A RIGID CURVILINEAR POLYGONAL CORE IN AN INFINITE PLATE UNDER TENSIONS AT INFINITY AND SHEAR

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ABSTRACT. The function theoretic method of Muschelisvili for solving two dimensional problems in elasticity is employed to obtain solutions to the problems of a rigid curvilinear polygonal core in an infinite plate under (i) an all-round tension at infinity, (ii) a uniform tension at infinity at an inclination to the x-axis and (iii) a uniform shear in the plane of the plate.

INTRODUCTION

Muschelisvili (1933) has developed a method of solving plane problems in elasticity by discarding the stress function and introducing two functions of complex variable z in terms of which all the relevant physical quantities are expressed. He has indicated how, in certain cases, these two functions of z can be determined easily with the help of the theory of functions of a complex variable. In the present paper this method has been applied to obtain solutions to the problems of a rigid curvilinear polygonal core in an infinite plate acted upon by (*i*) a uniform all-round tension at infinity, (*ii*) a uniform tension at infinity in a direction making an angle α with the x-axis and (*iii*) a uniform shear in the plane of the plate.

It has been shown by Muschelisvili that in the state of generalised plane stress the stress combinations

$$\widehat{xx}+\widehat{yy}=4\times \text{ real part of } \phi_1'(z) \qquad \dots \quad (1)$$

$$=2\times\left[\phi_1'(z)+\overline{\phi_1'(z)}\right]$$

and

$$\widehat{yy}-\widehat{xx}+2i\widehat{xy}=2\times\left[\bar{z}\phi_1''(z)+\psi_1'(z)\right] \qquad \dots \quad (2)$$

where $\phi_1(z)$ and $\psi_1(z)$ are two analytic functions of $z(=x+iy)$ and a bar over a function represents the complex conjugate of the function.

In terms of the above two functions the displacements are obtained from the relation

$$2\mu(u+iv)=K\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)} \qquad \dots \quad (3)$$

where

$$K=\frac{\lambda'+3\mu}{\lambda'+\mu}$$

To find the parts of $\phi_1(z)$ and $\psi_1(z)$ which give the stresses $\widehat{xx}=T_1$, $\widehat{yy}=T_2$, $\widehat{xy}=S$ at infinity, we get from (1) and (2)

$$\left.\begin{aligned} T_1+T_2&=4\times \text{ real part of } \phi_1'(z) \\ T_2-T_1+2iS&=2\times\left[\bar{z}\phi_1''(z)+\psi_1'(z)\right] \end{aligned}\right] \quad \dots \quad (4)$$

These give

$$\left.\begin{aligned} \phi_1(z)&=\tfrac{1}{4}(T_1+T_2)z \\ \psi_1(z)&=\tfrac{1}{2}(T_2-T_1+2iS)z \end{aligned}\right] \quad \dots \quad (5)$$

The imaginary part of the coefficient of z in $\phi_1(z)$ is omitted as it gives only a rigid body displacement.

Hence in an infinite plate with a core which exerts no force on the rest of the plate, we can write

$$\left.\begin{aligned} \phi_1(z)&=Bz+\phi_1^0(z) \\ \psi_1(z)&=(B'+iC')z+\psi_1^0(z) \end{aligned}\right] \quad \dots \quad (6)$$

where $\phi_1^0(z)$ and $\psi_1^0(z)$ are analytic outside the boundary of the core.

THE SOLUTION

Let an infinite plate containing a rigid core, whose boundary is a curvilinear polygon, be subjected to the prescribed stresses at infinity and let the displacement of the core be a translation u and a rotation through a small angle ϵ. By superposing on the plate an equal and opposite translation we do not alter either the magnitude or the directions of the stresses at infinity. The displacement of any point on the boundary of the core is, therefore, given by $u=-\epsilon y$, $v=\epsilon x$, so that $u+iv=i\epsilon z$ at a point on the boundary of the core. As the resultant traction exerted by the core on the remainder of the plate is zero, the values of B, B', C' in (6) depend only on ths stresses at infinity.

