ON THE DISTRIBUTION OF INITIAL STRESS DUE TO DISLOCATION IN AN INFINITE PLATE CONTAINING TWO UNEQUAL CIRCULAR HOLES*

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ABSTRACT. Solutions are obtained for two problems on state of initial stress due to dislocation in an infinite plate containing two unstressed unequal circular holes. The dislocations are due to (1) a parallel fissure joining the two circular holes or joining each hole to infinity and (2) two opposite kinds of wedge shaped fissures of the same apex-angles, on two sides of the x-axis. Cases of equal holes in each of these problems are treated as particular examples and tables for numerical values of the constant coefficients when the hole boundaries are $\alpha=0.8$ and $\alpha=-0.8$ are given. Graphs showing the stresses are drawn and discussed.

INTRODUCTION

Let an elastic body, occupying a multiply connected region, be cut along a system of barriers which would make the region occupied by the body simply connected and let the two faces of each barrier be rejoined after removal or insertion of thin slices of the same material. Then the state of initial stress of the body is given by a solution of equations of elastic equilibrium which gives rise to multiple valued displacements, the displacements on one side of the barrier relative to the other being possible in a rigid body. In the present paper solutions are given of two problems of dislocation in an infinite elastic plate containing two unequal circular holes. In the first, dislocation is due to a parallel fissure joining the two circular holes or joining each hole to infinity, while in the second, dislocation is due to a wedge shaped fissure on one side of the x-axis with its apex at the origin and insertion of a similar wedge shaped strip below the x-axis. The solutions are obtained in bipolar co-ordinates, defined by the substitution (Jeffery, 1921)

$$\alpha+i\beta=\log\frac{x+i(y+a)}{x+i(y-a)}$$

Here

$$x=\frac{a\sin\beta}{\cosh\alpha-\cos\beta},\quad y=\frac{a\sinh\alpha}{\cosh\alpha-\cos\beta}$$

and

$$h=\frac{d(\alpha+i\beta)}{d(x+iy)}=\cosh\alpha-\cos\beta$$

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In terms of the stress function χ, the displacements are given by

$$2\mu u = \frac{\mu}{\lambda+\mu} \cdot h. \frac{\partial \chi}{\partial \alpha} - h \frac{\partial Q}{\partial \beta} \qquad \ldots \quad (1)$$

$$2\mu v = \frac{\mu}{\lambda+\mu} h \frac{\partial \chi}{\partial \beta} + h \frac{\partial Q}{\partial \alpha}$$

where

$$hQ = \frac{\lambda+2\mu}{2(\lambda+\mu)} \iint \left\{ \frac{\partial^2 (h\chi)}{\partial \alpha^2} - \frac{\partial^2 (h\chi)}{\partial \beta^2} - h\chi \right\} d\alpha . d\beta \qquad .. \quad (2)$$

and the stresses are given by

$$a\alpha\alpha = \left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \beta^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha \right\} (h\chi)$$

$$\left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cos \beta \right\} (h\chi) \qquad \ldots \quad (3)$$

$$a\overline{\alpha\beta} = -(\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha \partial \beta} (h\chi)$$

DISLOCATION DUE TO A PARALLEL FISSURE

Let $\alpha = \alpha_1$ and $\alpha = -\alpha_2$ be the boundaries of the two unequal circular holes in an infinite plate.

To obtain many valued terms in expressions for displacements, let us choose a stress function

$$h\chi_0 = A\alpha \sinh \alpha \qquad (4)$$

Writing only the many valued terms in the displacements, we have

$$u = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)} \cdot \frac{A\beta \sinh \alpha \sin \beta}{\cosh \alpha - \cos \beta}$$

$$v = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)} \cdot \frac{A\beta(1 - \cosh \alpha \cos \beta)}{\cosh \alpha - \cos \beta} \qquad (5)$$

when $\beta = \pi$

$$u_+ = 0 \; ; \; v_+ = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)} \cdot \pi A \qquad (6)$$

and when $\beta = -\pi$

$$u_- = 0 \; ; \; v_- = -\frac{\lambda-2\mu}{2\mu(\lambda+\mu)} \cdot \pi A \qquad (7)$$

Thus u remains continuous across the barrier $\beta = \pm\pi$, *i.e.*, the y-axis and v suddenly decreases by the constant amount

$$\frac{\lambda+2\mu}{\mu(\lambda+\mu)} \cdot \pi A$$

Therefore, the chosen stress function $h\chi_0$ will suit the state of dislocation due to a parallel fissure of that thickness along the line joining the centres of the two holes or any other line parallel to this line cutting both the circles $\alpha=\alpha_1$ and $\alpha=-\alpha_2$.

