STRESS DISTRIBUTION IN AN INFINITE PLATE WITH AN ELLIPTIC HOLE ACTED UPON BY A FORCE AND A COUPLE AT AN INTERNAL POINT *

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ABSTRACT. The Function theoretic method of Muschelisvili (1933) for solving two dimensional problems in elasticity is employed to obtain solutions of the problems of stress distribution in a large plate containing an elliptic hole under the action of an isolated force or a couple at any point inside the plate From these solutions particular cases of the force and couple acting on the boundary of the hole are derived and the results are compared with those obtained by Rothman (1950), who used complex potentials as introduced by Stevenson (1945).

INTRODUCTION

In a recent paper Rothman (1950) has obtained the solution of the problems of stress distribution in an infinite plate with elliptic and circular holes under the actions of isolated forces on the boundary of the hole. Following Stevenson (1945), he has introduced two functions $\Omega(z)$ and $\omega(z)$, in terms of which the stresses can be expressed. In the present paper the solution is given of the problems of stress distribution in an infinite plate with an elliptic hole under the action of an isolated force or a couple at any point inside the plate. The method of Muschelisvili (1933) has been followed here. It may be mentioned that the complex stress functions introduced by Muschelisvili are connected with the complex potentials introduced by Stevenson. One of the functions of Stevenson is a constant multiple of one of the functions of Muschelisvili and the derivative of the second function of Stevenson is a constant multiple of the second function of Muschelisvili. Rothman's problems appear as particular cases of the problems solved here, if we make the point where the force is applied tend to the boundary of the hole. Another particular case is the problem of an infinite plate with a circular hole acted upon by a force at an internal point. This problem has been solved by Sen (1945) by a method which he has developed in the case of a force acting along a radius of the hole.

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It has been shown by Muschelisvili (1933) that in state of generalised plane stress the stress combinations $\widehat{xx}+\widehat{yy}$ and $\widehat{yy}-\widehat{xx}+2i\,\widehat{xy}$ of the average stresses are given by

$$\widehat{xx}+\widehat{yy}=4\times \text{ real part of } \phi_1'(z)$$

$$2=[\phi_1'(z)+\bar{\phi}_1'(\bar{z})] \qquad \ldots (1)$$

and

$$\widehat{yy}-\widehat{xx}+2i\,\widehat{xy}=2[z\phi_1''(z)+\psi_1'(z)] \qquad \ldots (2)$$

where $\phi_1(z)$ and $\psi_1(z)$ are two analytic functions of $z(=x+iy)$ and a bar over a function represents the complex conjugate of the function.

The resultant traction across an arc AB (which the material on the right exerts on the left) is given by

$$X+iY=-i[\phi_1(z)+z\bar{\phi}_1'(\bar{z})-\bar{\psi}_1(\bar{z})]_A^B \qquad \ldots (3)$$

and the moment of the traction across AB about the origin is given by

$$M_0=\text{real part of } [-z\bar{z}\phi_1'(z)-z\psi_1(z)+\psi_2(z)]_A^B \qquad \ldots (4)$$

The displacements are obtained from the relation

$$2\mu(u+iv)=K\phi_1(z)-z\bar{\phi}_1'(\bar{z})-\bar{\psi}'_1(\bar{z}) \qquad \ldots (5)$$

where

$$K=\frac{\lambda'+3\mu}{\lambda'+\mu}.$$

To determine $\phi_1(z)$ and $\psi_1(z)$ when stresses are prescribed on the boundary, we have from (3)

$$\left.\begin{aligned}\phi_1(z)+z\bar{\phi}_1'(\bar{z})+\bar{\psi}_1(\bar{z})&=f_1+if_2\\ \bar{\phi}_1(\bar{z})+\bar{z}\phi_1'(z)+\psi_1(z)&=f_1-if_2\end{aligned}\right] \qquad \ldots (6)$$

where

$$f_1+if_2=i\int_{B_0}^{B}(F_1+iF_2)dS+\text{constant}.$$

$F_1(S)\,dS$ and $F_2(S)\,dS$ being the x and y components of tractions across the arc dS.

For an isolated force $X+iY$ at the origin supposed to be in the interior of the plate, the parts of $\phi_1(z)$ and $\psi_1(z)$ which give rise to the force are

$$\begin{aligned}\phi_1(z)&=-\frac{X+iY}{2\pi(K+1)}\log z\\ \psi_1(z)&=\frac{K(X+iY)}{2\pi(K+1)}\log z\end{aligned} \qquad \ldots (7)$$

For a couple of moment M, we have them as

$$\left.\begin{aligned} \phi_1(z) &= 0 \\ \psi_1(z) &= -\frac{iM}{2\pi Z}. \end{aligned}\right] \quad \dots \quad (8)$$

When the force $X+iY$ acts at the point z_0, the relevant parts of $\phi_1(z)$ and $\psi_1(z)$ are

$$\left.\begin{aligned} \phi_1(z) &= -\frac{X+iY}{2\pi(K+1)} \log (z-z_0) \\ \psi_1(z) &= \frac{K(X-iY)}{2\pi(K+1)} \log (z-z_0) + \frac{X+iY}{2\pi(K+1)} \cdot \frac{\bar{z}_0}{z-z_0}. \end{aligned}\right] \quad \dots \quad (9)$$

and when the couple acts at the point z_0, we have,

$$\left.\begin{aligned} \phi_1(z) &= 0 \\ \psi_1(z) &= \frac{iM}{2\pi} \cdot \frac{1}{z-z_0} \end{aligned}\right] \quad \dots \quad (10)$$