We have on the boundary of the core (3)

$$K\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)}=2\mu i\epsilon z \quad \dots \quad (7)$$

Taking complex conjugate of both sides we get

$$K\overline{\phi_1(z)}-\bar{z}\phi_1'(z)-\psi_1(z)=-2\mu i\epsilon\bar{z}. \quad \dots \quad (8)$$

on the boundary of the core.

Let the region outside the boundary of the core in the z-plane be represented on the region outside the unit circle on the ζ-plane by the conformal transformation $z=\omega(\zeta)$. Then we get on the unit circle γ

$$\left.\begin{aligned} K\phi(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi'(\sigma)}-\overline{\psi(\sigma)}&=2\mu i\epsilon\omega(\sigma) \\ K\overline{\phi(\sigma)}-\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\phi'(\sigma)-\psi(\sigma)&=-2\mu i\epsilon\overline{\omega(\sigma)} \end{aligned}\right\} \quad \dots \quad (9)$$

where $\phi(\zeta)=\phi_1\{\omega(\zeta)\}$, $\psi(\rho)=\psi_1\{\omega(\zeta)\}$

If

$$\omega(\zeta) = b(\zeta + a\zeta^{-n})$$

where n is a positive integer and $0 \leq na \leq 1$, the boundary of the core is a curvilinear polygon.

Substituting for z in (6), we get

$$\left.\begin{aligned} \phi(\zeta) &= Bb\zeta + \phi^0(\zeta) \\ \psi(\zeta) &= (B' + iC')b\zeta + \psi^0(\zeta) \end{aligned}\right\} \quad \ldots (10)$$

where $\phi^0(\rho)$ and $\psi^0(\rho)$ are analytic outside the unit circle γ, and can be written as

$$\left.\begin{aligned} \phi^0(\zeta) &= \frac{\alpha_1 + i\beta_1}{\zeta} + \frac{\alpha_2 + i\beta_2}{\zeta^2} + \ldots \\ \psi^0(\zeta) &= \frac{\alpha_1' + i\beta_1'}{\zeta} + \frac{\alpha_2' + i\beta_2'}{\zeta^2} + \ldots \end{aligned}\right\} \quad \ldots (11)$$

From equations (9) we have on the boundary of the core, where $\sigma\bar{\sigma} = 1$,

$$\left.\begin{aligned} K\phi^0(\sigma) - \frac{\sigma^{n+1} + a}{\sigma^n(1 - an\sigma^{n+1})}\bar{\phi}^{0\prime}\left(\frac{1}{\sigma}\right) - \bar{\psi}^0\left(\frac{1}{\sigma}\right) \\ = 2\mu i\epsilon b(\sigma + a\sigma^{-n}) - KBb\sigma + (B' - iC')b/\sigma \\ + Bb \cdot \frac{\sigma^{n+1} + a}{\sigma_n(1 - an\sigma^{n+1})} \end{aligned}\right\} \quad \ldots (12)$$

and

$$\left.\begin{aligned} K\bar{\phi}^0\left(\frac{1}{\sigma}\right) - \frac{\sigma^n(1 + a\sigma^{n+1})}{\sigma^{n+1} - an}\phi^{0\prime}(\sigma) - \psi^0(\sigma) \\ = -2\mu i\epsilon b\left(\frac{1}{\sigma} + a\sigma^n\right) - \frac{KBb}{\sigma} + (B' + iC')b\sigma + Bb\frac{1 + a\sigma^{n+1}}{\sigma - an\sigma^{-n}} \end{aligned}\right\} \quad \ldots (13)$$

Multiplying (12) and (13) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - \zeta}$ and integrating along γ, we get

$$\left.\begin{aligned} -\frac{1}{2\pi i}\int_\gamma \frac{\sigma + a\sigma^{-n}}{1 - an\sigma^{n+1}}\bar{\phi}^{0\prime}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma - \zeta} - K\phi^0(\zeta) \\ = -2\mu i\epsilon b\frac{a}{\zeta^n} - Bb\frac{a}{\zeta^n} - (B' - iC')\frac{b}{\zeta} \end{aligned}\right\} \quad \ldots (14)$$

