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THE THEORY OF COMPTON EFFECT

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ABSTRACT. The wavestatistical theory of interaction between radiation and matter previously developed is extended by taking into account the energy and momentum of recoil of the scattering electron The well-known Einstein-Dirac formula for the intensity of Compton scattering is derived

While discussing recently certain problems[1] on the interaction between radiation and matter from the standpoint of wavestatistics, I have derived, among others, Thomson's classical formula for the scattering of radiation by an electron. In so doing I have neglected the change of frequency that is produced by the recoil of the scattering electron. The object of the present paper is to show that the intensity of Compton scattering may be easily derived by the above method if the change of frequency mentioned above is taken into account.

Now the fundamental assumption made in the previous paper is that as soon as the incident radiation $h\nu_i$ approaches the scattering electron up to a critical distance r_{os}, both the q- and p- components of the phase space for the region surrounding the electron ($r \leq r_{os}$) temporarily acquires negative viscosity relative to the region outside ($r \geq r_{os}$). As a result of that, $h\nu_i$ enters the inner region thereby considerably increasing its phase density. Consequently the region immediately develops positive viscosity relative to that outside and the radiation is scattered away in different directions. This is in short the mechanism of scattering as imagined in wavestatistics. It may, however, be noted that the critical distances of approach and scattering may not in general be the same.

As already shown the wave equations are

$$\Delta\chi_1 + \frac{8\pi^2 m}{h^2}\left(E + \frac{b^2h^2}{4\pi^2 E}\right)\chi_1 = 0, \quad \text{inside} \qquad \dots \quad (1)$$

$$\Delta\chi_1 + \frac{8\pi^2 m}{h^2}(E + \bar{U}_2)\chi_1 = 0, \quad \text{outside} \qquad \dots \quad (2)$$

and similarly for χ_2-wave. In the above b denotes the viscosity and $\bar{U}_2$ the average interaction potential between radiation and the free electron. It is evident that $\bar{U}_2 = 0$ for the region inside, while $\bar{U}_2 \neq 0$ for the region outside. This may be taken as the condition which helps the growth of viscosity in a given

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phase space. It is shown before that the interaction potential for s-type radiation is given by

$$U_2^s = \frac{8\pi c^2}{\Omega m} \cdot \overline{a_s^2 Q_s^2 \sin^2\gamma_s} \quad \ldots \quad (3)$$

where m is the mass of the electron, Ω the volume, a_s unit vector in the direction of the electric field of the electromagnetic wave, Q_s the coordinate of radiation and γ_s the phase. As eqs. (1) and (2) must be identical at the boundary we have for s-type radiation

$$b_1^s = b_2^s = \frac{2\pi}{h}\sqrt{h\nu_s \cdot \overline{U_2^s}} \quad \ldots \quad (4)$$

where b_1^s and b_2^s are of the nature of the viscosity coefficients of the q- and p-spaces respectively.

It is shown before that the wavestatistical average

$$\overline{Q}_s = e^{-2\pi i\nu_s t}\left(\frac{hn_s}{8\pi^2\nu_s}\right)^{\frac{1}{2}} \quad \ldots \quad (5)$$

and as we have approximately $\overline{Q_s^2} = (\overline{Q}_s)^2$ the interaction potential in (3) becomes

$$\overline{U_2^s} = \frac{hc^2}{\pi\Omega m} \cdot \frac{n_s}{\nu_s} \cdot \overline{a_s^2 \sin^2\gamma_s} \cdot e^{-4\pi i\nu_s t} \quad \ldots \quad (6)$$

Let ν_s be the frequency of the primary radiation. Its number densities before and after scattering are evidently $\frac{n_s}{\Omega}$ and $\frac{n_s - 1}{\Omega}$ respectively. Consequently the number densities of the scattered radiation of frequency ν_σ before and after scattering would be 0 and $\frac{1}{\Omega}$ respectively. Hence we have for the scattered radiation corresponding to (6)

$$\overline{U_2^\sigma} = \frac{hc^2}{\pi\Omega m} \cdot \frac{1}{\nu_\sigma} \cdot \overline{a_\sigma^2 \sin^2\gamma_\sigma}\; e^{4\pi i\nu_\sigma t} \quad \ldots \quad (7)$$

On substituting in (4) the value of $\overline{U_2^s}$ from (6) we get for the damping coefficients of s-type radiation

$$b_1^s = b_2^s = 2\pi\left\{\frac{n_s c^2}{\pi\Omega m} \cdot \overline{a_s^2 \sin^2\gamma_s}\right\}^{\frac{1}{2}} e^{-2\pi i\nu_s t} \quad \ldots \quad (8)$$

Similarly we have for the scattered radiation

$$b_1^\sigma = b_2^\sigma = 2\pi\left\{\frac{e^2}{\pi\Omega m} \cdot \overline{a_\sigma^2 \sin^2\gamma_\sigma}\right\}^{\frac{1}{2}} e^{2\pi i\nu_\sigma t}. \quad \ldots \quad (9)$$