The stress function $h\chi_0$ produces normal stresses over the circular boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_2$ and they are given by

$$a\,\widehat{\alpha\alpha}_1=-A\sinh^2\alpha_1\,;\qquad a\,\widehat{\alpha\alpha}_{-2}=-A\sinh^2\alpha_2 \tag{8}$$

To reduce these stresses to zero, for the case of stress-free boundaries, add a stress function $h\chi_1$ to $h\chi_0$, where

$$h\chi_1=B_{01}\alpha(\cosh\alpha-\cos\beta)+(A_{11}\cosh 2\alpha+B_{11}+C_{11}\sinh 2\alpha)\cos\beta \quad \ldots \tag{9}$$

and it gives stresses over the circular boundaries, equal and opposite to those given by $h\chi_0$, *i.e.*

$$\widehat{\alpha\alpha}_1=\frac{A}{a}\sinh^2\alpha_1\,;\quad \widehat{\alpha\alpha}_{-2}\qquad\sinh^2\alpha_2 \tag{10}$$

From the solution given by Jeffery (1921) for an eccentrically bored pipe under constant pressure over its boundaries, where he used the same stress function as $h\chi_1$, we obtain by suitable substitutions

$$\left.\begin{aligned}
B_{01}&=2AM(\sinh^2\alpha_2-\sinh^2\alpha_1)\cosh(\alpha_1+\alpha_2)\\
A_{11}&=-AM(\sinh^2\alpha_2-\sinh^2\alpha_1)\sinh(\alpha_1-\alpha_2)\\
C_{11}&=AM(\sinh^2\alpha_2-\sinh^2\alpha_1)\cosh(\alpha_1-\alpha_2)\\
B_{11}&=AM\{\sinh^2\alpha_1\cosh(\alpha_1+\alpha_2)\sinh 2\alpha_2\\
&\qquad+\sinh^2\alpha_2\cosh(\alpha_1+\alpha_2)\sinh 2\alpha_1\\
&\qquad-(\sinh^2\alpha_2+\sinh^2\alpha_1)\sinh(\alpha_1+\alpha_2)\}
\end{aligned}\right\} \tag{11}$$

where

$$M=\tfrac{1}{2}\operatorname{cosech}(\alpha_1+\alpha_2)\{\sinh^2\alpha_2+\sinh^2\alpha_1\}^{-1} \tag{12}$$

From the sum of the stress functions $h\chi_0$ and $h\chi_1$, we get no stress over the boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_2$, but a constant all round stress is obtained at infinity ($\alpha=0$, $\beta=0$). The value of this stress is given by

$$\begin{aligned}a\,\widehat{\alpha\alpha}=a\,\widehat{\beta\beta}&=AM\sinh\alpha_1\sinh\alpha_2\{\sinh 2(\alpha_1+\alpha_2)-\sinh 2\alpha_1-\sinh 2\alpha_2\}\\&=\text{constant}=S\ \text{(say)}\end{aligned} \quad \ldots \tag{13}$$

For the complete solution of the problem we must add to $(h\chi_0+h\chi_1)$ another stress function $h\chi_2$ such that $h\chi_2$ produces stresses equal and opposite to (S/a) at infinity and no stress over the boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_2$. To get the required all round constant stress at a great distance from the holes, we may choose

$$h\chi_{20}=-\tfrac{1}{2}S(\cosh\alpha+\cos\beta) \quad \ldots \tag{14}$$

and add to it another stress function

$$h\chi_{21}=\tfrac{1}{2}S\left[\left\{B_0\alpha+K\log(\cosh\alpha-\cos\beta)\right\}(\cosh\alpha-\cos\beta)+\sum_{n=1}^{\infty}\phi_n(\alpha)\cos n\beta\right] \tag{15}$$

where

$$\phi_n(\alpha) = A_n \cosh (n+1)\alpha + B_n \cosh (n-1)\alpha + C_n \sinh (n+1)\alpha + D_n \sinh (n-1)\alpha$$

for $n \geq 2$, and ... (16)

$$\phi_1(\alpha) = A_1 \cosh 2\alpha + B_1 + C_1 \sinh 2\alpha$$

$h\chi_{21}$ is such that it produces no stress at infinity and the sum $(h\chi_{20} + h\chi_{21}) = h\chi_2$ produces no stress over the holes $\alpha = \alpha_1$ and $\alpha = -\alpha_2$.

