ON THE DISTRIBUTION OF STRESS ROUND THE EDGE OF A HOLE IN A DEEP BEAM UNDER A UNIFORM BENDING MOMENT *

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ABSTRACT. A solution to the problem of stress concentration due to the presence of an unstressed hole of a fairly general shape in a deep plate beam under a uniform bending moment is obtained. The solution is verified for the cases of circular and elliptic holes which are already known, and is applied to obtain some new results.

INTRODUCTION

The problems of stress distribution in an infinite plate containing an unstressed hole, under various types of load have been widely studied by finding the stress function χ in suitable curvilinear coordinates, satisfying the biharmonic equation $\nabla^4\chi=0$. A method of solution to the problems of stress distribution in an infinite plate containing a hole of a fairly general shape has been developed by Green (1945). In the present paper Green's method has been applied to obtain the stress distribution round the edge of a hole in a deep plate beam under uniform bending moment.

This method can be applied when the hole is given by a curve $\eta=0$, defined by the conformal transformation

$$Z=F(\zeta) \quad \ldots \quad (1)$$

where $$Z=x+iy,\ \zeta=\xi+i\eta$$

and $$F'(\zeta)=a_0e^{-i\zeta}+b_ne^{in\zeta}$$

it being assumed that $\eta\to\infty$ when $|Z|\to\infty$. It is seen that the first term in $F'(\zeta)\to\infty$ and the second term $\to 0$ as $\eta\to\infty$.

It is known that the general solution of $\nabla^4\chi=0$ is given by the real part of

$$F(Z)+\bar{Z}g(Z) \quad \ldots \quad (2)$$

where $\bar{Z}=x-iy$ and where $f'(Z)$ and $g(Z)$ are regular functions of Z.

From (2) the stresses in curvilinear coordinates ξ, η, are found to be given by the real parts of (Green, 1945,)

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$$\widehat{\xi\xi} = 2g'(Z) - \frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}\left\{f''(Z) + \bar{F}(\bar{\zeta})g''(Z)\right\}$$

$$\widehat{\eta\eta} = 2g'(Z) + \frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}\left\{f''(Z) + \bar{F}(\bar{\zeta})g''(Z)\right\} \quad \ldots (3)$$

$$\widehat{\xi\eta} = -i\frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}\left\{f''(Z) + \bar{F}(\bar{\zeta})g''(Z)\right\}$$

where dashes attached to $g(Z)$ and $f(Z)$ denote differentiation with respect to Z and dashes attached to $F(\zeta)$ denote differentiation with respect to ζ

Introducing two functions of ζ, $V(\zeta)$ and $W(\zeta)$ which are finite at infinity and are of the form (Green, 1945,)

$$V(\zeta) = -2g'(\zeta) - \frac{F'(\zeta)}{F'(\zeta)}\left\{f''(Z) + F(\zeta)g''(Z)\right\} \quad \ldots (4)$$

$$W(\zeta) = \frac{F'(\zeta)}{\bar{F}'(\zeta)}\left\{f''(Z) + F(\zeta)g''(Z)\right\}$$

so that at $\eta = 0$, the hole boundary, the real part of $V(\zeta) = -\eta\eta_e$ and the imaginary part of $w(\zeta) = \widehat{\xi\eta}_e$, where $\widehat{\xi\eta}_e$, $\widehat{\eta\eta}_e$ denote stresses at the edge of the hole, we get the stresses in terms of $V(\zeta)$ and $W(\zeta)$ (Green, 1948) as the real parts of

$$\widehat{\xi\xi} = -V(\zeta) - W(\zeta) - W(\zeta)\frac{F'(\zeta)}{F'(\zeta)} - \tfrac{1}{2}\left\{V'(\zeta) + W'(\zeta)\right\}\frac{F(\zeta) - \bar{F}(\zeta)}{\bar{F}'(\bar{\zeta})}$$

