A NOTE ON DIRAC EQUATIONS AND THE ZEEMAN EFFECT

By S. N. BOSE

AND

K. BASU

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ABSTRACT. A new treatment has been given for solving Dirac's equations for hydrogenic atoms, and the radial functions are expressed in terms of a combination of Sonine's polynomials $T_n^{(\mu)}(\rho)$, $T_{n-1}^{(\mu)}(\rho)$ of only two consecutive degrees n, $n-1$; and the elementary properties of such polynomials have enabled us to tackle the Zeeman effect problem in general (homogeneous field) leading to the standard quadratic equation in energy for the effect

1. With the help of the two-dimensional matrices s_x, s_y, s_z of Pauli, the wave-equations of Dirac can be put in the well-known matrix form:

$$(1.1)\qquad \frac{i}{\mathbf{h}c}\left[E+E_0+\frac{Ze^2}{r}\right]X+DY=0,\quad \frac{i}{\mathbf{h}c}\left[E-E_0+\frac{Ze^2}{r}\right]Y+DX=0$$

where D is the operator $\qquad s_x\frac{\partial}{\partial x}+s_y\frac{\partial}{\partial y}+s_z\frac{\partial}{\partial z}$.

If, similarly $S=xs_x+ys_y+zs_z$, and $s=\frac{1}{r}(xs_x+ys_y+zs_z)$, then

$S.D.=r\frac{d}{dr}+L$, where $L=i(M_xs_x+M_ys_y+M_zs_z)$.

And the following commutation rules can be easily deduced:

$$s(L-1)+(L-1)s=0,\quad D(L-1)+(L-1)D=0;$$

also $\qquad s^2\equiv 1.$

Hence multiplying the equations on the left by S we have

$$(1.2)\qquad \frac{i}{\mathbf{h}c}\left[E+E_0+\frac{Ze^2}{r}\right]SX+\left[r\frac{d}{dr}+1\right]Y+(L-1)Y=0;$$

$$\frac{i}{\mathbf{h}c}\left[E-E_0+\frac{Ze^2}{r}\right]SY+\left[r\frac{d}{dr}+1\right]X+(L-1)X=0.$$

Assuming $X\equiv\begin{vmatrix}u_1\\u_2\end{vmatrix}=\frac{f(r)}{r}\Phi$; $Y\equiv\begin{vmatrix}u_3\\u_4\end{vmatrix}=\frac{g(r)}{r}\Psi$ where Φ, Ψ are similar onecolumn matrices and functions of (θ, ϕ), the equations can be rewritten as

$$(1.3) \qquad \frac{\iota}{\hbar c} f(r)\left[E + E_0 + \frac{Ze^2}{r}\right] s\Phi + \frac{dg(r)}{dr}\cdot\Psi + \frac{g(r)}{r}(L-1)\Psi = 0,$$

$$\frac{\iota}{\hbar c} g(r)\left[E - E_0 + \frac{Ze^2}{r}\right] s\Psi + \frac{df(r)}{dr}\cdot\Phi + \frac{f(r)}{r}(L-1)\Phi = 0.$$

The equations become easily separable if the matrices Φ and Ψ are so chosen that at first $(L-1)\Psi = k\Psi$ and $s\Phi = \Psi$; and as $s^2 \equiv 1$, it follows therefore from commutation rules that

$$\Phi = s\Psi \text{ and } (L-1)\, s\Psi = -k s\Psi.$$

We observe at once that $L(L-1)\Psi = -\nabla_2\Psi = k(k+1)\Psi$, where

$$\nabla_2 \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2};$$

and therefore k can be either a positive or a negative integer.

Secondly, if (Φ, Ψ) are the matrices for positive k, then (Ψ, Φ) are the matrices for negative k. Also remembering that the operator

$$-\left[\iota\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{2}s_z\right]$$

commutes with the equation-system, the angular matrices can be expressed in terms of spherical harmonics of order k and $k-1$ in the following form:

$$(1.4) \qquad \Phi \equiv \begin{vmatrix} \sqrt{\dfrac{k+m}{2k-1}}\; Y^{\mu}_{k-1} \\ \sqrt{\dfrac{k-m-1}{2k-1}}\; Y^{\mu+1}_{k-1} \end{vmatrix}, \quad \Psi \equiv \begin{vmatrix} \sqrt{\dfrac{k-m}{2k+1}}\; Y^{\mu}_{k} \\ \sqrt{\dfrac{k+m+1}{2k+1}}\; Y^{\mu+1}_{k} \end{vmatrix} \quad (k, \text{ a positive integer})$$

where $Y^{\mu}_{k} \equiv \sqrt{\dfrac{2k+1}{2}\cdot\dfrac{(k-\mu)!}{(k+\mu)!}}\cdot\dfrac{1}{2^k k!}\cdot\dfrac{d^{k+\mu}}{dx^{k+\mu}}(x^2-1)^k\cdot\dfrac{e^{i\mu\phi}}{\sqrt{2\pi}}$. $(x = \cos\theta)$