We get, when $\alpha > 0$,

$$h\chi_2 = -\tfrac{1}{2}S\Big[\cosh \alpha + (B_0 + K)\alpha \cosh \alpha - K \log 2 \cosh \alpha + Ke$$
$$+ \Big\{1 - (B_0 + K)\alpha + K \log 2 - 2K \cosh \alpha e^{-\alpha} + \frac{K}{2} e^{-2\alpha} + \phi_1(\alpha)\Big\} \cos \beta$$
$$+ K \sum_{n=2}^{\infty} \frac{1}{n(n^2-1)} \Big\{(n+1)e^{-(n-1)\alpha} - (n-1)e^{-(n+1)\alpha}\Big\} \cos n\beta \qquad (17)$$
$$\sum_{n=2}^{\infty} \phi_n(\alpha) \cos n\beta\Big]$$

and when $\alpha < 0$,

$$h\chi_2 = -\tfrac{1}{2}S\Big[\cosh \alpha + (B_0 - K)\alpha \cosh \alpha - K \log 2 \cosh \alpha + Ke^{\alpha}$$
$$+ \Big\{1 - (B_0 - K)\alpha + K \log 2 - 2K \cosh \alpha e^{\alpha} + \frac{K}{2} e^{2\alpha} + \phi_1(\alpha)\Big\} \cos \beta$$
$$+ K \sum_{n=2}^{\infty} \frac{1}{n(n^2-1)} \Big\{(n+1)e^{(n-1)\alpha} - (n-1)e^{(n+1)\alpha}\Big\} \cos n\beta \qquad (18)$$
$$+ \sum_{n=2}^{\infty} \phi_n(\alpha) \cos n\beta\Big]$$

To calculate the values of the constant coefficients, the boundary conditions for no stress (Jeffery, 1921),

$$\frac{\partial}{\partial\alpha}(h\chi) = \text{constant} = p$$

and ... (19)

$$h\chi = p \tanh \alpha + \sigma(\cosh \alpha \cos \beta - 1) + \tau \sin \beta$$

are applied separately on equations (17) and (18) for the boundaries $\alpha = \alpha_1$ and $\alpha = -\alpha_2$ respectively. We obtain

$$B_0 = KN_1\{\cosh 2(\alpha_1 + \alpha_2) - 1\}(\cosh 2\alpha_2 - \cosh 2\alpha_1)$$
$$A_1 = -KN_1(\cosh 2\alpha_1 - 1)(\cosh 2\alpha_2 - 1)(\sinh 2\alpha_2 + \sinh 2\alpha_1) + K/2$$
$$B_1 = KN_1(\cosh 2\alpha_1 - 1)(\cosh 2\alpha_2 - 1) \sinh 2(\alpha_1 + \alpha_2) - 2 \qquad \ldots (20)$$
$$C_1 = -\tfrac{1}{2}KN_1[(\cosh 2\alpha_2 - 1)(\cosh 2\alpha_1 - \cosh 2\alpha_2 - \sinh 2\alpha_1 \sinh 2\alpha_2 - \sinh^2 2\alpha_1) + (\cosh 2\alpha_1 - 1)(\cosh 2\alpha_1 - \cosh 2\alpha_2 + \sinh 2\alpha_1 \sinh 2\alpha_2 + \sinh^2 2\alpha_1)]$$

where

$$N_1=\{\cosh 2(\alpha_1+\alpha_2)-1\}^{-1}(\cosh 2\alpha_1+\cosh 2\alpha_2-2)^{-1} \qquad \dots \quad (21)$$

and for $n \geq 2$,

$$A_n=\frac{KN}{n(n+1)}\Big[(n\sinh 2\alpha_2-\sinh 2n\alpha_2)\{-(n+1)+n\cosh 2\alpha_1+\cosh 2n\alpha_1\}$$
$$+(n\sinh 2\alpha_1-\sinh 2n\alpha_1)\{-(n+1)+n\cosh 2\alpha_2+\cosh 2n\alpha_2\}+N^{-1}\Big]$$

