# ON THE PROPAGATION OF E. M. WAVES THROUGH THE UPPER ATMOSPHERE

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**ABSTRACT.** This paper reports a comprehensive working of the problems of an ionised atmosphere, traversed by a magnetic field, as in the case of the Earth's atmosphere. Expressions are deduced for electrical polarisation and complex conductivity for such an atmosphere when traversed by radio waves, in a tensor-form, as first suggested by Darwin. The equations of propagation of radio frequency waves through such a medium are obtained by the use of cardinal axes, and then the equations of vertical propagation are deduced. Expressions are obtained for refractive indices of ordinary and extraordinary waves, which agree with the expressions given by Appleton. Expressions are obtained for polarisation, absorption etc. of the radio waves travelling in the ionosphere. Curves are given for the polarisation ratio and refractive indices of the two waves as functions of the magnetic latitude of the place of observation.

## 1 INTRODUCTION

Ever since the classical works of Appleton (1932) and Hartree (1932), the problem of the propagation of e.m. waves in the ionosphere has received attention from numerous workers. Summaries of these works are available in various reports. Recently B. K. Banerjea (1947) made a critical and comparative study of the fundamental methods of Appleton (1932), Hartree (1932), Saha, Rai and Mathur (1937) and Saha and Banerjea (1945) and showed that these various methods can be deduced as special cases of a general method developed according to Darwin's (1925) suggestion of treating the e.m. properties of the medium as tensor quantities. The present paper continues the treatment further and aims at giving a true wave formulation of the general problem. For the convenience of the reader some results of the previous works carried out by the senior author and his early collaborators are included so that no further references to these papers are needed. Part of the results mentioned in the earlier parts are not new, but have been derived in a novel, easier and unitary way.

*The Displacement of the Ions in the Ionosphere*

The equation of motion of the charged ions referred to any system of co-ordinates can be written as:

$$\frac{d^2\rho}{dt^2}+\nu\frac{d\rho}{dt}+\frac{e}{mc}\left[\mathrm{H}\times\frac{d\rho}{dt}\right]=\frac{e}{m}\boldsymbol{E} \qquad (1.1)$$

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where $\rho$ = displacement vector with components $(\xi, \eta, \zeta)$

$e, m$ = magnitude of the charge and mass of the ion respectively.

$\nu$ = collision frequency of the ions.

$H$ = Earth's magnetic field.

$\boldsymbol{E} = \boldsymbol{E}_0 \cos pt$, electric vector of the incident electromagnetic wave.

The effect of the magnetic vector and the space charges have been omitted as usual. The notation conforms as closely as possible to those used by Appleton (1932) and Saha, Rai and Mathur (1937) and B. K. Banerjea (1947).

It can easily be verified that the solution of the above equation with $\boldsymbol{E} = \boldsymbol{E}_0 \cos pt$ is the real part of the solution obtained with $\boldsymbol{E} = \boldsymbol{E}_0 e^{ipt}$; we use $\boldsymbol{E}$ in this latter form because solution is then easy to obtain. The quantity analogous to the static conductivity now comes out as complex (Stratton, 1942), whose real part gives ordinary refractive index and the imaginary part gives deviation of the refractive index from unity.

Introducing the polarisation vector $\boldsymbol{P} = 4\pi Ne\rho$ where N is the ion-concentration and using the abbreviations,

$$\frac{mp^2}{Ne^2} = \frac{4\pi p^2}{p_0^2} = \frac{4\pi}{r}, \quad p_0^2 = \frac{4\pi Ne^2}{m}, \quad r = \frac{p_0^2}{p^2},$$

$$\nu/p = \delta, \qquad 1 - i\delta = \beta, \qquad \ldots \quad (1.2)$$

$$\frac{eH}{mc} = \mathbf{p}_h, \qquad \frac{\mathbf{p}_h}{p} = \omega \text{ with components } \omega_x, \omega_y, \omega_z.$$

We get from equation (1.1) replacing $\rho\ (\xi, \eta, \zeta)$ by $\frac{1}{4\pi Ne}(P_x, P_y, P_z)$

$$\begin{aligned} \beta P_x + i\omega_z P_y - i\omega_y P_z &= -E_x \\ -i\omega_z P_x + \beta P_y + i\omega_x P_z &= -E_y \\ i\omega_y P_x - i\omega_x P_y + \beta P_z &= -E_z \end{aligned} \qquad \ldots \quad (1.3)$$

The solution of these equations can be briefly written as

$$P = \mathrm{A}\,\Delta . E \qquad \ldots \quad (1.4)$$

where $\mathrm{A} = \frac{r}{\beta(\beta^2 - \omega^2)}$ and $\Delta$ is a tensor given by the matrix,

$$\Delta = \begin{vmatrix} \omega_x^2 - \beta^2 & \omega_x\omega_y + i\beta\omega_z & \omega_x\omega_z - i\beta\omega_y \\ \omega_y\omega_x - i\beta\omega_z & \omega_y^2 - \beta^2 & \omega_y\omega_z + i\beta\omega_x \\ \omega_z\omega_x + i\beta\omega_y & \omega_z\omega_y - i\beta\omega_x & \omega_z^2 - \beta^2 \end{vmatrix} \qquad \ldots \quad (1.5)$$

It has been shown by Saha and Banerjea (1945) that the tensor possesses certain "Cardinal Axes" which may be denoted by 1, 2, 3. "1" is the direction of the earth's magnetic field, "2" is the line perpendicular to the magnetic meridian, and "3" is the line perpendicular to "1" lying in the magnetic meridian. The relation between these axes and the axes commonly used in ionospheric problems with XZ as magnetic meridian and OZ as vertical is shown in the diagram below :

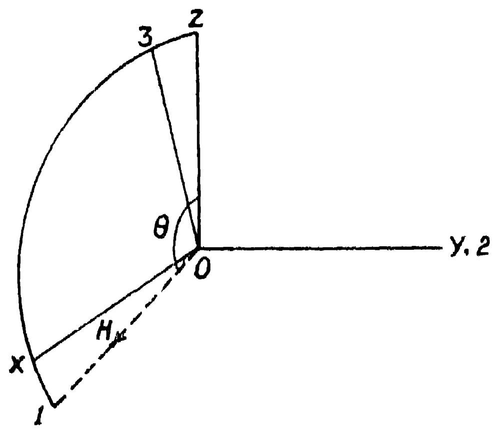


FIG. 1

Shows disposition of cardinal axes (1, 2, 3) with axes used generally in considering vertical propagation.

In this figure $\theta = \angle ZO1$ is called the angle of propagation. The axis O1 is always along the positive direction of H. In general literature on ionospheric problems, the positive direction of H is generally not expressed quite clearly, with the result that the sense of rotation of the electric and magnetic vectors of the returning radio wave is left unclarified. In what follows the positive direction of H is along the positive direction of the magnetic lines of force, *i.e.* in the northern hemisphere it is downward and in the southern the reverse is the case.

