FORMULAE CONNECTING SELF-RECIPROCAL FUNCTIONS

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1. In 1931 Hardy[1] and Titchmarsh[2] started the question as to how functions, which were self-reciprocal for Hankel Transforms of different orders, were connected with one another. They gave certain rules for deriving self-reciprocal functions for transforms of different orders from those which were self-reciprocal for transforms of a given order. In 1932 and 1934, I [3,4] gave some more rules of a similar nature. The object of this note is to add a few more to the list of these rules. In the end I use these rules to derive some new self-reciprocal functions.

I will say that a function is R_ν if it is self-reciprocal for J_ν transforms. For $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

I will make use of the following result given by me elsewhere (3: §8) :—

If $f(x)$ is R_μ, the function

$$g(x)=\int_0^\infty P(xy)f(y)dy$$

is R_ν, provided that

$$P(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\lambda(s)x^{-s}ds, \quad \ldots \quad (1.1)$$

where

$$0<k<1,$$

and

$$\lambda(s)=\lambda(1-s). \quad \ldots \quad (1.2)$$

2. *Rule I.*—The kernel $x^{-\frac{1}{2}}$ transforms R_μ into R_ν, where $\mu>-1$, $\nu>-1$.

This is almost obvious. For, in this case,

$$g(x)=\frac{1}{\sqrt{x}}\int_0^\infty\frac{1}{\sqrt{y}}f(y)dy.$$

This is only a constant multiple of the function $x^{-\frac{1}{2}}$ which is known to be R_ν.

3. *Rule II.*—The kernel $x^{\frac{3}{2}-\nu}J_{\nu-\frac{1}{2}}(\frac{1}{2}x)J_{\nu-\frac{3}{2}}(\frac{1}{2}x)$ transforms $R_{3\nu-3}$ into R_ν where $\nu>\frac{2}{3}$.

To prove this rule, I start with the Weber-Schafheitlin [6 : 13.41 (2)] integral

$$\int_0^\infty \frac{J_m(ax)J_n(ax)}{x^\lambda}dx = \frac{(\frac{1}{2}a)^{\lambda-1}\Gamma(\lambda)}{2\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}\lambda)} \times \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}\lambda)}{\Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}\lambda)\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2}\lambda)},$$

where $R(m+n+1) > R(\lambda) > 0$.

This formula is the same as

$$\int_0^\infty x^{l-1}J_m(ax)J_n(ax)dx = \frac{(\frac{1}{2}a)^{-l}\Gamma(1-l)}{2\Gamma(1+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)} \times \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l)}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)\Gamma(1+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)}$$

$$= \frac{a^{-l}\Gamma(\frac{1}{2}-\frac{1}{2}l)\Gamma(1-\frac{1}{2}l)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l)}{2\sqrt{\pi}\Gamma(1+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)\Gamma(1-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)\Gamma(1+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)},$$

where $R(m+n) > R(-l) > -1$.

By Mellin's Inversion Formula (1) we have

$$J_m(ax)J_n(ax) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}l)}{\Gamma(1+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)} \times \frac{\Gamma(1-\frac{1}{2}l)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l)}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)\Gamma(1+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)}(ax)^{-l}dl,$$

so that

$$x^a J_m(\tfrac{1}{2}x)J_n(\tfrac{1}{2}x) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} 2^l \frac{\Gamma(\frac{1}{2}-\frac{1}{2}l)}{\Gamma(1+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)} \times \frac{\Gamma(1-\frac{1}{2}l)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l)}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)\Gamma(1+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)}x^{a-l}dl$$

$$= \frac{2^{a-1}}{\sqrt{\pi}} \cdot \frac{1}{2\pi i}\int_{b-a-i\infty}^{b-a+i\infty} 2^s \frac{\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}s)\Gamma(1-\frac{1}{2}a-\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}a-\frac{1}{2}s)} \times \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}a+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}a-\frac{1}{2}s)\Gamma(1+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}a-\frac{1}{2}s)}x^{-s}ds, \qquad (3.1)$$

where $-R(m+n) < b < 1$.

