ON THE CONCENTRATION OF STRESS ROUND THE EDGE OF A HOLE BOUNDED BY TWO INTERSECTING CIRCLES IN A LARGE PLATE

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ABSTRACT. A stress function in bipolar coordinates has been obtained to give the distribution of stress round the edge of a hole bounded by two intersecting circles in an infinite plate under uniform shear in the plane of the plate; and some particular cases have been discussed.

INTRODUCTION

In a recent paper, Ling (1948) has studied the concentration of stress in an infinite plate containing a hole bounded by two equal intersecting circles when the plate is under (*i*) a uniform all round tension, (*ii*) a uniform longitudinal tension, and (*iii*) a uniform transverse tension, at infinity. We shall here consider the effect of the hole on the stress distribution in the plate when it is under a uniform shear S at infinity.

Using bipolar coordinates

$$\alpha + i\beta = \log \frac{x + i(y+a)}{x + i(y-a)} \qquad \dots \quad (1)$$

we take the boundary of the hole to be given by two equal intersecting circles $\beta = \pm\beta_1$, where β_1 is a constant. The points of intersection of the two circles are given by $\alpha = \pm\infty$, and on each circle α varies from $-\infty$ to $+\infty$.

The stress function χ satisfies the equation (Jeffery, 1921)

$$\left(\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right)(h\chi) = 0 \qquad \dots \quad (2)$$

The boundary conditions of no stress over this new type of holes are established in terms of Michell's constants of the boundary, as has been done by Jeffery for circular holes given by α=constant.

THE STRESS FUNCTION

Let us choose a solution of (2) of the type

$$h\chi = f(\beta) \cos n\alpha, \quad f(\beta) \sin n\alpha \qquad \dots \quad (3)$$

Then from (2) we get the differential equation for $f(\beta)$ as

$$\left[\frac{\partial^4}{\partial\beta^4} - 2(n^2-1)\frac{\partial^2}{\partial\beta^2} + (n^2+1)^2\right] f(\beta) = 0 \qquad (4)$$

the solution of which is (Ling, 1918)

$$\left.\begin{aligned} f(\beta = K_n \cos\beta \cosh n\beta + L_n \cos\beta \sinh n\beta \\ + M_n \sin\beta \cosh n\beta + N_n \sin\beta \sinh n\beta \end{aligned}\right\} \qquad (5)$$

Hence we obtain an expression for $h\chi$ in the form

$$\begin{aligned} h\chi = \int^{\alpha} & \{(A_n \cosh n\beta + B_n \sinh n\beta)\cos\beta \\ & + (C_n \cosh n\beta + D_n \sinh n\beta)\sin\beta\} \cos n\alpha \\ & + \{(E_n \cosh n\beta + F_n \sinh n\beta)\cos\beta \\ & + (G_n \cosh n\beta + H_n \sinh n\beta)\sin\beta\} \sin n\alpha \; dn \end{aligned} \qquad (6)$$

CONDITIONS FOR NO STRESS OVER A BOUNDARY

We have the stresses in bipolar coordinates given by (Jeffery, 1921,)

$$\begin{aligned} a\widehat{\alpha\alpha} &= \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\beta^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha\right\}(h\chi) \\ a\widehat{\beta\beta} &= \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cos\beta\right\}(h\chi) \\ a\widehat{\alpha\beta} &= -(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha\partial\beta}(h\chi) \end{aligned} \quad \ldots \; (7)$$

If a boundary $\beta = \pm\beta_1$ has to be free from stress, $\widehat{\alpha\beta} = 0$ and $\widehat{\beta\beta} = 0$, and we have on the boundary

$$\frac{\partial^2(h\chi)}{\partial\alpha\delta\beta} = 0$$

therefore

$$\frac{\partial(h\lambda)}{\partial\beta} = \text{constant} = \rho(\text{say}) \quad \ldots \; (8)$$

Also on the boundary

$$(\cosh\alpha - \cos\beta)\frac{\partial^2(h\chi)}{\partial\alpha^2} - \sinh\alpha\frac{\partial(h\chi)}{\partial\alpha} + \cos\beta(h\chi) = \rho\sin\beta$$

the solution of which is

$$h\chi = \rho\tan\beta + \sigma(\cosh\alpha\cos\beta - 1) + \tau\sinh\alpha \quad \ldots \; (9)$$

ρ, σ and τ are Michell's three constants of the boundary.