Choice of these axes is equivalent to putting

$$\omega_1 = \omega, \quad \omega_2 = \omega_3 = 0,$$

where $\omega_1$, $\omega_2$, $\omega_3$ are the components of $\omega$ along (1, 2, 3) axes. We have then

$$\Delta = -\begin{vmatrix} \beta^2 - \omega^2 & 0 & 0 \\ 0 & \beta^2 & i\beta w \\ 0 & -i\beta w & \beta^2 \end{vmatrix} \qquad \ldots \quad (1.6)$$

The complex conductivity $\sigma$ of the medium, defined by the equation

$$\boldsymbol{\sigma}.\boldsymbol{E} = \text{current} = -\text{N}e\frac{d\rho}{dt} = -ip\text{N}e\rho = -\frac{ip}{4\pi}\boldsymbol{P} = -\frac{ip\text{A}}{4\pi}\Delta.\boldsymbol{E}$$ is a tensor

quantity defined by the matrix,

$$\sigma = \frac{iNe^2}{m}\begin{vmatrix} \frac{1}{p-i\nu} & 0 & 0 \\ 0 & \frac{p-i\nu}{(p-i\nu)^2-p_h^2} & \frac{-ip_h}{(p-i\nu)^2-p_h^2} \\ 0 & \frac{ip_h}{(p-i\nu)^2-p_h^2} & \frac{p-i\nu}{(p-i\nu)^2-p_h^2} \end{vmatrix} \quad \ldots \quad (1.7)$$

The steady current conductivity $\sigma^*$ is obtained from above by putting $p=0$. We have

$$\sigma^* = \frac{Ne^2}{m}\begin{vmatrix} \frac{1}{\nu} & 0 & 0 \\ 0 & \frac{\nu}{p_h^2+\nu^2} & \frac{p_n}{p_h^2+\nu^2} \\ 0 & \frac{-p_n}{p_h^2+\nu^2} & \frac{\nu}{p_h^2+\nu^2} \end{vmatrix} \quad \ldots \quad (1.8)$$

Thus in the direction of the magnetic field, the steady current conductivity is $\frac{Ne^2}{m\nu}$. We have the components of current as

$$i_1 = \frac{Ne^2}{m\nu}E_1$$

$$i_2 = \frac{Ne^2}{m(p_h^2+\nu^2)}(\nu E_2 + p_h E_3) \quad \ldots \quad (1.9)$$

$$i_3 = \frac{Ne^2}{m(p_h^2+\nu^2)}(-p_h E_2 + \nu E_3)$$

If $E_3=0$, we have $i_2 = \frac{Ne^2}{m(p_h^2+\nu^2)}\nu E_2$. The quantity $\frac{Ne^2\nu}{m(p_h^2+\nu^2)}$ is known as transverse conductivity. We have besides, the current $i_3 = -\frac{Ne^2p_h}{m(p_h^2+\nu^2)}E_2$ along the Z axis, though there may be no e.m.f. in that direction

*The Polarisation Vector.*—The polarisation vector $\boldsymbol{P}$ is defined as

$$\boldsymbol{P} = \frac{4\pi i}{p}\boldsymbol{\sigma}.\boldsymbol{E}$$

and we can easily deduce that

$$E_1 = -\frac{\beta}{\tau}P_1, \quad E_2 \pm iE_3 = -\frac{\beta \mp w}{\tau}(P_2 \pm iP_3) \quad \ldots \quad (1.10)$$

*The Electric Displacement Vector and the Complex Dielectric Tensor.*—The electric displacement vector $\boldsymbol{D}=\boldsymbol{E}+\boldsymbol{P}$ may be expressed as $\boldsymbol{D}=\mathbf{K}.\boldsymbol{E}$., where $\mathbf{K}$ is the complex dielectric tensor given by the matrix,

$$\mathbf{K}=\begin{vmatrix} 1-r/\beta & 0 & 0 \\ 0 & 1-r\beta/(\beta^2-\omega^2) & ir\omega/(\beta^2-\omega^2) \\ 0 & -ir\omega/(\beta^2-\omega^2) & 1-r\beta/(\beta^2-\omega^2) \end{vmatrix} \quad \dots \quad (1.11)$$

## 2. THE MAXWELLIAN EQUATIONS

From the Maxwellian equations :

$$\nabla\times\boldsymbol{H}=\frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t},\quad \nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t},\quad \nabla.\boldsymbol{D}=\nabla.\boldsymbol{H}=0 \quad \dots \quad (2.1)$$

We get by the usual methods, the equations of propagation for the electric and magnetic vectors in the form :

$$\Delta^2 E_1+\frac{p^2}{c^2}(1-r/\beta)E_1=0$$

$$\nabla^2(E_2\pm iE_3)+\frac{p^2}{c^2}\left(1-r/(\beta\pm\omega)\right)(E_2\pm iE_3)=0 \quad \dots \quad (2.2)$$

$$\nabla^2\boldsymbol{H}+\frac{p^2}{c^2}\boldsymbol{H}=-\frac{4\pi}{c}\nabla\times(\boldsymbol{\sigma}.\boldsymbol{E})$$

*The Wave Equations for Vertical Propagation in any Latitude:*—Let us first confine ourselves to the propagation along the vertical $Z$-axis, so that $\nabla$ and $\nabla^2$ simply reduce to $\frac{d}{dz}$ and $\frac{d^2}{dz^2}$. Introducing the new variable $u=pz/c$, we get from (2.2)

$$\frac{d^2E_1}{du^2}+\left(1-\frac{r}{\beta}\right)E_1=0 \quad \dots \quad (2.3a)$$

$$\frac{d^2}{du^2}(E_2+iE_3)+\left(1-\frac{r}{\beta-\omega}\right)(E_2+iE_3)=0 \quad \dots \quad (2.3b)$$

$$\frac{d^2}{du^2}(E_2-iE_3)+\left(1-\frac{r}{\beta+\omega}\right)(E_2-iE_3)=0 \quad \dots \quad (2.3c)$$

The components of the vector E in two systems (1, 2, 3) and with $XZ$-plane as magnetic meridian and $OZ$ as vertical are related as :

$$\left.\begin{array}{ll} E_1=E_x\sin\theta+E_z\cos\theta; & E_x=E_1\sin\theta-E_3\cos\theta. \\ E_2=E_y; & E_y=E_2. \\ E_3=-E_x\cos\theta+E_z\sin\theta; & E_z=E_1\cos\theta+E_3\sin\theta. \\ \omega_x=\omega\sin\theta,\quad \omega_y=0, & \omega_z=-\omega\cos\theta \end{array}\right\} \quad \dots \quad (2.4)$$

The equations (2.3) as such are not suitable for use when we consider the propagation of plane waves, for such cases we have to use in conjunction with (2.2), the Maxwellian condition $\nabla.\boldsymbol{D}=0$. For vertical propagation, this reduces to $\frac{d}{dz} D_z = 0$, *i.e.*, $D_z = 0$, since the steady components of $\boldsymbol{D}$, if any, are unimportant in the study of the wave propagation :

From $D_z = 0$ and (1.4) and (1.5) we have eliminating $P_x$, $P_y$, $P_z$ and putting $w_y = 0$.

$$E_z = \frac{i w_x}{C'} \left( -w_z E_x + i\beta E_y \right) \quad \ldots \quad (2.5)$$

where $C' = \beta(\beta^2 - w^2) - r(\beta^2 - w_z^2)$.

