ABSTRACT

BLASCHKE PRODUCTS AND PARAMETER SPACES

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Northern Illinois University, 2014
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In this thesis, we discuss certain aspects of complex dynamics. We will introduce the important concepts in iteration theory, discuss examples of families of holomorphic mappings, and their dynamics. In particular, we will discuss the family of quadratic functions, the family of Möbius mappings of the disk, and a certain subclass of Blaschke products. We will exhibit some of the ideas of Fletcher [3] which show how the dynamics depend on the parameters.

NORTHERN ILLINOIS UNIVERSITY
DE KALB, ILLINOIS

DECEMBER 2014

BLASCHKE PRODUCTS AND PARAMETER SPACES

BY

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A THESIS SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICAL SCIENCES

Thesis Director:
Alastair Fletcher

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor Dr. Alastair Fletcher for the numerous, interesting, and enlightening discussions during my time at NIU, and in particular helping me to learn about some aspects of complex dynamics. I'd also like to thank Dr. Ye for helping me around every corner these past few years, laughing at all of my jokes, and for not giving up on me when many professors would have.

DEDICATION

To my parents, for all the love and support throughout my education. And to Candice Nielsen, who helped me to keep my sanity throughout graduate school. Thank you for being my partner in tears, laughs, and especially donuts.

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CHAPTER 1

OVERVIEW

The aim of this thesis is to discuss the results contained in the paper [3]. We will discuss the background to the material in this paper and explain the ideas behind the proofs of the main results.

In [3], a certain subclass of Blaschke products of degree n is considered and their dynamics are classified according to a decomposition of the corresponding parameter space. To motivate this, we will discuss some of the fundamental concepts from complex dynamics, in particular, definitions and properties of the Fatou set and the Julia set of a rational function. These concepts depend on the dynamics of a particular function, but very often we want to be able to say something about a whole family of functions all at once.

Given a family of functions depending on a certain parameter, we will discuss how they can be classified according to the values of the parameter. As a specific example to illustrate this idea, we will discuss the well-known family $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$ and how the famous Mandelbrot set $\mathcal{M}$ encodes certain features of the dynamics of this quadratic family.

To motivate Blaschke products, we will give a classification of the different dynamical behaviors that can occur for Möbius mappings of the unit disk $\mathbb{D}$. We will define Blaschke products, discuss some of their basic properties and talk about their dynamics. They can be classified in a similar way to Möbius mappings of the disk, but this classification, in terms of parameter space, has not been extensively studied.

We will state the main result of the paper [3], which classifies Blaschke products of a certain form, and discuss the ideas contained in the proofs of the main results. The key ingredient here is recognizing that a description of the set of points where $|f'(z)| < 1$ for a given Blaschke product f and $z \in \partial\mathbb{D}$ is crucial for classifying f.

CHAPTER 2

COMPLEX DYNAMICS

In this section, we briefly introduce some of the important concepts from complex dynamics. We start by defining what we mean by iteration.

Definition 2.0.1. If X is a set and $f : X \to X$ is a function, then we can *iterate* f. We define $f^1 = f$ and for $n \geq 2$, denote by f^n the *n'th iterate of* f, given by $f^n = f \circ f^{n-1}$.

We will be interested in the case when X is the complex plane and f is holomorphic or meromorphic.

Example 2.0.2. *Let* $f(z) = z^2$*, then* $f^2(z) = (z^2)^2 = z^4$ *and in general,* $f^n(z) = z^{2^n}$.

An important notion from complex analysis is that of a normal family.

Definition 2.0.3. Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of holomorphic functions defined on D. We say that $\mathcal{F}$ is a *normal* family if for every sequence of functions in $\mathcal{F}$ there is a subsequence which converges uniformly on compact subsets to a holomorphic function or a constant.

We will be interested in the case when $\mathcal{F}$ is a family of iterates of a given function. Montel's Theorem gives us an important criterion to tell when a family is normal.

Theorem 2.0.4. (Montel's Theorem) *Suppose* $\mathcal{F}$ *is a family of holomorphic functions defined on a domain* $D \subset \mathbb{C}$*. If the functions from* $\mathcal{F}$ *all omit the same two values* $a, b \in \mathbb{C}$*, then* $\mathcal{F}$ *is a normal family in* D*.*

Now we are in a position to define the set where the iterates of a holomorphic function behave nicely.

Definition 2.0.5. The *Fatou set* $F(f)$ is the set of good dynamical behavior, i.e. $z \in F(f)$ if and only if the family of iterates on some neighborhood of z forms a normal family.

One way of viewing this is that two nearby points in the Fatou set stay near to each other under iteration.

Definition 2.0.6. The *Julia set* $J(f)$ is the set of chaotic behavior, i.e. $z \in J(f)$ if and only if the family of iterates does not form a normal family on any neighborhood of z.

One way of viewing this is that typically two nearby points in the Julia set will eventually end up relatively far apart under iteration. It is clear from the definitions that the Fatou set and Julia set are disjoint and $\mathbb{C} = F(f) \cup J(f)$.

Example 2.0.7. *Let $f(z) = z^2$. Then the Fatou set has two components: the unit disk and $\{z : |z| > 1\}$ and the Julia set is $\{z : |z| = 1\}$.*

The escaping set $I(f)$ is defined to be the set of points for which $f^n(z) \to \infty$ as $n \to \infty$. In the above example, we see that $I(f) = \{z : |z| > 1\}$ and so we have $\partial I(f) = J(f)$. This is an equality which holds in general, see [2].

Some more well-known properties which illustrate the chaos of the Julia set are:

(i) The Julia set is the closure of the repelling periodic points.

(ii) The Julia set satisfies a blowing-up property: if U is any open set that intersects $J(f)$, then the forward orbit of U covers everything except possibly one point in $\mathbb{C}$. This follows since if the family f^n defined on U all omit the same two points, then by Montel's Theorem the family would be normal on U and hence the point would be in the Fatou set.

CHAPTER 3

PARAMETER SPACES IN DYNAMICS

Given a holomorphic function f, we saw in the previous section that there are sets, the Julia set and the Fatou set, which describe the dynamics of f. These are sets that are defined by iterating f and seeing what happens; i.e. given $z \in \mathbb{C}$, we see what happens to the sequence $f(z), f^2(z), f^3(z), \ldots$. The dynamical plane is the z-plane which contains information about the iterates of one function f.

