WAVE-TREATMENT OF PROPAGATION OF ELECTRO-MAGNETIC WAVES IN THE IONOSPHERE

By M. N. SAHA* and B. K. BANERJEA

ABSTRACT. Wave-equations for the propagation of e m. waves through the ionosphere have been obtained by the use of a new mathematical method involving the use of dyadic analysis introduced by Gibbs Expressions for steady current conductivity of the ionosphere have been obtained by this method and the results are concordant with those of Chapman; an extra term for the conductivity, which is more prominent in the F_2-layer has been obtained.

It has been shown that the wave is split up into three waves, as in Zeeman effect, one of which is ordinary, the other two extraordinary, in accordance with observations by Toshniwal, and Harang.

1 INTRODUCTION

The subject of propagation of electromagnetic waves in the ionosphere appears to be at the present time in a rather confused state. Appleton (1932), in his pioneering work, used what is now commonly known the ray treatment, *i.e.*, starting from Maxwell's equations, he obtained a value of the refractive index of the e.m. waves in terms of the electron concentration, the earth's magnetic field and the damping coefficient of electrons. He further postulated that the wave gets reflected when the refractive index vanishes. From the two values of refractive index it was deduced that the wave splits up into two, one ordinary and the other extraordinary and the sense of polarisation of each wave was determined. The condition of reflection of the extraordinary wave is, however, satisfied, at two distinct levels given by the condition $p_0^2 = p^2 \pm pp_h$. It appears to have been assumed that only one of these waves, corresponding to the negative sign existed. Toshniwal (1935) and Harang (1936) have however, obtained at times reflections corresponding to the conditions $p_0^2 = p^2 + pp_h$, so that it is legitimate to think that the wave really gets split into three components on entry into the ionosphere, one of which fails usually to get reflected owing to heavy absorption. Further, we have to explain the phenomena of M-reflections, which prove that the wave does not get completely reflected even when $\mu = 0$, but may leak through the ion-layer in considerable intensity, and get reflected from a higher layer.

The wave treatment was first attempted by Hartree (1929, 1931) in three important papers. The papers of Hartree are extremely difficult to follow on account of the difficult notations used and some unnecessary complications introduced. He used throughout the notation of dyadics, introduced by Gibbs.

* Fellow of the Indian Physical Society.

This notation, though much convenient for mathematical working is not generally familiar and to make the deductions intelligible the results have to be transcribed to ordinary notations which was not carried out by Hartree. Hartree obtained the displacement of the electron or the ion as **S**.**E** where S is a tensor, **E** = Electric field. This part is rendered rather complicated because the electron is regarded as bound by a quasi-elastic force. From the expression for S, he obtained an expression for σ called the scattering tensor. The underlying physical idea is borrowed from a paper by Darwin (1925), who has shown that almost all optical phenomena, *e.g.* reflection and refraction can be explained in terms of scattering by elementary constituents of the medium. Hartree has shown from the equivalence of two different processes that the equation of propagation of the electromagnetic waves in the ionosphere continues to obey the Maxwellian form. The treatment was also rendered complicated by the introduction of the term β, the Lorentz polarisation term which he took not much far from 1/3. It has, however, been shown by Darwin (1934) that $\beta = 0$, and this considerably simplifies Hartree's method. The expression for refractive index was obtained by considering the case of normal incidence in a stratified medium where μ is supposed to be constant. He ultimately obtains the same result as Appleton. So far his treatment led to a justification rather than laying the foundations of a rigorous wave treatment. In a later paper he takes the wave equation with a variable μ and tries to solve this equation for a few simple cases but it is obvious none of these assumptions corresponds to reality.

Saha, Rai and Mathur (1937) expressed the displacement of the ions in simpler analytical form, which may be shown identical with those of Hartree in spite of the apparent differences in form. From this displacement they obtained the value of the dielectric tensor for a stratified medium and ultimately obtained the same expression for μ as that of Appleton. The wave treatment was applied in a simple case for the O-wave and the penetrability of the electron barrier for a simple case was deduced.