Multiplying (2.3a) by $\sin\theta$ and the difference of (2.3b) and (2.3c) by $-\cos\theta$, adding the results and then replacing $E_1$, $E_2$, $E_3$ by their equivalent expressions in terms of $E_x$, $E_y$, $E_z$ from (2.4) we get after some simplification,

$$\frac{d^2 E_x}{du^2} + K_1 E_x - iL E_y = 0 \quad \ldots \quad (2.6)$$

where

$$\left. \begin{aligned} K_1 &= 1 - r\frac{\beta^2 - r\beta - \omega^2 \sin^2\theta}{C'} \\ L &= \frac{-r(\beta - r)\omega \cos\theta}{C'} \end{aligned} \right\} \quad \ldots \quad (2.7)$$

Again replacing $E_2$ and $E_3$ by $E_x$, $E_y$, $E_z$ in equation (2.3b) from (2.4) and $E_z$ by $E_x$ and $E_y$ from (2.2), we have after some work,

$$\frac{d^2 E_y}{du^2} + K_2 E_y + iL E_x = 0. \quad \ldots \quad (2.8)$$

where

$$K_2 = 1 - r\frac{\beta^2 - r\beta}{C'} \quad \ldots \quad (2.9)$$

Equation (2.6) and (2.8) were obtained explicitly in this form by Saha, Rai and Mathur (1937). Equivalent equations with vector components of the ordinary and extraordinary waves intermixed in each equation were obtained by Rydbeck (1944). But equations in this form do not help much in the understanding of the phenomena, unless the coupling term L between the variables vanishes. This takes place at $\theta = \pi/2$, *i.e.*, at the magnetic equator, where the equations of propagation become,

$$\frac{d^2 E_x}{du^2} + \left( 1 - \frac{r}{\beta} \right) E_x = 0 \quad \ldots \quad (2.10)$$

$$\frac{d^2 E_y}{du^2} + \left( 1 - \frac{r}{\beta - \frac{\omega}{\beta - r}} \right) E_y = 0$$

For the magnetic poles, $\theta = \pi$ and 0, and for these values of $\theta$, $K_1 = K_2$; for $\theta = \pi$, *i.e.*, mag N-pole, the equation of propagation takes the form :

$$\frac{d^2}{du^2} (E_x \pm iE_y) + \left( 1 - \frac{r}{\beta \mp \omega} \right) (E_x \pm iE_y) = 0 \quad \ldots \quad (2.11)$$

For $\theta=0$, *i.e.*, mag. S-Pole, the equation similarly reduces to

$$\frac{d^2}{du^2}(E_x \pm iE_y)+\left(1-\frac{\ }{\beta+\omega}\right)(E_x \pm iE_y)=0 \qquad (2.11a)$$

Equation in these forms were studied by Saha and Rai (1037), for the case when damping is negligible, *i.e.*, $\beta=1$, from the wave mechanical point of view. The Chapman layer of ion-distribution was treated as a potential barrier and the penetration of the waves under certain simplifying assumptions were studied in the same way as Gamow did in his famous work on the "Penetration of the Potential Barrier of Nuclei of Atoms by High Energy Particles." Recently Rydbeck (1012) has studied these equations when the coupling term L vanishes; he has given an elaborate treatment of the wave equations for magnetic equator and taking a parabolic ion-layer and using Weber's parabolic functions he has obtained expressions for the reflection co-efficient, transmission co-efficient and phase retardation of the wave in a thin friction free parabolic layer. In the ray treatment of Appleton we practically confine ourselves to these two limiting cases, *viz.*, their quasi-longitudinal case is for $\theta=\pi$, 0, *i.e.*, $K_1=K_2$ and their quasi-transverse case, *i.e.*, $\theta=\frac{\pi}{2}$, $L=0$.

The following method will be found applicable to all stations. Multiplying both sides of (2.8) by "$iF$," and adding to (2.6), where F is an indeterminate multiplier to be presently determined, we have

$$\frac{d^2E_x}{du^2}+iF\frac{d^2E_y}{du^2}+(K_1-FL)E_x+\left(K_2-\frac{L}{F}\right)iFE_x=0 \quad \ldots \quad (2.12)$$

Now choose F in such a way that $K_1-FL=K_2-\frac{L}{F}$

so that F is given by the equation

$$F^2-\frac{K_1-K_2}{L}F-1=0 \quad \ldots \quad (2.13)$$

Put

$$\frac{K_1-K_2}{L}=\frac{\omega\sin^2\theta}{(r-\beta)\cos\theta}=2G=2g\cos\alpha e^{i\alpha},$$

where

$$g=\frac{\omega\sin^2\theta}{(r-1)\cos\theta},\quad \tan\alpha=\frac{\delta}{1-r} \quad \ldots \quad (2.14)$$

Let $F_1$, $F_2$ be the roots of equation (2.13). Then

$$F_1, F_2=G\pm\sqrt{1+G^2}$$

$$=g\pm\sqrt{1+g^2} \qquad \text{for } \delta=0 \quad \ldots \quad (2.15)$$

Now turning to equation (2.12) we can rewrite it in the form

$$\frac{d^2}{du^2}(E_x+iFE_y)+q^2(E_x+iFE_y)-2i\frac{dF}{du}\frac{dE_y}{du}-i\frac{d^2F}{du^2}E_y=0 \ldots \quad (2.16)$$

where $q$ has the two values given by

$$q_1^2 = K_2 - \frac{L}{F_1} = K_2 + LF_2 = 1 - \frac{r}{C'}(\beta - \gamma)(\beta + \omega \cos\theta\, F_2)$$

$$= 1 - \frac{r}{\beta + \omega \cos\theta\, F_1}. \quad \ldots \quad (2.17)$$

$$q_2^2 = K_1 - \frac{L}{F_2} = K_1 + LF_1 = 1 - \frac{r}{C'}(\beta - r)(\beta + \omega \cos\theta\, F_1)$$

$$= 1 - \frac{r}{\beta + \omega \cos\theta\, F_2}. \quad \ldots \quad (2.18)$$

for $C' = (\beta - r)(\beta + \omega \cos\theta\, F_1)(\beta + \omega \cos\theta\, F_2)$.

