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THE SCATTERING OF FAST ELECTRONS BY ATOMS

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ABSTRACT. The wave statistical theory of scattering of fast electron by atomic nucleus is developed by taking into account the effect of relativity and spin-orbit interaction. The correction term due to the spin-orbit interaction is derived from entirely statistical consideration and is in exact agreement with Mott's second correction obtained from Dirac's theory. However, Mott's first correction which is independent of the atomic number is not derived by the present method. It is suggested that this correction is due to the spin-spin interaction between the incident electron and the nucleus

The problem of the scattering of fast electrons by atoms has been treated theoretically by Mott (1929) using Dirac's equations. Accordingly Mott's formula is valid for electron spin $\pm(h/4\pi)$ and it takes account of the effect of relativity. The relative intensity of scattering at an angle between θ and $\theta + d\theta$ and per unit solid angle is according to Mott

$$I = \left(\frac{ze^2}{2m_0v^2}\right)^2\left(1-\frac{v^2}{c^2}\right)\left(\text{cosec}^4\tfrac{1}{2}\theta-\frac{v^2}{c^2}\text{cosec}^2\tfrac{1}{2}\theta+\pi\frac{v}{c}\cdot\frac{2\pi ze^2}{hc}\frac{\cos^2\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta}\right.$$

$$\left. + \text{higher terms}\right) \qquad (1)$$

It is evident that the second factor $1-(v^2/c^2)$ gives a general correction for relativity. In the last factor, the first term, *viz.*, $\text{cosec}^4\frac{1}{2}\theta$ gives the Rutherford value of intensity while the second and third terms, *viz.*, $-(v^2/c^2)\,\text{cosec}^2\frac{1}{2}\theta$ and $\pi\frac{v}{c}\frac{2\pi zc^2}{hc}\frac{\cos^2\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta}$ give the two corrections for spin-relativity. These two corrections will be hereafter called the *first* and *second spin* corrections of Mott. Of the two corrections, the second is appreciable only for heavy elements, *i.e.*, for high value of z, whereas the first is independent of z.

It should be noted that in deriving the above formula Mott has taken the interaction potential to be $-(ze^2/r)$ due to the nuclear charge $+ze$ of the atom. In other words, the repulsive potential of the orbital electrons for the incident electron is completely neglected. This may be justified only when the incident velocity is very high and the interaction is very close.

In the present paper I propose to develop the wave-statistical theory of the above problem.

It is well-known in wave-statistics (Kar and Mukherjee, 1934) that on taking into account the effect of relativity and spin, the wave equations are

$$\Delta\chi+\frac{4\pi^2}{h^2c^2}\left[(E-V)^2-E_0{}^2-\frac{V^2}{2(l+1)}\right]\chi=0, \quad +\text{spin} \qquad \dots \quad (2.1)$$

and

$$\Delta\chi+\frac{4\pi^2}{h^2c^2}\left[(E-V)^2-E_0{}^2-\frac{V^2}{2l}\right]\chi=0, \quad -\text{spin} \qquad \dots \quad (2.2)$$

where for positive spin $l_{min}=0$ while for negative spin $l_{min}=1$. It is evident that the term $-\frac{V^2}{2(l+1)}$ or $-\frac{V^2}{2l}$ in the above equations represents the spin-orbit interaction potential for positive or negative spin. On neglecting it one gets the usual Schrödinger's relativistic wave equation which does not give the correct eigen-value of energy for fine-structure. On the other hand, it has been already shown (Kar and Mukherjee, 1934) that the wave-statistical equations (2.1) and (2.2) do give the correct eigen-energy. Therefore, in developing the theory of the present problem, we should use these fundamental equations. Now, we take for l, the minimum value for both the spin. Thus the two equations (2.1) and (2.2) become identical and we have

$$\Delta\chi+\frac{4\pi^2}{h^2c^2}\left[(E-V)^2-E_0{}^2-\left(\frac{V^2}{2}\right)_{spin}\right]\chi=0 \qquad \dots \quad (2.3)$$

the last term being written $(V^2/2)_{spin}$ in order to distinguish it from the other terms not connected with spin but involving V.

