ON THE DISTRIBUTION OF STRESS IN A DEEP BEAM CONTAINING TWO EQUAL CIRCULAR HOLES*

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ABSTRACT. The influence of two equal circular holes, placed symmetrically above and below the neutral axis of a deep plate beam on the distribution of stress in it, has been discussed. The values of the circumferential stress over the boundaries of the holes, $\alpha=0.8$ and $\alpha=-0.8$ are shown in a graph and discussed.

INTRODUCTION

The solution to the problem of stress distribution in a deep plate beam containing two equal circular holes placed symmetrically above and below the neutral axis is here obtained in bipolar co-ordinates as introduced by Jeffery (1921). The corresponding problem, when the centres of the holes lie on the neutral axis, has been solved by Sengupta (1952). In this paper the notations of the co-ordinates etc. are kept the same as of Jeffery and the following equations, as given by him, are made use of.

$$h\chi=\{B_0\alpha+K\log(\cosh\alpha-\cos\beta)\}(\cosh\alpha-\cos\beta) + \sum_{n=1}^{\infty}\left\{\phi_n(\alpha)\cos n\beta+\psi_n(\alpha)\sin n\beta\right\} \quad \ldots (1)$$

where

$$\left.\begin{aligned}\phi_1(\alpha)&=A_1\cosh 2\alpha+B_1+C_1\sinh 2\alpha\\ \psi_1(\alpha)&=A_1'\cosh 2\alpha+C_1'\sinh 2\alpha\end{aligned}\right\} \quad \ldots (2)$$

and for $n\geq 2$

$$\left.\begin{aligned}\phi_n(\alpha)&=A_n\cosh(n+1)\alpha+B_n\cosh(n-1)\alpha\\ &\quad+C_n\sinh(n+1)\alpha+D_n\sinh(n-1)\alpha\\ \psi_n(\alpha)&=A_n'\cosh(n+1)\alpha+B_n'\cosh(n-1)\alpha\\ &\quad+C_n'\sinh(n+1)\alpha+D_n'\sinh(n-1)\alpha\end{aligned}\right\} \quad \ldots (3)$$

$$a(\widehat{\alpha\alpha}-\widehat{\beta\beta})=(\cosh\alpha-\cos\beta)\left\{\frac{\partial^2}{\partial\beta^2}-\frac{\partial^2}{\partial\alpha^2}+1\right\}(h\chi) \quad \ldots (4)$$

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THE SOLUTION

Let $2b$ be the depth of the beam and $2c$ be its thickness. Using bipolor co-ordinates, let $\alpha=0$ *i.e.*, the x-axis of the relevent Cartesian co-ordinates be the neutral axis and $\alpha=\alpha_1$ and $\alpha=-\alpha_1$, while $\alpha_1>0$, be the boundaries of the two equal circular holes. If r be the radius and d be the distance of the centre of the holes from the neutral axis, we have,

$$r=a\ \mathrm{cosech}\ \alpha_1 \qquad\qquad d=a\coth\alpha_1$$
$$d/r=\cosh\alpha_1$$

Let M be the applied bending moment. Then at a great distance from the holes the stress function may be taken as

$$\chi_0=Ay^3$$

where

$$A=\frac{M}{8b^3c}$$

Transforming in bipolar co-ordinates we obtain,

$$h\chi_0=Aa^2\frac{\sinh^3\alpha}{(\cosh\alpha-\cos\beta)^2} \qquad \dots \quad (5)$$

When $\alpha>0$

$$h\chi_0=Aa^2\left\{\cosh\alpha+2\sum_{n=1}^{\infty}(n\sinh\alpha+\cosh\alpha)e^{-n\alpha}\cos n\beta\right\} \qquad \dots \quad (6)$$

and when $\alpha<0$

$$h\chi_0=Aa^2\left\{-\cosh\alpha+2\sum_{n=1}^{\infty}\left(n\sinh\alpha-\cosh\alpha\right)e^{n\alpha}\cos n\beta\right\} \qquad \dots \quad (7)$$

We have to add to $h\chi_0$ another stress function $h\chi_1$ such that $h\chi_1$ gives no stress at infinity ($\alpha=0$, $\beta=0$) and the complete stress function $(h\chi_0+h\chi_1)$ gives no stress over the boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_1$.

Obviously the required stress function is odd in α and even in β. Hence we may omit the terms in even functions of α and odd functions of β in the general solution of $h\chi$ and choose

$$h\chi_1=Aa^2\Big[B_0\alpha(\cosh\alpha-\cos\beta)+C_1\sinh2\alpha\cos\beta$$
$$+\sum_{n=2}^{\infty}\left\{C_n\sinh(n+1)\alpha+D_n\sinh(n-1)\alpha\right\}\cos n\beta\Big] \qquad \dots \quad (8)$$

It is clear that at infinity ($\alpha=0$, $\beta=0$) $h\chi_1$ vanishes and $h\chi_0$ becomes equal to the complete stress function.