In those cases where the quantities $\frac{dF}{du}$, $\frac{d^2F}{du^2}$ can be neglected, the equations can be written as

$$\frac{d^2}{du^2}(E_x + iF_1E_y) + q_1^2(E_x + iF_1E_y) = 0 \quad \ldots \quad (2.16a)$$

$$\frac{d^2}{du^2}(E_x + iF_2E_y) + q_2^2(E_x + iF_2E_y) = 0$$

These signify that the beam is broken up into two, with the refractive indices $q_1$, and $q_2$, and polarisations determined by $F_1$ and $F_2$ (*Vide* §4).

We next proceed to discuss the case of friction free atmosphere. In this case we have

$$q_1^2 = 1 - \frac{r}{1 + \omega \cos\theta\, F_1} \quad \ldots \quad (2.17a)$$

$$q_2^2 = 1 - \frac{r}{1 + \omega \cos\theta\, F_2} \quad \ldots \quad (2.18a)$$

Both $q_1$, $q_2$ are to be continuous functions of $r$. We find from the expression for $g$, that for $r \to 1$, $g \to \infty$. At this point, $q_1^2$, $q_2^2$ should obey the condition of continuity, *i.e.*,

$$\underset{r=1-0}{Lt}(q_1^2, q_2^2) = \underset{r=1+0}{Lt}(q_1^2, q_2^2)$$

Taking first $q_1$, we find that if we take for the region $r=0$ to $r=1$

$$F_1 = g - \sqrt{1+g^2}, \text{ consequently } q_1^2 = 1 - \frac{r}{1 - \omega \cos\theta\,(\sqrt{1+g^2} - g)} \quad (2.17b)$$

Then $q_1$ varies from 1 to 0 in the domain $r=0$ to 1. As the value is to be continuous, and since $g$ on crossing over to $r = 1+0$, becomes negative, we find that for this region ($r > 0$) we should put

$$F_1 = \sqrt{1+g^2} - |\,g$$

$$i.e., \; q_1^2 = 1 - \frac{r}{1 + \omega \cos\theta\,(\sqrt{1+g^2} - |\,g\,|)} \quad \ldots \quad (2.17c)$$

These expressions for $q_1^2$ has no singularity at any point and it is identical with the expression for the refractive index of the ordinary wave as given by Appleton. A $(1-q_1^2)$ curve for different values of $\theta$ from expressions (2.17 *b*, *c*) is given in Fig. 2, for $\omega < 1$, and $\omega > 1$.

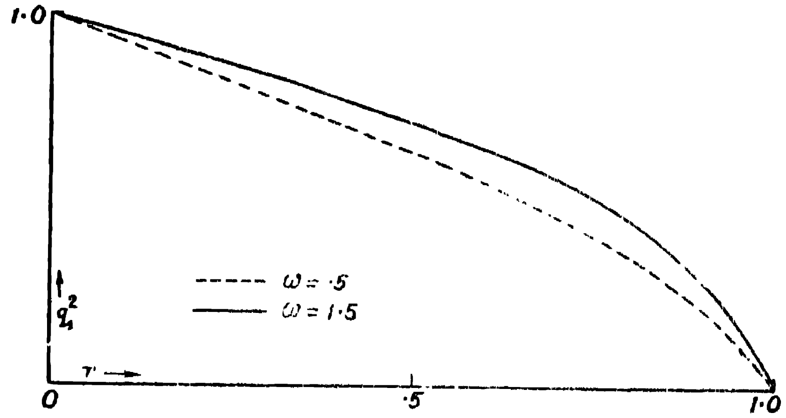


FIG. 2

Variation of the square of the refractive index for the o-wave with electron concentration $\left(r = \frac{4\pi e^2}{mp^2} N\right)$, for $p_H/p = \omega < 1$ (here ·5) and $\omega > 1$ (here 1.5)

For the other beam we can now substitute the corresponding value of $F_2$, we obtain :

$$q_2^2 = 1 - \frac{r}{1+\omega \cos\theta\, F_2} = 1 - \frac{r}{1+\omega\cos\theta\,(\sqrt{1+g^2}+g)} \quad \text{for } r < 1 \quad \ldots (2.18b)$$

$$= 1 - \frac{r}{1-\omega\cos\theta\,(\sqrt{1+g^2}+|g|)} \quad \text{for } r > 1 \quad \ldots (2.18c)$$

It can be easily shown that for $\omega < 1$, $(1-q_2^2)$-curve starts from (0, 1) passes through $(1-\omega, 0)$ and a point of infinite singularity at $r = \frac{1-\omega^2\cos^2\theta}{1-\omega^2}$ where it passes from $-\infty$ to $+\infty$, passes through the point (1, 1) and $(1+\omega, 0)$ for all values of $\theta$.

$q_2^2$ has therefore to be identified with the square of the refractive index of the extraordinary wave (Fig. 3).

For $\omega > 1$, we find that the curve passes through (0, 1) and (1, 1), between $r = 0$, and 1, $q_2^2 > 1$ but after (1, 1) the value of $q_2^2$ becomes less than unity and gradually tends to the value zero at $r = 1+\omega$, after which it is negative (Fig. 3).

The quantities $g(\omega, r, \theta)$, $-F = \sqrt{1+g^2} - |g|$ which occur in this work are functions of $\omega$, $r$ and $\theta$.

In Table I, the function $g(\omega, 0, \theta)$ has been given for various values of $\omega$ and $\theta$. To obtain $g(\omega, r, \theta)$ we have to divide $g(\omega, 0, \theta)$ by $(r-1)$.