The functions are interchanged for negative values of k. Writing $f(r) = \iota F(r)$ so as to remove the imaginary from the radial equations, we get

$$(1.5) \qquad \left[\frac{dG}{dr} + k\frac{G}{r}\right] - \frac{1}{\hbar c}\left[E + E_0 + \frac{Ze^2}{r}\right]F = 0,$$

$$\left[\frac{dF}{dr} - k\frac{F}{r}\right] + \frac{1}{\hbar c}\left[E - E_0 + \frac{Ze^2}{r}\right]G = 0.$$

2. To solve the equations we assume

$$(2.1) \qquad F = f_0 e^{-\lambda r} r^{\mu} F_1(r),\; G = g_0 e^{-\lambda r} r^{\mu} G_1(r), \text{ and } N = \alpha E/(E_0^2 - E^2)^{\frac{1}{2}}$$

where $Ze^2/\hbar c = \alpha$.

Substituting in (1.5), F_1 and G_1 are easily seen to satisfy the following equations:

$$(2.2)\qquad \begin{aligned} &\frac{dG_1}{dr} - \lambda G_1 + \lambda F_1 + \frac{k+\mu}{r} G_1 + \frac{N}{E}(E_0 - E)\frac{F_1}{r} = 0, \\ &\frac{dF_1}{dr} - \lambda F_1 + \lambda G_1 - \frac{k-\mu}{r} F_1 - \frac{N}{E}(E_0 + E)\frac{G_1}{r} = 0, \end{aligned}$$

(2.3) where $f_0/g_0 = -(E_0 - E)/hc\lambda = \dfrac{\lambda hc}{E_0 + E} = -\left(\dfrac{E_0 - E}{E_0 + E}\right)^{\frac{1}{2}}$ and $\lambda = \left(\dfrac{E_0^2 - E^2}{hc}\right)^{\frac{1}{2}}$

If $F_1 + G_1 = 2\chi^+$, $G_1 - F_1 = 2\chi^-$, elimination leads easily to the following equations:

$$(2.4)\qquad r\frac{d^2\chi^+}{dr^2} + (2\mu + 1 - 2\lambda r)\frac{d\chi^+}{dr} + 2\lambda(N - \mu)\chi^+ = 0,$$

$$(2.5)\qquad r\frac{d\chi^+}{dr} - (N - \mu)\chi^+ = \left(N\frac{E_0}{E} - k\right)\chi^-,$$

provided $\qquad k^2 - \mu^2 = N^2(E_0^2 - E^2)/E^2 = \alpha^2.$

Taking the first equation it can be easily seen that it admits of polynomial solutions if $N - \mu = n$ (an integer). Writing ρ for $2\lambda r$, and writing a Sonine polynomial in the form

$$T_n^{(\mu)}(\rho) = B_n^{(\mu)}\left[\rho^n - \frac{n(n+2\mu)}{1!}\rho^{n-1} + \frac{n(n-1)(n+2\mu)(n+2\mu-1)}{2!}\rho^{n-2} - \ldots\ldots\right]$$

$$(2.6)\qquad = B_n^{(\mu)}(-1)^n e^{\rho}\rho^{-2\mu}\frac{d^n}{d\rho^n}\left[e^{-\rho}\rho^{n+2\mu}\right]$$

with constant $B_n^{(\mu)}$ so chosen that it is normalized according to

$$\int_0^\infty e^{-\rho}\rho^{2\mu}T_n^{(\mu)}(\rho)T_n^{(\mu)}(\rho)d\rho = 1$$

we get $B_n^{(\mu)} = 1/\{n!\Gamma(n + 2\mu + 1)\}^{\frac{1}{2}}$.