When we talk about parameter space, we are dealing with a whole family of functions at once. We assume we can write our family as

$$\mathcal{F} = \{f_y : y \in Y\},$$

where Y is a set we call parameter space. We say Y parameterizes the family $\mathcal{F}$. To illustrate this fairly abstract idea, we use the family

$$\mathcal{F} = \{f_c(z) = z^2 + c : c \in \mathbb{C}\}$$

as an example. The parameter space here is the set of c-values, i.e. $Y = \mathbb{C}$. Every c-value in $\mathbb{C}$ gives rise to a different function in $\mathcal{F}$. A difficulty with this example is that the dynamical plane is the plane of z-values for any particular f_c and the parameter space is also a plane, but the plane of c-values that gives rise to the various f_c.

To recap, given a family of functions, its parameter space is the set such that every element in it gives rise to a different function, but for each function, we can look at its dynamics, and that happens in the dynamical plane.

We are very often interested in situations where the parameter space splits up into two sets, i.e. $Y = Y_1 \cup Y_2$, where if the parameter is in Y_1 we get one type of behavior in the dynamics of the corresponding function and if the parameter is in Y_2, we get a different behavior. With the family of quadratic polynomials $\mathcal{F}$, it is wellknown that the Julia set, which lives in the dynamical plane, can be one of two things: either it is connected or it is totally disconnected, i.e. it is a Cantor set and every component is a point. In Figure 3.1, we see an example of a connected Julia set, for $z^2 - 1$, and in Figure 3.2, we see an example of a totally disconnected Julia set, for $z^2 + \frac{1}{2}$.

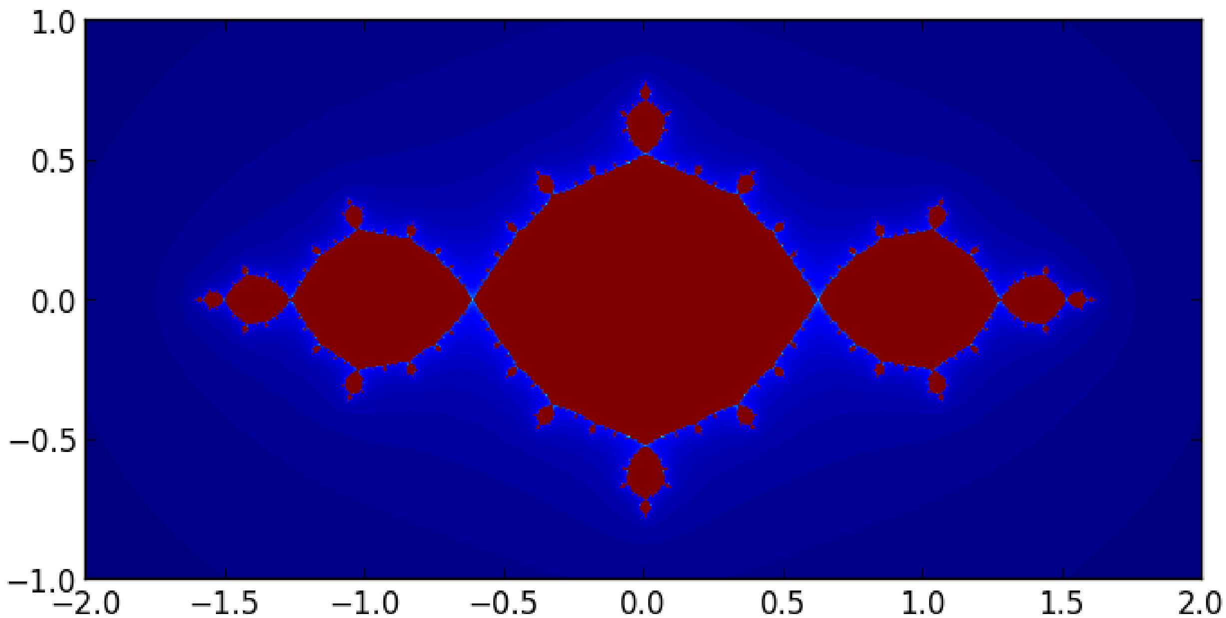

Figure 3.1: The Julia set of $z^2 - 1$.

Since for the family of quadratic polynomials, we have two different types of dynamical behavior, the parameter space breaks up into two sets: one of parameters which give rise

