation, namely that for a variable polytrope,

$$d(\log P)=\left(1+\frac{1}{n}\right)d(\log \rho) \qquad \dots \quad (1)$$

where the polytropic index n is variable from point to point of the star. Candler[1] has recently investigated the physical properties of such a polytrope. The estimates of upper limits made in this paper are dependent on Candler's[1] results. The next paragraph recapitulates the relevant results of Candler.

If P be the pressure, $\rho(r)$ the density, and $\bar{\rho}(r)$ the mean density within a sphere of radius r, two variables X and Y can be defined thus

$$X=\left(\frac{2\pi G}{3}\right)^{\frac{1}{2}}\frac{r\bar{\rho}(r)}{P^{\frac{1}{2}}},\quad Y=\left(\frac{2\pi G}{3}\right)^{\frac{1}{2}}\frac{r\rho(r)}{P^{\frac{1}{2}}}. \qquad \dots \quad (2)$$

Taken in conjunction with

$$\frac{dP}{dr}=-\frac{GM(r)}{r^2}\rho(r) \qquad \dots \quad (3)$$

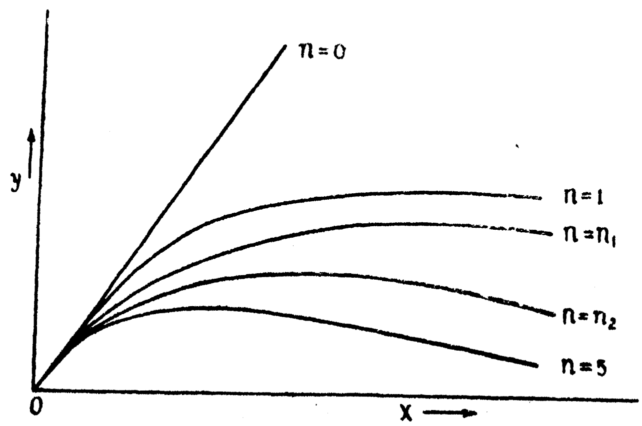
and

$$\frac{dM(r)}{dr}=4\pi\rho(r)r^2,\quad M(r)=\frac{4\pi}{3}\bar{\rho}(r)r^3 \qquad \dots \quad (4)$$

and (1), it may be shown that X and Y satisfy a differential equation of the first order as follows :

$$\frac{dY}{dX}=\frac{Y-\frac{n-1}{n+1}XY^2}{(3+X^2)Y-2X}. \qquad \dots \quad (5)$$

The solution curves of this equation have a very important characteristic. They all emerge from the origin and have unit slope there. The solution curves for $n=1$, $n=5$ are shown in the figure, as well as those for $n_1=$const., $n_2=$const. If $-1<n_1<n_2<5$, the solution curve for $n_1=$const. lies entirely above that for $n_2=$const., and both above that for $n=5$.



FIGURE

Now, if in a certain stellar configuration n is variable, and lies between n_1 and n_2, then the solution curve for the star in X, Y plane will lie entirely between the curves for n_1 = const. and n_2 = const. From this important characteristic of the solution curves it can be concluded that many of the characteristics of the variable polytrope are intermediate between the corresponding characteristics of the polytropes n_1 = const. and n_2 = const. at corresponding points (points with same value of X). For instance, if β_c be the ratio of gas pressure to total pressure at the centre of a star whose polytropic indices lie between n_1 and n_2, then

$$D(n_2)\,\frac{\pi a G^3 H^4}{18\kappa^4}\,\mu_c{}^4 M^2 < \frac{1-\beta_c}{\beta_c{}^4} < D(n_1)\,\frac{\pi a G^3 H^4}{18\kappa^4}\,\mu_c{}^4 M^2 \qquad \ldots \quad (6)$$

where $D(n_1)$ and $D(n_2)$ are constants whose values have been tabulated by Candler[1], and all other symbols have their usual meanings.