$$\widehat{\eta\eta} = -V(\zeta) - W(\zeta) + W(\zeta)\frac{\bar{F}'(\zeta)}{\bar{F}'(\zeta)} + \tfrac{1}{2}\left\{V'(\zeta) + W'(\zeta)\right\}\frac{F(\zeta) - \bar{F}(\zeta)}{\bar{F}'(\bar{\zeta})} \quad \ldots (5)$$

$$\widehat{\xi\eta} = -iW(\zeta)\frac{\bar{F}'(\zeta)}{\bar{F}'(\zeta)} - \tfrac{1}{2}i\left\{V'(\zeta) + W'(\zeta)\right\}\frac{\bar{F}(\zeta) - F(\bar{\zeta})}{F'(\zeta)}$$

where dashes attached to $V(\zeta)$ and $W(\zeta)$ denote differention with respect to ζ. From (5) we get the circumferential stress over the edge of the hole boundary given by the real part of

$$\widehat{\xi\xi}_e = -V(\zeta) - 2W(\zeta) \quad \ldots (6)$$

The solution of an individual problem depends on finding the suitable $V(\zeta)$ and $W'(\zeta)$.

THE SOLUTION

Let a bending moment M be applied to a plate beam of depth $2b$ and thickness $2c$. When there is no hole in the plate we may take

$$\chi = R[f(Z) + \bar{Z}g(Z)] = Ay^3 \qquad \dots (7)$$

where

$$A = \frac{M}{\cdot\cdot} \qquad \dots (8)$$

so that

$$f(Z) = \frac{Ai}{4}Z^3, \; g(Z) = -\frac{3Ai}{4}Z^2 \qquad \dots (9)$$

Then we shall have the stresses in the plate given by the real parts of

$$\begin{aligned} \widehat{\xi\xi} &= -3Ai\,F(\zeta) - \frac{3Ai}{2} \cdot \frac{F'(\zeta)}{\bar{F}'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \\ \widehat{\eta\eta} &= -3Ai\,F(\zeta) + \frac{3Ai}{2} \cdot \frac{F'(\zeta)}{\bar{F}'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \\ \widehat{\xi\eta} &= -\frac{3A}{2} \cdot \frac{F'(\zeta)}{F'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \end{aligned} \qquad \dots (10)$$

On the boundary $\eta = 0$, these stresses have values given by the real parts of

$$\begin{aligned} \widehat{\xi\xi}_0 &= -3AiF(\zeta) - \frac{3Ai}{2} \cdot \frac{F'(\zeta)}{F'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \\ \widehat{\eta\eta}_0 &= -3AiF(\zeta) + \frac{3Ai}{2} \cdot \frac{F'(\zeta)}{\bar{F}'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \\ \widehat{\xi\eta}_0 &= \frac{3A}{2} \cdot \frac{F'(\zeta)}{F'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \end{aligned} \qquad \dots (11)$$

Superposing on the stress system (10) another which gives $\widehat{\eta\eta}_e = -\widehat{\eta\eta}_0$ and $\widehat{\xi\eta}_e = -\widehat{\xi\eta}_0$ on the boundary $\eta = 0$ and which tends to zero at infinity, we shall get the stress system in the plate beam containing the stress free hole $\eta = 0$ under the uniform bending moment M.

To obtain the required superposed stress system we first take

$$V_1(\zeta) = -3AiF(\zeta) + \frac{3Ai}{2} \cdot \frac{F'(\zeta)}{\bar{F}'(\zeta)} \left\{ F(\zeta) - \bar{F}(\zeta) \right\} \qquad (12)$$

Its real part $= \widehat{\eta\eta}_0 = -\widehat{\eta\eta}_e$ on the boundary $\eta = 0$.