In the present paper the foundations of a rigorous wave treatment have been laid down and the expressions for refractive index, conductivity and direct current conductivity have been deduced. The solution of the wave-equations has not yet been tried.

2. THE DISPLACEMENT OF THE IONS IN THE IONOSPHERE

The equation of motion of the charged ions referred to any system of axes can be written in the form

$$\frac{d^2\mathbf{r}}{dt^2} + \nu\frac{d\mathbf{r}}{dt} + \frac{e}{m_0 c}\left[\mathbf{H} \times \frac{d\mathbf{r}}{dt}\right] = \frac{e\mathbf{E}}{m_0} \qquad (1)$$

where $\mathbf{r}$ = displacement vector of the ions with components (ξ, η, ζ)

m_0 = mass of the ions

ν = collision frequency of the ions

$\mathbf{H}$ = Earth's magnetic field with direction cosines l, m, n.

$\mathbf{E} = \mathbf{E}_0 \cos pt$, the electric vector of the incident e.m. wave with direction cosines l', m', n'.

The effect of the magnetic vector and space charge have been omitted as usual. The notation conforms as closely as possible to those used by Appleton (1932) and Saha, Rai and Mathur (1937).

It can be easily verified that the solution of the above equation with $\mathbf{E} = \mathbf{E}_0 \cos pt$ is the real part of the solution when we put $\mathbf{E} = \mathbf{E}_0 e^{ipt}$; we use $\mathbf{E}$ in this form, because the solution is now easy to obtain. The quantity analogous to static conductivity now comes out to be complex (Stratton, 1939) whose real part gives ordinary conductivity, imaginary part gives the deviation of the refractive index from unity.

Starting with $\mathbf{E} = \mathbf{E}_0 e^{ipt}$ and introducing the notations

$$p_0^2 = \frac{4\pi N e^2}{m_0}\ ; \qquad \alpha = i\nu p - p^2\ ; \qquad p_h = \frac{eH}{m_0 c}\ ;$$

$$\beta = i p p_h\ ; \qquad \mathbf{r} = (\xi, \eta, \zeta) e^{ipt}$$

and breaking up the above equation into components we get

$$\begin{aligned} \alpha\xi + i p p_h (m\zeta - n\eta) &= \frac{e}{m_0} l' E_0 \\ \alpha\eta + i p p_h (n\xi - l\zeta) &= \frac{e}{m_0} m' E_0 \qquad \ldots \quad (2) \\ \alpha\zeta + i p p_h (l\eta - m\xi) &= \frac{e}{m_0} n' E_0 \end{aligned}$$

Solving these equations by the usual determinant method, we have

$$\xi = \frac{eE_0}{m_0} \left[l'(\alpha^2 + \beta^2 l^2) + m'(\alpha\beta n + \beta^2 m l) + n'((\beta^2 n l - \beta\alpha m) \right] / \left[\alpha(\alpha^2 + \beta^2) \right] \quad \ldots \quad (3)$$

$$\eta = \frac{eE_0}{m_0} \left[l'(\beta^2 m l - \beta\alpha n) + m'(\alpha^2 + \beta^2 m^2) + n'(\beta^2 m n + \beta\alpha l) \right] / \left[\alpha(\alpha^2 + \beta^2) \right]$$

$$\zeta = \frac{eE_0}{m_0} \left[l'(\beta^2 n l + \beta\alpha m) + m'(\beta^2 m n - \beta\alpha l) + n'(\alpha^2 + \beta^2 n^2) \right] / \left[\alpha(\alpha^2 + \beta^2) \right]$$

Let $\boldsymbol{\sigma}$ denote the complex conductivity. We have

$$\boldsymbol{\sigma}.\mathbf{E} = \text{current} = -Ne\frac{d\mathbf{r}}{dt} = -ipNe\mathbf{r} \qquad \ldots \quad (4)$$