$$B_n=-\frac{KN}{n(n-1)}\Big[(n\sinh 2\alpha_2-\sinh 2n\alpha_2)\{(n-1)-n\cosh 2\alpha_1+\cosh 2n\alpha_1\}$$
$$+(n\sinh 2\alpha_1-\sinh 2n\alpha_1)\{(n-1)-n\cosh 2\alpha_2+\cosh 2n\alpha_2\}+N^{-1}\Big] \qquad (22)$$

$$C_n=\frac{KN}{n(n+1)}\Big[n\{\cosh 2(n\alpha_1-\alpha_2)-\cosh 2(n\alpha_2-\alpha_1)\}$$
$$+(n+1)\{n(\cosh 2\alpha_1-\cosh 2\alpha_2)-(\cosh 2n\alpha_1-\cosh 2n\alpha_2)\}\Big]$$

$$D_n=\frac{KN}{n(n-1)}\Big[n\{\cosh 2(n\alpha_1+\alpha_2)-\cosh 2(n\alpha_2+\alpha_1)\}$$
$$-(n-1)\{n(\cosh 2\alpha_1-\cosh 2\alpha_2)+(\cosh 2n\alpha_1-\cosh 2n\alpha_2)\}\Big]$$

where

$$N=\{\cosh 2n(\alpha_1+\alpha_2)-n^2\cosh 2(\alpha_1+\alpha_2)+(n^2-1)\}^{-1} \qquad \dots \quad (23)$$

To obtain the values of K, we have

$$\sum_{n=1}^{\infty}\phi_n(0)=0 \qquad \dots \quad (24)$$

as $h\chi_1$ is to produce no stress at infinity ($\alpha=0$, $\beta=0$).
Putting

$$A_1=K/2+Ka_1 \qquad B_1=-2+Kb_1$$

and for $n\geq 2$

$$A_n=\frac{K}{n(n+1)}+Ka_n \qquad B_n=-\frac{K}{n(n-1)}+Kb_n$$

we get from (24)

$$K(a_1+b_1)+K\sum_{n=2}^{\infty}(a_n+b_n)=2 \qquad \dots \quad (25)$$

a_1, b_1, a_n, b_n being known from (22), the values of K can be calculated.

Therefore, the complete solution for the state of initial stress caused by dislocation due to a parallel fissure between the unequal circular holes is given by the stress function

$$
\begin{aligned}
h\chi &= h\chi_0 + h\chi_1 + h\chi_2 \\
&= A\alpha \sinh \alpha + B_{01}\alpha(\cosh \alpha - \cos \beta) \\
&\quad + (A_{11} \cosh 2\alpha + B_{11} + C_{11} \sinh 2\alpha) \cos \beta \\
&\quad - \tfrac{1}{2}S\Big[\cosh \alpha + \cos \beta + \{B_0\alpha + K \log (\cosh \alpha - \cos \beta)\}(\cosh \alpha - \cos \beta \\
&\qquad + \sum_{n=1}^{\infty} \phi_n(\alpha) \cos n\beta\Big]
\end{aligned} \tag{26}
$$

The circumferential stress $\widehat{\beta\beta}$ over the hole boundaries $\alpha = \alpha_1$ and $\alpha = -\alpha_2$ are calculated from the complete stress function $h\chi$ as

$$
\begin{aligned}
a\widehat{\beta\beta}_1 &= 2A \cosh \alpha_1(\cosh \alpha_1 - \cos \beta) \\
&\quad + JAM(\cosh \alpha_1 - \cos \beta)(\sinh^2\alpha_2 - \sinh^2 \alpha_1) \\
&\quad \times \{\sinh (\alpha_1 + \alpha_1) \cos \beta + \sinh \alpha_1 \cosh (\alpha_1 + \alpha_2)\} \\
&\quad - \tfrac{1}{2}S(\cosh \alpha_1 - \cos \beta)\Big[2K e^{-\alpha_1} + 2(B_0 + K) \sinh \alpha_1 \\
&\quad + \phi_1''(\alpha_1) \cos \beta - 2K \sin^2\beta(\cosh \alpha_1 - \cos \beta)^{-1} \\
&\quad + \sum_{n=2}^{\infty} \{\phi_n''(\alpha_1) + n^2\phi_n(\alpha_1) - \phi_n(\alpha_1)\} \cos n\beta \Big]
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
\widehat{\beta\beta}_2 &= 2A \cosh \alpha_2(\cosh \alpha_2 - \cos \beta) \\
&\quad - JAM(\cosh \alpha_2 - \cos \beta)(\sinh^2\alpha_2 - \sinh^2\alpha_1) \\
&\quad \times \{\sinh (\alpha_1 + \alpha_2) \cos \beta + \sinh \alpha_2 \cosh (\alpha_1 + \alpha_2)\} \\
&\quad - \tfrac{1}{2}S(\cosh \alpha_1 - \cos \beta)\Big[2Ke^{-\alpha_2} - 2(B_0 - K)\sinh \alpha_2 \\
&\quad + \phi_1''(-\alpha_2)\cos \beta - 2K \sin^2\beta(\cosh \alpha_2 - \cos \beta)^{-1} \\
&\quad + \sum_{n=2}^{\infty} \{\phi_n''(-\alpha_2) + n^2\phi_n(-\alpha_2) - \phi_n(-\alpha_2)\} \cos n\beta \Big]
\end{aligned} \tag{28}
$$

