PLAUSIBLE SPEED LIMIT FOR METALLIC PROJECTILES

By K. R. DIXIT

(Received for publication, Aug. 18, 1949)

ABSTRACT The flight of a metallic projectile is discussed in two separate cases (*i*) in the Earth's atmosphere and (*ii*) in the interstellar space. It is shown that in the Earth's atmosphere 7 Mach numbers appear to be the plausible speed limit for long duration flight. In the interstellar space it is shown that when flying with a speed, which is greater than the velocity of escape from the surface of the Earth, the body will be warmed up in a short interval of time by the power plant to such an extent that we have to apply to it the equations governing the stability of hot bodies in interstellar space. Here we find that our metallic projectile is too small to be stable. Lastly, starting from the equations of Wigner and Seitz (1933) and considering the connection between the elastic properties of ordinary metals and the velocity of sound in those metals, it is shown that at speeds greater than the velocity of sound in metals, the Fermi energy of the electron cloud will become very large and thus make the metal relatively less stable in this state.

The flight of metallic projectiles at high speeds is assuming increasing importance. In the consideration of the movement of such projectiles two separate cases have to be considered; (1) the flight in the Earth's atmosphere and (2) the flight in the interstellar space. We shall consider these two in that order. Further, as the velocity of sound in a metal is connected with the elastic properties, some new phenomena might be expected to occur when the metal object moves with a velocity comparable with the velocity of sound propogation in the metal. Some aspects of this are considered in the last section.

1. FLIGHT IN THE EARTH'S ATMOSPHERE

We are quite familiar with the laws which govern the movement of ordinary propeller planes moving at speeds of about 500 m.p.h. and we are gradually gaining some knowledge of the flight at higher speeds. In this connection we often hear of supersonic speeds, that is speeds which are greater than the speed of sound in air at N.T.P., namely 700.5 m.p.h. When a plane travels at a speed greater than this, it is travelling faster than the pressure waves and would be able to catch them up. This results in a furious clash of air-currents, which throws the whole plane into violent vibrations. As a result of these vibrations some conventional planes have been actually torn apart. Wind tunnel experiments have been made going up to Mach number 4.4, (4.4 times the velocity of sound in air). Experiments at higher speeds, with air rushing past a model at one mile per second and more, have not yet been performed. It is, however, possible to draw some conclusions from our knowledge of the meteoric flights, about what we may expect to happen, when a projectile is moving very fast in the atmosphere of the Earth. In this connection it must be remembered that a meteor does not require a power plant for its propulsion. The presence of the power plant and the heat produced by it will introduce more complications. We shall have to consider these, specially in the case of interstellar flight, where no cooling effect will be produced by the air currents and the cooling due to radiation becomes appreciable only at very high temperatures.

Meteors usually come from distances beyond the Moon and have velocities of about 20 miles per second at the time of entering the ionosphere. The motion of the meteors is produced by gravity, and their temperature is very low, a few degrees absolute, at the time of entering the Earth's atmosphere. The meteors during their flight become visible at a height of about 100 kilometres above the surface of the Earth. At this height the air pressure is of the order of 10^{-3} mm. of Hg and the temperature is about $-100°$ C. Thus the meteor up to the time it becomes visible, has been flying only for a few seconds through cold rarefied atmosphere; and yet it is warmed up to such an extent as to become visible. The high temperature is the result of the movement of the meteor. The meteor is moving faster than the air molecules, it therefore traps and compresses the slow moving air molecules in front of it. A cap of compressed air is created in front of the meteor. Such a cap of compressed air becomes intensely hot and melts the surface material of the meteor. The behaviour of the meteor is not so very peculiar, as any projectile moving through air with a speed, which is greater than the speed of sound in air, will be trapping air in front of it. These trapped molecules will be pushed from behind by the projectile and will acquire the velocity of the projectile. This is an additional velocity acquired by the molecules. The effect of the added velocity is to increase the kinetic energy and hence raise the temperature of the trapped air molecules. By applying the laws of the kinetic theory of gases, we find that a projectile moving with a speed of 41 metres per second, will increase the temperature of the molecules trapped in front of it by 1°C. The increase in the temperature of the cap, as determined by the kinetic theory, is proportional to the square of the velocity and we could expect that the temperature of the aircap trapped in front of a projectile, moving with the speed of sound in air would be about 70°C. Thus we could expect a jet plane moving at a speed of 760.5 m.p.h., to be preceded by an aircap, whose temperature is greater than the temperature of the surrounding air by about 70°C. The projectile is in contact with the aircap and hence its temperature could also be expected to be greater than that of the surrounding by 70°C. The large amount of fuel consumed in a jet plane will further increase the temperature, but it may to some extent be compensated for by the partial vacuum and the corresponding cooling created behind the projectile. There will also be loss of energy due to radiation, calculations are made for such changes in the next section, but in general the cooling produced by radiation for a flight in the Earth's atmosphere will be smaller than the cooling produced by other causes.