TABLE I

$g = \omega \cos^2\theta/2 \sin\theta, \quad r = 0$

| $\theta°$ | $\omega$ = .1 | $\omega$ = .2 | $\omega$ = .3 | $\omega$ = .5 | $\omega$ = .8 | $\omega$ = 1.0 | $\omega$ = 1.5 | $\omega$ = 2 | $\omega$ = 5 | $\omega$ = 10 | $\omega$ = 20 | $\omega$ = 50 | $\omega$ 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 91° | 2.8640 | 5.7270 | 8.591 | 14.315 | 22.908 | 28.635 | 42.955 | 57.270 | 143.15 | 286.35 | 572.70 | 1431.5 | 2863.5 |
| 92° | 1.4310 | 2.8620 | 4.293 | 7.155 | 11.448 | 14.310 | 21.465 | 28.620 | 71.55 | 143.10 | 286.20 | 715.5 | 1431.0 |
| 93° | .9525 | 1.9050 | 2.858 | 4.763 | 7.620 | 9.525 | 14.290 | 19.050 | 47.63 | 95.25 | 190.50 | 476.3 | 952.5 |
| 94° | .7135 | 1.4270 | 2.141 | 3.568 | 5.708 | 7.135 | 10.705 | 14.270 | 35.68 | 71.35 | 142.70 | 356.8 | 713.5 |
| 95° | .5695 | 1.1390 | 1.709 | 2.848 | 4.556 | 5.695 | 8.545 | 11.390 | 28.48 | 56.95 | 113.90 | 284.8 | 569.5 |
| 100° | .2793 | .5586 | .838 | 1.397 | 2.234 | 2.793 | 4.190 | 5.586 | 13.97 | 27.93 | 55.86 | 139.7 | 279.3 |
| 105° | .1802 | .3604 | .541 | .901 | 1.442 | 1.802 | 2.795 | 3.604 | 9.01 | 18.02 | 36.04 | 90.1 | 180.2 |
| 110° | .1291 | .2581 | .387 | .645 | 1.032 | 1.291 | 1.935 | 2.581 | 6.45 | 12.91 | 25.81 | 64.5 | 120.1 |
| 115° | .0972 | .1944 | .292 | .486 | .777 | .972 | 1.460 | 1.944 | 4.86 | 9.72 | 19.44 | 48.6 | 97.2 |
| 120° | .0750 | .1500 | .225 | .375 | .600 | .750 | 1.113 | 1.500 | 3.75 | 7.50 | 15.00 | 37.5 | 75.0 |
| 125° | .0585 | .1170 | .176 | .293 | .468 | .585 | .880 | 1.170 | 2.93 | 5.85 | 11.70 | 29.3 | 58.5 |
| 130° | .0457 | .0913 | .137 | .228 | .365 | .457 | .685 | .913 | 2.28 | 4.57 | 9.13 | 22.8 | 45.7 |
| 135° | .0354 | .0707 | .106 | .177 | .283 | .354 | .530 | .707 | 1.77 | 3.54 | 7.07 | 17.7 | 35.4 |
| 140° | .0270 | .0539 | .081 | .135 | .216 | .270 | .405 | .539 | 1.35 | 2.70 | 5.39 | 13.5 | 27.0 |
| 145° | .0201 | .0402 | .060 | .100 | .161 | .201 | .300 | .402 | 1.00 | 2.01 | 4.02 | 10.0 | 20.1 |
| 150° | .0144 | .0289 | .043 | .072 | .115 | .144 | .215 | .289 | .72 | 1.44 | 2.89 | 7.20 | 14.4 |
| 155° | .0099 | .0197 | .030 | .049 | .079 | .099 | .150 | .197 | .40 | .99 | 1.97 | 4.90 | 9.9 |
| 160° | .0062 | .0125 | .019 | .031 | .05 | .062 | .095 | .125 | .31 | .62 | 1.25 | 3.11 | 6.2 |
| 165° | .0035 | .0069 | .010 | .017 | .028 | .035 | .050 | .069 | .17 | .35 | .69 | 1.72 | 3.5 |
| 170° | .0015 | .0031 | .005 | .008 | .012 | .015 | .025 | .031 | .08 | .15 | .31 | .80 | 1.5 |
| 175° | .9004 | .0008 | .001 | .002 | .003 | .004 | .005 | .008 | .02 | .04 | .08 | .21 | .4 |
| 176° | .0002 | .0004 | .001 | .001 | .002 | .002 | .004 | .004 | .01 | .02 | .04 | .10 | .2 |
| 177° | .0001 | .0003 | .000 | .001 | .001 | .001 | .002 | .003 | 99 5) | .01 | .02 | .05 | .1 |
| 178° | .0001 | .0001 | .000 | .000 | .000 | .001 | .001 | .001 | .00(3) | .00(6) | .01 | .03 | .016 |
| 179° | .0000 | .0000 | .000 | .000 | .000 | .000 | .000 | .000 | .00 | .00 | .00 | .00 | .00 |
| 180° | .0000 | .0000 | .000 | .000 | .000 | .000 | .000 | .00 | .00 | .00 | .00 | .00 | .00 |

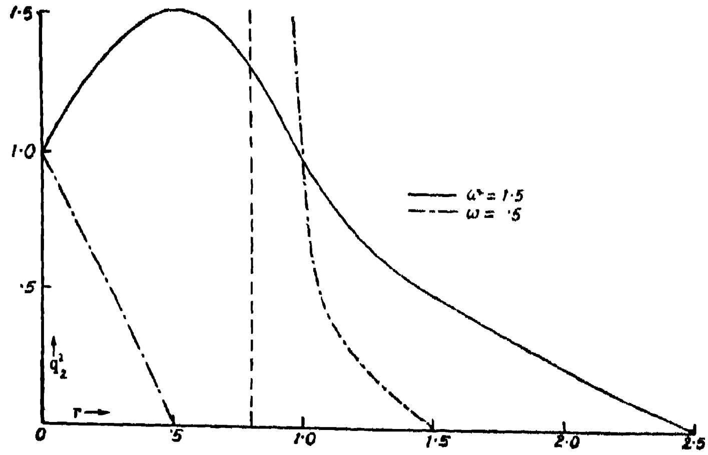


FIG. 3

Variation of the square of the refractive index for the $e$-wave with electron concentration $\left(r = \frac{4\pi e^2}{m p^2} N\right)$ for $p_0/p = \omega < 1$ (here ·5) and $p_0/p = \omega > 1$ (here 1·5)

## 3 FINITE DAMPING

We next discuss the case when $\delta > 0$, and in so doing we have to formulate the expressions for polarisation ratios and refractive indices in such a way that if $\delta \to 0$, these general expressions should reduce to those discussed in the previous section.

In this case $F_1$ and $F_2$ are complex roots of the equation (2.13). Let us put

$$F_1 = -\rho\, e^{i\phi}, \text{ and consequently } F_2 = \frac{1}{\rho} e^{-i\phi} \qquad \ldots \quad (3.1)$$

since $F_1F_2 = -1$, with the condition that $\rho$ is always positive. In the particular case $\delta = 0$, we have $\phi = 0$ or $\pi$. Since $F_1$ is negative for $r < 1$, therefore $\phi = 0$ for $r < 1$. Again $F_1$ is positive for $r > 1$, therefore, $\phi = \pi$ for $r > 1$. So we get for $\delta = 0$,

$$\rho = \sqrt{1+g^2} - |g| \text{ for } r >. =, < 1.$$

Now

$$F_1 + F_2 = \frac{1}{\rho} e^{i\phi} - \rho\, e^{-i\phi} = 2g \cos \alpha\, e^{i\alpha} = 2G$$