If the two holes be equal, *i.e.* $\alpha = \alpha_1$ and $\alpha = -\alpha_1$, we have

$$
\begin{aligned}
h\chi &= A\alpha \sinh \alpha + B_{11} \cos \beta \\
&\quad - \tfrac{1}{2}S\Big[\cosh \alpha + \cos \beta + K(\cosh \alpha - \cos \beta)\log(\cosh \alpha - \cos \beta) \\
&\quad + \sum_{n=1}^{\infty} \phi_n(\alpha) \cos n\beta\Big]
\end{aligned} \tag{29}
$$

where

$$\phi_n(\alpha) = A_n \cosh (n+1)\alpha + B_n \cosh (n-1)\alpha$$

for $n \geq 2$, and

$$\phi_1(\alpha) = A_1 \cosh 2\alpha + B_1 \tag{30}$$

and

$$S = A \sinh^2 \alpha_1$$

The values of the constant coefficients are

$$B_{11}=A\sinh^2\alpha_1$$
$$A_1=\tfrac{1}{2}K(1-\tanh\alpha_1)$$
$$B_1=\tfrac{1}{2}K\tanh\alpha_1\cosh 2\alpha_1-2$$

and for $n\geq 2$,

$$A_n=\frac{K}{n(n+1)}\left[1-\frac{n\cosh 2\alpha_1+\cosh 2n\alpha_1-(n+1)}{\sinh 2n\alpha_1+n\sinh 2\alpha_1}\right]$$

$$B_n=\frac{-K}{n(n-1)}\left[1+\frac{n\cosh 2\alpha_1-\cosh 2n\alpha_1-(n-1)}{\sinh 2n\alpha_1+n\sinh 2\alpha_1}\right] \qquad (32)$$

while K is determined from the equation

$$K=2\left[\frac{(\cosh 2\alpha_1-1)^2}{2\sinh 2\alpha_1}+2\sum_{n=2}^{\infty}\frac{(\cosh 2n\alpha_1-1)-n^2(\cosh 2\alpha_1-1)}{n(n^2-1)(\sinh 2n\alpha_1+n\sinh 2\alpha_1)}\right. \qquad (33)$$

The stresses over the circular boundaries $\alpha=\alpha_1$ and $\alpha=-\alpha_1$ are given by

$$a\widehat{\beta\beta}_1=a\widehat{\beta\beta}_{-1}$$
$$=2A\cosh\alpha_1(\cosh\alpha_1-\cos\beta)-\tfrac{1}{2}S(\cosh\alpha_1-\cos\beta)$$
$$\times\left[2K\cosh\alpha_1+4A_1\cosh 2\alpha_1\cos\beta-2K\sin^2\beta(\cosh\alpha_1-\cos\beta)^{-1}\right. \quad \ldots (34)$$
$$\left.+\sum_{n=2}^{\infty}2n\{(n+1)A_n\cosh(n+1)\alpha_1+(n-1)B_n\cosh(n-1)\alpha_1\}\cos n\beta\right]$$

The numerical values of the constants K, A_n and B_n for several values of α_1 are given in the Tables I, II and III.

