ON THE APPROXIMATE SOLUTIONS OF MAXWELL'S EQUATIONS IN AN INFINITE MEDIUM WITH REGIONS OF FINITE CONDUCTIVITY

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ABSTRACT. The approximate solutions of Maxwell's equations lead to solutions of the normal mode type for the propagation of electromagnetic waves inside wave guides. The failure of such normal mode type solutions to describe the entire field of a given source has been discussed. The discussion is restricted to the case of propagation along a tube of circular cross-section.

INTRODUCTION

The propagation of electromagnetic waves through a wave guide, with walls assumed to be of infinite conductivity, has been discussed in detail in recent years by Condon (1942) and Slater (1942). In a case like this, the problem is relatively simple and resolves itself to the solution of the wave equation in a finite region bounded by the walls of the wave guide. An infinite number of normal mode solutions, which are orthogonal, can be easily found and a linear combination of these solutions is the most general solution possible. Even if the conductivity of the walls is large, but not infinite, an approximate solution in fair agreement with practical results can be easily found by assuming that the above normal mode solutions are applicable and that the tangential magnetic field at the boundary is essentially unchanged. For a bounded region with discontinuities in the properties of the medium however, solutions of the normal mode type can be found but they will not be orthogonal.

In the case of semi conductors and dielectric wave guides, which have become important in the last two or three years, the conductivity is small and the wave equation has to be solved in an infinite region. For an infinite region having zero conductivity, Sommerfeld (1912) has shown in a general way that solutions of the normal mode type do not exist. Hondros and Debye (1910) have shown that for a dielectric wire embedded in another dielectric, there can be only a finite number of normal mode solutions.

The purpose of the present paper is to show that these solutions of the normal mode type are not enough to describe the complete field of a given source. As an example, the propagation of electromagnetic waves along a tube of circular cross-section will now be considered.

THEORY

Maxwell's equations for a homogeneous isotropic medium are :

$$\left.\begin{aligned} &\nabla . E = 0 \\ &\nabla . H = 0 \\ &\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \\ &\nabla \times \bar{H} = \sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t} \end{aligned}\right\} \quad (1)^*$$

where $\bar{E}$ and $\bar{H}$ are respectively the electric and magnetic field vectors, σ the conductivity, ϵ, the dielectric constant and μ, the permeability of the medium.

Using cylindrical coordinates r, θ, z, and confining the discussion to sinusoidal fields, the components of the electric and magnetic fields resulting from equation (1) are :

$$\left.\begin{aligned} E_r &= \frac{\partial^2 \phi^{(e)}}{\partial z \partial r} + \frac{i\mu\omega}{r} \frac{\partial}{\partial r} \phi^{(m)} \\ E_\theta &= \frac{1}{r} \frac{\partial^2 \phi^{(e)}}{\partial z \partial e} - i\mu\omega \frac{\partial}{\partial r} \phi^{(m)} \\ E_z &= \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \phi^{(e)} \end{aligned}\right\} \quad (2)$$

and

$$\left.\begin{aligned} H_r &= \frac{\partial^2 \phi^{(m)}}{\partial z \partial r} + \frac{k^2}{i\mu\omega r} \frac{\partial}{\partial \theta} \phi^{(e)} \\ H_\theta &= \frac{1}{r} \frac{\partial^2}{\partial z \partial \theta} \phi^{(m)} - \frac{k^2}{i\mu\omega} \frac{\partial}{\partial r} \phi^{(e)} \\ H_z &= \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \phi^{(m)} \end{aligned}\right\} \quad (3)$$

where the ϕs are related to the $\bar{E}$ and $\bar{H}$ vectors by the equations :

$$\left.\begin{aligned} \bar{E} &= \nabla \times \nabla \times (\bar{a}_3 \phi^{(e)}) + i\mu\omega \nabla \times (\bar{a}_3 \phi^{(m)}) \\ \bar{H} &= \frac{k^2}{i\mu\omega} \nabla \times (\bar{a}_3 \phi^{(e)}) + \nabla \times \nabla \times (\bar{a}_3 \phi^{(m)}) \end{aligned}\right\} \quad (4)$$