Hence it easily follows that χ^+ and χ^- will have solutions as given below:

$$(2.7)\qquad \chi^+ = \delta\left(N\frac{E_0}{E} - k\right)^{\frac{1}{2}} T_n^{(\mu)}(\rho)$$

$$(2.8)\qquad \chi^- = \delta\left(N\frac{E_0}{E} + k\right)^{\frac{1}{2}} T_{n-1}^{(\mu)}(\rho)$$

$$\left[N = n + \mu = \frac{\alpha\epsilon}{\sqrt{1 - \epsilon^2}}\ ;\ \epsilon = \frac{E}{E_0}\right]$$

where the normalising factor δ is to be determined presently.

Our radial functions of eqs. (2.1) stand thus

$$(2.9)\qquad F = -\delta\sqrt{\left(1 - \frac{E}{E_0}\right)}e^{-\frac{\rho}{2}}\rho^{\mu}\left[\left(N\frac{E_0}{E} - k\right)^{\frac{1}{2}}T_n^{(\mu)}(\rho) - \left(N\frac{E_0}{E} + k\right)^{\frac{1}{2}}T_{n-1}^{(\mu)}(\rho)\right]$$

$$(2.10)\qquad G = \delta\sqrt{\left(1 + \frac{E_0}{E_0}\right)}e^{-\frac{\rho}{2}}\rho^{\mu}\left[\left(N\frac{E_0}{E} - k\right)^{\frac{1}{2}}T_n^{(\mu)}(\rho) + \left(N\frac{E_0}{E} + k\right)^{\frac{1}{2}}T_{n-1}^{(\mu)}(\rho)\right]$$

Normalisation requires

$$\int_0^{\infty} (F^2 + G^2)\, dr = 1,$$

which gives
$$\delta = \sqrt{\frac{\lambda}{2N} \frac{E}{E_0}}.$$

We can write the two solutions (A) and (B), corresponding to positive and negative values of k respectively.

$$(A) \quad \begin{aligned} u_1 &= \sqrt{\frac{k+\mu}{2k-1}}\, i\, \frac{f_+(r)}{r} Y^{\mu}_{k-1}, & u_3 &= \sqrt{\frac{k-\mu}{2k+1}}\, \frac{g_+(r)}{r} Y^{\mu}_{k} \\ u_2 &= -\sqrt{\frac{k-\mu-1}{2k-1}}\, i\, \frac{f_+(r)}{r} Y^{\mu+1}_{k-1}, & u_4 &= \sqrt{\frac{k+\mu+1}{2k+1}}\, \frac{g_+(r)}{r} Y^{\mu+1}_{k} \end{aligned}$$

$$(B) \quad \begin{aligned} v_1 &= \sqrt{\frac{k-\mu}{2k+1}}\, i\, \frac{f_-(r)}{r} Y^{\mu}_{k}, & v_3 &= \sqrt{\frac{k+\mu}{2k-1}}\, \frac{g_-(r)}{r} Y^{\mu}_{k-1} \\ v_2 &= \sqrt{\frac{k+\mu+1}{2k+1}}\, i\, \frac{f_-(r)}{r} Y^{\mu+1}_{k}, & v_4 &= -\sqrt{\frac{k-\mu-1}{2k-1}}\, \frac{g_-(r)}{r} Y^{\mu+1}_{k-1} \end{aligned}$$

wherein it is understood that f_- and g_- are obtained by changing the sign of k, in the expressions for F and G in (1.5) above.

3. When the atom is in an electromagnetic field defined by the vector-potential $(A_x A_y A_z)$, the Dirac-equations become

$$(3.1) \qquad \frac{i}{hc}\left[E + E_0 + \frac{Ze^2}{r}\right] X + DY + \frac{e}{hc} AY = 0$$

$$(3.2) \qquad \frac{i}{hc}\left[E - E_0 + \frac{Ze^2}{r}\right] Y + DX + \frac{e}{hc} AX = 0$$

where
$$A = i\left[A_x s_x + A_y s_y + A_z s_z\right]$$

In the case of a constant magnetic field **H**, in the direction of Z-axis

$$(3.3) \qquad A = \tfrac{1}{2}Hr \begin{vmatrix} 0 & e^{-i\mu\phi}\sin\theta \\ -e^{i\mu\phi}\sin\theta & 0 \end{vmatrix} = \tfrac{1}{2}Hr\mathbf{a}.$$

We observe in passing that in the absence of A, the wave function X is generally small compared with Y—the radial component of $F(r)$ has the factor $\sqrt{1-\frac{E}{E_0}}$, while $G(r)$ has $\sqrt{1+\frac{E}{E_0}}$; so that the perturbation effect of $\frac{e}{hc}AX$ is small as compared with effect due to the term $\frac{e}{hc}AY$ in eq. (3.1).