Using the boundary conditions for no stress (Jeffery, 1921) over the boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_1$ seperately, we obtain

$$B_0=-\frac{6\cosh2\alpha_1}{\sinh2\alpha_1(\cosh2\alpha_1-1)}, \quad C_1=\frac{3}{\sinh2\alpha_1(\cosh2\alpha_1-1)}$$

$$C_n=\frac{2(n-1)e^{-n\alpha_1}[n\sinh\alpha_1e^{(n-1)\alpha_1}+\cosh n\alpha_1]}{\sinh2n\alpha_1-n\sinh2\alpha_1}$$

$$D_n=-\frac{2(n+1)e^{-n\alpha_1}[n\sinh\alpha_1e^{(n+1)\alpha_1}+\cosh n\alpha_1]}{\sinh2n\alpha_1-n\sinh2\alpha_1}$$

Substituting these values in (8), the stresses $\beta\beta_1$ and $\beta\beta_{-1}$ over the circular boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_1$ are calculated from the sum of the stress functions $h\chi_1$ and the respective $h\chi_0$ as

$$\widehat{\beta\beta}_1 = -\widehat{\beta\beta}_{-1} = Aa(\cosh\alpha_1-\cos\beta)$$

$$\times\left(\frac{6\cosh2\alpha_1}{\cosh\alpha_1(\cosh2\alpha_1-1)}+\frac{12\cos\beta}{\cosh2\alpha_1-1}+8\sum_{n=2}^{\infty}M_n\cos n\beta\right) \quad \ldots \quad (9)$$

where

$$M_n=\frac{n(n^2-1)\sinh\alpha_1\cosh n\alpha_1}{\sinh2n\alpha_1-n\sinh2\alpha_1} \quad (10)$$

The series in (9) converges only slowly, unless α_1 is large, so for convenience in numerical calculations the more slowly converging part in it is separated out by setting

$$M_n=n(n^2-1)\sinh\alpha_1 e^{-n\alpha_1}+N_n \quad \ldots \quad (11)$$

We can readily obtain

$$8\sum_{n=2}^{\infty}n(n^2-1)\sinh\alpha_1 e^{-n\alpha_1}\cos n\beta=\frac{12\{(1-\cosh\alpha_1\cos\beta)^2-\sinh^2\alpha_1\sin^2\beta\}\sinh\alpha_1}{(\cosh\alpha_1-\cos\beta)^4}$$

Substituting in (9) we have

$$\widehat{\beta\beta}_1=-\widehat{\beta\beta}_{-1}=Aa\left[\frac{12\sinh\alpha_1}{(\cosh\alpha-\cos\beta)^3}\left\{(1-\cosh\alpha_1\cos\beta)^2-\sinh^2\alpha_1\sin^2\beta\right\}\right.$$

$$+\frac{\cosh\alpha_1-\cos\beta}{\cosh\alpha_1(\cosh2\alpha_1-1)}(6\cosh2\alpha_1+12\cosh\alpha_1\cos\beta)$$

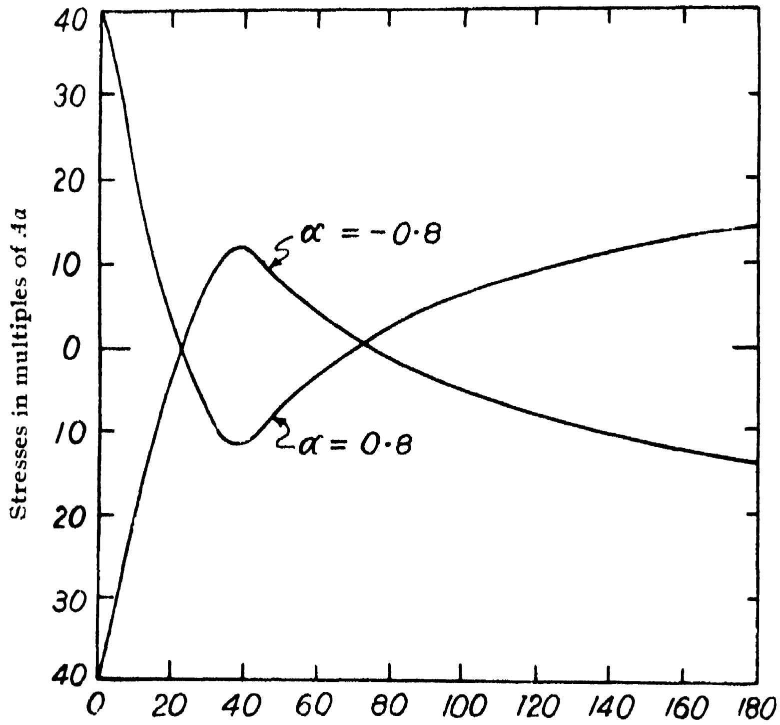
$$\left.+8\sum_{n=2}^{\infty}(\cosh\alpha_1-\cos\beta)N_n\cos n\beta\right] \quad \ldots \quad (12)$$

The numerical values of the coefficients N_n are given in Table I.

TABLE I

α_1	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
N_2	1.677	0.755	0.370	0.188	0.099	0.053	0.028	0.015	0.001
N_3	0.921	0.274	0.082	0.020	0.008	0. 02	0.001		
N_4	0.416	0.072	0.013	0.002	0.001				
N_5	0.159	0.016	0.001						
N_6	0.048	0.003							
N_7	0.015								
N_8	0.004								

The graphs of the stresses on the boundaries are plotted in figure 1 for a case in which the holes are fairly close to the neutral axis, $\alpha = \pm 0.8$ for which the shortest distance between the boundaries of the holes and the neutral axis is approximately one-third of the radius of the circle. The maximum stresses are, as expected, on the points furthest from the neutral axis. But they quickly fall to zero at $\beta = \pm 24°$ and change sign there. They again change their signs at $\beta = \pm 77°$



Positive or negative values of β

FIG. 1

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REFERENCES

Jeffery, G. B., 1921, *Phil. Trans. Roy. Soc.*, *A*, **221**, 265.
Sengupta, A. M., 1952, *Proc. 39th Ind. Sc. Cong.*, Part **III**, 11.