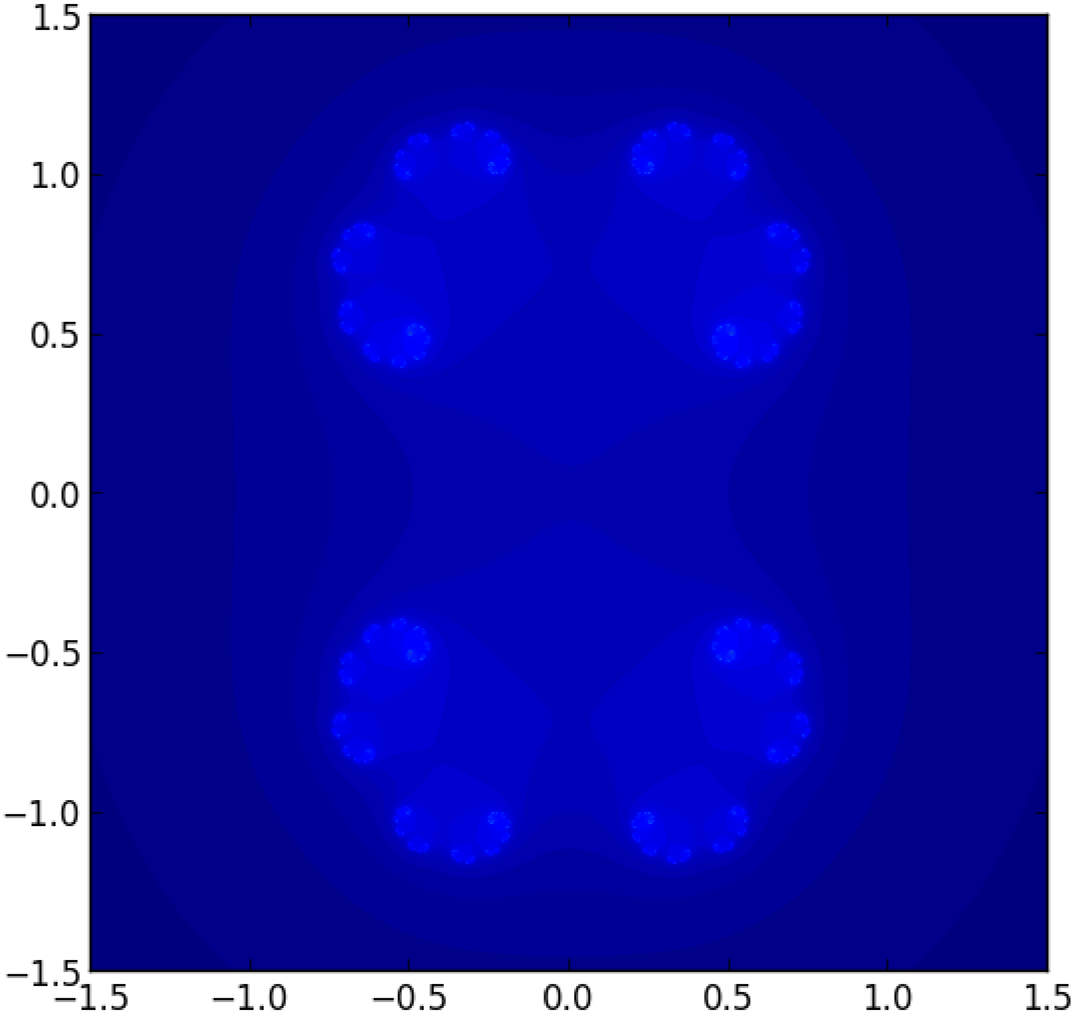

Figure 3.2: The Julia set of $z^2 + \frac{1}{2}$.

to connected Julia sets and one of parameters which give rise to totally disconnected Julia sets. The Mandelbrot set $\mathcal{M}$ is defined to be the set in parameter space which gives rise to connected Julia sets. In other words, if $c \in \mathcal{M}$, then $J(f_c)$ is connected, and if $c \notin \mathcal{M}$, then $J(f_c)$ is totally disconnected. See Figure 3.3 for a picture of the Mandelbrot set.

We briefly describe how the dynamical pictures were produced. Recall the escaping set $I(f)$ consists of those points in the dynamical plane that escape to infinity under iteration by f. It's a well-known fact that $J(f) = \partial I(f)$, so dynamical plots Figures 3.1 and 3.2 are

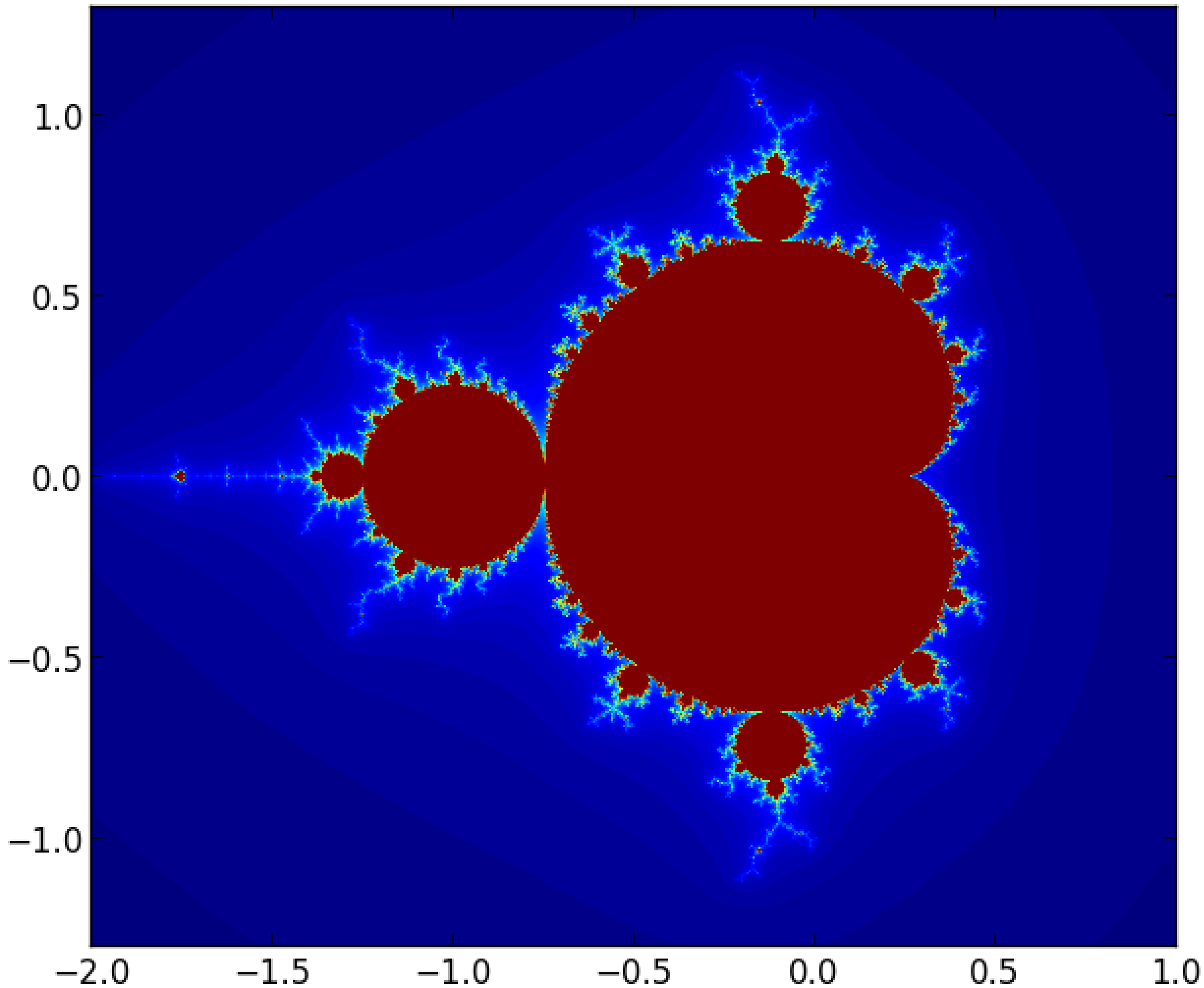

Figure 3.3: The Mandelbrot set.

produced by plotting those points which escape to infinity. Since computers can only handle finite objects, it is computed by iterating finitely many times and seeing if the iterates ever lie outside some big ball. In the figures, the blue section is the escaping set, and so the boundary of the blue part is the Julia set.