To the same approximation, the temperature of a projectile made up of thin sheets of metal, and moving at a speed which is 3.5 times the speed of sound in air, the increase in temperature above the surroundings would be $70°C \times 3.5^2 = 850°C$. If the temperature of the surrounding air is $-100°C$, temperature of the projectile would be 750°C; and if made of aluminium it would melt before this speed is reached. Thus 3.5 Mach numbers appear to be the speed limit, for a flight of thin aluminium projectile, even through rarefied air, for an interval sufficiently long for the establishment of equilibrium conditions. There will be a slightly higher speed limit for other metals.

Thus we are forced to conclude, that it would be almost impossible to have a metal projectile, flying for an appreciable distance at speeds represented by Mach numbers beyond 7, without deformation of the projectile caused by melting of some of its parts. Such deformation changes the shape of the fins of the projectile and reduces the speed. It should be pointed out that these calculations consider the warming up due to compression of air in front of the projectile, partial cooling due to the vacuum created at the back and the increase in temperature due to the combustion of the fuel. These considerations, therefore, apply to the flight in the Earth's atmosphere, be it in troposphere, stratosphere or ionosphere. To this approximation there must be a plausible speed limit, about 7 Mach numbers, for long duration ($\sim$100 seconds) flights, even in a rarefied atmosphere (pressure $\sim 10^{-3}$ mm. of Hg).

2 INTERSTELLAR FLIGHT

In the case of flight in interstellar space, we have not to consider the warming or cooling of the aircap or the airtail. We have to consider the warming up of the plane by the combustion of the fuel and the radiation of heat, to and from the projectile. But in order to reach interstellar space, it is necessary for a projectile to acquire a speed which is greater than, (at least equal to) the velocity of escape from surface of the Earth. This velocity is given by the relation, $V = (2GM/a)^{\frac{1}{2}}$, where G is the universal constant of Gravitation, M is the mass of the Earth and a the radius. This velocity is about 7.1 miles or 11.3 kilometres per second, and is about 2.2 times the velocity of sound in metals like aluminium and steel. But the velocity of sound in a metal is connected with the elastic properties and some new phenomena might be expected to occur, when a metal object moves first through air and then in interstellar space with a velocity comparable with the velocity of sound propagation in the metal. We shall defer this theoretical discussion to the next section. Here we shall assume that such speeds are permissible without producing an elastic deformation of the metallic object, and proceed to consider the thermal conditions governing the flight in interstellar space.

Let us consider a metallic projectile, mass 10 tons (10,000 kilograms) and surface area 100 square metres. This projectile starts from rest on the surface of the Earth and acquires a speed of 11.3 kilometres per second before it leaves the Earth's atmosphere, in about 100 seconds. During this period it has acquired a kinetic energy $\sim 6.4 \times 10^{18}$ ergs. This energy is supplied by the rapid combustion of the fuel. Now every unit of fuel utilised may be split up in two parts, a and b. a is used in producing motion and in imparting kinetic energy $\frac{1}{2}mv^2$, whereas b is wasted in producing an increase of temperature. The ratio $a/a+b$ is known as the efficiency of the fuel. The ratio $a : b$, is independent of the velocity, but it may be slightly modified by the temperature of the fuel, which modifies the properties of the fuel and hence its efficiency. But to a first approximation we may assume that the ratio $a : b$ is constant, then the increase of temperature of the object, will be proportional to b and hence to a and thus to the square of the velocity. Even the best fuel is not 100% efficient and a certain percentage of it will be wasted in heating the power plant and our projectile, and to a first approximation the change of temperature produced by this wasteful combustion can be taken proportional to the square of the velocity. It should be noted that this heating is not the same as produced by frictional resistance, which is not proportional to the square of the velocity at high velocities.

To get an idea of the changes of temperature produced, by the heat supplied or radiated, let us calculate the energy required to heat the projectile from 300°K to 1000°K. The average specific heat of the projectile could be assumed to be 0.2 calories per degree C. The required energy will be $0.2 \times 700 \times 10^7 = 1.4 \times 10^9$ calories or 5.9×10^{16} ergs, which is only one per cent of the kinetic energy, and could be easily supplied even by a fuel with an efficiency of 99%. But no fuel has reached this efficiency, and the actual heating produced while imparting the kinetic energy corresponding to the velocity of escape would be much greater than that indicated by the range 300°K to 1000°K.

