THE RELATIVISTIC THEORY OF SCATTERING IN COULOMB FIELD BY ATOMS

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ABSTRACT Relativistic theory of nuclear scattering of electrons has been considered from the wavestatistical point of view It has been shown that to a first order relativistic approximation the ordinary hydrodynamical wave equation is slightly modified On deriving the wellknown χ-equations with the help of the modified equation we get a new term in the interaction energy which together with the other wellknown interaction terms gives the correct scattering formula

INTRODUCTION

Relativistic theory of electron scattering has first been given by Mott (1929), using Dirac's linear equation for the electron. In addition to the wellknown $\operatorname{cosec}^4 \frac{\theta}{2}$ term he has obtained two correction terms, the second of which is proportional to $(Ze^2)^3$. Later on Sexl (1933) has considered the problem afresh starting with the quadratic form of Dirac's equation (*vide* Dirac, 1947) and has obtained a formula for the scattering intensity differing from Mott's formula only in the last correction term The controversy over the second correction term has been finally, settled by Urban (1942), who, on checking up the calculations of Mott, finds that Mott's method, after proper approximation, also gives exactly the same formula as that obtained by Sexl (*loc. cit.*). It may also be mentioned that Sauter (1933) and recently Sengupta and Chatterji (1950) have obtained the above Mott-Sexl formula without the second correction term by application of Born's method of approximation to the linear equation of Dirac.

Recently Kar (1945, 1947) has considered the problem of high velocity scattering from the wave-statistical point of view and has derived Mott's formula using some new ideas regarding spin-spin interaction. In the present paper we shall show that the introduction of these ideas is not at all necessary and the correct Mott-Sexl formula can be deduced wave-statistically in a perfectly straightforward manner

It is wellknown that in wavestatistics we take for the phase waves the general hydrodynamical wave equation in the form

$$\frac{\partial^2 D}{\partial t^2} = v^2 \Delta D \qquad (1)$$

and from it we obtain the wellknown differential equations for the χ_1- and χ_2- waves. We shall presently show that equation (1) is only a first approximation of the actual equation satisfied by the density function in a compressible medium. If we carry the approximation a stage further, we get an equation slightly different from (1). On deriving χ_1- and χ_2- equations from it we get a new term in the interaction energy which together with the other wellknown interaction energies gives the correct scattering formulæ.

WAVE EQUATION

The wellknown Bernoulli's equation and the equation of continuity for a fluid in motion can be written in the form

$$\frac{\partial\phi}{\partial t}+\tfrac{1}{2}w^2+\frac{\delta P}{\rho_0}=\text{const.} \qquad \ldots (2.1)$$

$$\frac{\partial\rho}{\partial t}+\text{div}\,(\rho w)=0 \qquad \ldots (2.2)$$

where ϕ is the velocity potential and w the velocity of the fluid.

From (2.2) we get

$$\frac{\partial\rho}{\partial t}+\rho\,\text{div}\,w+(w\,\text{grad}\,\rho)=0$$

or,
$$\frac{\partial^2\rho}{\partial t^2}+\rho\Delta\dot{\phi}+(w\,\text{grad}\,\dot{\rho})+\text{div}\,\rho\dot{w}=0$$

Substituting for $\dot{\phi}$ and $\dot{\rho}$ from (2) we get

$$\frac{\partial^2\rho}{\partial t^2}-\rho\Delta\frac{\delta P}{\rho_0}-\frac{\rho}{2}\Delta w^2+(w\,\text{grad}\,\dot{\rho})-\text{div}\,(w\,\text{div}\,\rho w)=0$$

or,
$$\frac{\partial^2\rho}{\partial t^2}-v^2\Delta\rho+(w\,\text{grad}\,\dot{\rho})-\frac{\rho}{2}\Delta w^2-\text{div}\,(w\,\text{div}\,\rho w)=0 \qquad \ldots (3)$$

because,
$$\frac{\delta P}{\rho_0}=\frac{\delta P}{\delta\rho}\cdot\frac{\delta\rho}{\rho_0}=v^2\cdot\frac{\rho-\rho_0}{\rho_0} \qquad \ldots (3.1)$$

Neglecting the last two terms in (3) we get

$$\frac{\partial^2\rho}{\partial t^2}-v^2\Delta\rho+(\dot{w}\,\text{grad})\,(\bar{\rho}-\rho_0)=0 \qquad \ldots (4)$$