To make $V(\zeta)$ tend to zero when $\eta \to \infty$, we add the teams

$$3A(\bar{a}_0 e^{i\zeta} - a_0 e^{-i\zeta}) + \frac{3Ai}{2}\left\{\frac{a_0 e^{i\zeta}}{F'(\zeta)} - \frac{a_0 e^{-i\zeta}}{\bar{F}'(\zeta)}\right\}\left\{F(\zeta) - \bar{F}(\zeta)\right\} \qquad \dots (13)$$

which sum up to give an imaginary quantity at $\eta = 0$, so that on $\eta = 0$ the real part of $V(\zeta)$ is the same as that of $V_1(\zeta)$ and is therefore equal to $-\widehat{\eta\eta}_e$. In choosing the terms in (13) care is taken not to include any term which produces no stress either at infinity or over the hole boundary.

If we take

$$W_1(\zeta) = -\frac{3Ai}{2}\frac{F'(\zeta)}{\bar{F}'(\zeta)}\left\{F(\zeta) - \bar{F}(\zeta)\right\} \tag{14}$$

its imaginary part $=\widehat{\xi\eta}_e = -\widehat{\xi\eta}_0$ over the hole boundary.

To make $W(\zeta)$ tend to zero as η tends to infinity, we add the terms

$$3Ai\left\{\frac{a_0e^{-i\zeta}}{F'(\zeta)} + \frac{\bar{a}_0e^{i\zeta}}{F'(\zeta)}\right\}\left\{F(\zeta) - \bar{F}(\zeta)\right\} \tag{15}$$

which sum up to give a real quantity on $\eta=0$, so that on $\eta=0$ the imaginary part of $W(\zeta)$ is equal to that of $W_1(\zeta)$ and is therefore equal to $\widehat{\xi\eta}_e$.

We get from (12), (13), (14) and (15)

$$\begin{aligned} V(\zeta) &= 3A\{\bar{a}_0e^{i\zeta} - iF(\zeta)a_0e^{-i\zeta}\} \\ &\quad + \frac{3Ai}{2}\left\{\frac{F'(\zeta)}{\bar{F}'(\zeta)} - \frac{a_0e^{-i\zeta}}{\bar{F}'(\zeta)} + \frac{\bar{a}_0e^{i\zeta}}{\bar{F}'(\zeta)}\right\}\left\{F(\zeta) - \bar{F}(\zeta)\right\} \\ W(\rho) &= \frac{3Ai}{2}\left\{-\frac{F'(\zeta)}{\bar{F}'(\zeta)} + \frac{a_0e^{-i\zeta}}{F'(\zeta)} + \frac{\bar{a}_0e^{i\zeta}}{F'(\zeta)}\right\}\left\{F(\zeta) - \bar{F}(\zeta)\right\} \end{aligned} \tag{16}$$

$V(\zeta)$ and $W(\zeta)$ have finite values at infinity and their sum contains no poles.

The complete stress system is obtained from (5) and (16) together with the stresses (11) which are transmitted from infinity.

Calculating the stresses from (5) with the above values of $V(\zeta)$ and $W(\zeta)$ it is seen that the stresses tend to zero as $\eta \to \infty$ only when $n < \frac{3}{2}$. Therefore (16) can be used for $n=0$ or $n=1$. The circumferential stress over the edge of the hole, as calculated with the help of (6), is given by the real part of

$$\widehat{\xi\xi}_e = 6Ai\frac{\bar{a}_0e^{i\zeta}}{F'(\zeta)}\left\{\bar{F}(\zeta) - F(\zeta)\right\} \quad \ldots \tag{17}$$