Substituting the value of $\mathbf{r}$ from (3) we get $\boldsymbol{\sigma}$ in the tensor form

$$\boldsymbol{\sigma} = \frac{-m_0 / ipNe^2}{\alpha(\alpha^2 + \beta^2)} \begin{vmatrix} \alpha^2 + \beta^2 l^2 & \beta^2 m l - \beta\alpha n & \beta^2 n l + \beta\alpha m \\ \beta^2 l m + \beta\alpha n & \alpha^2 + \beta^2 m^2 & \beta^2 m n - \beta\alpha l \\ \beta^2 n l - \beta\alpha m & \beta^2 m n + \beta\alpha l & \alpha^2 + \beta^2 n^2 \end{vmatrix} \qquad \ldots \quad (5)$$

Let us next find out the principal axes of the tensor ellipsoid by using the Hamilton-Cayley method. The principal components are given by the roots of the equation,

$$\lambda^3 - \Phi_1\lambda^2 + \Phi_2\lambda - \Phi_3 = o \qquad \ldots \quad (6)$$

where Φ_1 = spur of the tensor, σ, $q = x, y, z$

$\Phi_2 = \Sigma(\sigma_{qq}\sigma_{ss} - \sigma_{qs}\sigma_{sq})$ s, $q = x, y, z$, $s \neq q$

Φ_3 = Determinant of σ.

We obtain from (5)

$$\Phi_1 = 3\alpha^2 + \beta^2;\ \Phi_2 = 3\alpha^2(\alpha^2 + \beta^2);\ \Phi_3 = \alpha^2(\alpha^2 + \beta^2)^2$$

Hence the cubic equation reduces to

$$\lambda^3 - (3\alpha^2 + \beta^2)\lambda^2 + 3\alpha^2(\alpha^2 + \beta^2)\lambda - \alpha^2(\alpha^2 + \beta^2)^2 = 0$$

or

$$[\lambda - (\alpha^2 + \beta^2)]\,[\lambda - (\alpha^2 + i\alpha\beta)]\,[\lambda - (\alpha^2 - i\alpha\beta)] = 0 \qquad \ldots \quad (7)$$

If $\nu = 0$, α, $i\alpha\beta$, and β^2 are all real, and the roots are all real, such a dyad has been classified by Gibbs as a tonic dyad.

In general case, for $\nu \neq 0$, we separate the complex conductivity tensor $\boldsymbol{\sigma}$ in (5) into real, and imaginary parts. Then forming the corresponding Hamilton-Cayley equations for the real and the imaginary parts we get the roots for $R_e\boldsymbol{\sigma}$ as

$$\lambda_1 = \frac{Nc^2}{mp}\cdot\frac{\nu}{p^2+\nu^2};\ \lambda_2 = \frac{Nc^2}{mp}\cdot\frac{\nu + ip_h}{p^2 + (\nu + ip_h)^2};\ \lambda_3 = \frac{Nc^2}{mp}\cdot\frac{\nu - ip_h}{p^2 + (\nu - ip_h)^2} \qquad \ldots \quad (8)$$

and for $I_m\boldsymbol{\sigma}$

$$\lambda_1 = \frac{Nc^2}{m}\cdot\frac{1}{p^2+\nu^2};\ \lambda_2 = \frac{Nc^2}{m}\cdot\frac{1}{p^2 + (\nu + ip_h)^2};\ \lambda_3 = \frac{Nc^3}{m}\cdot\frac{1}{p^2 + (\nu - ip_h)^2} \qquad \ldots \quad (9)$$

Dyads of this type, having two complex conjugate roots of the H. C. equation are classified by Gibbs as cyclotonic dyad. All the properties of the ionosphere are of this type.