Outside the potential field the spin-orbit interaction evidently vanishes and so (2.3) becomes

$$\Delta\chi_0+\frac{4\pi^2}{h^2c^2}[E^2-E_0{}^2]\chi_0=0 \qquad \dots \quad (2.4)$$

On proceeding in the usual way (Kar, Ghosh and Mukherjee, 1937) we have for the differential equation satisfied by the first order scattering function $(\lambda_1\chi_1)$

$$\Delta(\lambda_1\chi_1)+k^2(\lambda_1\chi_1)=\frac{4\pi^2}{h^2c^2}\chi_0\left[2EV-V^2+\left(\frac{V^2}{2}\right)_{spin}\right] \qquad \dots \quad (2.5)$$

$$k^2=\frac{4\pi^2}{h^2c^2}[E^2-E_0{}^2] \qquad \dots \quad (2.6)$$

Of the three perturbation terms on the righthand side of Eq. (2.5), the first is evidently the most important, the other two give only higher order corrections. On solving (2.5) as before (Kar, 1937) with only the first perturbation term we have for the scattering function

$$\lambda_1\chi_1=-\frac{2\pi^2E}{h^2c^2k^2}\operatorname{cosec}^2\tfrac{1}{2}\theta A\,\frac{e^{ikr}}{r}\,F(r_0) \qquad \dots \quad (3)$$

where A is the amplitude of the incident wave and

$$F(r_0)=k'\int_{r_0}^{\infty}V(r)\sin k'r\,rdr \qquad \dots \quad (3.1)$$

in which

$$k'=2k\sin\tfrac{1}{2}\theta \qquad \dots \quad (3.2)$$

and r_0 the critical approach. Now, because the interaction potential $V(r)=-(ze^2/r)$, we have from (3.1)

$$F(r_0)=-ze^2\cos k'r_0.$$

On substituting the above value of $F(r_0)$ in (3) and also remembering that $E=E_0/(1-\beta^2)^{\frac{1}{2}}$, we have from (3) after simplification

$$\lambda_1\chi_1 = + \frac{ze^2}{2m_0v^2}(1-\beta^2)^{\frac{1}{2}}\text{cosec}^2\,\tfrac{1}{2}\theta\,A\,\frac{e^{ikr}}{r}\cos k'r_0 \qquad \ldots \quad (3.3)$$

On solving (2.5) with only the second perturbation term on the righthand side, we have for the scattering function [*vide* (3)]

$$\lambda_1\chi_1 = + \frac{\pi^2}{h^2c^2k^2}\text{cosec}^2\,\tfrac{1}{2}\theta\,A\,\frac{e^{ikr}}{r}F(r_0) \qquad \ldots \quad (4)$$

where

$$F(r_0) = k'\int_{r_0}^{r} V^2 \sin k'r\, dr \qquad \ldots \quad (4.1)$$

On substituting for V and integrating we get from (4.1)

$$F(r_0) = k'z^2e^4\left(\frac{\pi}{2} - k'r_0\right) \qquad \ldots \quad (4.2)$$

On combining (4) and (4.2) and after simplification we get

$$\lambda_1\chi_1 = \frac{(\pi-2k'r_0)}{2m_0v^2}\cdot\frac{v}{c}\cdot\frac{\pi z^2e^4}{hc}(1-\beta)^{\frac{1}{2}}\text{cosec}\,\tfrac{1}{2}\theta A\,\frac{e^{ikr}}{r} \qquad \ldots \quad (4.3)$$

It may be easily seen that the last spin perturbation term in (2.5) will give for the scattering function, $-\frac{1}{2}$ the value given in (4.3). However, because it is the case of a spin-orbit perturbation there may arise two cases. The scattered electron may have the same or the opposite spin-direction as the incident. Of course, in assigning the spin direction one should not give much importance to the geometry. It should be rather looked at from the statistical point of view. Now, in order to take account of the two events mentioned above we have to multiply the scattering function by a spin-factor (δ), which may be evaluated in the following way.