The terms in $e^{(2-n)i\zeta}$ in $V(\zeta)$ and $W(\zeta)$ produce infinite stresses at $n > \frac{3}{2}$, so for the cases where $n \geq 2$ we subtract these terms $V(\zeta)$ and $W(\zeta)$ and add such new terms as to keep the stresses over the hole boundary unchanged. We get

$$\begin{aligned} V_n(\zeta) &= V(\zeta) - \frac{3A}{2}\,\frac{b_n}{n}e^{-in\zeta}\frac{\bar{a}_0e^{i\zeta}}{F'(\zeta)} + \frac{3A}{2}\,\frac{b_n}{n}e^{in\zeta}\frac{a_0e^{-i\zeta}}{\bar{F}'(\zeta)} \\ W_n(\zeta) &= W(\zeta) - \frac{3A}{2}\,\frac{b_n}{n}e^{-in\zeta}\frac{\bar{a}_0e^{i\zeta}}{F'(\zeta)} - \frac{3A}{2}\,\frac{b_n}{n}e^{in\zeta}\frac{a_0e^{-i\zeta}}{\bar{F}'(\zeta)} \end{aligned} \quad \ldots \tag{18}$$

$V_n(\zeta)$ and $W_n(\zeta)$ and the stresses produced by them tend to zero when $\eta \to \infty$ for $n > 0$. But the new terms in them produce no stress either at infinity or over the hole boundary for $n < \frac{3}{2}$. So the use of these functions $V_n(\zeta)$ and $W_n(\zeta)$ will be valid only for $n \geq 2$. In this case the circumferential stress round the edge of the hole is the real part of

$$\widehat{\xi\xi}_e = 6Ai\frac{a_0 e^{i\zeta}}{F'(\zeta)}\left\{F(\zeta) - F(\zeta) - \frac{ib_n}{n}e^{-in\zeta}\right\} \qquad (19)$$

APPLICATIONS

When $n=1$ and $b_n=0$, we have a circular hole of radius ia_0 when $\eta=0$. The circumferential stress round the edge of this hole, as calculated with the help of (17) is given by

$$\widehat{\xi\xi}_e = 12Aia_0 \sin\xi \cos 2\xi. \qquad \ldots \quad (20)$$

which is in agreement with the results obtained by other authors.

When $n=1$ and

$$a_0 = \frac{c}{2i}e^{\alpha+i\beta}, \quad b_n = -\frac{c}{2i}e^{-\alpha+i\beta}$$

we have an elliptic hole of semi-axes $c\cosh\alpha$ and $c\sinh\alpha$ with its major axis inclined at an angle β to the x-axis. The circumferential stress round the edge of this hole is obtained from (17) as

$$\widehat{\xi\xi}_e = 3Ac\frac{e^{\alpha}\sin(\xi-\beta) - e^{-\alpha}\sin(\xi+\beta)}{\cosh 2\alpha - \cos 2\xi}\left\{e^{2\alpha}\cos 2(\xi-\beta) - \cos 2\beta\right\} \qquad (21)$$

When $\beta=0$, it becomes

$$\widehat{\xi\xi}_e = 6Ac\frac{\sinh\alpha\sin\xi}{\cosh 2\alpha - \cos 2\xi}\left\{e^{2\alpha}\cos 2\xi - 1\right\} \qquad (22)$$

and when $\beta = \frac{\pi}{2}$, it becomes

$$\widehat{\xi\xi}_e = 6Ac\frac{\cosh\alpha\cos\xi}{\cosh 2\alpha - \cos 2\xi}\left\{e^{2\alpha}\cos 2\xi - 1\right\} \qquad (23)$$

These results also agree with those obtained by other authors.

When $n=2$ and $a_0=-2b_n$, $\eta=0$ represents approximately an equilateral triangular hole with rounded corners. The circumferential stress over the edge of this hole is calculated from (19) as

$$\widehat{\xi\xi}_e = \frac{3Ai\,a_0}{4\cos 3\xi - 5}(2\sin 4\xi - 8\sin 3\xi + 4\sin 2\xi - \sin\xi) \qquad (24)$$

When $n=3$ and $a_0=-3b_n$, $\eta=0$ represents approximately a square hole with rounded corners. The circumferential stress of this hole is given by

$$\widehat{\xi\xi}_e = \frac{Aia_0}{3\cos 4\xi - 5}(3\sin 5\xi - 18\sin 3\xi + 17\sin\xi) \qquad \ldots \quad (25)$$

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REFERENCE

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