Putting $m-1=n=-a=\nu-\frac{3}{2}$, we get

$$x^{\frac{3}{2}-\nu} J_{\nu-\frac{1}{2}}(\tfrac{1}{2}x) J_{\nu-\frac{3}{2}}(\tfrac{1}{2}x) = \frac{2^{\frac{1}{2}-\nu}}{\sqrt{\pi}} \cdot \frac{1}{2\pi i} \int_{b-\frac{3}{2}+\nu-i\infty}^{b-\frac{3}{2}+\nu+i\infty} 2^{s}$$

$$\times \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(-\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(-\frac{3}{4}+\frac{3}{2}\nu-\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)} x^{-s}\, ds,$$

where $\qquad 2-2\nu < b < 1.$

This integral is of the same form as (1.1) with $\mu=3\nu-3$,

$$x(s) = \frac{2^{\frac{1}{2}-\nu}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(-\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(-\frac{5}{4}+\frac{3}{2}\nu+\frac{1}{2}s)\Gamma(-\frac{3}{4}+\frac{3}{2}\nu-\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)},$$

which evidently satisfies (1.2).

Hence follows the result.

Putting $\nu=1$ we get

Rule III.—The kernel $\dfrac{\sin x}{\sqrt{x}}$ transforms R_0 into R_1.

In 1934, I (4 : §3) gave the rule that

The kernel $x^{-\frac{1}{2}\nu} H_{\frac{1}{2}\nu-\frac{1}{2}}(x)$ transforms R_1 into R_ν, where $H_n(x)$ is Struve's function [6 : 10.4(2)] of order ν.

As μ and ν are inter-changeable in (1.1), this rule may also be written thus :

The kernel $x^{-\frac{1}{2}\nu} H_{\frac{1}{2}\nu-\frac{1}{2}}(x)$ transforms R_ν into R_1.

As $H_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$,

it follows that Rule III is a particular case of this rule for $\nu=0$.

If, in (3.1), we take $m=n=-d=\nu-\frac{1}{2}$ and proceed as before, we obtain

Rule IV.—The kernel $x^{\frac{1}{2}-\nu} J_{\nu-\frac{1}{2}}(\frac{1}{2}x) J_{\nu-\frac{1}{2}}(\frac{1}{2}x)$ transforms $R_{3\nu-1}$ into R_ν for $\nu > \frac{1}{2}$.

If here we take $\nu=1$, we get

Rule V.—The kernel $x^{-\frac{3}{2}}(1-\cos x)$ transforms R_2 into R_1.

If, in (3.1), we take $m=-n=\frac{1}{2}\nu-\frac{1}{4}$, $d=0$, and proceed as above, we derive

Rule VI.—The kernel $J_{\frac{1}{2}\nu-\frac{1}{4}}(\frac{1}{2}x)\,J_{\frac{1}{4}-\frac{1}{2}\nu}(\frac{1}{2}x)$ transforms $R_{\nu-1}$ into R_ν for $\nu>0$.

For the particular case $\nu=\frac{3}{2}$, we have

Rule VII.—The kernel $\frac{\sin x}{x}$ transforms $R_{\frac{1}{2}}$ into $R_{\frac{3}{2}}$.

4. Titchmarsh (5 : 7.10.4) has given the pair of Mellin Transforms

$$J_n(x)\gamma_n(x), -\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+n)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)\Gamma(1+n-\frac{1}{2}s)}.$$

If we start with the integral formula for this pair, and proceed as above, we arrive at

Rule VIII.—The kernel $x^{\frac{1}{3}\nu+\frac{1}{6}}J_{\frac{1}{3}\nu+\frac{1}{6}}(\frac{1}{2}x)\,\gamma_{\frac{1}{3}\nu+\frac{1}{6}}(\frac{1}{2}x)$ transforms $R_{\frac{1}{3}\nu-\frac{1}{3}}$ into $R_{\frac{2}{3}\nu}$, where $\nu>-\frac{1}{2}$.