Figure 3.3 of the Mandelbrot set is produced in a similar and slightly different way by using another way of characterizing the Mandelbrot set. If $c \in \mathcal{M}$, then 0 turns out to not be in the escaping set for f_c, whereas if $c \notin \mathcal{M}$, 0 is in $I(f_c)$. Given c, it is enough to conclude that $c \notin \mathcal{M}$ if $|f_c^n(0)| > 2$ for any $n \in \mathbb{N}$. Using this condition, computer programs can be written to check if, for any large finite n, this condition is satisfied.

3.1 Möbius Mappings and the Hyperbolic Disk

In this section, we discuss function theory on the unit disk. In particular, we focus on isometries in the hyperbolic metric, i.e. Möbius transformations. Some of the presentation in this section follows [1].

3.2 The Poincaré Disk

Definition 3.2.1. The *Poincaré disk* model of hyperbolic space is defined as the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ equipped with the hyperbolic metric density

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2} \tag{3.2.1}$$

and the hyperbolic distance

$$d_{\mathbb{D}}(z, w) = \log\left(\frac{1 + |\frac{z-w}{1-\bar{w}z}|}{1 - |\frac{z-w}{1-\bar{w}z}|}\right) \tag{3.2.2}$$

for $z,w \in \mathbb{D}$.

Definition 3.2.2. An *isometry* of the hyperbolic metric, in the unit disk, is a map A such that

$$d_{\mathbb{D}}(A(z), A(w)) = d_{\mathbb{D}}(z, w), \quad \forall z, w \in \mathbb{D}.$$

Definition 3.2.3. A *Möbius transformation* is a function of the form

$$A(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Every Möbius transformation can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The condition $ad - bc \neq 0$ means that the matrix is non-singular.

Now, every isometry of the hyperbolic metric in the unit disk is a Möbius transformation and can be written in the form

$$A(z) = e^{i\theta}\left(\frac{z-w}{1-\overline{w}z}\right)$$

for some $e^{i\theta} \in \partial\mathbb{D}$ and $w \in \mathbb{D}$. We can divide through by $e^{i\theta/2}$ to obtain

$$A(z) = \frac{Ce^{i\theta/2}z - Ce^{i\theta/2}w}{Ce^{-i\theta/2} - Ce^{-i\theta/2}\overline{w}z},$$

where $C = (1-|w|^2)^{-1/2}$. The reason for writing A in this form is that the matrix representing A, given by

$$\begin{pmatrix} Ce^{i\theta/2} & -Ce^{i\theta/2}w \\ -Ce^{-i\theta/2}\overline{w} & Ce^{-i\theta/2} \end{pmatrix},$$

has determinant 1 and trace-squared equal to

$$\tau(A) = \left(\frac{e^{i\theta/2}}{\sqrt{1-|w|^2}} + \frac{e^{-i\theta/2}}{\sqrt{1-|w|^2}}\right)^2 = \frac{2(1+\cos\theta)}{1-|w|^2}. \tag{3.2.3}$$

We note that τ is invariant under conjugation, i.e. $\tau(P^{-1}AP) = \tau(A)$.

3.3 The Schwarz Lemma

The Schwarz Lemma is a key tool in function theory on the unit disk. It will have some important consequences in the study of Blaschke products.

Theorem 3.3.1. (Schwarz's Lemma). *Suppose that* $f\colon \mathbb{D} \to \mathbb{D}$ *is holomorphic and satisfies* $f(0) = 0$. *Then either*

(a) $|f(z)| < |z|$ *for every non-zero* $z \in \mathbb{D}$, *and* $|f'(0)| < 1$, *or*

(b) *for some real constant* θ, $f(z) = e^{i\theta}z$ *and* $|f'(0)| = 1$.

Proof. Define $g\colon \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z)/z & , \quad z \neq 0 \\ f'(0) & , \quad z = 0 \end{cases} .$$

Then g is analytic in $\mathbb{D}$. For

$$0 < r < 1 \quad \text{and} \quad |z| < r,$$

$$|g(z)| \leq \frac{|f(z)|}{r} \leq \frac{1}{r},$$

by the Maximum Modulus Theorem. Letting $r \to 1$ implies $|g(z)| \leq 1, \quad \forall z \in \mathbb{D}$. Hence $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$. If either equality is achieved, $|g|$ attains its maximum inside $\mathbb{D}$, which implies $g(z) \equiv c$ for some c with $|c| = 1$. Thus, $f(z) = cz$. $\square$

3.4 The Schwarz-Pick Lemma

The Schwarz-Pick Lemma is a generalization of the Schwarz Lemma for functions that do not fix 0.

Theorem 3.4.1. (Schwarz-Pick Lemma). *Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic. Then either*

(a) *f is a hyperbolic contraction; that is, for all z and w in $\mathbb{D}$,*

$$d_{\mathbb{D}}(f(z), f(w)) < d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}(f(z))|f'(z)| < \lambda_{\mathbb{D}}(z), \tag{3.4.1}$$

or

(b) *f is a hyperbolic isometry; that is, for all z and w in $\mathbb{D}$,*

$$d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z). \tag{3.4.2}$$

Proof. We have that f is an isometry if and only if one, and hence both, of the conditions in (3.4.2) hold.

Suppose now the $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic but not an isometry. Select any two points z_1 and z_2 in $\mathbb{D}$. Here is the intuitive idea behind the proof. Because we can find Möbius transformations that send any point to any other point, we may assume without loss of generality that both z_1 and $f(z_1)$ are at the origin. In this special situation, (3.4.1) follows directly from the Schwarz Lemma.