It is evident from the above derivation that the change of ρ in the last term of (4) is due to motion of the fluid. It is therefore negligible, except for very large velocity, *i.e.*, when w is comparable with c, the velocity of light. Hence for this ρ we can write (in the relativistic region)

$$\rho=\frac{\rho_0}{(1-\beta^2)^{\frac{1}{2}}}\ ;\ \beta=\frac{w}{c} \qquad (4.1)$$

or approximately,

$$\rho\left(1-\frac{\beta^2}{2}\right)=\rho_0 \quad \text{i.e.,} \quad \rho-\rho_0=\tfrac{1}{2}\beta^2\rho \qquad (4.2)$$

Substituting this value of $\rho-\rho_0$ in the last term of (4) we get

$$\frac{\partial^2\rho}{\partial t^2}-v^2\Delta\rho+\tfrac{1}{2}\beta^2(\dot{w}\ \text{grad}\ \rho)=0 \qquad (5)$$

It is evident that for small velocities, the last term can be neglected and then (5) becomes identical with the equation (1)

If we now take the general hydrodynamical wave equation in the form (5), the equation for the phase waves becomes

$$\frac{\partial^2 D}{\partial t^2}-v^2\Delta D+\frac{\beta^2}{2}(\dot{w}\ \text{grad}\ D)=0 \qquad (6)$$

Eliminating time we get for the χ-waves

$$\Delta\chi+\frac{4\pi^2\nu^2}{v^2}\chi-\frac{\beta^2}{2v^2}(\dot{w}\ \text{grad}\ \chi)=0 \qquad (7)$$

Now for χ_1-waves $v\equiv v_1=w$ and the relativistic frequency ν_1 is given by [Kar and Sengupta (1949)]

$$\nu_1=\frac{1}{h}\cdot\frac{(E-E_{s-0}+E_0-V)^2-E_0^2}{E-E_{s-0}+E_0-V} \qquad (7.1)$$

where E is the total energy, E_{s-0} the well known spin-orbit interaction energy being given by Thomas (1927)

$$E_{s-0}=\frac{Ze^2}{2m_0^2c^2r^3}\ (\mathbf{LS}) \qquad (7.2)$$

and V is the electrostatic potential energy. Substituting in (7) and remembering that $\dot{w}=-\frac{1}{m_0}\ \text{grad}\ V=-\frac{V'}{m_0}\frac{\mathbf{r}}{r}$, we get,

$$\Delta\chi_1+\frac{8\pi^2m_0}{h^2}\left\{E-V-E_{s-0}+\frac{1}{2E_0}(E-V)^2\right\}\chi_1+\frac{V'}{2E_0}\frac{\partial\chi_1}{\partial r}=0 \qquad (8)$$

For χ_2-waves, we have,

$$\beta=\frac{v_2}{c} \qquad (9)$$

where v_2 is the velocity of the χ_2-waves projected in the q-space being given by [Kar and Sengupta (*l. c.*)],

$$v_2=\frac{cE}{\{(E-E_{s-0}+E_0-V)^2-E_0^2\}^{\frac{1}{2}}} \qquad (9.1)$$

Remembering that $\nu_2=\frac{E}{h}$, we get for the χ_2-waves the same equation as (8).

Thus equation (8) may be written in the general form

$$\Delta\chi+\frac{8\pi^2m_0}{h^2}\left\{E-V-E_{s=0}+\frac{V'h^2}{16\pi^2m_0^2c^2}\frac{\partial}{\partial r}+\frac{(E-V)^2}{2E_0}\right\}\chi=0 \qquad (10)$$

On comparing (10) with the equation deduced in the earlier paper [Kar and Sengupta (1949)] we notice that in addition to the well known relativistic and spin-orbit corrections to energy, we have here a new correction term whose value is given by

$$V_r=\frac{h^2}{16\pi^2m_0^2c^2}\cdot\frac{Ze^2}{r^2}\cdot\frac{\partial}{\partial r} \qquad \ldots \quad (10.1)$$

It may here be noted that this particular interaction term has also been obtained by Condon and Shortley (1935) by forming, to a first approximation, the relativistic quadratic wave equation from Dirac's linear equation for the electron. They have showed the importance of the term in spectroscopy. We shall presently show that this term has also great importance in scattering and that its contribution to the scattering formula corresponds to Mott-Sexl's first correction.

