LIMIT OF INTERFERENCE IN OPTICAL INSTRUMENTS

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ABSTRACT. A rigorous theory has been given for the limit of interference in Lummer Gehrcke plate, transmission and reflection echelons and the grating on the lines of Saha's treatment for Fabry-Perot interferometer.

INTRODUCTION

The theory of the limit of interference in optical instruments has been worked out by Lippich (1870), Schönrock (1906) and Rayleigh (1915). They have shown that when the pressure is small, the critical distance D (or the limit of interference) is connected by the following formula with the wavelength λ of light, the temperature T of the tube and mass M of the emission centres.

$$\frac{D}{\lambda}=A\sqrt{\frac{M}{T}} \qquad \ldots \quad (1)$$

However, their value of A was deduced from approximate and not altogether satisfactory theoretical considerations. Saha (1917) gave a rigorous solution for the case of Fabry-Perot interferometer taking into account the infinite number of interfering beams and the effect of reflection. Gilchrist (1926) has discussed the case of Lummer Gehrcke plate but has neglected the term $4a^4 \sin^2 (N\delta/2)$, thereby getting the same result as Saha. In the present communication a rigorous theory has been developed for the cases of Lummer Gehrcke plate, transmission and reflection echelons and the grating.

DEDUCTION OF THE FORMULÆ

The general expression for intensity in case of the above mentioned instruments is given by

$$I=I_0\,\frac{(1-a^N)^2+4a^N \sin^2(N\delta/2)}{(1-a)^2+4a \sin^2(\delta/2)} \qquad \ldots \quad (2)$$

where
$$\delta=\frac{2\pi}{\lambda}\Delta$$

Δ is the path difference between two consecutive interfering beams.

N is the number of interfering beams.

and a is equal to

(*i*) $Re^{-k't \sec r}$ in case of Lummer Gehrcke Plate (Sodha, 1952).

R = reflection coefficient

k' = intensity absorption coefficient.

t = thickness of plate

r = angle of refraction.

(*ii*) $e^{-k't/2}$ in case of Transmission echelon (Sodha, 1953)

k' = intensity absorption coefficient

t = height of the step of the echelon

(*iii*) 1 in case of reflection echelon

(*iv*) 1 in case of grating.

The number of particles having their velocity between v and $v+dv$ is $Ae^{-\beta v^2}dv$, and the frequency of radiation emitted by these particles is $\nu(1+v/c)$ where ν is the frequency of light emitted by the particles at rest. In the expression for retardation in phase, we must, therefore, replace λ by $\lambda/\left(1+\frac{v}{c}\right)$ and write $\frac{2\pi\Delta}{\lambda}\left(1+\frac{v}{c}\right)$ in place of $\frac{2\pi\Delta}{\lambda}$.

The intensity of light emitted by molecules having their velocity between v and $v+dv$ is

$$dI = B.\frac{(1-a^N)^2+4a^N\sin^2\left\{\frac{N\delta}{2}(1+v/c)\right\}}{(1-a)^2+4a\sin^2\{(\delta/2)(1+v/c)\}}.e^{-\beta v^2}dv \quad \ldots \quad (3)$$

The total intensity

$$I = B\int_{-\infty}^{+\infty}\frac{(1-a^N)^2+4a^N\sin^2\left\{\frac{N\delta}{2}(1+v/c)\right\}}{(1-a)^2+4a\sin^2\{(\delta/2)(1+v/c)\}}.e^{-\beta v^2}dv \quad (4)$$

We have by trigonometric expansion

$$\frac{1-a^2}{(1-2a\cos\delta+a^2)} = 1+2a\cos\delta+2a^2\cos 2\delta+\ \ldots \quad \ldots \quad (5)$$

Further, we have

$$\int_{-\infty}^{+\infty}e^{-\beta v^2}dv\sin\frac{n\delta v}{c} = 0 \quad \ldots \quad (6)$$

$$\int_{-\infty}^{+\infty}e^{-\beta v^2}dv\cos\frac{n\delta v}{c} = \sqrt{\frac{\pi}{\beta}}e^{-\frac{1}{\beta}\left(\frac{n\delta}{c}\right)^2} \quad \ldots \quad (7)$$

The integral (4) can be broken up into two integrals.