TABLE I

α_1	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
K	8.356	2.655	1.485	0.889	0.555	0.355	0.231	0.152	0.101

TABLE II

α_1	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
A_1	1.240	0.446	0.177	0.074	0.032	0.014	0.006	0.002	0.001
A_2	0.243	0.066	0.019	0.006	0.002	0.001			
A_3	0.060	0.011	0.002						
A_4	0.015	0.002							
A_5	0.004								
A_6	0.001								

TABLE III

α_1	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
B_1	0 605	0.271	0.127	0.059	0.029	0.013	0.005	0.003	0.001
$-B_2$	1 783	0.692	0.296	0.131	0.058	0 026	0,012	0 005	0.001
$-B_3$	0 310	0 083	0.023	0.007	0.002	0.001			
$-B_4$	0.075	0.012	0.002						
$-B_5$	0.016	0-002							
$-B_6$	0.001								

Stresses over the circular boundaries $\alpha=0.8$ and $\alpha=-0.8$ are shown in figure 1, in multiples of A/a, between $\beta=0°$ and $\beta=180°$ only, because

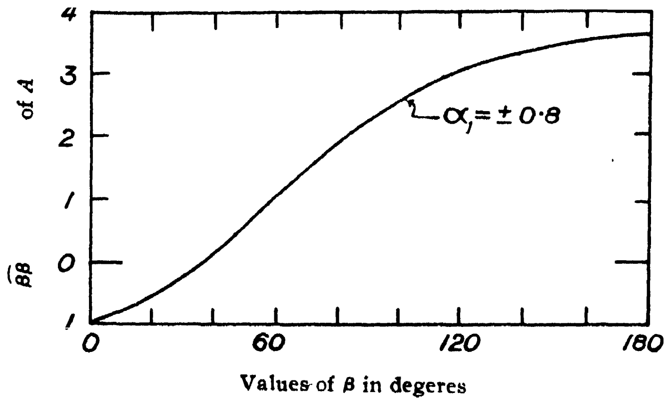

FIG. 1

the stress distribution is symmetrical about the y-axis. The maximum stresses are, as expected, at $\beta=\pm 180°$ *i.e.* at those points on the holes which are nearest to each other. The maximum stresses have the value $3{\cdot}63\ A/a$. At $\beta=0$ the stresses are negative on both the circles. The stress change from negative to positive at $\beta\simeq\pm 37°$, *i.e.* at about the middle of the top quadrants of the holes.

DISLOCATION DUE TO WEDGE SHAPED FISSURE

To obtain such many valued terms in expressions for displacements so as to suit the case of wedge shaped fissure between two uneqal circular holes in an infinite plate, choose

$$hX_0=A\alpha\cosh\alpha$$

The many valued terms in the displacements are given by

$$u=\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot A\cdot\frac{\beta\cosh\alpha\sin\beta}{\cosh\alpha-\cos\beta}$$

$$v=-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot A\cdot\frac{\beta\sinh\alpha\cos\beta}{\cosh\alpha-\cos\beta} \qquad (36)$$

If $\alpha > 0$, we have when $\beta=\pi$

$$u_{+}=0\ ;\ v_{+}=\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\pi A\frac{\sinh\alpha}{\cosh\alpha+1} \qquad \dots \quad (37)$$

and when $\beta=-\pi$

$$u_{-}=0\ ;\ v_{-}=-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\pi A\frac{\sinh\alpha}{\cosh\alpha+1} \qquad \dots \quad (38)$$

The discontinuity over the barrier $\beta=\pm\pi$ is

$$v_{+}-v_{-}=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi A\frac{\sinh\alpha}{\cosh\alpha+1}=v_0 \text{ (say)}$$

we have, when $\beta=\pm\pi$

$$y=y_0=\frac{a\sinh\alpha}{\cosh\alpha+1}$$

Therefore

$$v_0=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\cdot\frac{\pi A}{a}\cdot y_0 \qquad \dots \quad (39)$$

And if $\alpha < 0$, we get

$$v_0=-\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\cdot\frac{\pi A}{a}y_0 \qquad \dots \quad (40)$$

So the chosen stress function $h\chi_0$ will suit the state of dislocation due to the removal of a wedge shaped strip bounded by the planes

$$x=\pm\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\cdot\frac{\pi A}{a}\cdot y \qquad \dots \quad (41)$$

above the x-axis, and insertion of another wedge shaped strip of the same description below the x-axis, or *vice-versa*.