If $z=\omega(\zeta)$ transforms the region outside the hole in the z-plane conformally on the region outside the unit circle γ in the ζ-plane, we have on γ

$$\left.\begin{aligned} \phi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \phi'(\sigma) + \psi(\sigma) &= f_1 + if_2 \\ \bar{\phi}(\bar{\sigma}) + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \phi'(\sigma) + \psi(\sigma) &= f_1 - if_2 \end{aligned}\right] \quad (11)$$

where

$$\phi(\zeta) = \phi_1\{\omega(\zeta)\} \text{ and } \psi(\zeta) = \psi_1\{\omega(\zeta)\}.$$

and $\zeta=\sigma$ gives a point on γ, so that $\sigma\bar{\sigma}=1$.

INFINITE PLATE WITH AN ELLIPTIC HOLE ACTED UPON BY AN ISOLATED FORCE

Let the boundary of the hole be given by

$$\frac{}{R^2(1+m)^2} + \frac{}{R^2(1-m)^2} = 1 \quad \dots \quad (12)$$

The transformation

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \qquad 0 < m < 1,\ R > 0 \quad \dots \quad (13)$$

transforms the region outside the hole in the z-plane, to the region outside the circle γ, $|\zeta|=1$ in the ζ-plane and the point at infinity in the two planes correspond. Substituting from (13) in (11) and replacing σ by $1/\sigma$, we get on γ

$$\begin{aligned} \phi(\sigma)+\frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\,\phi'\left(\frac{1}{\sigma}\right)+\psi\left(\frac{1}{\sigma}\right)&=f_1+if_2 \\ \bar{\phi}\left(\frac{1}{\sigma}\right)+\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\phi'(\sigma)+\psi(\sigma)&=f_1-if_2 \end{aligned} \qquad \ldots \quad (14)$$

If a force $X+iY$ act at the point z_0 of the plate, we may take

$$\begin{aligned} \phi_1(z)&=-\frac{X+iY}{2\pi(K+1)}\log(z-z_0)+\phi_1^{0}(z) \\ \psi_1(z)&=\frac{K(X-iY)}{2\pi(K+1)}\log(z-z_0)+\frac{X+iY}{2\pi(K+1)}\,\frac{\bar{z}_0}{z-z_0}+\psi_1^{0}(z). \end{aligned} \qquad \ldots \quad (15)$$

where $\phi_1^0(z)$ and $\psi_1^0(z)$ are analytic outside the elliptic hole (12),

Using the transformation (13) we write

$$\begin{aligned} \phi(\zeta)&=-\frac{X+iY}{2\pi(K+1)}\log\frac{(\zeta-\zeta_0)(\zeta\zeta_0-m)}{\zeta\zeta_0}+\phi^0(\zeta) \\ \psi(\zeta)&=\frac{K(X-iY)}{2\pi(K+1)}\log\frac{(\zeta-\zeta_0)(\zeta\zeta_0-m)}{\zeta\zeta_0} \\ &\quad+\frac{X+iY}{2\pi(K+1)}\,\frac{\zeta\zeta_0(\bar{\zeta}_0^2+m)}{\bar{\zeta}_0(\zeta-\zeta_0)(\zeta\zeta_0-m)}+\psi^0(\zeta) \end{aligned} \qquad \ldots \quad (16)$$

where $\phi^0(\zeta)$ and $\psi^0(\zeta)$ are analytic outside the unit circle γ in the ζ-plane. Since the boundary of the hole is free from tractions, substituting in (14) $f_1=0$, $f_2=0$, we get on the unit circle γ

$$\begin{aligned} &\phi^0(\sigma)+\frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\,\phi^{0\prime}\left(\frac{1}{\sigma}\right)+\psi^0\left(\frac{1}{\sigma}\right)=\frac{X+iY}{2\pi(K+1)}\log\frac{(\sigma-\zeta_0)(\sigma\zeta_0-m)}{\sigma\zeta_0} \\ &-\frac{K(X+iY)}{2\pi(K+1)}\log\frac{(1-\sigma\bar{\zeta}_0)(\bar{\zeta}_0-m\sigma)}{\sigma\bar{\zeta}_0}+\frac{X-iY}{2\pi(K+1)}\cdot\frac{\zeta_0(\sigma-\zeta_0)(\sigma\zeta_0-m)}{\bar{\zeta}_0(1-\sigma\bar{\zeta}_0)(\bar{\zeta}_0-m\sigma)} \\ &\bar{\phi}^0\left(\frac{1}{\sigma}\right)+\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\phi^{0\prime}(\sigma)+\psi^0(\sigma)=\frac{X-iY}{2\pi(K+1)}\log\frac{(1-\sigma\bar{\zeta}_0)(\bar{\zeta}_0-m\sigma)}{\sigma\bar{\zeta}_0} \\ &-\frac{K(X-iY)}{2\pi(K+1)}\log\frac{(\sigma-\zeta_0)(\sigma\zeta_0-m)}{\sigma\zeta_0}+\frac{X+iY}{2\pi(K+1)}\,\frac{\bar{\zeta}_0(1-\sigma\bar{\zeta}_0)(\bar{\zeta}_0-m\sigma)}{\zeta_0(\sigma-\zeta_0)(\sigma\zeta_0-m)}. \end{aligned} \qquad (17)$$