DEDUCTION OF AN UPPER BOUND

It is to be noticed first of all that, for a uniform polytrope (n = constant throughout), the variables X and Y are the two invariants of the Emden equation involving the Emden function and its first derivative, with respect to a Lane-transformation. It may thus be expected that in terms of these two variables the usual second-order equation will reduce to a first-order one. In fact, if we transform X and Y to usual polytropic variables for n = constant, we obtain

$$Y = \sqrt{\frac{n+1}{6}}\, I_1, \quad X = \sqrt{\frac{3(n+1)}{2}} \cdot \frac{I_2}{I_1^{\frac{n+1}{n-1}}} \qquad \ldots \quad (7)$$

where I_1, I_2 are the two invairants

$$I_1 = \xi u^{\frac{n-1}{2}}, \qquad I_2 = -\xi^{\frac{n+1}{n-2}}\frac{du}{d\xi}, \qquad \ldots \quad (8)$$

ξ and u being the variables of the normalised Emden equation.[2] The property which is important in the present discussion is that of I_1. It possesses *a single maximum for given n.*[2] The values of these maxima for different n are taken from the Tables of Emden functions[3] and shown in Table 1.

TABLE 1

n	$\sqrt{6}\ Y_{max}$
1.5	3.02
2	2.65
3	2.34
4	2.20
4.5	2.46₈

We can easily construct a function involving radius, temperature, and the ratio of the radiation to gas pressure, which is a function of Y only.

Putting

$$p_g = \beta P = \frac{\kappa}{\mu H}\rho T, \quad p_r = \tfrac{1}{3}aT^4 \qquad \ldots \quad (9)$$

we get

$$q = \frac{1-\beta}{\beta} = \frac{a}{3}\,\frac{\mu H}{\kappa}\,\frac{T^3}{\rho}$$

and

$$\frac{q^{\frac{1}{2}}}{T} = \left(\frac{a}{3}\,\frac{\mu H}{\kappa}\right)^{\frac{1}{2}} \frac{T^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} = \left(\frac{a}{3}\right)^{\frac{1}{2}} \frac{\mu H}{\kappa}\,\frac{\beta^{\frac{1}{2}} P^{\frac{1}{2}}}{\rho}.$$

Thus

$$\left(\frac{q}{\beta}\right)^{\frac{1}{2}} \frac{1}{rT} = \left(\frac{a}{3}\right)^{\frac{1}{2}} \frac{\mu H}{\kappa}\left(\frac{2\pi G}{3}\right)^{\frac{1}{2}} Y^{-1}$$

or

$$\frac{rT}{\sqrt{q(1+q)}} = \frac{rT}{(1-\beta)^{\frac{1}{2}}/\beta} = \left(\frac{9}{2\pi a G}\right)^{\frac{1}{2}} \frac{\kappa}{\mu H}\,Y. \qquad \ldots \quad (10)$$

Regarding the stellar body as a variable polytrope, let us suppose that its polytropic index n lies between two limits n_1 and n_2, so that $n_1 \leq n \leq n_2$. In case of central degeneracy (non-relativistic), or of stars whose masses are not large we may take $n_1 = 1.5$, while n_2 may be taken to be less than 5. Thus, if the complicated pressure-density relation within a star be represented by a variable n [defined by (1)] with 1.5 as its lowest value, then by Candler's theorem the value of Y at any point within the star will be less than that for the curve $n = 1.5$ at the corresponding point. In the general case $n_1 \leq n \leq n_2$, this value will be less than that for the curve n_1 at the corresponding point. But we have seen that for a fixed n_1, the relation between Y and I is given by (7), and then Y has a single maximum whose value is shown in Table I. Hence the value of Y anywhere within the star will be less than this maximum value of Y for the said fixed value of n_1 (which is the minimum value of n within the star). Hence from (10) we obtain

$$\frac{rT}{\sqrt{q(1+q)}} = \frac{rT}{(1-\beta)^{\frac{1}{2}}/\beta} < \frac{\kappa}{\mu H}\left(\frac{3}{4\pi a G}\right)^{\frac{1}{2}} \left[(n_1+1)^{\frac{1}{2}} \xi u^{(n-1)/2}\right]_{\max} \equiv A. \quad \ldots \quad (11)$$