If there is no change in the direction of spin due to scattering the spin factor should evidently be unity. The same result may also be obtained for $l=\frac{1}{2}$, $m=\frac{1}{2}$, if we take the spin factor as the product

$$\delta = P_{+\frac{1}{2}}^{-\frac{1}{2}}(\theta)\,P_{-\frac{1}{2}}^{+\frac{1}{2}}(\theta)\,e^{-i\frac{1}{2}\phi}e^{+i\frac{1}{2}\phi} \qquad \ldots \quad (5)$$

where the Legendre functions (*vide* Appendix)

$$\left.\begin{aligned} P_{+\frac{1}{2}}^{-\frac{1}{2}}(\theta) &= \text{const. } (\sin\theta)^{\frac{1}{2}} \\ P_{\frac{1}{2}}^{+\frac{1}{2}}(\theta) &= \text{const. } (\sin\theta)^{-\frac{1}{2}} \end{aligned}\right\} \qquad \ldots \quad (5.1)$$

It should be noted that the possible values of l and m taken in (5) are symmetrical and form a complete group.

Now, of the three spin functions, namely,

$$P^{-\frac{1}{2}}_{+\frac{1}{2}}(\theta)e^{-\frac{1}{2}i\phi},\ P^{+\frac{1}{2}}_{-\frac{1}{2}}(\theta)e^{+\frac{1}{2}i\phi},\ P^{+\frac{1}{2}}_{+\frac{1}{2}}(\theta)e^{+i\frac{1}{2}\phi} \quad \dots \quad (5.2)$$

the product of any two gives the corresponding spin factor. Again, because the scattering has axial symmetry, the spin factor must be taken independent of ϕ. From the above considerations, we find for the spin-factor

$$\delta = P^{+\frac{1}{2}}_{+\frac{1}{2}}(\theta)P^{-\frac{1}{2}}_{+\frac{1}{2}}(\theta)e^{+\frac{1}{2}i\phi}e^{-\frac{1}{2}i\phi} \quad \dots \quad (5.3)$$

where (vide Appendix)

$$P^{+\frac{1}{2}}_{+\frac{1}{2}}(\theta) = \text{const.} \frac{\cos\theta}{(\sin\theta)^{\frac{1}{2}}} \quad \dots \quad (5.4)$$

and $P^{-\frac{1}{2}}_{+\frac{1}{2}}(\theta)$ is given in (5.1). Now, to the observer of scattering the spin factor in (5.3) should be obtained by putting $\pi-\theta$ for θ. Thus, on substituting for the Legendre functions we have from (5.3)

$$\delta = -\cos\theta \quad \dots \quad (5.5)$$

The spin-factor in (5) is not changed by putting $\pi-\theta$ for θ. Hence, we have for the total spin factor

$$\delta = 1-\cos\theta \quad \dots \quad (5.6)$$

Therefore, we have for the scattering function due to the spin-orbit perturbation

$$(\lambda_1\lambda_1)_{\text{spin}} = -\frac{\pi-2k'r_0}{2m_0v^2}\cdot\frac{v}{c}\cdot\frac{\pi z^2e^4}{hc}(1-\beta^2)^{\frac{1}{2}}\text{cosec}\,\tfrac{1}{2}\theta.\frac{1-\cos\theta}{2}A\frac{e^{ikr}}{r} \quad \dots \quad (6)$$

On adding the scattering functions in (3.3), (4.3) and (6) we have for its total value

$$\lambda_1\lambda_1 = \frac{ze^2}{2m_0v^2}(1-\beta^2)^{\frac{1}{2}}A\frac{e^{ikr}}{r}\Big\{\text{cosec}^2\tfrac{1}{2}\theta\cos k'r_0 + (\pi-2k'r_0)\frac{v}{c}.$$

$$\frac{\pi z e^2}{hc}\text{cosec}\,\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\Big\} \quad \dots \quad (7)$$

Let B be the amplitude of the incident wave with opposite spin. The scattering function corresponding to it is obtained from (7) by simply putting B for A. Thus remembering that the total intensity of the incident wave is (A^2+B^2) we have for the relative intensity of scattering

$$I = \left(\frac{ze^2}{2m_0v^2}\right)^2(1-\beta^2)\Big\{\text{cosec}^4\tfrac{1}{2}\theta\cos^2k'r_0 + (\pi-2k'r_0)\frac{v}{c}.\frac{2\pi ze^2}{hc}\cos k'r_0$$

$$\times\frac{\cos^2\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta}\Big\} \quad \dots \quad (8)$$