Now we write out a formal argument. Let g and h be Möbius transformations of $\mathbb{D}$ such that $g(z_1) = 0$ and $h(f(z_1)) = 0$. Let $F = hfg^{-1}$; then F is a holomorphic self-map of $\mathbb{D}$ that fixes 0. As g and h are isometries, F is not an isometry or else f would be too. Therefore, by the Schwarz Lemma, $d_{\mathbb{D}}(0, F(z)) < d_{\mathbb{D}}(0, z)$ and $|F'(0)| < 1, \forall z \in \mathbb{D}$. Thus, as $Fg = hf$ and g, h are hyperbolic isometries:

$$\begin{aligned}
d_{\mathbb{D}}(f(z_1), f(z_2)) &= d_{\mathbb{D}}(hf(z_1), hf(z_2)) \\
&= d_{\mathbb{D}}(Fg(z_1), Fg(z_2)) \\
&= d_{\mathbb{D}}(0, Fg(z_2)) \\
&< d_{\mathbb{D}}(0, g(z_2)) \\
&= d_{\mathbb{D}}(g(z_1), g(z_2)) \\
&= d_{\mathbb{D}}(z_1, z_2).
\end{aligned}$$

This is the first inequality in (3.4.1). To obtain the second inequality, we apply the Chain Rule to each side of $Fg = hf$ and obtain

$$|F'(0)| = \frac{|f'(z_1)|(1 - |z_1|^2)}{1 - |f(z_1)|^2} < 1.$$

This gives us the second inequality in (3.4.1) at an arbitrary point z_1. □

The Schwarz-Pick Lemma is often stated in the following form: Every holomorphic self-map of the unit disk is a contraction relative to the hyperbolic metric. That is, if f is a holomorphic self-map of $\mathbb{D}$, then

$$d_{\mathbb{D}}(f(z), f(w)) \leq d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}(f(z))|f'(z)| \leq \lambda_{\mathbb{D}}(z). \tag{3.4.3}$$

The work of Pick in geometric function theory has shown that the hyperbolic metric, not the Euclidean metric, is the natural metric for most of the subject. Although the definition of the hyperbolic metric may seem arbitrary, in fact, up to multiplication by a positive scalar, it is the only metric on the unit disk that makes every holomorphic self-map a contraction.

The following consequence of the Schwarz-Pick Lemma will be important later in classifying Blaschke products.

Theorem 3.4.2. *If a holomorphic self-map of $\mathbb{D}$ fixes two points in $\mathbb{D}$, then it is the identity.*

Proof. Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map such that $f(w_1) = w_1$ and $f(w_2) = w_2$. Also, let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a Möbius map defined by $\varphi(z) = \frac{z-w_1}{1-\bar{w}_1 z}$, where $\varphi(w_2) = \mu$ and we have $\varphi(w_1) = 0$. Define $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$. Note that $\varphi^{-1}(z) = \frac{z+w_1}{1+\bar{w}_1 z}$. So, we have

$$\tilde{f}(0) = \varphi[f(\varphi^{-1}(0))] = \varphi[f(w_1)] = \varphi(w_1) = 0$$

and

$$\tilde{f}(\mu) = \varphi[f(\varphi^{-1}(\mu))] = \varphi[f(w_2)] = \varphi(w_2) = \mu.$$

Therefore, since $\tilde{f}$ is a holomorphic map such that $\tilde{f}(0) = 0$ and $|\tilde{f}(w_1)| = |w_1|$, by Schwarz's Lemma part (b), we must have $\tilde{f}(z) = e^{i\theta} z$. Consider when $z = w_1$. Then $\tilde{f}(z) = \tilde{f}(w_1) = e^{i\theta} w_1 = w_1$ and so $\theta = 0$. Thus, $\tilde{f} = id$, the identity. Therefore, we have $f = \varphi^{-1} \circ \tilde{f} \circ \varphi = \varphi^{-1} \circ \varphi = id$. Hence, if $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map that fixes two points, f is the identity. $\square$

3.5 Classification of Möbius Maps

To find fixed points of a Möbius map, we need to solve $\left(\frac{az+b}{cz+d}\right) = z$. This simplifies to solving $cz^2 + (d-a)z - b = 0$, which has either one or two solutions, and so a Möbius map has one or two fixed points.

We recall that Möbius transformations of $\mathbb{D}$ can be classified as follows:

(i) A is called *hyperbolic* if A has two fixed points on $\partial\mathbb{D}$ and none in $\mathbb{D}$ (Figure 3.4),

(ii) A is called *parabolic* if A has one fixed point on $\partial\mathbb{D}$ and none in $\mathbb{D}$ (Figure 3.5),

(iii) A is called *elliptic* if A has no fixed points on $\partial\mathbb{D}$ and one in $\mathbb{D}$ (Figure 3.6).

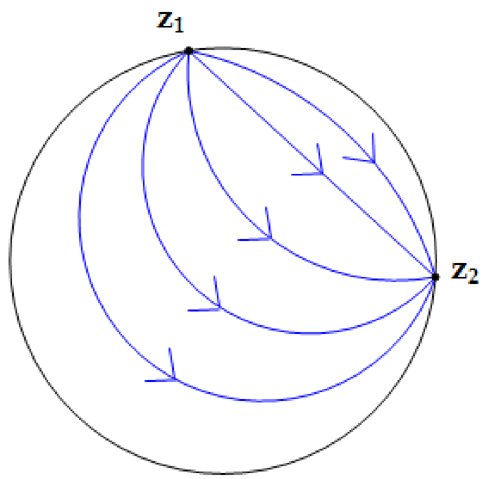

Figure 3.4: Hyperbolic Möbius transformation.

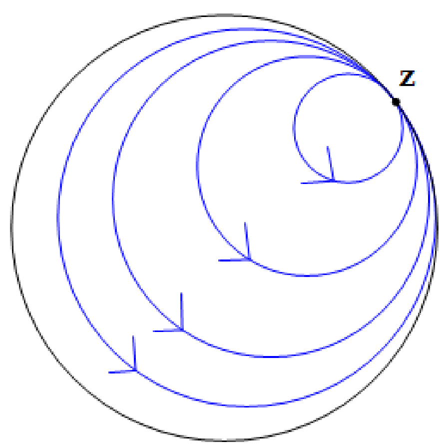

Figure 3.5: Parabolic Möbius transformation.