Let us next find out the orientation of the characteristic principal axis of the tensor σ, to the earth's magnetic field. We consider a vector $\boldsymbol{\rho}$ such that

$$\sigma\boldsymbol{\rho} = \lambda\boldsymbol{\rho}$$

where λ is a proportionality factor. If I_0 be the idem factor

$$\sigma\rho = \lambda I_0\rho \qquad \text{or} \quad (\boldsymbol{\sigma} - \lambda I_0)\rho = 0$$

i.e.

$$\left.\begin{aligned} (\sigma_{11} - \lambda_1)L + \sigma_{12}M + \sigma_{13}N &= 0 \\ \sigma_{21}L + (\sigma_{22} - \lambda_1)M + \sigma_{23}N_0 &= 0 \\ \sigma_{31}L + \sigma_{32}M + (\sigma_{33} - \lambda)N &= 0 \end{aligned}\right\} \qquad \ldots \quad (10)$$

where L, M, N are the direction cosines of the principal axis. Substituting $\lambda = \alpha^2 + \beta^2$ and the corresponding values of σ_{11}, σ_{12}, etc., and remembering that

$$L^2 + M^2 + N^2 = 1$$

we get

$$L = \pm l, \quad M = \pm m, \quad N = \pm n \qquad \ldots \quad (11)$$

We thus see that the real characteristic principal axis of the tensor $\boldsymbol{\sigma}$ coincides with the earth's magnetic field. Similarly we can show that the other properties like ordinary conductivity, dielectric constant, etc., have the same characteristic principal axis. The other two axes of the cyclotonic dyad are in a plane perpendicular to this axis, and may be oriented arbitrarily.

Let us next choose a new system of axes with the direction of the earth's magnetic field as the X-axis, Z-axis being in the magnetic meridian and Y axis horizontal perpendicular to the magnetic meridian. Referred to this new coordinate system, let us now express the displacements and the tensor $\boldsymbol{\sigma}$ in the new coordinates, We put $l = 1$, $m = n = 0$ in (5). Then

$$\boldsymbol{\sigma} = \frac{-ipNe^2/m_0}{\alpha(\alpha^2+\beta^2)} \begin{vmatrix} \alpha^2+\beta^2 & 0 & 0 \\ 0 & \alpha^2 & -\alpha\beta \\ 0 & \alpha\beta & \alpha^2 \end{vmatrix} \qquad \ldots \quad (12)$$

and from (3)

$$\xi = \frac{eE}{m_0}\left[l'(\alpha^2+\beta^2)\right] / \left[\alpha(\alpha^2+\beta^2)\right] \qquad \ldots \quad (13)$$

$$\eta = \frac{eE}{m_0}\left[\alpha^2 m' + \beta\alpha n'\right] / \left[\alpha(\alpha^2+\beta^2)\right]$$

$$\zeta = \frac{eE}{m_0}\left[\alpha^2 n' - \beta\alpha n'\right] / \left[\alpha(\alpha^2+\beta^2)\right]$$

Let us next form that the real parts of the displacements, which are the true solutions of equation (1) with $E = E_0 \cos pt$. We have, putting $E(l', m', n') = (E_x, E_y, E_z,)$

$$\text{Re}\,\xi = -\frac{eE_x}{m_0(p^2+\nu^2)}\left[\cos pt - \frac{\nu}{p}\sin pt\right] \qquad \ldots \quad (14)$$

$$(\text{Re}\,\eta \pm i\,\text{Re}\,\zeta) = -\frac{e(E_y \pm iE_z)}{m_0\left[p^2+(\nu \pm ip_h)^2\right]}\left\{\cos pt - \frac{\nu \pm ip_h}{p}\sin pt\right\}$$

From the above expressions, we can easily obtain expressions for the steady current electrical conductivity in the ionosphere as obtained by Schuster, Chapman and Pedersen. We have to put $p = 0$