CALCULATION OF THE SCATTERING INTENSITY

By separating the interaction energies involving V and V^2 we can write (10) in the form

$$\Delta\chi+\frac{8\pi^2m_0}{h^2}\left\{E\left(1+\frac{E}{2E_0}\right)-V\left(1+\frac{E}{E_0}\right)-E_{s=0}+V_r+\frac{V^2}{2E_0}\right\}\chi=0 \qquad (11)$$

Remembering that E in the above does not contain E_0, we get from (11), the rest energy of the electron,

$$\Delta\chi+\frac{4\pi^2}{h^2}\left\{p^2-2m_0V-2m_0E_{s=0}+2m_0V_r+\frac{V^2}{c^2}\right\}\chi=0 \quad \ldots \quad (11\cdot1)$$

where $p=\dfrac{m_0v}{(1-\beta^2)^{\frac{1}{2}}}$, the relativistic momentum of the electron.

We now consider the scattering of a beam of fast electrons incident on a nucleus by usual Born's approximation method Outside the potential field the wave function for the incident beam is

$$\chi_0=e^{2\pi ip\,(\mathbf{n}_0\mathbf{r})/h}. \qquad \ldots \quad (12)$$

where $\mathbf{n}_0$ is the unit vector along the direction of incidence. χ_0 in (12) is averaged for unit volume. Hence it represents an intensity of v electrons crossing unit area per unit time. Near the nucleus, the wave equation is given by (11.1). This equation can be written in the form

$$\Delta\chi+\frac{4\pi^2p^2}{h^2}\chi=F\chi \qquad \ldots \quad (13)$$

where $$F = \frac{4\pi^2}{h^2}\left\{2mV - 2m_0V_r - \frac{V^2}{c^2} + 2m_0E_{r-0}\right\} \quad \ldots \quad (13.1)$$

Now we try to find a solution of (13) of the form

$$\chi = \chi_0 + \lambda_1\chi_1 \quad (13.2)$$

where χ_0 represents the incident wave and $\lambda_1\chi_1$ the scattered wave.
From (13) and (13.2) we get at once the well known solution

$$\lambda_1\chi_1 = -\frac{1}{4\pi}\int \frac{e^{2\pi i/h\, p|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} F(r')\chi\, d\tau' \quad \ldots \quad (14)$$

which has the asymptotic form (for large r)

$$\lambda_1\chi_1 = -\frac{1}{4\pi}\cdot\frac{e^{2\pi i\, pr/h}}{r}\int e^{-2\pi i p(\mathbf{nr})/h} F\chi\, d\tau \quad (14.1)$$

where $\mathbf{n}$ is the unit vector along the direction of scattering. In the following we shall replace χ in the integral by χ_0, the incident wave function, as is usual in the Born's first approximation.

Now the integral in (14.1) consists of four different integrals corresponding to the four different interaction terms in F. The first of the integrals is, remembering that $V = -\frac{Ze^2}{r}$,

$$I_1 = -\frac{8\pi^2 mZe^2}{h^2}\int e^{2\pi i\, p(\mathbf{n}_0-\mathbf{n},\, \mathbf{r})/h}\,\frac{1}{r}\cdot d\tau \quad (15)$$

Now taking polar angle $\theta_1 = 0$ along the direction of the vector $\mathbf{n}_0 - \mathbf{n}$ the integration (15) can be easily performed. If we take into account the correction for critical approach (Kar, 1945), then evidently the lower limit of the $\mathbf{r}$-integration should be taken r_0 instead of zero, where r_0 is the distance of critical approach. Then we get

$$I_1 = -\frac{8\pi^2 mZe^2}{h^2}\cdot\frac{h^2}{\pi p^2|\mathbf{n}_0-\mathbf{n}|^2}\cdot\cos kr_0$$

$$= -\frac{2\pi Ze^2}{m_0v^2\sin^2\frac{\theta}{2}}(1-\beta^2)^{\frac{1}{2}}\cos kr_0 \quad \ldots \quad (15.1)$$

θ being the angle of scattering and $k = \frac{4\pi mv\sin\frac{\theta}{2}}{h}$. The value of r_0 as has been derived wave-statistically by Kar (1945) is

$$r_0 = 1.35\cdot\frac{Ze^2}{m_0v^2}(1-\beta^2)\left(\operatorname{cosec}\frac{\theta}{2} - 1\right) \quad \ldots \quad (15.2)$$

It is evident from above that kr_0 is generally small for high velocity of incidence and $\cos kr_0$ is only slightly different from one.