Remembering

$$e^{-i\phi}Y_k^{\mu+1}\sin\theta = -\sqrt{\frac{(k-\mu)(k-\mu+1)}{(2k+1)(2k+3)}}Y_{k+1}^{\mu} + \sqrt{\frac{(k+\mu)(k+\mu+1)}{(2k+1)(2k-1)}}Y_{k-1}^{\mu};$$

$$e^{i\phi}Y_k^{\mu}\sin\theta = \sqrt{\frac{(k+\mu+1)(k+\mu+2)}{(2k+1)(2k+3)}}Y_{k+1}^{\mu+1} - \sqrt{\frac{(k-\mu)(k-\mu-1)}{(2k-1)(2k+1)}}Y_{k-1}^{\mu+1};$$

we seek an approximate solution of the equation by choosing one set of angular function $(Y_k^{\mu}, Y_k^{\mu+1})$, and assuming the existence of both the sets $(Y_{k-1}^{\mu}, Y_{k-1}^{\mu+1})$ and $(Y_{k+1}^{\mu}, Y_{k+1}^{\mu+1})$ in X. This can be done by suitably combining the (A) and (B) types, thus

Take $u_j = C_1^+ u_j^+ + C_{-(l+1)}^- v_j^-$, $(j = 1, 2, 3, 4)$, where $(u_1^+, u_2^+, u_3^+, u_4^+)$ corresponds to $k = +l$; and $(v_1^-, v_2^-, v_3^-, v_4^-)$ corresponds to $k = -(l+1)$. More explicitly, their values are

$$u_1^+ = \sqrt{\frac{l+\mu}{2l-1}}\, i\frac{f_+(r)}{r}Y_{l-1}^{\mu}; \qquad v_1^- = \sqrt{\frac{l-\mu+1}{2l+3}}\, Y_{l+1}^{\mu}\, i\frac{f_-(r)}{r};$$

$$u_2^+ = -\sqrt{\frac{l-\mu-1}{2l-1}}\, i\frac{f_+(r)}{r}Y_{l-1}^{\mu+1}, \qquad v_2^- = \sqrt{\frac{l+\mu+2}{2l+3}}\, Y_{l+1}^{\mu+1}\, i\frac{f_-(r)}{r};$$

$$u_3^+ = \sqrt{\frac{l-\mu}{2l+1}}\,\frac{g_+(r)}{r}Y_l^{\mu}; \qquad v_3^- = \sqrt{\frac{l+\mu+1}{2l+1}}\,\frac{g_-(r)}{r}Y_l^{\mu};$$

$$u_4^+ = \sqrt{\frac{l+\mu+1}{2l+1}}\,\frac{g_+(r)}{r}Y_l^{\mu+1}; \qquad v_4^- = -\sqrt{\frac{l-\mu}{2l+1}}\,\frac{g_-(r)}{r}Y_l^{\mu+1}.$$

The constants λ, N, μ of the two types of solutions are different and are expressed by the following relations

(3.4) $\lambda_+ = (E_0^2 - E_+^2)^{\frac{1}{2}}/\mathbf{h}c$, $l^2 - \alpha^2 = \mu_+^2$, $N_+ = \alpha E_+/[E_0^2 - E_+^2]^{\frac{1}{2}}$,
and $N_+ = n_+ + \mu_+$ [for (A) − type with $k = l$].

(3.5) $\lambda_- = (E_0^2 - E_-^2)^{\frac{1}{2}}/\mathbf{h}c)$, $(l+1)^2 - \alpha^2 = \mu_-^2$, $N_- = \alpha E_-/[E_0^2 - E_-^2]^{\frac{1}{2}}$,
and $N_- = n_- + \mu_-$ [for (B) − type with $k = -(l+1)$].

If E_+ and E_- differ slightly from one another, we have the following approximate relations

(3.6) $\lambda_+ = \lambda_-$, $\mu_- = \mu_+ + 1$, $N_+ = N_-$, $n_- = n_+ - 1$, and $\frac{E_{+-}}{E_0} \to 1$.