On putting $b_1^s = \dot{a}_1^s$ we get from (8) after integration with respect to t

$$a_1^s = 2\pi \left\{ \frac{n_s e^2}{\pi\Omega m} \cdot \overline{\alpha_s^2 \sin^2\gamma_s} \right\}^{\frac{1}{2}} \cdot \frac{e^{-2\pi i\nu_s t}}{-2\pi i\nu_s} \quad \ldots \quad (10)$$

Hence we get for the rate of scattering in the q-space

$$b_1^{s\sigma} = a_1^s b_1^\sigma = \frac{2e^2}{\Omega m} \left\{ \overline{\alpha_s^2 \alpha_\sigma^2 \sin^2\gamma_s \sin^2\gamma_\sigma} \right\}^{\frac{1}{2}} n_s^{\frac{1}{2}} \cdot \frac{e^{-2\pi i(\nu_s - \nu_\sigma)t}}{-i\nu_s} \quad \ldots \quad (11)$$

It should be noted that the above represents the rate of scattering by *one* electron. So to get the rate of scattering by a volume-element $d\tau$, where the *instantaneous* number density of the scattering electron is given by

$$D d\tau = \chi_{1,m}^{*} \chi_{2,n}^{*} \cdot e^{\frac{2\pi i E_m t}{h}} \cdot e^{-\frac{4\pi i E_m(\text{kin.})}{h} t} d\tau, \quad \ldots \quad (12)$$

one has to take the product of (11) and (12). Now, putting $E_n - E_m = h\nu'$ we find from (12)

$$D d\tau = e^{-2\pi i\nu' t} \cdot \chi_{1,m}^{*} \chi_{2,n}^{*} \cdot e^{\frac{2\pi i E_n}{h} t} e^{-\frac{4\pi i E_n(\text{kin.})}{h} t} d\tau \quad \ldots \quad (12{\cdot}1)$$

It is well-known in wavestatistics[1] that because the phase waves are stationary, the last two exponential time factors may be dropped while integrating over the phase space.

Thus the rate of scattering per electron when there is change of momentum of the scattering electron, is given by

$$b_1^{s\sigma} = \frac{2e^2}{\Omega m} \left\{ \overline{\alpha_s^2 \alpha_\sigma^2 \sin^2\gamma_s \sin^2\gamma_\sigma} \right\}^{\frac{1}{2}} \cdot n_s^{\frac{1}{2}} \frac{e^{-2\pi i(\nu_s + \nu' - \nu_\sigma)t}}{-i\nu_s} \times \int \chi_{1,m}^{*} \chi_{2,n}^{*} d\tau . \quad \ldots \quad (13)$$

It is evident that the integral in (13) representing the scattering electron is unity. It would be, however, noted that the above integral for a free electron does not vanish because the electron before and after scattering moves in different directions. We have then from (13)

$$b_1^{s\sigma} = \frac{2e^2}{\Omega m} \left\{ \overline{\alpha_s^2 \alpha_\sigma^2 \sin^2\gamma_s \sin^2\gamma_\sigma} \right\}^{\frac{1}{2}} \cdot n_s^{\frac{1}{2}} \cdot \frac{e^{-2\pi i(\nu_s + \nu' - \nu_\sigma)t}}{-i\nu_s} . \quad \ldots \quad (13{\cdot}1)$$

Now as $b_1^{s\sigma} = \dot{a}_1^{s\sigma}$ we have on integrating (13·1) with respect to time and remembering that $a_1^{s\sigma} = 0$ at $t = 0$,

$$a_1^{s\sigma} = \frac{e^2}{\pi\Omega m} \left\{ \overline{\alpha_s^2 \alpha_\sigma^2 \sin^2\gamma_s \sin^2\gamma_\sigma} \right\}^{\frac{1}{2}} \cdot n_s^{\frac{1}{2}} \cdot \frac{1 - e^{-2\pi i(\nu_s + \nu' - \nu_\sigma)t}}{\nu_s(\nu_s + \nu' - \nu_\sigma)} . \quad \ldots \quad (14)$$

Similarly

$$a_2^{s\sigma} = \frac{e^2}{\pi\Omega m}\left\{\overline{a_s^2 a_\sigma^2 \sin^2\gamma_s \sin^2\gamma_\sigma}\right\}^{\frac{1}{2}} \cdot n_s^{\frac{1}{2}} \cdot \frac{1 - e^{2\pi i(\nu_s + \nu' - \nu_\sigma)t}}{\nu_s(\nu_s + \nu' - \nu_\sigma)} \qquad \dots (15)$$

Now the total number of quanta scattered in any component is

$$N^s = \sum_\sigma a_1^{s\sigma} a_2^{s\sigma} . \qquad \dots (16)$$

In order to replace the summation by integration one has to multiply (16) by $\frac{2\pi\Omega\nu_\sigma^2 d\nu_\sigma \sin\theta d\theta}{c^3}$ giving the number of $h\nu_\sigma$-quanta of one kind of spin in the volume Ω, moving at an angle θ with the direction of the incident radiation. Moreover, since γ_σ and γ_s are the values of the phases at the boundaries, we may take

$$\frac{\sin\gamma_\sigma}{\sin\gamma_s} = \frac{\sin\left(\frac{2\pi\nu_\sigma}{c} r_{o\sigma} + \delta\right)}{\sin\left(\frac{2\pi\nu_s}{c} r_{os} + \delta\right)} = \frac{\sin \frac{2\pi\nu_\sigma}{c}\left(r_{o\sigma} + a_\sigma\right)}{\sin \frac{2\pi\nu_s}{c}\left(r_{os} + a_s\right)} \qquad \dots (17)$$

where $r_{o\sigma}$, r_{os} are the boundaries for σ- and s-type radiations.