$$\text{First integral} = B(1-a^N)^2\int_{-\infty}^{+\infty}\frac{e^{-\beta v^2}dv}{(1-a)^2+4a\sin^2\left\{\frac{\delta}{2}(1+v/c)\right\}}$$

$$= B(1-a^N)^2 \int_{-\infty}^{+\infty} \frac{e^{-\beta v^2}dv}{1-2a\cos\delta\left(1+\frac{v}{c}\right)+a^2}$$

$$= \frac{B(1-a^N)^2}{1-a^2}\sqrt{\frac{\pi}{\beta}}\left[1+2\sum a^n e^{-(1/\beta)(n\delta/c)^2}\cos n\delta\right] \tag{8}$$

by using (5), (6) and (7).

$$\text{Second integral} = 4Ba^N \int \frac{\sin^2 \frac{N\delta}{2}\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv}{1-2a\cos\delta\left(1+\frac{v}{c}\right)+a^2}$$

$$= \frac{4Ba^N}{1-a^2}\int\left\{1+2\sum a^n \cos n\delta\left(1+\frac{v}{c}\right)\right\}\sin^2\frac{N\delta}{2}\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv$$

$$= \frac{4Ba^N}{1-a^2}\left[\int \sin^2\frac{N\delta}{2}\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv\right.$$

$$\left.+2\int\sum a^n\cos n\delta\left(1+\frac{v}{c}\right)\sin^2\frac{N\delta}{2}\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv\right]$$

$$= \frac{4Ba^N}{1-a^2}\left[\frac{1}{2}\int\left\{1-\cos N\delta\left(1+\frac{v}{c}\right)\right\}e^{-\beta v^2}dv\right.$$

$$\left.+\sum a^n\int\cos n\delta\left(1+\frac{v}{c}\right)\left\{1-\cos N\delta\left(1+\frac{v}{c}\right)\right\}e^{-\beta v^2}dv\right]$$

$$= \frac{4Ba^N}{1-a^2}\left[\frac{1}{2}\int e^{-\beta v^2}dv - \frac{1}{2}\int\cos N\delta\cos\frac{N\delta v}{c}e^{-\beta v^2}dv\right.$$

$$+\frac{1}{2}\int\sin N\delta\sin\frac{N\delta v}{c}e^{-\beta v^2}dv + \sum a^n\int\cos n\delta\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv$$

$$\left.-\sum a^n\int\cos n\delta\left(1+\frac{v}{c}\right)\cos N\delta\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv\right]$$

$$= \frac{4Ba^N}{1-a^2}\left[\frac{1}{2}\sqrt{\frac{\pi}{\beta}} - \frac{1}{2}\cos N\delta.\sqrt{\frac{\pi}{\beta}}e^{-(1/\beta)(N\delta/c)^2} + \sum a^n\cos n\delta\sqrt{\frac{\pi}{\beta}}e^{-(1/\beta)(n\delta/c)^2}\right.$$

$$-\frac{1}{2}\sum a^n\int\cos(N+n)\delta\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv$$

$$\left.-\frac{1}{2}\sum a^n\int\cos(N-n)\delta\left(1+\frac{v}{c}\right)e^{-\beta v^2}dv\right].$$

$$=\frac{4Ba^N}{1-a^2}\left[\tfrac{1}{2}\sqrt{\frac{\pi}{\beta}}-\tfrac{1}{2}\sqrt{\frac{\pi}{\beta}}\cos N\delta\, e^{-(1/\beta)(N\delta/c)^2}+\sqrt{\frac{\pi}{\beta}}\Sigma a^n\cos n\delta\, e^{-(1/\beta)(n\delta/c)^2}\right.$$

$$-\tfrac{1}{2}\Sigma a^n\cos(N+n)\delta\sqrt{\frac{\pi}{\beta}}e^{-(1/\beta)\{(N+n)\delta/c\}^2}$$

$$\left.-\tfrac{1}{2}\Sigma a^n\cos(N-n)\delta\sqrt{\frac{\pi}{\beta}}\,e^{-(1/\beta)\{(N-n)\delta/c\}^2}\right] \quad \ldots (9)$$