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{2}bT\left(\zeta + \frac{a}{K\zeta}\right)\\ \psi(\zeta) &= \tfrac{1}{2}bT\left\{\frac{K}{\zeta} + \frac{a(1+a\zeta^2)}{K\zeta(\zeta^2-a)} - \frac{\zeta(1+a^2)}{\zeta^2-a}\right\}\end{aligned}\right] \quad \ldots \quad (28)$$

For $n=2$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{2}bT\left(\zeta + \frac{a}{K\zeta^2}\right)\\ \psi(\zeta) &= \tfrac{1}{2}bT\left\{\frac{K}{\zeta} + \frac{2a(1+a\zeta^3)}{K\zeta(\zeta^3-2a)} - \frac{\zeta^2(1+2a^2)}{\zeta^3-2a}\right\}\end{aligned}\right] \quad \ldots \quad (29)$$

For $n=3$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{2}bT\left(\zeta + \frac{a}{K\zeta^3}\right)\\ \psi(\zeta) &= \tfrac{1}{2}bT\left\{\frac{K}{\zeta} + \frac{3a(1+a\zeta^4)}{K\zeta(\zeta^4-3a)} - \frac{\zeta^3(1+3a^2)}{\zeta^4+3a}\right\}\end{aligned}\right] \quad \ldots \quad (30)$$

For $n \geq 4$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{2}bT\left(\zeta + \frac{a}{K\zeta^n}\right)\\ \psi(\zeta) &= \tfrac{1}{2}bT\left\{\frac{K}{\zeta} + \frac{na(1+a\zeta^{n+1})}{K\zeta(\zeta^{n+1}-na)} - \frac{\zeta^n(1+na^2)}{\zeta^{n+1}-na}\right\}\end{aligned}\right] \quad \ldots \quad (31)$$

Case 2. Uniform tension T at infinity in a direction making an angle α with the x-axis.

Here

$$B = \tfrac{1}{4}T$$

$$B' + iC' = -\tfrac{1}{2}Te^{-2i\alpha}$$

and

$$\epsilon = \tfrac{1}{2}T\,\frac{a(1+K)\sin 2\alpha}{2\mu(a^2+K)}, \quad \text{for } n=1$$

$$\epsilon = 0 \qquad \text{for } n \geq 2$$

We get when $n=1$

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{4}bT\left\{\zeta + \left(1 + \frac{2ia(1+K)\sin 2\alpha}{a^2+K} - \frac{2e^{2i\alpha}}{a}\right)\frac{a}{K\zeta}\right\}\\ \psi(\zeta) &= \tfrac{1}{4}bT\left\{\left(K + \frac{2ia(1+K)\sin 2\alpha}{a^2+K}\right)\frac{1}{\zeta} - \frac{\zeta(1+a^2)}{\zeta^2-a}\,2e^{-2i\alpha}\zeta\right.\\ &\quad \left. + \left(1 + \frac{2ia(1+K)\sin 2\alpha}{a^2+K} - \frac{2e^{2i\alpha}}{a}\right)\frac{a(1+a\zeta^2)}{K\zeta(\zeta^2-a)}\right\}\end{aligned}\right] \quad \ldots \quad (32)$$

hen $n=2$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{4}bT\left\{\zeta + \frac{a}{K\zeta^2} - \frac{2e^{2i\alpha}}{K\zeta}\right\}\\ \psi(\zeta) &= \tfrac{1}{4}bT\left\{\frac{K}{\zeta} - \frac{\zeta^2(1+2a^2)}{\zeta^3-2a} - \frac{2e^{2i\alpha}(1+a\zeta^3)}{K(\zeta^3-2a)}\right.\\ &\qquad\left. + \frac{2a(1+a\zeta^3)}{K\zeta(\zeta^3-2a)} - 2e^{-2i\alpha}\zeta\right\}\end{aligned}\right] \quad \cdots \quad (33)$$