Let us assume that for the q-space

$$\nu_\sigma r_{o\sigma} = \nu_s r_{os} \qquad \dots (17\cdot1)$$

which means that the harder ray is more penetrating. So from (17) we get

$$\sin\gamma_\sigma / \sin\gamma_s = 1. \qquad \dots (17\cdot11)$$

The above assumption cannot be valid in the p-space. It is assumed that in this space

$$r_{o\sigma} + a_\sigma = r_{os} + a_s \qquad \dots (17\cdot2)$$

and so $r_{o\sigma}$ is, as it should be, nearly equal to r_{os}. Consequently from (17) we get approximately

$$\sin\gamma_\sigma / \sin\gamma_s = \frac{\nu_\sigma}{\nu_s} . \qquad \dots (17\cdot21)$$

Again we have

$$\overline{\sin^2\gamma_s} = \tfrac{1}{2} . \qquad \dots (18)$$

Hence, on substituting the values given in (17·11) and (17·21), on remembering (18) and replacing the summation by integration, we get from (14), (15) and (16)

$$N_s = \frac{2e^4}{\pi\Omega m^2 c^3} \cdot \frac{n_s}{\nu_s^3} \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} a_s^2 a_\sigma^2 \cdot \frac{\sin^2\pi(\nu_s + \nu' - \nu_\sigma)t}{(\nu_s + \nu' - \nu_\sigma)^2} \nu_\sigma^3 d\nu_\sigma \ . \qquad (19)$$

If the incident beam moving along x-axis is unpolarised, then for scattered radiation at an angle θ we have[2]

$$\overline{a_s^2 a_\sigma^2} = \tfrac{1}{2}(1 + \cos^2\theta) \ . \qquad \ldots \ (20)$$

Now in Compton scattering, the loss of energy of the free electron, *viz.*, $E_n - E_m = h\nu'$, depends on the direction of its recoil, *i.e.*, from the conservation of momentum, on the angle of scattering of the radiation. Thus we may put

$$\nu_\theta = \nu_s + \nu' \qquad \ldots \ (21)$$

where ν_θ is the frequency of scattered radiation at angle θ and from conservation of energy and momentum it may be easily shown that

$$\nu_\theta = \frac{\nu_s}{1 + \alpha(1 - \cos\theta)} \ ; \ \alpha = \frac{h\nu_s}{m_0 c^2} \ . \qquad \ldots \ (21 \cdot 1)$$

Hence (19) takes the form

$$N_s = \frac{e^4}{\pi\Omega m^2 c^3} \cdot \frac{n_s}{\nu_s^3} \int_0^{\pi} (1 + \cos^2\theta)\sin\theta d\theta \int_0^{\infty} \frac{\sin^2\pi(\nu_\theta - \nu_\sigma)t}{(\nu_\theta - \nu_\sigma)^2} \nu_\sigma^3 d\nu_\sigma \qquad \ldots \ (22)$$

Because the value of the expression, within the sign of integration with respect to ν_σ, is concentrated near $\nu_\sigma = \nu_\theta$, one may take ν_σ^3 out of the integral as ν_θ^3. The integral that is left can be readily transformed to the standard form

$$\int_{-\infty}^{+\infty} \frac{\sin^2 kx}{x^2} dx = \pi k \ .$$

Hence we have from (22)

$$N_s = \frac{\pi e^4}{\Omega m^2 c^3} \cdot \frac{n_s}{\nu_s^3} t \int_0^{\pi} \nu_\theta^3 (1 + \cos^2\theta)\sin\theta d\theta \ . \qquad \ldots \ (23)$$

Let I_θ denote the intensity of Compton scattering at angle θ at a distance r from the scattering electron and $I_0 = \frac{n_s}{\Omega}$ the intensity of incident radiation.

We have from (23)

$$I_\theta = I_0 \cdot \frac{e^4}{2r^2 m^2 c^4} (1 + \cos^2\theta) \cdot \frac{\nu_\theta^3}{\nu_s^3} \qquad \ldots \quad (24)$$

which is the formula for the intensity of Compton scattering, as given by Einstein, Dirac and others.

If we neglect the energy of the recoil of the scattering electron, $\nu' = 0$ and so from (21) $\nu_\theta = \nu_s$. Hence from (23) after integrating with respect to θ

$$I = \frac{8\pi}{3} \cdot \frac{e^4}{m^2 c^4} \cdot I_0 \qquad (25)$$

which is the well-known Thomson formula of classical scattering.

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REFERENCES

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[2] A. H. Compton, *X-rays and Electrons*, p. 60.