For stars of small and moderate masses, we take $n_1 = 1.5$, for which we get

$$A = 5.4 \times 10^{18} \mu^{-1} (cm.)(deg.). \qquad \ldots \quad (12)$$

For most small stars, the ratio of the radiation to gas pressure increases inwards, and may be taken to be maximum at the centre. This is always true if n does not exceed 3.[4] For stars of small masses, n does not probably exceed 3.25. We then write (11) and (12) as

$$rT < \gamma \frac{(1-\beta_c)^{\frac{1}{2}}}{\beta_c} \cdot \frac{1}{\mu} \cdot 5.4 \times 10^{18} \equiv B \qquad \ldots \quad (13)$$

where γ is a numerical constant introduced as a compensating factor for the contingency that the ratio of radiation to gas pressure may not be maximum at the centre. This factor, however, we put equal to 1 for stars of not large masses, which is not far from truth.

For stars in which the radiation pressure is not small compared to gas pressure, n_1 (the minimum value of n) can probably be put much higher than 1.5. For instance, for stars whose masses are such that $\mu^2(M/\odot)$ is greater than 5.7, no complete degeneracy is possible, and the radiation pressure will also be considerable. For such wholly gaseous stars calculations have been made with $n_1 = 3$ (figures for $n_1 = 1.5$ are also given). If there are convection zones near the centre where adiabatic equilibrium may be assumed, then for stars of the main sequence $n_1 \sim 1.5$ will be quite good, while for more massive stars with higher radiation pressure an intermediate value between 1.5 and 3 will be probably nearer the mark. In case of $n_1 = 3$, (13) should be replaced by

$$rT < \gamma \frac{(1-\beta_c^{\frac{1}{2}})}{\beta_c} \cdot \frac{1}{\mu} \cdot 4.2 \times 10^{18} \equiv B'. \qquad \ldots \quad (13)'$$

Now as n may generally exceed 3, the maximum value of q may not occur exactly at the centre, so γ is expected to exceed the value unity. But the variation of q in the *interior* is not considerable (the outer portion of the star is left out of consideration), and γ will always remain of the order of unity. But if by q_c we mean *for the present* the *maximum* value of the ratio of radiation to gas pressure within the star, we may always put $\gamma = 1$. According to the recent theory of energy generation due to Atkinson, Gamow and Bethe, the energy of the main sequence stars is generated in the extreme central region, so that the stars have mostly approximately point sources of energy. In such cases, as the numerical integrations by Eddington and Biermann show the ratio of radiation to gas pressure has the maximum value near the centre, and this characteristic is expected to be retained in most stars where energy is generated by the process suggested by Atkinson, Gamow and Bethe, and the radiative gradient is replaced by an adiabatic one.

The following Table gives an upper bound of the product rT against the value of $q_c = (1-\beta_c)/\beta_c$, the ratio of the two pressures within the star at the centre. We shall rather put $\gamma = 1$, and, for the present, mean by q_c the maximum value of q in the stellar interior.

TABLE 2

q_c	$B\mu$ (cm.) (deg.)	q_c	$B.\mu$. (cm.) (deg.)
4	25×10^{18}	$\frac{1}{4}$	3.1×10^{18}
2	13×10^{18}	$\frac{1}{9}$	1.9×10^{18}
1	7.8×10^{18}	$\frac{1}{24}$	1.2×10^{18}
$\frac{1}{2}$	4.8×10^{18}	$\frac{1}{99}$	0.56×10^{18}

We may generally put $\mu = 1$ for stars in which hydrogen is abundant.

INTRODUCTION OF MASS IN THE UPPER BOUND

Under certain circumstances we can put the upper bounds in (13) and (13′) in terms of the masses of the stars instead of the maximum value of the ratio of radiation to gas pressure.