The normal stresses on the holes produced by $h\chi_0$ are given by

$$a\,\widehat{\alpha\alpha}_1=A(\alpha_1-\sinh\alpha_1\cosh\alpha_1),\ \ a\widehat{\alpha\alpha}_{-2}=-A(\alpha_2-\sinh\alpha_2\cosh\alpha_2) \quad \dots \quad (42)$$

To reduce these stresses to zero, a stress function $h\chi_1$ of the same form as in (9) is added to $h\chi_0$. In this case $h\chi_1$ has to produce stresses over the circular boundaries as given by

$$a\,\widehat{\alpha\alpha}_1=-A(\alpha_1-\sinh\alpha_1\cosh\alpha_1),\ \ a\widehat{\alpha\alpha}_{-2}=A(\alpha_2-\sinh\alpha_2\cosh\alpha_2) \quad \dots \quad (43)$$

Adjusting the constant coefficients to suit this requirement, we get

$$\begin{aligned}
B_{01} &= AM\{2(\alpha_1+\alpha_2) - \sinh 2\alpha_1 - \sinh 2\alpha_2\} \cosh(\alpha_1+\alpha_2) \\
A_{11} &= -\tfrac{1}{2}AM\{2(\alpha_1+\alpha_2) - \sinh 2\alpha_1 - \sinh {}^2\alpha_2\} \sinh(\alpha_1-\alpha_2) \\
C_{11} &= \tfrac{1}{2}AM\{2(\alpha_1+\alpha_2) - \sinh 2\alpha_1 - \sinh 2\alpha_2\} \cosh(\alpha_1-\alpha_2) \\
B_{11} &= \tfrac{1}{2}AM[2(\alpha_2 \sinh 2\alpha_1 - \alpha_1 \sinh 2\alpha_2)\cosh(\alpha_1+\alpha_2) \\
&\quad + \{2(\alpha_1-\alpha_2) - \sinh 2\alpha_1 + \sinh 2\alpha_2\} \sinh(\alpha_1+\alpha_2)]
\end{aligned} \quad \cdots \quad (44)$$

where

$$M = \tfrac{1}{2} \operatorname{cosech}(\alpha_1+\alpha_2)(\sinh^2\alpha_2 + \sinh^2\alpha_1)^{-1} \quad \cdots \quad (45)$$

As in the previous case here too we get a constant uniform all round stress at infinity due to the stress function $h\chi_1$. The value of this stress is

$$\begin{aligned}
&AM[(2\alpha_1 - \sinh 2\alpha_1)\cosh\alpha_1 \sinh\alpha_2 - (2\alpha - \sinh 2\alpha_2)\sinh\alpha_1 \cosh\alpha_2 \\
&\quad + (\alpha_2 \sinh 2\alpha_1 - \alpha_1 \sinh 2\alpha_2)\cosh \alpha_1+\alpha_2)] \\
&= a\,\widehat{\alpha\alpha} = a\,\widehat{\beta\beta} = S'(\text{say})
\end{aligned} \quad \cdots \quad (46)$$

To reduce this stress at infinity to zero, a stress function $h\chi_2$ of the same form and nature as in (15), with the substitution of S' in place of S, is added to $(h\chi_0 + h\chi_1)$. The complete solution is therefore given by

$$\begin{aligned}
h\chi &= h\chi_0 + h\chi_1 + h\chi_2 \\
&= A\alpha\cosh\alpha + B_{01}\,\alpha(\cosh\alpha - \cos\beta) \\
&\quad + \{A_{11}\cosh 2\alpha + B_{11} + C_{11}\sinh 2\alpha\}\cos\beta \\
&\quad - \tfrac{1}{2}S'[\cosh\alpha + \cos\beta + \{B_0\alpha + K\log(\cosh\alpha - \cos\beta)\}(\cosh\alpha - \cos\beta) \\
&\qquad + \sum_{n=1}^{\infty} \phi_n(\alpha)\cos n\beta]
\end{aligned} \quad \cdots \quad (47)$$

The circumferential stress $\widehat{\beta\beta}$ over the circular boundaries are now easily calculated from (47) and (3) as follows.