As
$$\gamma_{\frac{1}{2}}(x)=-\sqrt{\frac{2}{\pi x}}\cos x,$$

if we put $\nu=1$ in Rule VIII, we arrive at Rule III again.

If we use the pair

$$e^x k_n(x),\quad \Pi^{-\frac{1}{2}}\cos n\pi.2^{-s}\Gamma(\tfrac{1}{2}-s)\Gamma(s+n)\Gamma(s-n)$$

given by Titchmarsh (5 : 7.10.9) and proceed in the same way, we obtain

Rule IX.—The kernel $x^{\frac{1}{3}\nu-\frac{1}{3}}e^{\frac{1}{2}x}k_{\frac{1}{3}\nu-\frac{1}{6}}(\frac{1}{2}x)$ transforms $R_{\nu-1}$ into R_ν for $\nu>\frac{1}{4}$.

When $\nu=1$, we obtain the rule that the kernel $x^{-\frac{1}{2}}$ transforms R_0 into R_1. This is already contained in Rule I.

5. I now proceed to derive certain self-reciprocal functions with the help of the above rules.

We know that the function

$$x^{\mu+\frac{1}{2}}e^{-\frac{1}{2}x^2} \qquad \ldots \quad (5.1)$$

is R_μ; in other words, that the function

$$x^{3\nu-\frac{5}{2}}e^{-\frac{1}{2}x^2}$$

is $R_{3\nu-1}$.

Applying Rule II to this function, we arrive at the R_ν function $g(x)$ given by

$$g(x)=\int_0^\infty (xt)^{\frac{3}{2}-\nu} J_{\nu-\frac{1}{2}}(\tfrac{1}{2}xt)\, J_{\nu-\frac{3}{2}}(\tfrac{1}{2}xt)\, t^{3\nu-\frac{5}{2}} e^{-\frac{1}{2}t^2}\, dt$$

$$=x^{\frac{3}{2}-\nu}\int_0^\infty t^{2\nu-1} e^{-\frac{1}{2}t^2} J_{\nu-\frac{1}{2}}(\tfrac{1}{2}xt) J_{\nu-\frac{3}{2}}(\tfrac{1}{2}xt)\, dt.$$

This integral may be evaluated by a formula given by Macdonald (6 : 13.32) for $\nu > \frac{1}{2}$. We thus find that $g(x)$ is a constant multiple of

$$x^{\frac{3}{2}-\nu} . x^{2\nu-2}\, {}_3F_3\left(\begin{matrix}\nu-\frac{1}{2},\ \nu,\ 2\nu-1 \\ \nu+\frac{1}{2},\ \nu-\frac{1}{2},\ 2\nu-1\end{matrix}\ ;\ -\tfrac{1}{2}x^2\right)$$

which is the same as

$$x^{\nu-\frac{1}{2}}\, {}_1F_1(\nu\ ;\ \nu+\tfrac{1}{2}\ ;\ -\tfrac{1}{2}x^2).$$

Using Kummer's Transformation Formula

$${}_1F_1(a\ ;\ \rho\ ;\ z)=e^z\, {}_1F_1(\rho-d\ ;\ \rho\ ;\ -z),$$

we find that the function

$$x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x^2}\, {}_1F_1(\tfrac{1}{2}\ ;\ \nu+\tfrac{1}{2}\ ;\ \tfrac{1}{2}x^2)$$

is R_ν for $\nu > \frac{2}{3}$.

If we apply Rule IV to the function (5.1) we arrive at the same function again.

If we apply Rule VI to the function we find that the function

$${}_1F_1(\tfrac{1}{2}\ ;\ \tfrac{5}{4}-\tfrac{1}{2}\nu\ ;\ -\tfrac{1}{2}x^2)$$

is R_ν for $\nu > 0$.

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