When $n=3$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{4}bT\left\{\zeta + \frac{a}{K\zeta^3} - \frac{2(Ke^{2i\alpha}-ae^{-2i\alpha})}{(K^2-a^2)\zeta}\right\}\\ \psi(\zeta) &= \tfrac{1}{4}bT\left\{\frac{K}{\zeta} + \frac{\zeta^3(1+3a^2)}{\zeta^4-3a} - \frac{2(Ke^{2i\alpha}-ae^{-2i\alpha})\zeta(1+3a^2)}{(K^2-a^2)(\zeta^4-3a)}\right.\\ &\qquad\left. + \frac{3a(1+a\zeta^4)}{K\zeta(\zeta^4-3a)} - 2e^{-2i\alpha}\zeta\right\}\end{aligned}\right] \quad \cdots \quad (34)$$

When $n \geq 4$,

$$\left.\begin{aligned}\phi(\zeta) &= \tfrac{1}{4}bT\left\{p + \frac{a}{K\zeta^n} - \frac{2Ke^{2i\alpha}}{\{K^2-(n-2)a^2\}\zeta} + \frac{2ae^{-2i\alpha}}{\{K^2-(n-2)a^2\}\zeta^{n-2}}\right\}\\ \psi(\zeta) &= \tfrac{1}{4}bT\left\{\frac{K}{\zeta} - \frac{\zeta^n(1+na^2)}{\zeta^{n+1}-na} - \frac{2Ke^{2i\alpha}(1+na^2)\zeta^{n-2}}{\{K^2-(n-2)a^2\}\{\zeta^{n+1}-na\}}\right.\\ &\qquad\left. + \frac{2(n-2)ae^{-2i\alpha}\zeta(1+na^2)}{\{K^2-(n-2)a^2\}\{\zeta^{n+1}-na\}} + \frac{na(1+a\zeta^{n+1})}{K\zeta(\zeta^{n+1}-na)} - 2e^{-2i\alpha}\zeta\right\}\end{aligned}\right] \quad \cdots \quad (35)$$

Case 3. Uniform shear S in the plane of the plate.

Here

$$B=0,\ B'=0,\ C'=S$$

Therefore

$$\epsilon = \frac{a(1+K)S}{2\mu(a^2+K)} \quad \text{for } n=1$$

$$\epsilon = 0 \qquad \text{for } n \geq 2$$

We get for $n=1$,

$$\left.\begin{aligned}\phi(\zeta) &= \frac{ibS}{\zeta}\cdot\frac{a^2-1}{a^2+K}\\ \psi(\zeta) &= ibS\left\{\frac{a(1+K)}{(a^2+K)\zeta} + \frac{(a^2-1)(1+a\zeta^2)}{(a^2+K)\zeta(\zeta^2-a)} + \zeta\right\}\end{aligned}\right] \quad \cdots \quad (36)$$

For $n=2$,

$$\left.\begin{aligned} \phi(\zeta) &= -\frac{ibS}{K\zeta} \\ \psi(\zeta) &= ibS\left\{\zeta - \frac{1+a\zeta^3}{K(\zeta^3-2a)}\right\} \end{aligned}\right] \quad \ldots (37)$$

For $n=3$.

$$\left.\begin{aligned} \phi(\zeta) &= -ibS\frac{1}{(K-a)\zeta} \\ \psi(\zeta) &= ibS\left\{\rho - \frac{(1+3a^2)\zeta}{(K-a)(\zeta^4-3a)}\right\} \end{aligned}\right] \quad \ldots (38)$$

And for $n \geq 4$,

$$\left.\begin{aligned} \phi(\zeta) &= -ibS\frac{1}{K^2-(n-2)a^2}\left(\frac{K}{\zeta}+\frac{a}{\zeta^{n-2}}\right) \\ \psi(\zeta) &= ibS\left\{\zeta - \frac{(1+na^2)\{(n-2)a\zeta+K\zeta^{n-2}\}}{\{K^2-(n-2)a^2\}(\zeta^{n+1}-na)}\right\} \end{aligned}\right] \quad \ldots (39)$$

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REFERENCE

Muschelisvili, N. I., 1933, *Zeits. Angew. Math. u. Mech*,. **13**, 264