THE SHEAR PROBLEM

Let the hole boundary be defined by $\beta = \pm\beta_1$, $\left(\beta_1 < \frac{\pi}{2}\right)$ so that if r be the radius of the intersecting circles which give the boundary of the hole and if $2d$ be the distance between the centres of the circles, we have

$$r = \alpha \operatorname{cosec} \beta_1, \quad d = \alpha \cot \beta_1$$
$$d/r = \cos \beta_1$$

If S be the applied shear in the plane of the plate, we may take the stress function at a great distance from the hole as

$$\chi_0 = -Sxy \qquad \dots \quad (10)$$

so that

$$h\chi_0 = -aS \frac{\sinh \alpha \sin \beta}{\cosh \alpha - \cos \beta}. \qquad \dots \quad (11)$$

To obtain the condition of no stress over the boundary $\beta = \pm\beta_1$ we need add another fnnction χ_1 to χ_0 such that χ_1 produces no stress at infinity $(\alpha = 0, \beta = 0)$ and the sum of the two stress functions $(\chi_0 + \chi_1)$ produces no stress over $\beta = \pm\beta_1$.

Since $h\chi_0$ is odd both in α and in β, we may take only the terms in F_n and G_n in (6) and write for our complete solution

$$\left.\begin{aligned} h\chi = & -aS \frac{\sinh \alpha \sin \beta}{\cosh \alpha - \cos \beta} \\ & + aS \int_0^\infty [F_n \cos \beta \sinh n\beta + G_n \sin \beta \cosh n\beta] \sin n\alpha . \, dn \end{aligned}\right] \qquad \dots \quad (12)$$

From (12) it is clear that at infinity $(\alpha = 0, \beta = 0)$, $h\chi = h\chi_0$.

We have (Haan, 1867)

$$\int_0^\infty \frac{\sinh \alpha \sin n\alpha \, d\alpha}{\cosh \alpha - \cos \beta} = \pi \frac{\cosh n(\pi - \beta)}{\sinh n\pi}, \quad -\pi \leq \beta \leq \pi \qquad \dots \quad (13)$$

whose Fourier Transform is

$$\frac{\sinh \alpha}{\cosh \alpha - \cos \beta} = 2 \int_0^\infty \frac{\cosh n(\pi - \beta) \sin n\alpha \, dn}{\sinh n\pi} \qquad \dots \quad (14)$$

Substituting this value in (12) we get after a little reduction

$$\begin{aligned} h\chi = aS \int_0^\infty \{ & F_n \cos \beta \sinh n\beta + (G_n - 2 \coth n\pi) \sin \beta \cosh n\beta \\ & + 2 \sin \beta \sinh n\beta \} \sin n\alpha \, dn. \end{aligned} \qquad \dots \quad (15)$$

We now apply the boundary conditions (8) and (9) to (15) to calculate tbe values of F_n and G_n for no stress over the boundary $\beta = \pm\beta_1$ and get

$$\left.\begin{aligned} F_n &= \frac{4n^2 \sin^2 \beta_1}{\sinh 2n\beta_1 - n \sin 2\beta_1} \\ G_n &= -\frac{4 \sinh^2 n\beta}{\sinh 2n\beta_1 - n \sin 2\beta_1} + 2 \coth n\pi \end{aligned}\right] \quad \ldots \quad (16)$$

Substituting these values in (15) we have the stress function

$$h\chi = 4aS \int_0^\infty \{n \sin \beta_1 \sinh n\beta \sin (\beta_1 - \beta)$$

$$- \sin \beta \sinh n\beta_1 \sinh n(\beta_1 - \beta)\} \frac{\sin n\alpha \, dn}{\sinh 2n\beta_1 - n \sin 2\beta_1} \quad \ldots \quad (17)$$