We get

$$\dot{\xi} = -\frac{eE_x}{m_0}\,; \quad \dot{\eta} = \frac{e}{m_0(p_h^2+\nu^2)}\left[\nu E_y + p_h E_z\right]; \quad \dot{\zeta} = \frac{e}{m_0(p_h^2+\nu^2)}\left[\nu E_z - p_h E_y\right)$$

We have therefore $I_x = Ne\dot{\xi} = \frac{Ne^2}{m\nu} E_x$ and

$$\sigma_x = I_x/E_x = \text{conductivity parallel to the magnetic field} = \frac{Ne^2}{m\nu} \qquad \ldots \quad (15)$$

If now $E_z = o$, *i.e.* the e.m.f. is in the horizontal plane perpendicular to the magnetic meridian, we have

$$\sigma_{\parallel} = \frac{Ne^2}{m_0} \cdot \frac{\nu}{(\nu^2 + p_h^2)} \qquad \ldots \quad (16)$$

This is known as the transverse conductivity .

We have an additional current along the Z-axis, *i.e.*, in the meridian plane perpendicular to the lines of force, and the conductivity

$$\sigma_z = -\frac{Ne^2 p_h}{m_0(\nu^2 + p_h^2)} \qquad \ldots \quad (17)$$

If $E_z \neq o$, but $E_x = E_{\parallel} = o$, we have flow of currents both along Y and Z axes.

We observe from these results that even when ν tends to zero as in the F region, we have a conductivity transverse to the magnetic meridian $= \frac{Ne^2}{m_0} \cdot \frac{p_h}{(\nu^2 + p_h^2)}$ and this has a limiting value $\frac{Ne^2}{m_0 p_h}$

We have thus got an extra term for transverse conductivity, viz. $\frac{Ne^2}{m_0} \cdot \frac{p_h}{(\nu^2 + p_h^2)}$ in addition to those already known but we have not yet had time to examine its probable contribution to the theory of L and S terms in geomagnetism.

3 THE FUNDAMENTAL MAXWELLIAN EQUATIONS

The fundamental equations for the propagation of the e.m. waves in the ionosphere are

$$\nabla \times H = \frac{1}{c}\dot{E} + \frac{4\pi}{c} I. \qquad \ldots \quad (18)$$

$$\nabla \times E = -\frac{1}{c}\dot{H}.$$

$$\nabla \cdot E = 4\pi \rho$$
$$\nabla \cdot H = o$$

Here 'I' denotes total current, which requires some elucidation. We have $I = \sigma . \mathbf{E}$, where $\mathbf{E}$ = total field (*i.e.* sum of incident field plus radiation field due to surrounding ions. Also in the ionosphere, it is customary to take $\rho = o$. Thus (18.1) is modified as

$$\nabla \times H = \frac{1}{c}\dot{E} + \frac{4\pi\sigma . \mathbf{E}}{c}.$$

It can now be easily shown, taking curl of curl E, etc., that E satisfies the equation

$$\nabla^2 \mathbf{E} - \frac{\ddot{\mathbf{E}}}{c^2} = -\frac{4\pi}{c^2}.\boldsymbol{\sigma}\dot{\mathbf{E}}$$

or

$$\nabla^2 \mathbf{E} + \frac{p^2}{c^2}\left(\mathrm{I}_o + \frac{4\pi i\boldsymbol{\sigma}}{p}\right).\mathbf{E} = o \quad \ldots \quad (19)$$

and similarly H satisfied the the equation

$$\nabla^2 \mathbf{H} - \frac{1}{c^2}\ddot{\mathbf{H}} = -\frac{4\pi}{c}\nabla \times (\boldsymbol{\sigma}.\mathbf{E}) \quad \ldots \quad (20)$$

