A NOTE ON THE PRINCIPLE OF ADIABATIC INVARIANCE*

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The principle of adiabatic invariance[1] first formulated by Boltzmann, in connection with his studies on the Kinetic Theory of gases, acquired added importance when Ehrenfest incorporated it in the old quantum theory. It is usual [2] in discussing the principle, to illustrate it with the help of some simple examples of classical mechanics. The illustration of the simple pendulum, due to Einstein, has almost become historic. Andrade [3] has also considered the case of a conical-pendulum where for E/ν to be adiabatically invariant, it is no longer necessary to confine to small angles, a restriction which has to be imposed for a simple pendulum. In the above relation E represents the energy and ν the pulsatance of vibration, where the pulsatance is defined as 2π times the frequency.

Because of its importance and many applications, it is of some interest to have other simple illustrations of the principle of adiabatic invariance (besides those referred to above).

The examples of (*i*) a compound pendulum, (*ii*) an oscillating magnet[4]† and (*iii*) an electric oscillating circuit are given below. The case of simple pendulum is treated in §1, that of compound pendulum in §2, an elementary general treatment in §3 and the magnetic and electric cases are given in §4.

§1. *Simple pendulum.*—We consider a simple pendulum, its string passing through a ring which may be moved vertically up and down to change the point of suspension [Fig. 1 (*a*)]. If l be the length of the pendulum, ν is given by

$$\nu = \sqrt{\frac{g}{l}} \quad \text{and} \quad \frac{d\nu}{\nu} = -\frac{1}{2}\frac{dl}{l}.$$

The tension T in the string is

$$T = ml\dot{\theta}^2 + mg\cos\theta,$$

and the vibrational energy E is

$$E = \tfrac{1}{2}mgl\,\theta_0^2,$$

where θ_0 is the amplitude of the vibration

$$\theta = \theta_0 \sin \nu t.$$

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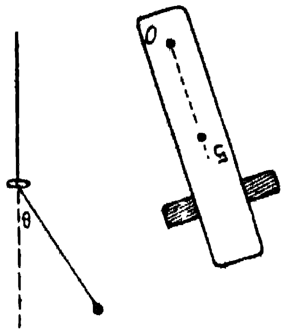
† Darwin (1940) has referred to a very interesting application of this case.

The reaction R of the string on the ring will be $2T \sin \theta/2$ and it will be inclined at an angle $(\pi/2 - \theta/2)$ to the vertical.



(a) (b)

FIG. 1

When the ring is displaced upwards, slowly and continuously, through a vertical height dl, the vertical component of R does work and the loss of energy of the system is equal to

$$R \sin \frac{\theta}{2} dl = \tfrac{1}{2} T\theta^2 dl = \tfrac{1}{2} mg\, \theta^2 dl,$$

neglecting terms higher than θ^2.

Therefore* the average loss of energy $d\bar{E}$ becomes

$$\tfrac{1}{2} mg\, \overline{\theta^2}\, dl = \tfrac{1}{4} mg\, \theta_0^2\, dl.$$

Hence
$$\frac{d\bar{E}}{\bar{E}} = -\tfrac{1}{2}\frac{dl}{l} = \frac{d\nu}{\nu}$$

or
$$\frac{\bar{E}}{\nu} = \text{constant}.$$

* If the displacement of the ring did not take place slowly and continuously, but say at the instant when $\theta = \theta_0$, the loss of energy would be $\tfrac{1}{2} mg\, \theta_0^2\, dl$ and on the other hand, if the displacements take place when $\theta = 0$, the energy loss would be zero.

An alteration in the length of the simple pendulum may be brought about in two ways: (*i*) by moving the ring as discussed above, and (*ii*) by suddenly clamping the string a little below the original point of suspension and then repeating this procedure of successively clamping at lower and lower points (the displacement of the pendulum remaining the same during the clamping). It is interesting to note that in the second case it is not $\bar{E}/\nu$, but $\bar{E}\nu$, that is, an adiabatic invariant.

Suppose that when the pendulum has a displacement θ, the string is suddenly clamped at a displacement dl from the initial point of suspension. The kinetic energy will remain the same during the process but the potential energy of vibration will be altered by

$$d\{mgl(1-\cos\theta)\} = \tfrac{1}{2} mg\, \theta^2\, dl,$$

and, therefore, the average change will be

$$\tfrac{1}{4} mg\, \theta_0^2\, dl = \tfrac{1}{2}\bar{E}\, dl/l,$$

and hence
$$\bar{E}\nu = \text{constant}.$$

§ 2. *Compound-pendulum.*—Consider a compound pendulum. The pulsatance of small oscillation is given by

$$\nu = \sqrt{\frac{gh}{k^2}} \quad \text{and} \quad \frac{d\nu}{\nu} = -\frac{dk}{k} + \frac{1}{2}\frac{dh}{h},$$

where k is the radius of gyration about the axis of suspension and h the distance between the centre of gravity and the axis. The pendulum carries a sliding mass, whose motion alters the position of the centre of gravity along the line OG [Fig. 1(*b*)].