Hence the total expression for intensity is

$$I=B.\frac{(1-a^N)^2}{1-a^2}\sqrt{\frac{\pi}{\beta}}\left[1+2\Sigma a^n e^{-(1/\beta)(n\delta/c)^2}\cos n\delta\right]$$

$$+\frac{2Ba^N}{1-a^2}\sqrt{\frac{\pi}{\beta}}\left[1-\cos N\delta\, e^{-(1/\beta)(N\delta/c)^2}+2\Sigma a^n\cos n\delta\, e^{-(1/\beta)(n\delta/c)^2}\right.$$

$$\left.-\Sigma a^n\cos(N+n)\delta\, e^{-(1/\beta)\{(N+n)\delta/c\}^2}-\Sigma a^n\cos(N-n)\delta\, e^{-(1/\beta)\{(N-n)\delta/c\}^2}\right] \quad \ldots (10)$$

I will be maximum, when $\frac{\delta}{2}=m\pi$ or $\delta=2m\pi$

I will be minimum, when $\frac{\delta}{2}=(2m+1)\ \frac{\pi}{2}$ or $\delta=(2m+1)\pi$

Putting $D=\frac{B(1-a^N)^2}{1-a^2}\sqrt{\frac{\pi}{\beta}}$ and $E=\frac{2Ba^N}{1-a^2}\sqrt{\frac{\pi}{\beta}}$ and neglecting higher terms in $ae^{-(1/\beta)(\delta/c)^2}$ and $e^{-(1/\beta)(\delta/c)^2}$ because $-(1/\beta)(\delta/c)^2 \sim 10^N$ and $a\leq 1$, we have

$$I_{max}=D[1+2ae^{-(1/\beta)(\delta/c)^2}]+E[1+2ae^{-(1/\beta)(\delta/c)^2}-a^{N-1}e^{-(1/\beta)(\delta/c)^2}$$
$$-a^N-a^{N+1}e^{-(1/\beta)(\delta/c)^2}]$$

$$I_{min}=D[1-2ae^{-(1/\beta)(\delta-c)^2}]+E[1-2ae^{-(1/\beta)(\delta/c)^2}+a^{N-1}e^{-(1/\beta)(\delta/c)^2}$$
$$-a^N+a^{N+1}e^{-(1/\beta)(\delta/c)^2}]$$

Hence the visibility

$$V=\frac{I_{max}-I_{min}}{I_{max}+I_{min}}$$

$$=\frac{D.4ae^{-(1/\beta)(\delta/c)^2}+E[4ae^{-(1/\beta)(\delta/c)^2}-2a^{N-1}e^{-(1/\beta)(\delta/c)^2}-2a^{N+1}e^{-(1/\beta)(\delta/c)^2}]}{2D+2E-2E.a^N}$$

$$=2ae^{-(1/\beta)(\delta/c)^2}\left[\frac{1-a^{2N-2}}{1-a^{2N}}\right] \quad \ldots (11)$$

Now $$\beta=\frac{m}{2kT}$$

m = weight of radiant atoms in gms.
k = Boltzmann constant
T = temperature in absolute scale.

Then we have

$$-\frac{1}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^{2} = \log_e \frac{V}{2a}\left\{\frac{1-a^{2N}}{1-a^{2N-2}}\right.$$

or

$$\frac{\Delta}{\lambda} = \frac{c}{2\pi}\sqrt{\frac{m}{2kT}\log_e \frac{2a}{V}\left(\frac{1-a^{2N-2}}{1-a^{2N}}\right)} \qquad \ldots \quad (12)$$

(*i*) Lummer Gehrcke plate: Putting $a = Re^{-k't \sec r}$ in expression (12), we get :

$$\frac{\Delta}{\lambda} = \frac{c}{2\pi}\sqrt{\frac{m}{2kT}\log_e \frac{2Re^{-k't \sec r}}{V}\left[\frac{1-R^{2N-2}e^{-(2N-2)k't \sec r}}{1-R^{2N}e^{-2Nk't \sec r}}\right]} \qquad (13)$$

(*ii*) Transmission echelon: $a = e^{-k't/2}$

$$\frac{\Delta}{\lambda} = \frac{c}{2\pi}\sqrt{\frac{m}{2kT}\log_e \frac{2e^{-k't/2}}{V}\left[\frac{1-e^{-(N-1)k't}}{1-e^{-Nk't}}\right]} \qquad (14)$$

(*iii*) Reflection echelon:

$$\frac{\Delta}{\lambda} = \frac{c}{2\pi}\sqrt{\frac{m}{2kT}\log_e \frac{2(N-1)}{N}} \qquad (15)$$

(*iv*) For grating also expression (15) holds.

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