Equating real and imaginary parts,

$$(1-r)\cos\phi\left(\frac{1}{\rho} - \rho\right) - \delta \sin\phi\left(\frac{1}{\rho} + \rho\right) = -\frac{\omega \sin^2\theta}{\cos\theta}$$

$$\delta \cos\phi\left(\frac{1}{\rho} - \rho\right) + (1-r)\sin\phi\left(\frac{1}{\rho} + \rho\right) = 0 \qquad (3.2)$$

or $\cos\phi\{(1/\rho)-\rho\} = 2g\cos^2\alpha, \quad \sin\phi\{(1/\rho)+\rho\} = -2g\sin\alpha\cos\alpha.$

Solving the above equations (3.2) we get

$$\sin^2\phi = \frac{2g^2\sin^2\alpha}{1+g^2+\sqrt{1+2g^2\cos 2\alpha+g^4}}, \qquad \tan\phi = -\frac{1-\rho^2}{1+\rho^2}\tan\alpha \quad \ldots \quad (3.3)$$

and

$$\rho = \pm\sqrt{1+g^2\cos^2\alpha-\sin^2\phi} \pm \sqrt{g^2\cos^2\alpha-\sin^2\phi} \quad \ldots \quad (3.4)$$

This expression for $\rho$ can reduce to the corresponding relation for $\delta=0$ only if we take

$$\rho = \sqrt{1+g^2\cos^2\alpha-\sin^2\phi} - \sqrt{g^2\cos^2\alpha-\sin^2\phi} \quad \ldots \quad (3.4a)$$

Hence $1/\rho-\rho = 2\sqrt{g^2\cos^2\alpha-\sin^2\phi} > 0$, for all values of $\alpha$ and $g$. Thus $\rho < 1$. Then returning to the equations (3.2), we have for northern hemisphere for the region $r < 1$, *i.e.*, $g > 0$

$$\cos\phi > 0, \quad \sin\phi < 0, \quad i.e., \; 3\pi/2 < \phi < 2\pi$$

and for the region $r > 1$, *i.e.*, $g < 0$

$$\cos\phi < 0, \quad \sin\phi < 0, \quad i.e., \; \pi < \phi < 3\pi/2.$$

The case in the southern hemisphere is just the opposite. The results can be tabulated as :

TABLE II

| $\theta$ | $r$ | $g = \dfrac{\omega\sin^2\theta}{2(r-1)\cos\theta}$ | $\delta=0$ | | $\delta>0$ | |
|---|---|---|---|---|---|---|
| | | | $\rho=-F_1$ | $\phi$ | $\rho$ | $\phi$ |
| N. H. | $<1$ | $(+)$ | $\sqrt{1+g^2}-g$ | $0$ | | $3/2\pi<\phi\leq 2\pi$ |
| $\pi/2\leq\theta\leq\pi$ | $>1$ | $(-)$ | $\sqrt{1+g^2}+\lvert g\rvert$ | $\pi$ | $\sqrt{1+g^2\cos^2\alpha-\sin^2\phi}$ | $\pi\leq\phi\leq 3/2\pi$ |
| S. H. | $<1$ | $(-)$ | $\sqrt{1+g^2}+\lvert g\rvert$ | $\pi$ | $-\sqrt{g^2\cos^2\alpha-\sin^2\phi}$ | $0<\phi<\pi/2$ |
| $0\leq\theta\leq\pi/2$ | $>1$ | $(+)$ | $\sqrt{1+g^2}-g$ | $0$ | | $\pi/2<\phi\leq\pi$ |

With these complex expressions for $F_1$ and $F_2$ which reduce to the expressions discussed in the previous chapter, we get the ordinary and extraordinary complex refractive indices as

$$q_o^2 = 1-\frac{r}{\beta+\omega\cos\theta F_1} = 1-\frac{r}{\beta-\omega\rho\cos\theta\, e^{i\phi}} = 1-\frac{r}{X_o-iY_o} \quad \ldots \quad (3.5)$$

where $X_o = 1-\omega\rho\cos\theta\cos\phi, \quad Y_o = \delta+\omega\rho\cos\theta\sin\phi.$

$$q_e^2 = 1-\frac{r}{\beta+\omega\cos\theta F_2} = 1-\frac{r}{\beta+\omega/\rho\cos\theta\, e^{-i\phi}} = 1-\frac{r}{X_e-iY_e} \quad \ldots \quad (3.6)$$

where $X_e = 1+\omega/\rho\cos\theta\cos\phi, \quad Y_e = \delta+\omega/\rho\cos\theta\sin\phi,$

Following Booker, we may put $q = \mu - (ick/p)$

Then

$$\mu^2 - \frac{c^2k^2}{p^2} = 1 - \frac{rX}{X^2+Y^2}, \quad \frac{2\mu ck}{p} = \frac{rX}{X^2+Y^2} \qquad \ldots \quad (3.7)$$

and for the non-deviating region, where $ck/p << 1$

$$k = \frac{p}{2c}\left(\frac{1}{\mu} - \mu\right)\frac{Y}{X}, \quad \mu^2 = 1 - \frac{rX}{X^2+Y^2}.$$

We have thus for the non-deviating region :

$$\left.\begin{aligned} \mu_0^2 &= 1 - r\frac{1-\omega\rho\cos\theta\cos\phi}{(1-\omega\rho\cos\theta\cos\phi)^2 + (\delta+\rho\omega\cos\theta\sin\phi)^2} \\ k_0 &= \frac{p}{2c}\left(\frac{1}{\mu} - \mu\right)\frac{\delta+\rho\omega\cos\theta\sin\phi}{1-\omega\rho\cos\theta\cos\phi} \end{aligned}\right\} \qquad \ldots \quad (3.8)$$

$$\left.\begin{aligned} \mu_e^2 &= 1 - r\frac{1+\omega/\rho\cos\theta\cos\phi}{(1+\omega/\rho\cos\theta\cos\phi)^2 + (\delta+\omega/\rho\cos\theta\sin\phi)^2} \\ k_e &= \frac{p}{2c}\left(\frac{1}{\mu_e} - \mu_e\right)\frac{\delta+\omega/\rho\cos\theta\sin\phi}{(1+\omega/\rho\cos\theta\cos\phi)} \end{aligned}\right\} \qquad \ldots \quad (3.9)$$

The correctness of the above expressions can be tested for special cases .

For the magnetic equator, $\theta = \pi/2$ we have from (3.4*a*), $\rho = 0$ and $\frac{\sin\theta}{\rho}e^{-i\phi} = \frac{\omega}{\beta-\gamma}$. Hence, we get the equations (2.10), as special cases of (2.16*a*), For the magnetic north pole, $\theta=\pi$, $\rho=1$, $\phi=\pi$, hence (2.16*a*) reduce to equations (2.11). For the magnetic south pole $\theta=0$, $\rho=1$, $\phi=0$, hence (2.16*a*) reduce to equations (2.11*a*).