Note that by Theorem 3.4.2 , if A is not the identity, A can have a maximum of one fixed point in $\mathbb{D}$, and so these three cases provide a complete classification of Möbius transformations of $\mathbb{D}$.

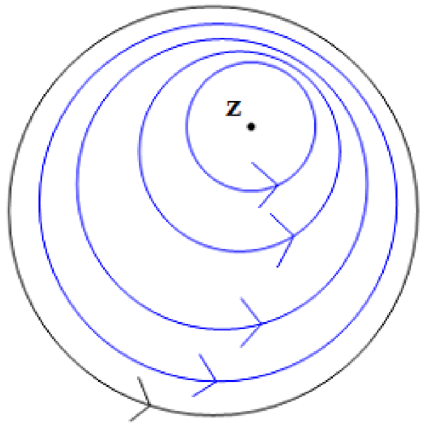

Figure 3.6: Elliptic Möbius transformation.

This classification can also be expressed in terms of τ:

(i) A is hyperbolic if and only if $\tau(A) > 4$.

(ii) A is parabolic if and only if $\tau(A) = 4$.

(iii) A is elliptic if and only if $0 \leq \tau(A) < 4$.

We see from (3.2.3) that $\tau(A)$ is real when A is represented in the normalized form as given above. Hence if we fix θ, the set of w-values for which A is elliptic is given by the disk

$$\left\{ w \in \mathbb{D} : |w| < \sqrt{\frac{1 - \cos\theta}{2}} \right\}.$$

Note this set is empty when $\theta = 0$. Since the set of parameters for Möbius transformations of $\mathbb{D}$ can be parameterized by the solid torus $S^1 \times \mathbb{D}$, the *domain of ellipticity* is given by the open set

$$E := \left\{ (e^{i\theta}, w) \in S^1 \times \mathbb{D} : |w| < \sqrt{\frac{1 - \cos\theta}{2}} \right\}. \tag{3.5.1}$$

The boundary of E gives the set of parabolic parameters by the classification in terms of τ.

CHAPTER 4

BLASCHKE PRODUCTS

4.1 Definition and Properties

A *finite Blaschke product* is a function $B : \mathbb{D} \to \mathbb{D}$ given by

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \left(\frac{z - w_i}{1 - \overline{w_i} z} \right), \tag{4.1.1}$$

for some $e^{i\theta} \in \partial\mathbb{D}$ and $w_i \in \mathbb{D}$ for $i = 1, \ldots, n$. We call a Blaschke product non-trivial if $n \geq 2$.

Every finite-degree self-mapping of $\mathbb{D}$ is a finite Blaschke product [1, p.19], and so they can be viewed as analogues for polynomials in the disk. Recalling Theorem 3.4.2, B can have at most one fixed point in $\mathbb{D}$. If z_0 is a fixed point of B, then we will show that $1/\overline{z_0}$ is also a fixed point of B.

Theorem 4.1.1. *Let B be a Blaschke product of the form*

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \left(\frac{z - w_i}{1 - \overline{w_i} z} \right),$$

where $w_i \in \mathbb{D}$ for $i = 1, 2, \ldots, n$. Then if $z_0 \in \mathbb{D}$ is a fixed point of B, so is $1/\overline{z_0}$.

Note that $1/\overline{z_0}$ is the reflection of z_0 in the unit circle.

Proof. Suppose that B is a Blaschke product and $B(z_0) = z_0$, then

$$\begin{aligned}
B(1/\overline{z_0}) &= e^{i\theta} \prod_{i=1}^{n} \left(\frac{1/\overline{z_0} - w_i}{1 - \overline{w_i}/\overline{z_0}} \right) \\
&= e^{i\theta} \prod_{i=1}^{n} \left(\frac{1 - w_i \overline{z_0}}{\overline{z_0} - \overline{w_i}} \right) \\
&= \overline{e^{-i\theta} \prod_{i=1}^{n} \left(\frac{1 - z_0 \overline{w_i}}{z_0 - w_i} \right)} \\
&= \frac{1}{\overline{B(z_0)}} \\
&= \frac{1}{\overline{z_0}}.
\end{aligned}$$

□

Hence all but possibly two (with the convention that infinity is a fixed point if some $w_i = 0$) of the fixed points of B must lie on $\partial\mathbb{D}$.

The following theorem helps us classify Blaschke products (see [4, p.58]).

Theorem 4.1.2 (Denjoy-Wolff Theorem)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and not a Möbius transformation. Then there exists some $z_0 \in \overline{\mathbb{D}}$ such that $f^n(z) \to z_0$ for every $z \in \mathbb{D}$. The point z_0 is called the Denjoy-Wolff point of f.*

There is a classification of finite Blaschke products in analogy with that for Möbius transformations:

(i) B is called *hyperbolic* if the Denjoy-Wolff point z_0 of B lies on $\partial\mathbb{D}$ and $|B'(z_0)| < 1$.

(ii) B is called *parabolic* if the Denjoy-Wolff point z_0 of B lies on $\partial\mathbb{D}$ and $|B'(z_0)| = 1$.

(iii) B is called *elliptic* if the Denjoy-Wolff point z_0 of B lies in $\mathbb{D}$. In this case, we must have $|B'(z_0)| < 1$.

Let B be a Blaschke product of the form

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \left(\frac{z - w_i}{1 - \overline{w_i} z} \right).$$

Suppose we want to find fixed points of B. Then we need to solve the equation $B(z) = z$, or in other words,

$$e^{i\theta} \prod_{i=1}^{n} \left(\frac{z - w_i}{1 - \overline{w_i} z} \right) = z.$$

By multiplying through by the denominator of the left-hand side and rearranging, this becomes

$$z \prod_{i=1}^{n} (1 - \overline{w_i} z) - e^{i\theta} \prod_{i=1}^{n} (z - w_i) = 0.$$

Since this is a polynomial of degree $n+1$, by the Fundamental Theorem of Algebra, there are $n+1$ solutions (counting multiplicity). By Theorem 3.4.2, there can be at most one fixed point in $\mathbb{D}$, and so at most one fixed point on $\mathbb{C} \setminus \overline{\mathbb{D}}$. All the others must be on $\partial\mathbb{D}$.