Let us now consider the radiation to and from the plane. The radiation received by the plane depends on the solar constant and the surface area. For the surface area of 100 sq. metres it is 1.35×10^{12} ergs per second and is negligible as compared to 5.9×10^{16} ergs. The loss of heat by radiation is governed by the Stefan's law and would depend on the temperature of the projectile. For an object at 1000°K and 100 sq. metres surface area it would be 5.73×10^{13} ergs per second, which means that only a thousandth part of the heat produced (5.9×10^{16} ergs) by a fuel which is 99% efficient, is lost in one second by radiation. The radiation loss per second will be comparable with the heat produced by the wasteful combustion of the fuel only at very high temperatures, comparable with the surface temperature of the stars. In interstellar space the loss of heat due to radiation is the only means available for reducing the temperature of the projectile. As seen above the rediation loss at low temperatures ($\sim 1000^{\circ}$K) is negligible as compared to the heat produced by the wasteful combustion of the fuel. Hence the temperature of the projectile will go on increasing until equilibrium conditions are established at very high temperatures. But long before this happens, the entire material of our projectile would be vapourised; we shall have to consider this object as a small hot gaseous body and apply to it equations similar to those given by S. Chandrasekhar, (1939)

$$\frac{d}{dr}(p_g + p_R) = -\frac{GM(r)}{r^2}\rho$$

This is the equation of hydrostatic, radiative equilibrium of a spherically symmetrical distribution of matter (Where p_g is the gas pressure, p_R, the radiation pressure, G, the constant of gravitation, $M(r)$ the mass enclosed inside r and ρ the density).

It will be seen from the equation, that the radiation pressure for our object will be large compared with the gravitational attraction, and the system will not be stable. In fact in an attempt to establish a state of equilibrium by making the loss of energy due to radiation equal to the heat produced, our projectile lands itself into a state, where it fails to satisfy the conditions of stability for heated bodies in interstellar space. Thus it appears that in interstellar space, the flight of a metallic object with a power plant, at speeds greater than the velocity of escape, for an appreciable time, will result in the deformation and the possible destruction of our object. But no object can reach the interstellar space, unless it has acquired in the Earth's atmosphere a velocity greater than the velocity of escape from the surface of the Earth. Even if it were possible to do so (and this is rather unlikely as 7 Mach numbers is a plausible speed limit for atmospheric flight of about 100 seconds duration), a flight in interstellar space with such a velocity for an appreciable time, will result in the deformation and possible destruction of the metallic object with a power plant. Thus even in interstellar space there appears to be a plausible speed limit (for a metallic object with a power plant), which must be less than the velocity of escape and hence must be comparable with the limit of about 7 Mach numbers for atmospheric flight of appreciable duration. The impossibility of interstellar flight at speeds less than the velocity of escape, need not concern us while considering the conditions of thermal equilibrium. Both these conditions, however, taken together give us the plausible speed limit as already mentioned.

3. THE VELOCITY OF SOUND IN A METAL AND ITS COHESIVE PROPERTIES

The first successful calculation of the cohesive forces in metals was made by Wigner and Seitz (1933) who obtained wave functions for electrons in metallic sodium and lithium by numerical integration. Wigner and Seitz divide the lattice into polyhedra, one polyhedron surrounding each atom. The wave functions of the valence electrons extend throughout the whole lattice and have periodic properties. This leads to boundary condition for the wave function at the boundary of any atomic polyhedron, $d\psi/dn=0$, where the differentiation is normal to the boundary. Fröhlich (1937) replaced the atomic polyhedron by a sphere of equal volume and radius r_0, so that,

$$\frac{4}{3}\pi r_0^3=\Omega=\text{ the atomic volume, or the volume of the atomic polyhedron.}$$

The polyhedron having been replaced by the sphere, the boundary condition assumed is that $\frac{d\psi}{dn}$ vanishes everywhere on the surface of the sphere. Within this sphere $V(r)$ is taken as the potential energy of an electron in the field of the ion. If $E(r_0)$ is the energy of an electron in the lowest state in the lattice : $E(r_0)$ is given by the Schrödinger equation,

$$\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}+[E(r_0)-V(r)]\psi=0$$

with the boundary condition,

$$\frac{d\psi}{dr}=0 \text{ for } r=r_0$$

Frohlich has calculated the values of $E(r_0)$ subject to certain simplifying assumptions (Hume-Rothery, 1947). He writes

$$E(r_0)=E_0+E_F+E_I$$

In this equation E_0 is the work required to break up into free electrons and positive ions a hypothetical crystal in which all the electrons are in the lowest energy state : E_F is the mean Fermi energy and in a free electron model it is the kinetic energy of the motion of translation of the electrons, whereas in the zone theories E_F is that part of the energy which is associated with the motion of the electrons in the periodic field of the lattice ; and E_I is the ionisation potential of a free atom and is constant for any given metal.