The inequality (6) gives an upper bound for $(1-\beta_c)/\beta_c^4$ in terms of the mass M of the star, strictly speaking in terms of $\mu_c^2 M$. Both $(1-\beta_c)/\beta_c^4$ and $(1-\beta_c)^{\frac{1}{2}}/\beta_c$ being monotone increasing functions of $(1-\beta_c)$, we can calculate from (6) an upper bound for $(1-\beta_c)^{\frac{1}{2}}/\beta_c$ as well. Calling this upper bound F(M), we write

$$(1-\beta_c)^{\frac{1}{2}}/\beta_c < F(M). \quad \ldots \quad (14)$$

This, taken with the two equations (13), gives

$$rT < \gamma . \frac{1}{\mu} F(M) . \begin{pmatrix} 5.2 \\ 4.4 \end{pmatrix} \times 10^{18} \text{ (cm.) (deg.)}. \quad \ldots \quad (15)$$

For main sequence stars with small and moderate masses we take the figure 5.4, and put $\gamma = 1$; for more massive stars we take 4.2, and γ to be of the order of unity.

From Table 2, as also from equation (15) an idea about the size of the configuration can generally be made, if by "radius" r' we mean a central distance where the temperature has fallen to a value which is a suitable (otherwise arbitrarily chosen) fraction of the central value. This step is rather delicate and difficult. As nearly the whole of the mass of the configuration will be included within a spherical surface on which the temperature is a fraction (say one-tenth) of the central value, the application of (14) does not produce any difficulty. For stars of small and moderate masses within which the radiation pressure is either very small, or not considerable, a variation limit for n between 1.5 and 3.25 will be quite appropriate. Let us now call that value of r' the "radius" of the configuration where the temperature has fallen to, say, *one million degrees*. There are certain difficulties in fixing this limit too low. Firstly, though stellar conditions in the inside are roughly of uniform character, they diverge widely in the exterior, where the approximations will be faulty to a great extent. Secondly, as the surface is approached, the gas pressure decreases very rapidly, and generally the ratio of radiation to gas pressure will increase. An adjustment of this by the γ factor may involve the use of such high values of γ as to render the approximations useless. It may be expected that for stars of small and moderate masses a temperature of about a million degrees will generally provide against this contingency. For this reason in Table 3 up to the value 5 in the first column, the factor γ has been put equal to 1. There will, of course, remain an outer envelope whose thickness is to be added to r' for the total radius R of the star (which is connected with the effective temperature). For non-massive stars, as Chandrashekhar's [5] investigation of stellar envelopes shows, the increase of the pressure ratio within the whole of the outer envelope up to about a million degrees does not at the utmost exceed 60% of its value at the bottom of the envelope, a consideration which suggests that it is not necessary to seriously modify our approximation $\gamma = 1$ up to about a million degrees temperature. For stars of small and moderate masses the addition to r' for the outer part of the envelope is also expected to be small, probably not above 20 to 25 per cent.

For stars of comparatively larger masses with larger radii, the results are much more uncertain. Firstly, the value of γ may be comparatively large; secondly, larger values of the polytropic index n will tend to make the envelope more extensive. But there may be one fact in favour of the approximation. If the recent theory of energy generation be applicable also within stars of larger masses (it does not certainly apply to all large masses), the stars will have nearly point sources of energy, for which, as we have remarked, the pressure ratio increases inside near the centre and γ will have a value not very much differing from 1. If the arbitrary limit of the temperature of 1 million degrees be inadequate, larger errors may arise. However, the question of the thickness of the outer part of the envelope will remain here uncertain.

CALCULATIONS

Corresponding to the definition of the radius r', we shall have

$$r'(M) < \gamma \frac{1}{\mu} F(M) \begin{pmatrix} 5.4 \\ 4.2 \end{pmatrix} \times 10^{12} \text{ cms.} \qquad \dots \quad (16)$$

Table 3 gives the maximum values of $\mu r'(M)=U$, corresponding to the values of $\mu_e{}^2M$ (in solar units) in the first column. $(1-\beta_c)_u$ in the second column is the upper bound of $1-\beta_c$ for corresponding values in the first column taken from Candler's Table [1]; the third column gives F(M), *i.e.*, the upper bound of $(1-\beta_c)^{\frac{1}{2}}/\beta_c$, calculated from the second column. The last column gives U the calculated upper bound of $\mu r'$ (r' in solar units) by (16). For values in the first column greater than 5, calculations have been made both for $n_1=1.5$, and $n_1=3$, and those for $n=1.5$ have been put within brackets.