Now for $\boldsymbol{\sigma}$ we have to substitute in the case of any coordinate system expressions (5), but it will simplify matters if we introduce the principal coordinates defined in (12). We have then

$$\mathrm{I}_o + \frac{4\pi i\boldsymbol{\sigma}}{p} = \begin{vmatrix} 1 + \frac{p_o^2}{\alpha} & o & o \\ o & 1 + \frac{p_o^2\alpha}{\alpha^2+\beta^2} & -\frac{p_o^2\beta}{\alpha^2+\beta^2} \\ o & +\frac{p_o^2\beta}{\alpha^2+\beta^2} & 1 + \frac{p_o^2\alpha}{\alpha^2+\beta^2} \end{vmatrix} \quad \ldots \quad (21)$$

The equations (10) can then be split up into three equations

$$\nabla^2 \mathrm{E}_x + \frac{p^2}{c^2}\left(1 - \frac{p_o^2}{p^2 - i\nu p}\right)\mathrm{E}_x = o \quad \ldots \quad (22)$$

$$\nabla^2 \mathrm{E}_y + \frac{p^2}{c^2}(b\mathrm{E}_y - c\mathrm{E}_z) = o$$

$$\nabla^2 \mathrm{E}_z + \frac{p^2}{c^2}(p\mathrm{E}_y + b\mathrm{E}_z) = o$$

where $\qquad b = 1 + \frac{p_o^2\alpha}{\alpha^2+\beta^2}, \qquad c = \frac{p_o^2\beta}{\alpha^2+\beta^2}$

The last two equations had better be written in the form

$$\nabla^2(\mathrm{E}_y \pm i\mathrm{E}_z) + \frac{p^2}{c^2}\left\{1 - \frac{p_o^2}{p^2 - i\nu p \mp pp_h}\right\}(\mathrm{E}_y \pm i\mathrm{E}_z) = o \quad \ldots \quad (28)$$

From these equations, we see clearly that on entrance into the ionosphere, the three components E_x, $\mathrm{E}_y + i\mathrm{E}_z$, $\mathrm{E}_y - i\mathrm{E}_z$ travel with different velocities, depending on p_o^2, ν, and p_h. If these quantities are slowly varying, we can talk of refractive index. The E_x-component (electric displacement parallel to the magnetic field) has the complex refractive index

$$\mu_o = 1 - \frac{p_o^2}{p^2 - ip\nu}$$

and the $E_y + iE_z$, $E_y - iE_z$ components have the refractive indices

$$\mu_e = 1 - \frac{p_o^2}{p^2 - i\nu p \mp pp_h}$$

The analogy with Zeemann-effect is obvious ; $(E_y + iE_z)$ denote anticlockwise circular polarisation and $(E_y - iE_z)$ denote clockwise circular polarisation. If we neglect ν and put $\mu_o = 0$ we get Appleton's conditions for the reflexion of the o-wave, $p_o^2 = p^2$, and if we put $\mu_e = 0$, we get the two conditions for the reflexion of the two extraordinary waves $p_0^2 = p^2 \mp pp_h$, which have different sense of polarisation.

The complete solution of the equations (21), however, are rather difficult, for we are using a coordinate system which, except at the magnetic equator, and at the magnetic poles, cannot be linked to the local coordinates in a simple manner.

At the magnetic equator, the X and Y axes are horizontal and Z-axis is vertical. In a vertical propagation of the e.m. wave, $E_z = 0$, and we have only E_x and E_y definite. The reflected wave will therefore have its o-component polarized parallel to the magnetic field, the X-component polarized parallel to the Y-axis, *i.e.*, perpendicular to the magnetic field in a horizontal direction. We have, however, not yet tried to evaluate E_x, E_y in terms of the amplitudes of the wave sent out by the antenna.

For the magnetic pole, the X-axis is vertical, and for a vertical propagation we have $E_x = 0$, and we have only $E_y \pm iE_z$, *i.e.*, two circularly polarized X-waves. We have to obtain the reflexion coefficient from a solution of (21), which will be attempted in a future paper.

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PALIT LABORATORY OF PHYSICS
CALCUTTA UNIVERSITY.

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