The circumferential stress $\widehat{\alpha\alpha}$ over the edge of the hole is easily calculated with the help of (7) after obtaining the simole form

$$a(\widehat{\alpha\alpha} - \widehat{\beta\beta}) = \cosh \alpha - \cos \beta)\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + 1\right)(h\chi) \quad \ldots \quad (18)$$

and putting $\widehat{\beta\beta} = 0$ for the boundary $\beta = \pm\beta_1$, we get

$$\widehat{\alpha\alpha} = 8S(\cosh \alpha - \cos \beta)$$

$$\times \int_0^\infty \frac{n \cos \beta_1 \sinh n\beta_1 - n^2 \sin \beta_1 \cosh n\beta_1}{\sinh 2n\beta_1 - n \sin 2\beta_1} \sin n\alpha \, dn \quad (19)$$

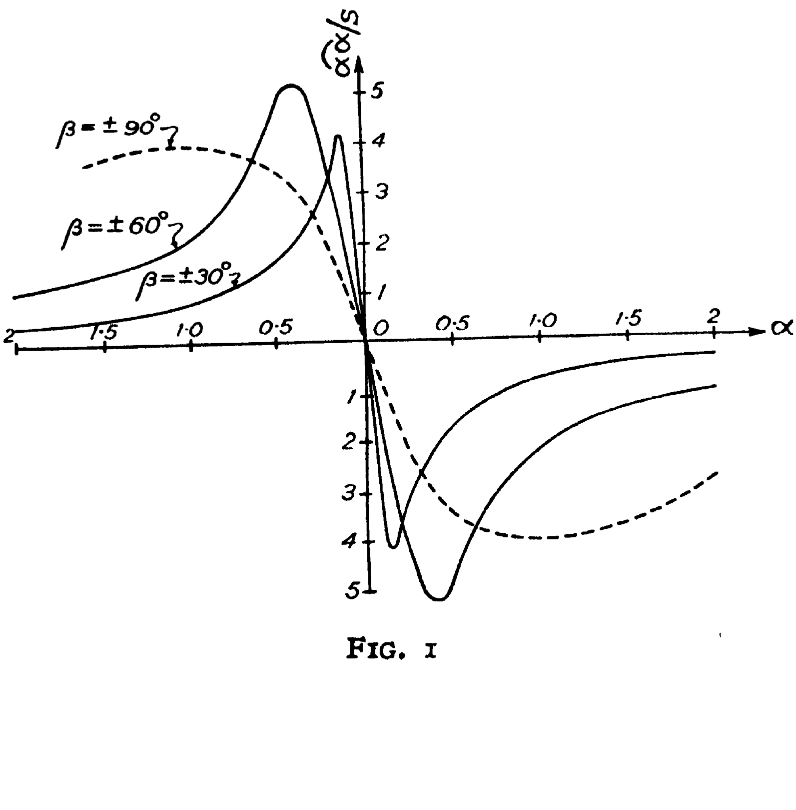

FIG. 1

In figure 1 we have plotted the graphs of edge stresses against different values of α for the cases where (*i*) $\beta=\pm 30^\circ$ when the distance between the centres of the intersecting circles is 1.732 times the radius of the circles, and (*ii*) $\beta=\pm 60^\circ$ when the distance between the centres of the intersecting circles is equal to the radius of the circles. The maximum stresses in these two cases are $\pm 4.2\,S$ and $\pm 5.3S$ at $\alpha=\pm 0.175$ and $\alpha=\pm 0.436$ respectively. We see as α tends to infinity the stresses tend to zero, which is obvious from the physical consideration that the points $\alpha=\pm\infty$ are projected outwards from the main body of the plate. The dotted graph in the figure shows the edge stress when $\beta=\pm 90^\circ$, *i.e.* the hole is one complete circle. The maximum stress in this case is equal to $4S$ which result is well known.

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