Here $\bar{a}_3$ is a unit vector in the z direction and the angular frequency of the sinusoidal field appearing in the expression $e^{-i\omega t}$, which has been omitted

* The units used are the rationalized M. K. S. system of units.

in the above equations for brevity. The scalar functions $\phi^{(e)}$ and $\phi^{(m)}$ must individually satisfy the wave equation

$$\nabla^2\phi + k^2\phi = 0 \quad \dots \quad (5)$$

where

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \quad \dots \quad (6)$$

k being the usual propagation constant with the convention that the imaginary part of k is always positive. Fields derived from $\phi^{(e)}$ and $\phi^{(m)}$ are usually referred to as Transverse Magnetic (TM) and Transverse Electric (TE) waves respctively.

Let the radius of the circular tube be a and let the discussion be confined to TM waves having radial symmetry for the sake of algebraic simplicity. Then the functions $\phi^{(e)}$ and $\phi^{(m)}$ must be of the form :

$$\phi_1^{(e)} = \frac{A}{\lambda_1^2}\,\frac{J_0(\lambda_1 r)e^{ihz}}{J_0(\lambda_1 a)}\,;\quad \phi_1^{(m)} = 0 \quad \dots \quad (7)$$

$$\phi_2^{(e)} = \frac{A}{\lambda_2^2}\,\frac{H_0^{(1)}(\lambda_2 r)e^{ihz}}{H_0^{(1)}(\lambda_2 a)}\,;\quad \phi_2^{(m)} = 0 \quad \dots \quad (8)$$

where the subscripts 1 and 2 refer to the regions inside $r < a$ and outside $r > a$ respectively and h is the progagation factor, J_0 is the Bessel function and $H_0^{(1)}$ is the Hankel function of the first kind. To satisfy the wave equation (5) these solutions satisfy the relations :

$$\lambda_1^2 + h^2 = k_1^2 \text{ and } \lambda_2^2 + h^2 = k_2^2 \quad \dots \quad (9)$$

and so,

$$\lambda_2^2 = \lambda_1^2 + (k_2^2 - k_1^2) \quad \dots \quad (10)$$

The continuity of the tangential components of H at $r=a$ requires that

$$\frac{k_1^2 J_1(u)}{\mu_1 u J_0(u)} = \frac{k_2^2 H_1^{(1)}(V)}{\mu_2 V H_0^{(1)}(V)} \quad \dots \quad (11)$$

where $u=\lambda_1 a$ and $v=\lambda_2 a$

The pair of equations (10) and (11) are satisfied only by certain discrete values of the parameters u and v. The physically feasible solutions require that the imaginary part of the roots V_n be positive.

Certain features of equation (11) in some special cases are interesting. When the conductivities of both the media inside as well as outside the cylinder are zero, so that k_1 and k_2 are both real, it can be easily shown that there are no proper roots if $k_1 < k_2$ and only a limited number of proper roots—roughly $a\sqrt{k_1^2 - k_2^2}/\pi$, if $k_1 > k_2$ This result agrees with that of Hondros and Debye (1910). But when one or both of the conductivities σ_1 or σ_2 are finite, the k's are no longer real, and the problem becomes a little complicated. To consider a typical case, let the medium outside the cylinder be a metal so that k_2 is very large. Then for the lower order modes it is easily seen that V_n has a positive imaginary part as required. But when n is sufficiently large, the imaginary part of V_n may or may not be positive. To ascertain this, consider the following relations which hold good when n is large :

$$\left.\begin{array}{l} |u_n| \gg 1 \\ |u_n^2| \gg |k_2^2 - k_1^2| \\ |u_n a| \gg |k_2^2\mu_1/8k_1^2\mu_2| \end{array}\right\} \qquad (12)$$