Thus it will be seen that $-(E_0+E_F+E_I)$ is the work required to evaporate the electrons and ions from the crystal and convert them into neutral atoms; or the energy of the crystal relative to the free atom is $+(E_0+E_F+E_I)$. Large values of this expression tend to make the energy of the crystal greater than that of free atoms and hence the crystal relatively less stable. E_I is the constant of the free atoms, while both E_0 and E_F are functions of the distance between the atoms. Frohlich assumed that the electron clouds do not overlap and that E_F is the same as given by the free electron theory ; and calculated E_0+E_F for different values of r_0. The curve for E_0+E_F passes through a minimum. The minimum indicates the equilibrium value of r_0, ($\frac{4}{3}\pi r_0^3=$ the atomic volume) or the lattice spacing and the curvature in the region of the minimum permits a calculation of the compressibility K,

$$\frac{1}{K} = -v\left(\frac{dp}{dv}\right) = v\left(\frac{d^2E}{dv^2}\right)_{v=\Omega}$$

where E is the total energy.

Fuchs (1936) has shown that the mutual attraction between the ionic cores cannot be neglected. Only by taking this into account it is possible to explain satisfactorily the elastic properties of the noble metals. But once this is done, the nature of the curve showing total energy as a function of r_0, is similar to the curve obtained by Fröhlich. Here also the minimum gives the equilibrium value of r_0 and the lattice spacing, the curvature as before gives the compressibility. This then may be assumed to be the general property for all metals, even aluminium and iron for which no exact calculations are available.

We now proceed to establish relations between the elastic properties as defined above with the velocity of sound in a metal.
We know that

$$V = \text{the velocity of sound} = \sqrt{\frac{\text{elasticity}}{\text{density}}} = \sqrt{\frac{\text{the Young's modulus}}{\text{density}}}.$$

We also know, that if K is the compressibility, k is the bulk modulus, Y is the Young's modulus and σ is the Poisson's ratio which is nearly equal to $\frac{1}{3}$ for most metals, then we have

$$k = \frac{Y}{3(1-2\sigma)} \simeq \frac{Y}{3(1-2/3)} \simeq Y.$$

This gives us the relation,

$$K = \frac{1}{k} \simeq \frac{1}{Y}, \text{ or } V^2 = \frac{Y}{\rho} \simeq \frac{1}{K\rho}.$$

Hence we get,

$$\frac{1}{K} \simeq V^2\rho = V^2\,\frac{\text{atomic weight}}{\text{atomic volume}} = V^2\,\frac{A}{\Omega},$$

but

$$\frac{1}{K} = \frac{1}{9\Omega}\,r^2\,\frac{d^2E}{dr^2}.$$

Therefore we finally come to the expression that

$$V \propto r\left(\frac{d^2E}{dr^2}\right)^{1/2}.$$

That is the velocity of sound in a metal is a function of the distance at which the total energy becomes a minimum (or the lattice spacing), as well as the curvature of the energy curve at that place. This will be strictly correct for the ideal metals of Fröhlch, for changes in which the atomic volume remains unaltered, and for metals for which the Poisson's ratio is about $\frac{1}{3}$; for all other metals we may assume it to be a good enough first approximation.

The question now arises as to what we might expect to happen when the metal begins to move with a velocity which is greater than the velocity of sound in that metal. As seen above the velocity of sound in a metal represents the minimum of the energy curve and the value of E_0 for which it is stable. When the velocity of a metal increases, it also increases the velocity of the electron cloud associated with it. As the velocity passes beyond the limit given by the velocity of sound in that metal, the Fermi energy E_0 of the electron cloud will begin to increase above the limit given by the minimum energy and the stability. Greater the deviation from the velocity of sound, the greater will be the increase in E_0 and greater will be the instability that results. This also appears to point out that the flight of a pure metallic object at velocities greater than that of sound in that metal increases the energy of the associated electron cloud and makes the metallic object relatively less stable.

The plausible speed limit suggested by these theoretical considerations is comparable with that given in sections 1 and 2. In this connection we should remember that meteors are not pieces of pure metals and hence their stability will be governed by other causes than the Fermi energy. For example, at velocities greater than the velocity of sound in a solid we can expect the characteristic frequencies, the Debye characteristic temperature and the melting point to be altered. We shall, however, defer this discussion to a later paper.

Royal Institute of Science,
Bombay, India.

REFERENCES

Chandrasekhar, S., 1939 'An Introduction to the Study of Stellar Structure', p. 216, Chicago.

Frohlich, H., 1937, *Proc. Roy. Soc.*, A, **158**, 97

Fuchs, K., 1935, *Proc. Roy. Soc. A.*, **153**, 622

Hume-Rothery, W., 1947, 'Atomic Theory for Students of Metallurgy', pp 217-224, London.

Wigner, E. and Seitz, F., 1933, *Phys. Rev.*, **43**, 804