By following the usual method of perturbation we see easily that the characteristic equation for determining the Eigen value E would be

$$(3.7) \qquad \begin{vmatrix} \frac{i}{\mathbf{h}c}\left[E - E_+\right] + pa_{11} & pa_{12} \\ pa_{21} & \frac{i}{\mathbf{h}c}\left[E - E_-\right] + pa_{22} \end{vmatrix} = 0$$

where $\rho = eH/2\mathbf{h}c$, and

$$
\text{(3.8)}\qquad
\begin{aligned}
a_{11} &= \int \left[X_+^* \, r\mathbf{a} Y_+ + Y_+^* \, r\mathbf{a} X_+ \right] dV \\
a_{22} &= \int \left[X_-^* \, r\mathbf{a} Y_- + Y_-^* \, r\mathbf{a} X_- \right] dV \\
a_{12} &= \int \left[X_+^* \, r\mathbf{a} Y_- + Y_+^* \, r\mathbf{a} X_- \right] dV \\
a_{21} &= \int \left[X_-^* \, r\mathbf{a} Y_+ + Y_-^* \, r\mathbf{a} X_+ \right] dV
\end{aligned}
$$

and X_+^* etc. are transposed conjugates of X_+ (the $\mathbf{a}$ is given in (3.3))

Carrying out integrations over the angle-variable-space we get

$$
\begin{aligned}
a_{11} &= -\frac{2i(4\mu+2)\,l}{(2l-1)(2l+1)} \int_0^\infty r f_+ g_+ \, dr \,; \\
a_{22} &= \frac{2i(4\mu+2)(l+1)}{(2l+1)(2l+3)} \int_0^\infty r f_- g_- \, dr \,; \\
a_{12} = a_{21} &= \frac{i\sqrt{(l-\mu)(l+\mu+1)}}{2l+1} \int_0^\infty r (f_- g_+ + f_+ g_-) \, dr.
\end{aligned}
$$

4. To evaluate the integral $\{I\}$ or $\{II\}$, we substitute the values of the corresponding f, g, and remembering $r = \rho/2\lambda$, we obtain

$$
\{I\} = -\frac{\lambda_+}{2N_+}\frac{E_+}{E_0^2}\frac{(E_0^2-E_+^2)^{\frac{1}{2}}}{4\lambda^2}\left\{\left(N_+\frac{E_0}{E_+}-k\right)\int_0^\infty e^{-\rho}\rho^{2\mu+1}\left[T_n^{(\mu)}(\rho)\right]^2 d\rho \right.
$$
$$
\left. -\left(N_+\frac{E_0}{E_+}+k\right)\int_0^\infty e^{-\rho}\rho^{2\mu+1}\left[T_{n-1}^{(\mu)}(\rho)\right]^2 d\rho\right\}.
$$

The integrals are evaluated quickly by repeated partial integrations ; thus

$$
\int_0^\infty e^{-\rho}\rho^{2\mu+1}\left[T_n^{(\mu)}(\rho)\right]^2 d\rho = \frac{(-)^n}{\sqrt{n!\,\Gamma(n+2\mu+1)}}\int_0^\infty \rho T_n^{(\mu)}(\rho)\frac{d^n}{d\rho^n}\left[e^{-\rho}\rho^{2\mu+n}\right]d\rho
$$
$$
= \int_0^\infty \frac{e^{-\rho}\rho^{2\mu+n}}{n!\,\Gamma(n+2\mu+1)}\,d\rho.\ \frac{d^n}{d\rho^n}\left[\rho\left(\rho^n - \frac{n(n+2\mu)}{1!}\rho^{n-1}+\ldots\right)\right]
$$
$$
= (n+1)(n+2\mu+1) - n(n+2\mu) = 2N_+ + 1.
$$

Similarly

$$
\int_0^\infty e^{-\rho}\rho^{2\mu+1}\left[T_{n-1}^{(\mu)}(\rho)\right]^2 d\rho = 2N_+ - 1.
$$

An easy substitution of the value of λ_+ in the foregoing leads to the following results :

$$
\text{(4.1)}\qquad \int_0^\infty r f_+ g_+ \, dr = -\frac{\mathbf{h}c}{4E_0}\left(1 - 2k\frac{E_0}{E_+}\right);
$$

(4.2) $$\int_0^\infty r f_- g_- dr = -\frac{\mathbf{h}c}{4E_0}\left(1+2(l+1)\frac{E_0}{E_-}\right).$$

The evaluation of the third integral cannot, however, be exactly expressed in a neat form ; for the simple reason that λ_+, N_+, E_+ as well as λ_-, N_-, E_- are different in the two solutions, as also $\rho_+ = 2\lambda_+ r$ and $\rho_- = 2\lambda_- r$. If, however, we introduce approximations in the beginning, and take