## 4. POLARISATION.

Let us next discuss the polarisation of the down-coming wave for any station for a stratified, slowly varying ionosphere with finite damping. Since the e.m. waves which are propagated in such a medium are not transverse in the electric vector ***E***, but are transverse in the magnetic vector ***H*** and in the method of detection, the ***H*** vector is utilized, it is customary to express the polarisation of the waves with respect to the latter. So we start with the equations of propagation of the magnetic vector, *viz.*,

$$\frac{d^2H_x}{du^2} + K_2H_x - iLH_y = 0 \qquad \ldots \quad (4.1)$$

$$\frac{d^2H_y}{du^2} + K_1H_y + iLH_x = 0 \qquad \ldots \quad (4.2)$$

in place of the corresponding equations (2.6) and (2.8) for the electric vector. Equations (4.1) and (4.2) follow immediately from (2.6) and (2.8) and (2.1). Eliminating $H_y$ and $H_x$ from (4.2) and (4.3) respectively we get

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$$\frac{d^4 H_x}{du^4} + (K_1 + K_2)\frac{d^2 H_x}{du^2} + (K_1 K_2 - L^2) H_x = 0 \quad \ldots \quad (4.3)$$

and

$$\frac{d^4 H_y}{du^4} + (K_1 + K_2)\frac{d^2 H_y}{du^2} + (K_1 K_2 - L^2) H_y = 0 \quad \ldots \quad (4.4)$$

where the derivatives of $K_1$, $K_2$ and L have been neglected as before. The general solutions of (4.3) and (4.4) are

$$H_x = A_1 e^{is_1 u} + A_2 e^{+is_2 u} + A_3 e^{-is_1 u} + A_4 e^{-is_2 u}$$

$$H_y = B_1 e^{is_1 u} + B_2 e^{+is_2 u} + B_3 e^{-is_1 u} + B_4 e^{-is_2 u}$$

where

$$s_1^2 = \frac{K_1 + K_2 - \sqrt{(K_1 - K_2)^2 + 4L^2}}{2}$$

$$s_2^2 = \frac{K_1 + K_2 + \sqrt{(K_1 - K_2)^2 + 4K^2}}{2}$$

It can be easily shown that

$$s_1^2 = q_1^2, \qquad s_2^2 = q_2^2$$

Retaining only the solutions for the down-coming waves, we get

$$H_x = A_1 e^{iq_2 u} + A_2 e^{iq_1 u} \quad \ldots \quad (4.5)$$

$$H_y = B_1 e^{iq_2 u} + B_2 e^{iq_1 u} \quad \ldots \quad (4.6)$$

where $q_1$ and $q_2$ are those roots of $s_1^2$ and $s_2^2$ respectively which have the imaginary parts positive. Substituting (4.5) and (4.6) in (4.1) we get,

$$(-q_1^2 A_2 + K_2 A_2 - iLB_2) e^{iq_1 u} + (-q_2^2 A_1 + K_2 A_1 - iLB_1) e^{iq_2 u} = 0$$

which being an identity in $u$ yields

$$-q_1^2 A_2 + K_2 A_2 - iL B_2 = 0$$

$$-q_2^2 A_1 + K_2 A_1 - iL B_1 = 0$$

whence we have, referring back to (4.1) and (4.2),

$$\frac{B_1}{A_1} = i\frac{q_2^2 - K_2}{L}\,; \quad \frac{B_2}{A_2} = i\frac{q_1^2 - K_2}{L} \quad \ldots \quad (4.7)$$

From the general solutions of (4.1) and (4.2) it is evident that these equations represent two waves given by

$$H_x^{(1)} = A_1 e^{iq_2 u}, \quad H_y^{(1)} = B_1 e^{iq_1 u}$$

$$H_x^{(2)} = A_2 e^{iq_1 u}, \quad H_y^{(2)} = B_2 e^{iq_1 u}$$

travelling with complex phase velocities $c/q_2$ and $c/q_1$ respectively. Following the nomenclature adopted before $H_x^{(1)}$ & $H_y^{(1)}$ combine to give the down coming extra-ordinary wave and $H_x^{(2)}$ & $H_y^{(2)}$ give the downcoming ordinary wave, and the polarisation ratios for the two waves are

$$\frac{H_y^{(0)}}{H_x^{(0)}} = \frac{B_2}{A_2} = i\,\frac{q_1^2 - K_2}{L} = iF_1$$

$$\frac{H_y^e}{H_x^e} = \frac{B_1}{A_1} = i\,\frac{q_2^2 - K_o}{L} = iF_2$$

Taking $\qquad H_x^{(0)} = R_{ox}\, e^{i(\gamma_{ox} + pt)}, \quad H_y^{(0)} = R_{oy}\, e^{i(\gamma_{oy} + pt)}$

where $R_{ox}$, $R_{oy}$, $\gamma_{ox}$ and $\gamma_{oy}$ are real functions of $u$, we get

$$H_x^{(0)} = R_{ox} \cos(\gamma_{ox} + pt), \quad H_y^{(0)} = R_{oy} \cos(\gamma_{oy} + pt)$$

as the two true solutions of the problem with

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos pt \text{ in place of } \boldsymbol{E} = \boldsymbol{E}_0\, e^{ipt}$$

Hence $\dfrac{H_y^{(0)}}{H_x^{(0)}} = \dfrac{R_{oy}}{R_{ox}} e^{i(\gamma_{oy} - \gamma_{ox})} = iF_1 = -i\rho\, e^{i\phi} = \rho\, e^{i(\phi - \pi/2)}$

whence
$$\rho = \frac{R_{oy}}{R_{ox}}$$

and
$$\gamma_{oy} - \gamma_{ox} = \phi - \frac{\pi}{2}.$$

are the ratio of the axes and the constant phase difference between the $y$ and $x$ components of the magnetic vector respectively. The equation of the polarisation ellipse for the ordinary wave follows immediately by eliminating "$pt$" between the two equations in (4.8) : We have

$$H_x^{(o)2} - \frac{2H_x^{(o)}H_y^{(o)}}{\rho}\sin\phi + \frac{H_y^{(o)2}}{\rho} = R_{ox}^2 \cos^2\phi \qquad \dots \quad (4.9)$$

This equation shows that the axes of the ellipse are tilted to the respective $y$ and $x$ axes, the amount $\psi_0$ of tilt to the $y$ axis being given by

$$\tan 2\psi_0 = -\frac{2\rho \sin\phi}{1 - \rho^2} \qquad \dots \quad (4.10)$$

The points of contact of this ellipse with the circumscribed rectangle are (FIG. 4) respectively $(\pm R_{ox} \sin\phi,\ \pm R_{ox}\,\rho)$ and $(\pm R_{ox},\ \pm\rho\, R_{ox} \sin\phi)$