This viewpoint illustrates the classification of Blaschke products:

(i) Elliptic: the Denjoy-Wolff point is the fixed point in $\mathbb{D}$; there is one other fixed point in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and there are $n-1$ distinct fixed points on $\partial\mathbb{D}$.

(ii) Parabolic: there are n distinct fixed points on $\partial\mathbb{D}$, but one is taken with multiplicity 2 (the Denjoy-Wolff point).

(iii) Hyperbolic: there are $n+1$ distinct fixed points on $\partial\mathbb{D}$, and one of them is the Denjoy-Wolff point.

4.2 Dynamics of Blaschke Products

Recall that the Fatou set of a rational map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the set

$$F(f) = \{z \in \overline{\mathbb{C}} : \text{ the family } (f^n)_{n=1}^{\infty} \text{ is normal in some neighborhood of } z\},$$

and the Julia set is the complement of the Fatou set.

Recall the blowing-up property of the Julia set: if $z \in J(f)$, then for any neighborhood U of z, the forward orbit of U, given by $\bigcup_{n \geq 0} f^n(U)$, covers everything in $\mathbb{C}$ except possibly one point (two points if we include ∞).

Now, since a Blaschke product is composed of various Möbius transformations all multiplied together, we have $|B(z)| < 1$ whenever $|z| < 1$ and $|B(z)| > 1$ whenever $|z| > 1$. This implies that $B(\mathbb{D}) = \mathbb{D}$ and $B(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Therefore, for any $z \in \mathbb{D}$, choose a neighborhood U contained in $\mathbb{D}$. The forward orbit of U can never leave $\mathbb{D}$, so z cannot be in the Julia set. Similarly, for any $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, choose a neighborhood U contained in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The forward orbit of U can never enter $\mathbb{D}$ and so again z cannot be in the Julia set. This means that the Julia set of a Blaschke product is contained in $\partial\mathbb{D}$.

The following is a rough description of why the Julia set takes the form it does for the classification of Blaschke products:

(i) Elliptic: any point on $\partial\mathbb{D}$ has points nearby in $\mathbb{D}$ which iterate to the fixed point z_0 and points nearby in $\mathbb{C} \setminus \overline{\mathbb{D}}$ which iterate to the fixed point $1/\overline{z_0}$, and so the behavior is chaotic.

(ii) Parabolic: depends on whether $B''(z_0)$ is 0 or not, but this case is complicated!

(iii) Hyperbolic: there is an attracting fixed point (the Denjoy-Wolff point) z_0 on $\partial\mathbb{D}$. Any attracting fixed point is contained in the Fatou set, so there is a neighborhood of z_0 in $\partial\mathbb{D}$ contained in the Fatou set (since the Fatou set is open). Taking the pre-images of this interval in $\partial\mathbb{D}$ leaves just a Cantor set.

In conclusion, for Blaschke products, the Julia set is the whole of $\partial\mathbb{D}$ or a Cantor subset of $\partial\mathbb{D}$. These two cases can be characterized as follows (see [3] for references to this characterization).

Proposition 4.2.1. *Let B be a non-trivial finite Blaschke product. Then:*

(i) if B is elliptic, $J(B) = \partial\mathbb{D}$;

(ii) if B is hyperbolic, $J(B)$ is a Cantor subset of $\mathbb{D}$;

(iii) if B is parabolic and $z_0 \in \partial\mathbb{D}$ is the Denjoy-Wolff point of B, $J(B) = \partial\mathbb{D}$ if $B''(z_0) = 0$ and $J(B)$ is a Cantor subset of $\partial\mathbb{D}$ if $B''(z_0) \neq 0$.

CHAPTER 5

THE MAIN THEOREM

We will now only deal with Blaschke products of the form

$$B(z) = e^{i\theta} \left(\frac{z - w}{1 - \overline{w} z} \right)^n \quad , n \geq 2,$$

where $e^{i\theta} \in \partial\mathbb{D}$ and$w \in \mathbb{D}$. So we have two parameters that specify the Blaschke product, $(e^{i\theta}, w) \in \partial\mathbb{D} \times \mathbb{D}$, the parameter space. Therefore the parameter space is a torus. Next we will describe the subset of this parameter space that gives Blaschke products with connected Julia sets. Recall $J(B) \subset \partial\mathbb{D}$ is either all of $\partial\mathbb{D}$ or a Cantor subset.

Theorem 5.0.2. *We have the following identification:* $\{(e^{i\theta}, w) : B \textit{ is elliptic}\} = \{(e^{i\theta}, w) : J(B) \textit{ is connected}\}$.

From Proposition 4.2.1, we know that when B is hyperbolic, we have that $J(B)$ is a Cantor subset of $\mathbb{D}$. For the particular subclass of Blaschke products we are interested in, if B is parabolic, then for the Denjoy-Wolff point z_0 we have $B''(z_0) \neq 0$. So we know that when B is parabolic we have that $J(B)$ is a Cantor subset of $\mathbb{D}$. Therefore, the only time we have $J(B)$ connected is when B is elliptic.

Now we want to find out exactly which of those Blaschke products are elliptic. Let us go back to our parameter space, the torus, and fix $e^{i\theta}$. This gives us a slice of the torus, so we now have a disk (Figure 5.1). Within the disk, let us fix a direction $e^{i\psi}$ and consider the ray

$se^{i\psi}, 0 \le s \le 1$. ($w = se^{i\psi}$ in polar coordinates.) We have now reduced down to only looking at Blaschke products of the form

$$B(z) = e^{i\theta}\left(\frac{z - se^{i\psi}}{1 - se^{-i\psi}z}\right). \tag{5.0.1}$$

When $s = 0, \quad B(z) = e^{i\theta}z^n$. So when $|z| = 1, \quad |B'(z)| = n|e^{i\theta}||z|^{n-1} = n > 1$. Thus, B must be elliptic.