(4.3) $$\lambda_+ = \lambda_- = |\lambda|,\ E_+ = E_-,\ \mu_- = \mu_+ + 1,\ n_- = n_+ - 1,\ \rho_+ = \rho_-,$$

we see the third integral

$$\{III\} = -\frac{2|\lambda|}{2N}\frac{|E|}{E_0^2}\frac{(E_0^2-|E|^2)^{\frac{1}{2}}}{4|\lambda|^2}\left\{\left(N\frac{E_0}{E}-l\right)^{\frac{1}{2}}\left(N\frac{E_0}{E}+l+1\right)^{\frac{1}{2}}\times\right.$$
$$\int_0^\infty e^{-\rho}\rho^{2\mu+2}T_n^{(\mu)}(\rho)T_{n-1}^{(\mu+1)}(\rho)d\rho-\left(N\frac{E_0}{E}+l\right)^{\frac{1}{2}}\left(N\frac{E_0}{E}-l-1\right)^{\frac{1}{2}}$$
$$\left.\times\int_0^\infty e^{-\rho}\rho^{2\mu+2}T_{n-1}^{(\mu)}(\rho)T_{n-2}^{(\mu+2)}(\rho)d\rho\right\}.$$

The two definite integrals can be easily evaluated in the same way as before. We give the results below .

(4.4) $$\int_0^\infty e^{-\rho}\rho^{2\mu+2}T_n^{(\mu)}(\rho)T_{n-1}^{(\mu+1)}(\rho)d\rho=\int_0^\infty\frac{e^{-\rho}\rho^{n+2\mu}}{\sqrt{n-1\,!\,n\,!\,\Gamma(n+2\mu+1)\Gamma(n+2\mu+2)}}$$
$$\times\frac{d^n}{d\rho^n}\left[\rho^{n+1}-(n-1)(n+2\mu+1)\rho^n+\ldots\right]=2\sqrt{(N-l)(N+l+1)}.$$

(4.5) $$\int_0^\infty e^{-\rho}\rho^{2\mu+2}T_{n-1}^{(\mu)}(\rho)T_{n-2}^{(\mu+1)}d\rho=2\sqrt{(N+l)(N-l-1)}.$$

Hence

$$\{III\}=-\frac{1}{2N}\frac{\mathbf{h}c}{E_0}\left[\sqrt{N\frac{E_0}{E}-l}\,\sqrt{N\frac{E_0}{E}+l+1}\,\sqrt{N-l}\,\sqrt{N+l+1}\right.$$
$$\left.-\sqrt{N\frac{E_0}{E}+l}\,\sqrt{N\frac{E_0}{E}-l-1}\,\sqrt{N+l}\,\sqrt{N-l-1}\right].$$

Observing $E_0/E \to 1$, we see

$$\{III\}=-\frac{1}{2N}\cdot\frac{\mathbf{h}c}{E_0}\,2N=-\frac{\mathbf{h}c}{E_0}.$$

Finally, putting $p = eH/2\mathbf{h}c$, and $e\mathbf{h}/2m_0c=\mu_0$ (Bohr-magneton) and substituting the values of the three integrals (and making $E_0/E \to 1$) we see that the determinantal equation (3.7) takes the form

(4.6) $$\begin{vmatrix} E-E_+-\frac{2l}{2l+1}m\mu_0H & -\frac{\mu_0H}{2l+1}\sqrt{(l-m+\frac{1}{2})(l+m+\frac{1}{2})} \\ -\frac{\mu_0H}{2l+1}\sqrt{(l-m+\frac{1}{2})(l+m+\frac{1}{2})} & E-E_--\frac{2l+2}{2l+1}m\mu_0H \end{vmatrix}=0$$

wherein we have put $\mu+\frac{1}{2}=m$ (magnetic quantum number).

The result (4.6) agrees completely with that quoted by Bethe (1933) supposed to have been worked out by him from Pauli's equations. Condon and Shortley (1935) obtained similar determinant from principles of quantum mechanics by applying two-fold perturbations (spin-orbit and magnetic) simultaneously. It may be noted that Darwin (1928) has many similar features with our mode of attack, and we can claim some elegance by our introduction of Sonine's properties, which exhibit our solutions in a good perspective.

Dacca University,
Ramna, Dacca.
Calcutta University,
92, Upper Circular Road, Calcutta.

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