The second term in (8) gives the spin-orbit correction for the relative intensity of scattering. If the critical approach is neglected, it reduces to the second correction of Mott [*vide* Eq. (1)]. We do not get Mott's first correction by considering the spin-orbit interaction. It appears that Mott's first correction which is independent of z, might be due to the spin-spin interaction between the incident electron and the nucleus

Critical Approach

Let us next find the corrected critical approach (r_0) appearing in the formula for the intensity (8). Using the well-known boundary conditions at $r=r_0$, *viz.*,

$$\left.\begin{aligned} \frac{d\chi_0}{dr}+\frac{d(\lambda_1\chi_1)}{dr}&=0 \\ \chi_0+\lambda_1\chi_1&=0 \end{aligned}\right\}_{r=r_0} \qquad \ldots \quad (9)$$

we have as before (Kar, 1937)

$$r_0=\rho\,\frac{ze^2}{m_0v^2}\,(1-\beta^2)^{\frac{1}{2}}(\mathrm{cosec}\,\tfrac{1}{2}\theta-1) \qquad \ldots \quad (9.1)$$

when we take only the scattering function (3.3), without the spin-orbit correction. The difference between (9.1) and the previous value of r_0 is the additional relativistic factor $(1-\beta^2)^{\frac{1}{2}}$ in (9.1). It is evident that the above value of the critical approach is only approximate, and it should be corrected for the spin-orbit interaction. This correction for the V^2-perturbation and the spin-orbit interaction may be evaluated in the following way. The total scattering function for the V^2-perturbation and the spin-orbit interaction may be taken in the form [*vide* Eqs. (4) and (6)]

$$\lambda_1\chi_1=\frac{\pi^2}{h^2c^2k^2}\,\mathrm{cosec}^2\,\tfrac{1}{2}\theta\,\cos^2\tfrac{1}{2}\theta\,A\,\frac{e^{ikr}}{r}\,F(r_0) \qquad \ldots \quad (10)$$

where $F(r_0)$ is given in (4.1). Now, on using the second boundary condition in (9) and proceeding in the usual manner (Kar, 1937) we get

$$r_0=-\frac{\pi^2}{h^2c^2k^2}\,\mathrm{cosec}^2\,\tfrac{1}{2}\theta\,\cos^2\tfrac{1}{2}\theta\,e^{2ikr_0\sin^2\frac{1}{2}\theta}\,F(r_0) \qquad \ldots \quad (11)$$

On remembering that r_0 cannot be negative, the above may be written in the form

$$r_0=\frac{\pi^2}{h^2c^2h^2}\,\mathrm{cosec}^2\,\tfrac{1}{2}\theta\,\cos^2\tfrac{1}{2}\theta\,e^{2ikr_0\sin\frac{1}{2}\theta-2ikr_0\sin^2\frac{1}{2}\theta}\,G(r_0) \qquad \ldots \quad (11.1)$$

Or, transforming as before with the help of the first boundary condition in (9), we get

$$r_0=\rho.\,\frac{ze^2}{m_0v^2}\,(1-\beta)^{\frac{1}{2}}(\mathrm{cosec}\,\tfrac{1}{2}\theta-1)\frac{2\pi ze^2}{hc}.\,\frac{v}{c}\,\cos^2\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta\,I \qquad \ldots \quad (11.2)$$

where

$$I=\cos k'r_0\left(\frac{\pi}{2}-k'r_0\right)+\sin k'r_0(C-\log\,(k'r_0)) \qquad \ldots \quad (11.2)$$

C being Euler constant.

On adding we get from (9.1) and (11.2)

$$r_0 = \rho . \frac{ze^2}{m_0 v^2}(1-\beta^2)^{\frac{1}{2}}(\operatorname{cosec} \tfrac{1}{2}\theta - 1)\left(1 + \frac{2\pi z e^2}{hc} . \frac{v}{c} \cos^2 \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta 1\right) \qquad (11.4)$$

being the corrected critical approach. It is obvious that (11.4) reduces to (9.1) when z is small. Thus, to a first approximation, one may use the uncorrected critical approach given in (9.1).