For large masses n_1 is certainly greater than 1.5 but the values corresponding to $n_1=1.5$ are shown to give an idea of the unattainable extreme value. γ has been put equal to 1 in the Table. It appears the bounds corresponding to $n_1=3$ are all sufficiently high to cover the cases of very large stars. As these approximations have been made without taking into account the flow of radiation, the opacity factor, ionisation, etc., these rough upper bounds, for stars of small and moderate masses at least, may not be considered unsatisfactory.

TABLE 3

$\mu_e{}^2M$ (M in ☉ units)	$(1-\beta_c)_u$	F(M)	U (in ☉ units)
0.5	0.0017	0.04	3
1	0.0068	0.08	6
2	0.025	0.16	12
5	0.109	0.37	29
10	(0.24) 0.16	(0.6) 0.48	(47) 29
20	(0.39) 0.30	(1.0) 0.77	(79) 46
50	(0.57) 0.50	(1.7) 0.70	(134) 84
100	(0.68) 0.62	(2.5) 2.0	(200) 120

The first column gives the values of $\mu_e{}^2$ (mass of the star in solar units), and the last column the upper limit of the radius in solar units. The figures within ( ) for large masses give the values of the upper limits on the supposition that the lowest polytropic index of such star is 1.5, while the figures not within ( ) are the values for lowest polytropic index 3.

The Table 3 shows that for small masses there is a linear rise in U for varying $\mu_e{}^2M$, which is approximately at the rate of 6 per unit increase in $\mu_e{}^2M$. For stars of small masses we may put $\mu_e \approx 1$.

We have collected the values of the masses and radii of some stars in Table 4. They are taken from Tables given by Stromgren, and in Russel, Dugan, Stewart's Astronomy, and are arranged into groups according to their masses.

TABLE 4

Name of the star	Mass	Radius
S. Ant.	0.75 0.42	1.66 1.29
W. U. Mα	0.69 0.49	0.78 0.78
Z. Hercules	1.63 1.3	1.77 3.29
ζ. Hercules	0.95	1.9
R. T. Lac br	1	4.9
R. T. Lac f	1.9	4.9
T. X. Her br	2.06	1.6
Sirius A	2.34	1.8
Capella A	4.18	15.8
Capella B	3.32	6.6
μ_1 Scorpii	12	5.5
S. Auriga (B. Comp)	8.1	5.1
(K. Comp)	14.8	200
V. V. Cephei (M.Comp)	40 102	2130 2630
A. O. Cassiopiae	40	19
29 Canis Majoris A	46	20

A comparison with Table 3 shows that except for small masses the calculated upper bounds are rather too high. They indeed correspond to only one steller parameter M, whereas by Vogt-Russel theorem a complete description is possible only in terms of two parameters. Hence values much higher than the maximum values in Table 4 are not altogether unexpected in our calculations. In this respect the cases of S. Ant., R. T. Lac, Capella A are interesting. Their values nearly touch, or are within hundred per cent. of the corresponding upper bounds. The calculated bounds appear to be exceeded by E-Auriga (infra-red component), V. V. Cephei which are distinguished by their exceptionally large radii. Chandrasekhar's investigation of their envelopes shows that the masses of these stars are very probably contained within 5 to 10 per cent. of their huge radii. Our approximations will not evidently apply to such deep atmospheres.

REFERENCES

[1] *M.N.R.A.S.*, **100**, 14 (1939).

[2] British Association Tables of Emden Functions; also proved analytically *Bull. Cal. Math. Soc.*, **30**, 11 (1938).

[3] Taken from British Association Tables of Emden Functions.

[4] This is discussed in a different paper by the author.

[5] An Introduction to the Study of Stellar Structure, Chap. VIII.