Then V_n should be of the form :

$$V_n = -\alpha u_n - \tfrac{1}{2}\frac{\alpha}{u_n}(k_2^2 - k_1^2) + f(u_n^{-3}) \qquad (13)$$

where $\alpha = \pm 1$

Using Hankel's asymptotic expressions for the Bessel functions valid for terms of order $\frac{1}{|u_n|} \ll 1$, the following relations are obtained for the higher order roots of equation (11) :

$$u_n a \approx -(n-\tfrac{1}{4})\pi - \tan^{-1}\left(\frac{\mu_1 k_2^2 \alpha}{i\mu_2 k_1^2}\right) + \frac{1}{8\pi n}\frac{3\mu_2 k_1^2 - \mu_1 k_2^2}{\mu_2 k_1^2 + \mu_1 k_2^2} + f(n^{-2}) \quad \ldots \quad (14)$$

and,

$$V_n a \approx \alpha\pi(n-\tfrac{1}{4}) + \alpha \tan^{-1}\left(\frac{\alpha\mu_1 k_2^2}{i\mu_2 k_1^2}\right) - \frac{\alpha}{8\pi m}\frac{3\mu_2 k_1^2 - \mu_1 k_2^2}{\mu_2 k_1^2 + \mu_1 k_2^2} + \frac{\alpha a^2}{2n\pi}(k_1^2 - k_2^2) + f(n^{-2}) \qquad (15)$$

The second term on the right hand side of equation (15) determines uniquely whether the imaginary part of V_n is positive or not. Since the imaginary part of $\tan^{-1}\theta$ has the same sign as the imaginary part of θ and since $\alpha^2 = 1$ it follows that

$$\left[\frac{\mu_1 k_2^2}{i\mu_2 k_1^2}\right] = -\frac{\sigma_1\sigma_2 + \epsilon_1\epsilon_2\omega^2}{\sigma_1^2 + \epsilon_1^2\omega^2} < 0$$

Thus the imaginary part of V_n will be negative instead of positive, as required.

For the case of a perfect dielectric inside a metal σ_1 will be zero and σ_2 will be very large so that equation (15) becomes :

$$V_n a \approx (n-\tfrac{1}{4})\pi + \left(\frac{\pi}{2} - \frac{\epsilon_1\omega}{\sigma_2} - i\frac{\epsilon_1\epsilon_2\omega^2}{\sigma_2^2}\right) + \frac{1}{8\pi m}\left(1 + \frac{4i\epsilon_1\omega}{\sigma_2}\right.$$
$$\left. + \frac{a^2\omega^2}{2m\pi}\left(\mu_2\epsilon_2 - \mu_1\epsilon_1 + i\frac{\mu_2\sigma_2}{\omega}\right)\right) \qquad (16)$$

The imaginary part of V_n will then be positive only if

$$m < \frac{a^2\mu_2\sigma_2^3}{2\pi\omega\epsilon_1\epsilon_2} \qquad (17)$$

Since the conductivity occurs in the third power, the number of permissible normal mode solutions will be very large for metallic wave guides, where as, for dieletric wave guides, the number will be very small. In fact in the limiting case of zero conductivity there will be no solutions of the normal mode type. Even in case of metallic wave guides, since there are only a finite number, though large, of normal mode solutions they cannot form a complete set or represent the complete field of a given source. The additional solutions required to form a complete solution of the problem will probably be negligible for all practical calculations on metallic wave guides but will certainly contribute an appreciable part in the case of dielectric wave guides.

Thus the usual methods of dealing with electromagnetic wave propagation in wave guides, leading to solutions just of the normal mode type are not adequate to describe the complete field of a given source. The inadequacy is particularly evident in case of dielectric wave guides The problem is whether the transition from the case in which there are no normal mode type of solutions for zero conductivity to that in which there exists a complete set of normal mode type of solutions for infinite conductivity occurs abruptly when the conductivity changes from zero to a small but finite value, or in steps as the conductivity is gradually increased, or abruptly when the conductivity changes from a large but finite value to infinity.

This question will be dealt with in detail in another paper dealing with an exact solution of the above problem

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REFERENCES

Condon, E. U., 1942, *Rev. Mod. Phys.*, **14**, 341.
Hondros, D., and Debye, P., 1910, *Ann. der Phys.*, (4) **32**, 465.
Slater, J. C., 1942, Microwave Transmission.
Sommerfeld, A., 1912, *Jahresbericht der Deutschen Math. Vereinigung*, **21**, 309