The vibrational energy of the pendulum is

$$E = \tfrac{1}{2}mk^2\dot{\theta}_0^2 = \tfrac{1}{2}mgh\theta_0^2.$$

At the instant when the angular displacement of the pendulum is θ, let a motion of the sliding weight lower the centre of gravity by a small amount dh, the change in the angle θ during this alteration of the position of c.g. being supposed to be negligible. As the angular momentum of the pendulum will remain constant during the above process, we have

$$mk^2\dot{\theta} = \text{constant},$$

and as the kinetic energy T is

$$T = \frac{1}{2}\frac{(mk^2\dot{\theta})^2}{mk^2}.$$

we have
$$\frac{dT}{T} = -\frac{2dk}{k},$$

or
$$d\overline{T} = -2\frac{dk}{k}\left(\tfrac{1}{2}mk^2\overline{\dot{\theta}^2}\right)$$

$$= -\frac{dk}{k}\left(\tfrac{1}{2}mk^2\dot{\theta}_0^2\right).$$

Hence we have
$$\frac{d\overline{T}}{E} = -\frac{dk}{k}.$$

The potential energy of the pendulum in the position θ is

$$mgh(1-\cos\theta) = mgh\frac{\theta^2}{2},$$

and the increase dh in h increases the potential energy by

$$dV = \tfrac{1}{2}mg\,\theta^2 dh,$$

and therefore
$$d\overline{V} = \tfrac{1}{2}mg\,\overline{\theta^2}\,dh = \tfrac{1}{4}mg\,\theta_0^2\,dh$$

or
$$\frac{d\overline{V}}{E} = \frac{1}{2}\frac{dh}{h}.$$

Therefore $$\frac{dE}{E}=-\frac{dk}{k}+\frac{1}{2}\frac{dh}{h}=\frac{d\nu}{\nu}$$

or $$\frac{E}{\nu}=\text{constant}.$$

§ 3. *An elementary general treatment* :—Consider any simple harmonic motion defined by

$$m\frac{d^2x}{dt^2}=-\lambda x$$

and $$\nu=\sqrt{\frac{\lambda}{m}}.$$

The energy of vibration E is given by

$$E=\tfrac{1}{2}\lambda a^2,$$

where a is the amplitude. When the displacement is x, let λ change to $\lambda+d\lambda$; then the increase in the restoring force is $xd\lambda$, and the energy of vibration is increased by

$$\int_0^x d\lambda . x\ dx=\tfrac{1}{2}d\lambda\ x^2$$

and averaging we have

$$\overline{dE}=\tfrac{1}{4}d\lambda a^2$$

and $$\frac{dE}{E}=\frac{1}{2}\frac{d\lambda}{\lambda}=\frac{d\nu}{\nu}$$

or $$\frac{E}{\nu}=\text{constant}.$$

§ 4 (*a*). *Small magnet* :—Consider a magnet of moment M oscillating in a field of strength H. When the deflection is θ, let H change to $H+dH$, then the increase in the restoring couple is $M\theta dH$ and, therefore, the increase in the vibrational energy will be

$$\int^{\theta}\theta M dH\ d\theta=\tfrac{1}{2}M\theta^2 dH$$

and averaging we have

$$dE=\tfrac{1}{4}M\theta_0^2 dH.$$

As the energy of vibration is

$$E=\tfrac{1}{2}MH\theta_0^2$$

and $$\nu=\sqrt{\frac{MH}{I}},$$

where I is the moment of inertia of the magnet about the axis of vibration.

We have
$$\frac{dE}{E}=\frac{\frac{1}{4}M\theta_0^2dH}{\frac{1}{2}M\theta_0^2H}=\frac{1}{2}\frac{dH}{H}=\frac{d\nu}{\nu}.$$

Therefore
$$\frac{E}{\nu}=\text{constant}.$$

(*b*) *Electric oscillating circuit* :—We consider an electric circuit consisting of a capacity c and a self-inductance l, then

$$\nu=\sqrt{\frac{1}{lc}}\quad\text{and}\quad\frac{d\nu}{\nu}=-\frac{1}{2}\frac{dc}{c}.$$

Let q be the charge on the condenser at instant t, then

$$q=q_0\sin\nu t.$$

The electrostatic energy at this instant is $\frac{1}{2}\frac{q^2}{c}$, and the total energy of the circuit E is

$$E=\frac{1}{2}\frac{q_0^2}{c}.$$

When the charge on the condenser is q, let its capacity be suddenly changed from c to $c+dc$, then the potential difference of the condenser will be changed by

$$dV=-\frac{q}{c^2}dc$$

and, therefore, the change in the energy will be

$$dE=\frac{1}{2}qdV=-\frac{1}{2}\frac{q^2}{c^2}dc.$$

The average change of energy is

$$dE=-\frac{1}{4}\frac{q_0^2}{c^2}dc,$$

and hence we have

$$\frac{dE}{E}=-\frac{1}{2}\frac{dc}{c}=\frac{d\nu}{\nu}$$

or
$$\frac{E}{\nu}=\text{constant}.$$

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