over $\alpha=\alpha_1$

$$\begin{aligned}
a\,\widehat{\beta\beta}_1 &= 2(\cosh\alpha_1 - \cos\beta)\{(A+B_{01})\sinh\alpha_1 - 2(A_{11}\cosh 2\alpha_1 \\
&\qquad + C_{11}\sinh 2\alpha_1)\cos\beta\} \\
&\quad - \tfrac{1}{2}S'(\cosh\alpha_1 - \cos\beta)[2Ke^{-\alpha_1} + 2(B_0+K)\sinh\alpha_1 + \phi_1''(\alpha_1)\cos\beta \\
&\quad - 2K\sin^2\beta(\cosh\alpha_1 - \cos\beta)^{-1} \\
&\qquad + \sum_{n=2}^{\infty}\{\phi_n''(\alpha_1) + n^2\phi_n(\alpha_1) - \phi_n(\alpha_1)\}\cos n\beta]
\end{aligned} \quad \cdots \quad (48)$$

over $\alpha = -\alpha_2$

$$
\begin{aligned}
a\widehat{\beta\beta}_{-2} = & -2(\cosh\alpha_2 - \cos\beta)\{(A+B_{01})\sinh\alpha_2 + 2(A_{11}\cosh 2\alpha_2 \\
& \qquad - C_{11}\sinh 2\alpha_2)\cos\beta\} \\
& -\tfrac{1}{2}S'(\cosh\alpha_2 - \cos\beta)[2Ke^{-\alpha_2} - 2(B_0 - K)\sinh\alpha_2 + \phi_1''(-\alpha_2)\cos\beta \\
& -2K\sin^2\beta(\cosh\alpha_2 - \cos\beta)^{-1} \\
& \qquad + \sum_{n=2}^{\infty}\{\phi_n''(-\alpha_2) + n^2\phi_n(-\alpha_2) - \phi_n(-\alpha_2)\}\cos n\beta]
\end{aligned} \quad \cdots \quad (49)
$$

If the holes be equal, we have $B_{11} = 0$, $A_{11} = 0$ and $S' = 0$. Therefore

$$hX = A\alpha\cosh\alpha + B_{01}\alpha(\cosh\alpha - \cos\beta) + C_{11}\sinh 2\alpha\cos\beta \quad \cdots \quad (50)$$

where

$$B_{01} = 2AM(2\alpha_1 - \sinh 2\alpha_1)\cosh 2\alpha_1,\ C_{11} = AM(2\alpha_1 - \sinh 2\alpha_1) \quad \cdots \quad (51)$$

The stresses over the hole boundaries are now given by

$$a\widehat{\beta\beta}_1 = -a\widehat{\beta\beta}_{-1} = 2(\cosh\alpha_1 - \cos\beta)\{(A+B_{01})\sinh\alpha_1 - 2C_{11}\sinh 2\alpha_1\cos\beta\} \quad \cdots \quad (52)$$

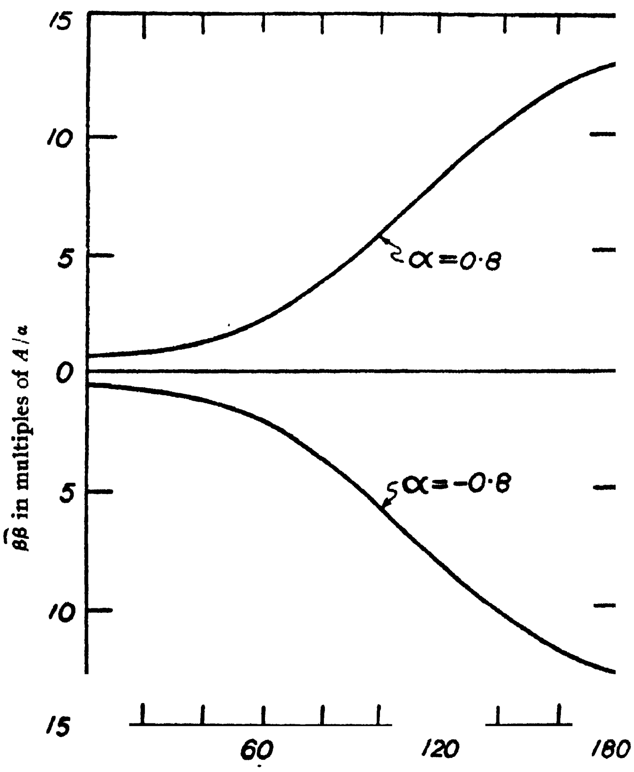

FIG. 2

In figure 2, the stresses in multiples A/a over the hole boundaries $\alpha = 0.8$ and $\alpha = -0.8$ are shown in a graph. Here again the maximum stresses are at $\beta = \pm 180°$ and the minima are at $\beta = 0$. The minimum and the maximum values are $0.506A/a$ and $12.59A/a$ and are of opposite signs on the two holes.

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