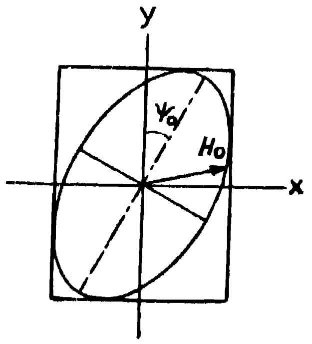


FIG. 4

**Polarisation ellipse for the reflected O–wave (northern hemisphere)**

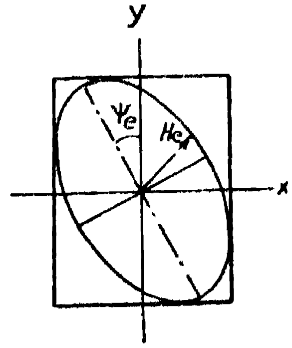


FIG. 5

**Polarisation ellipse for the reflected O–wave (southern hemisphere)**

For the other wave

$$\rho_e = \frac{R_{ey}}{R_{ex}}, \quad \gamma_{ex} - \gamma_{ex} = \phi + \frac{\pi}{2}$$

and it can easily be shown that

$$\rho_e = \frac{1}{\rho} \text{ and } (\gamma_{ey} - \gamma_{ex}) - (\gamma_{ox} - \gamma_{ox}) = \pi \qquad \dots \quad (4.11)$$

consequently the equation of the polarisation ellipse for the e-wave is

$$\frac{H_x'^2}{\rho^2} + \frac{2H_x' H_y'}{\rho} \sin\phi + H_y'^2 = \frac{R_{ex}^2}{\rho^2} \cos^2\phi$$

which shows the same ellipse rotated through an angle $\pi/2$. For this ellipse (FIG. 5) the angle of tilt and the points of contact with the circumscribed rectangle are given by :

$$\tan 2\psi_e = -\frac{2\rho \sin\phi}{1-\rho^2} = \tan(2\psi_o + \pi) \qquad \dots \quad (4.13)$$

and

$$\left(\pm R_e \sin\phi, \pm \frac{R_e}{\rho}\right), \quad \left(\pm R_e, \pm \frac{R_e}{\rho} \sin\phi\right),$$

In the experimental methods of determining the ratio of the axes of the polarisation ellipse, it is generally assumed that the polarisation of the downcoming wave is mainly determined by the lowest layers of the ionosphere where N the ion concentration tends to vanish. Recently Eckersly (1945) has determined the polarisation of the downcoming waves for $p = 6.1$, 6.4 and 7.6 Mc. and has remarked that in order to agree with his experimental results, the polarisation of the downcoming wave should be determined not by the lowest layer of the ionised strata but somewhere inside. Since there is as yet no definite and convincing evidence either experimental or theoretical, of the particular strata or the entire layer fixing the polarisation, we have plotted $\rho$, the ratio of the axes for the o-wave as a function of $\theta' = \theta - \frac{\pi}{2}$, the magnetic latitude of the place of observations for various values of $\omega$, for $r$, *i.e.*, $N \to 0$.

Sense of rotation of the polarisation ellipse can be inferred from equations (4.8). Since the damping has no effect on the sense of rotation of the magnetic vector, we infer the sense of rotation for the case where damping is absent. In this case for northern hemisphere $\phi = 0$ and hence equation (4.8) gives

$$H_x{}^o = R_{ox} \cos(\gamma_{ox} + pt)$$

$$H_y{}^{(o)} = \rho R_{ox} \sin(\gamma_{ox} + pt).$$

Hence as $t$ increases from 0, $H_x{}^o$ decreases from $R_{ox} \cos\gamma_{ox}$ to 0 and remains positive, while $H_y{}^{(o)}$ increases from $\rho R_{ox} \sin \gamma_{ox}$ to $\rho R_{ox}$ showing that the vector $H_T{}^o$, whose components are $H_x{}^o$ and $H_y{}^o$ and which describes the ellipse given by (4.9) is moving in the anticlockwise direction. Thus for all waves received in the northern hemisphere the down-

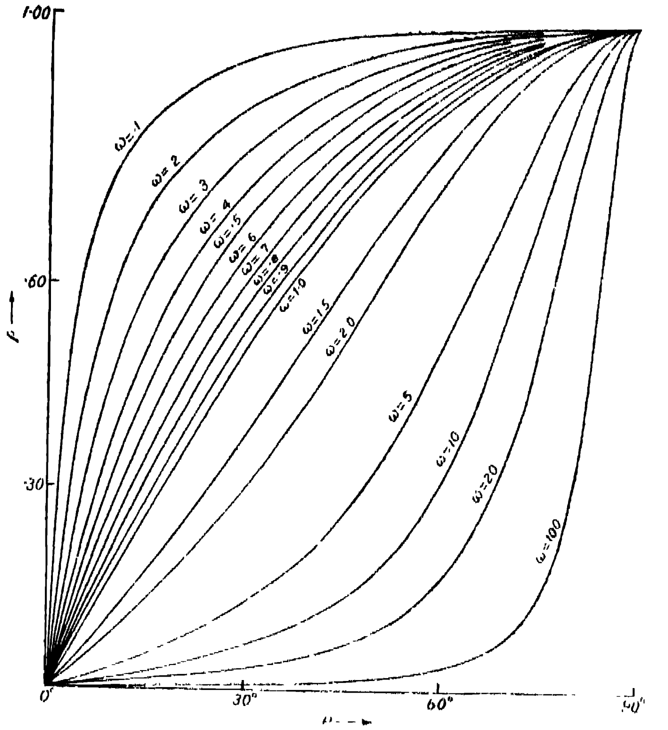


FIG. 9

Variation of the polarisation ratio $\rho = \sqrt{1+g^2} - |g|$ where $g = \frac{\omega \sin^2\theta}{2\cos\theta}$ for various values of $\omega = p_h/p$, for different angles of propagation $\rho \to 1$ means circular polarisation.

coming ordinary wave is polarised in the anticlock-wise direction as viewed along the direction of propagation. For the extraordinary wave,

$$H_x^e = R_{e,x}\cos(\gamma_{e,x} + pt)$$

$$H_y^e = -\frac{R_{e,x}}{\rho}\sin(\gamma_{e,x} + pt)$$

Hence as $t$ increases from o, $H_x^e$ decreases as before but $H_y^e$ becomes more and more negative showing that the vector $H^e$ whose components are $H_x^e$ and $H_y^e$ and which describes the ellipse given by (4.12) moves in the clock-wise direction. Thus for all stations in the northern hemisphere, the downcoming e-wave is polarised right handed as viewed along the direction of propagation.

For the southern hemisphere the sense of the rotation of the two ellipses will be just opposite since $\phi = \pi$ for $\delta = o$ in the southern hemisphere in place of $\phi = o$ for $\delta = o$ in the northern hemisphere.

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