APPENDIX

The general value of the Legendre function $P_l^m(\mu)$ is (Hobson, 1931)

$$P_l^m(\mu) = \frac{1}{\Gamma(1-m)} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F(-l, l+1;\ 1-m;\ \tfrac{1}{2}(1-\mu)) \qquad \ldots \quad (12)$$

where $\mu(=\cos\theta)$ is restricted to be real and l, m are real but unrestricted. F in (12) is the well-known hypergeometric function. Let us now consider some special cases.

Case I :—Put $l = +\frac{1}{2}$, $m = -\frac{1}{2}$ We have from (12)

$$P_{\frac{1}{2}}^{-\frac{1}{2}}(\mu) = \frac{1}{\Gamma(\frac{3}{2})}\left(\tan\frac{\theta}{2}\right)^{\frac{1}{2}} F(-\tfrac{1}{2}, \tfrac{3}{2};\ \tfrac{3}{2};\ \tfrac{1}{2}(1-\mu))$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})}(\sin\theta)^{\frac{1}{2}} \qquad \ldots \quad (12.1)$$

Case II :—Put $l = -\frac{1}{2}$, $m = +\frac{1}{2}$. We have from (12)

$$P_{-\frac{1}{2}}^{\frac{1}{2}}(\mu) = \frac{1}{\Gamma(\frac{1}{2})}\left(\cot\frac{\theta}{2}\right)^{\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2};\ \tfrac{1}{2},\ \tfrac{1}{2}(1-\mu))$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})}(\sin\theta)^{-\frac{1}{2}} \qquad \ldots \quad (12.2)$$

Case III :—Put $l = +\frac{1}{2}$, $m = +\frac{1}{2}$. We have from (12)

$$P_{\frac{1}{2}}^{\frac{1}{2}}(\mu) = \frac{1}{\Gamma(\frac{1}{2})}\left(\cot\frac{\theta}{2}\right)^{\frac{1}{2}} F(-\tfrac{1}{2}, \tfrac{3}{2};\ \tfrac{1}{2},\ \tfrac{1}{2}(1-\mu)) \qquad \ldots \quad (12.3)$$

Now, it may be easily proved from the definition of hypergeometric function that

$$F(-n, \beta+1;\ \beta;\ -z) = F(-n, \beta;\ \beta;\ -z) + \frac{nz}{\beta} F(-n+1, \beta;\ \beta;\ -z) \quad (12.4)$$

Thus from (12.3)

$$P_{\frac{1}{2}}^{\frac{1}{2}}(\mu) = \frac{\left(\cot\frac{\theta}{2}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\{F(-\tfrac{1}{2}, \tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}(1-\mu)) - \tfrac{1}{2}(1-\mu)F(+\tfrac{1}{2}, \tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}(1-\mu))\}$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{\cos\theta}{(\sin\theta)^{\frac{1}{2}}} \qquad \ldots \quad (12.5)$$

The above results may also be obtained from an alternative general formula given by Hobson (*l.c.*), *viz*,

$$P_l^m(\mu) = 2^m \cos \frac{l+m}{2}\pi \frac{\Gamma\left(\frac{l+m-1}{2}+1\right)}{\Gamma\left(\frac{l-m}{2}+1\right)\Gamma(\frac{1}{2})}(1-\mu^2)^{\frac{1}{2}m} F\left(\frac{l+m+1}{2}, \frac{m-l}{2}; \frac{1}{2}; \mu^2\right)$$

$$+ 2^{m+1} \sin \frac{l+m}{2}\pi \frac{\Gamma\left(\frac{l+m}{2}+1\right)}{\Gamma\left(\frac{l-m-1}{2}+1\right)\Gamma(\frac{1}{2})} \mu(1-\mu^2)^{\frac{1}{2}m} F\left(\frac{l+m+2}{2}, \frac{m-l+1}{2}; \frac{3}{2}; \mu^2\right)$$

where l and m are unrestricted. ... (13)

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