The second integration can be written in the form [*vide* (10.1) and (13.1)],

$$I_2 = -\frac{Ze^2}{2m_0c^2}\int e^{-2\pi i p(\mathbf{nr})/h}\cdot\frac{\partial}{\partial r}e^{2\pi i p/h\cdot(\mathbf{n_0 r})}\,dr\sin\theta_1 d\theta_1 d\phi_1 \quad \dots \quad (16)$$

If we remember that θ_1 is taken zero along $\mathbf{n}_0-\mathbf{n}$, then it can be easily shown that

$$(\mathbf{n}_0\mathbf{r}) = r(\sin\frac{\theta}{2}\cos\theta_1 + \cos\frac{\theta}{2}\sin\theta_1\cos\phi_1) \quad \dots \quad (16.1)$$

Substituting this in (16) and performing the differentiation, we get

$$I_2 = -\frac{\pi i p}{h}\cdot\frac{Ze^2}{m_0c^2}\left\{\sin\frac{\theta}{2}\int e^{ikr\cos\theta_1}\,dr\cos\theta_1\sin\theta_1 d\theta_1 d\phi_1\right.$$

$$\left.+\cos\frac{\theta}{2}\int e^{ikr\cos\theta_1}\,dr\sin^2\theta_1 d\theta_1\cos\phi_1 d\phi_1\right\} \quad \dots \quad (16.2)$$

The second term evidently vanishes through ϕ_1-integration. The first integration can be easily performed and we get,

$$I_2 = \frac{\pi Ze^2}{m_0c^2}\left(1-\frac{(kr_0)^2}{6}\right) \quad \dots \quad (16.3)$$

The third integration can be performed in the same way as the first one and we get,

$$I_3 = -\frac{2\pi^2(\pi-2kr_0)(Ze^2)^2}{hc^2m_0v\sin\frac{\theta}{2}}(1-\beta^2)^{\frac{1}{2}} \quad \dots \quad (17)$$

In the fourth integral there is a factor $(\mathbf{LS}) = L_xS_x + L_yS_y + L_zS_z$. If we write out the well-known operators for L_x, L_y etc., then with the help of (16·1) the integration can be easily performed and it may be seen that,

$$I_4 = 0 \quad \dots \quad (18)$$

Thus we get from (14.1), neglecting terms of the order of $\frac{v^4}{c^4}$,

$$\lambda_1\chi_1 = \frac{e^{2\pi i pr/h}}{r}\cdot\frac{Ze^2}{2m_0v^2}\cdot(1-\beta^2)^{\frac{1}{2}}\left\{\operatorname{cosec}^2\frac{\theta}{2}\cos kr_0 - \frac{1}{2}\frac{v^2}{c^2}\left(1-\frac{(kr_0)^2}{6}\right)\right.$$

$$\left.+\frac{\pi(\pi-2kr_0)Ze^2v}{hc^2}\operatorname{cosec}\frac{\theta}{2}\right\} \quad \dots \quad (19)$$

If the relative scattered intensity be given by $I(\theta)d\Omega$, then to a first approximation

$$I(\theta) = \left(\frac{Ze^2}{2m_0v^2}\right)^2(1-\beta^2)\left\{\operatorname{cosec}^4\frac{\theta}{2}\cos^2 kr_0 - \frac{v^2}{c^2}\cos kr_0\operatorname{cosec}^2\frac{\theta}{2}\right.$$

$$\left.+\frac{2\pi\cos kr_0(\pi-2kr_0)Ze^2v}{hc^2}\cdot\operatorname{cosec}^3\frac{\theta}{2}\right\} \dots \quad (19.1)$$

Neglecting the correction for the critical approach, we get

$$I(\theta)=\left(\frac{Ze^2}{2m_0v^2}\right)^2(1-\beta^2)\left(\operatorname{cosec}^4\frac{\theta}{2}-\frac{v^2}{c^2}\operatorname{cosec}^2\frac{\theta}{2}+\frac{2\pi^2Ze^2v}{hc^2}\operatorname{cosec}^3\frac{\theta}{2}\right) \tag{20}$$

which is identical with the formula obtained by Mott-Sexl.

In conclusion, it may be mentioned that in deriving the above formula of scattering we have found (Eqn. 18) that the spin-orbit interaction term in F has no contribution to the scattering intensity Since the other two interaction terms in F have a relativistic origin, it may be easily seen that both the corrections in Mott-Sexl formula (20) are really relativistic.

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