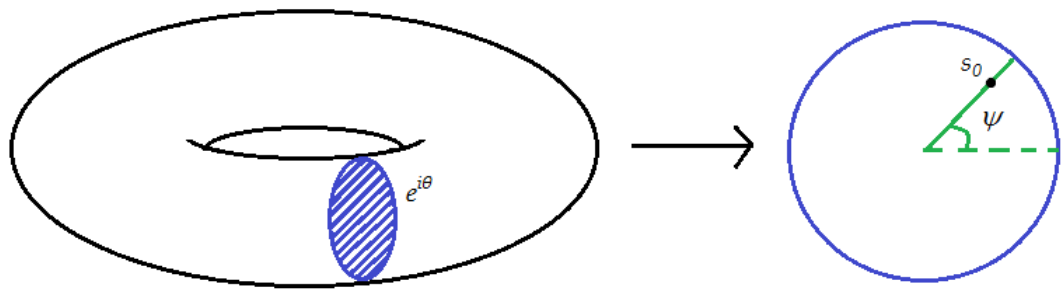

Figure 5.1: Parameter space for the subclass of Blaschke products.

Theorem 5.0.3. *There exists $s_0 \in (0, 1]$ so that:*

- $s < s_0$, *B is elliptic,*
- $s = s_0$, *B is parabolic,*
- *and* $s > s_0$, *B is hyperbolic.*

So $s < s_0$ gives elliptic Blaschke products of the form in (5.0.1). If we were to repeat this over all rays, we would see that the parameters that give elliptic Blaschke products are a star-like domain (Figure 5.2). Repeating over the entire torus, we see the elliptic Blaschke products form what is called the domain of ellipticity. The domain of ellipticity is the subset of parameter space for which $J(B)$ is connected. This set is open, contrasting with the Mandelbrot set which is closed.

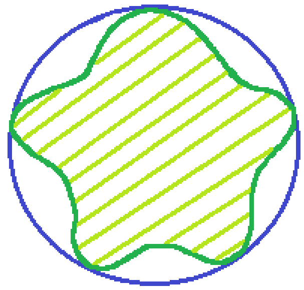

Figure 5.2: Star-like domain.

Now we will show the mechanics of Theorem 5.0.3.

If B is hyperbolic or parabolic, there is some Denjoy-Wolff point $z_0 \in \partial\mathbb{D}$, where $|B(z_0)| \leq 1$. So if we have $|B'(z)| > 1$ on $\partial\mathbb{D}$, B must be elliptic. The only way for B to be hyperbolic or parabolic is if $|B'(z)| \leq 1$ somewhere. Let $K = \{z \in \partial\mathbb{D} : |B'(z)| \leq 1\}$. We have calculations showing that $K = \emptyset$ if $s < \frac{n-1}{n+1}$. So B is guaranteed to be elliptic if $s < \frac{n-1}{n+1}$. Just because K is non-empty that does not mean we will have a Denjoy-Wolff point in K. The necessary conditions for B to be hyperbolic are to have K be non-empty and have K contain a fixed point which must be a Denjoy-Wolff point. Similarly, for parabolic, we need K non-empty, but we must also have a fixed point at one of the endpoints of K, and this fixed point will be the Denjoy-Wolff point.

From [3], we have these three lemmas:

Lemma 5.0.4. *The set K is empty for $s < \frac{n-1}{n+1}$, the single point $e^{i(\psi+\pi)}$ for $s = \frac{n-1}{n+1}$, and is an arc in $\partial\mathbb{D}$ centred at $e^{i(\psi+\pi)}$ for $s > \frac{n-1}{n+1}$.*

Lemma 5.0.5. (Figure 5.3) *If $s \geq \frac{n-1}{n+1}$, then $|K| = 2\pi - 2\cos^{-1}(t)$, where*

$$t = t(s) = \frac{1 - n + (1+n)s^2}{2s},$$

and $|B(K)| = 2\pi - 2\cos^{-1}(u)$, *where*

$$u = u(s) = \frac{1 - n - (1+n)s^2}{2ns}.$$

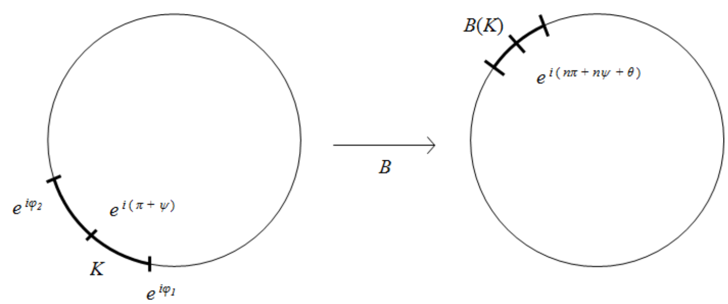

Figure 5.3: Diagram showing the action of B on K when $s > \frac{n-1}{n+1}$.

Lemma 5.0.6. *For* $s \in (\frac{n-1}{n+1}, 1)$, *let* $p(s) = |K|$ *and* $q(s) = |B(K)|$. *Then* $p'(s) > q'(s)$ *and so* $|K|$ *grows faster than* $|B(K)|$.

Lemmas 5.0.4, 5.0.5, and 5.0.6 together prove Theorem 5.0.3 in the following way (Figure 5.5). Once $s \geq \frac{n-1}{n+1}$, the set K is non-empty and $|K|$ grows as s grows, with $|K| \to 2\pi$ as $s \to 1$. Similarly, $|B(K)|$ grows to a certain point, then shrinks, with $|B(K)| \to 0$ as $s \to 1$. So unless we are in the special case where the center of $B(K)$ is directly opposite the center of K (Figure 5.4), eventually an endpoint of K will agree with an endpoint of $B(K)$. This value of s gives us s_0 in Theorem 5.0.3, which means that B is parabolic. As s increases beyond s_0, B will have a fixed point inside K, but not an endpoint of K, which means that B is hyperbolic.

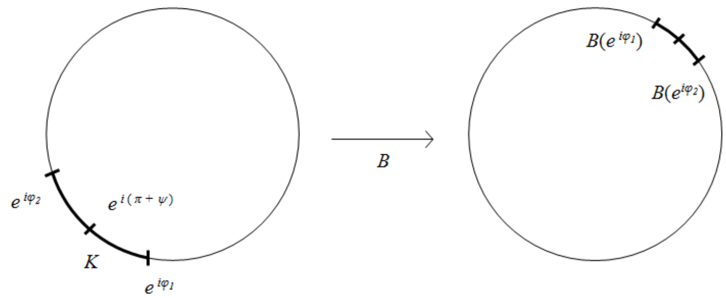

Figure 5.4: Diagram showing the special case where the center of $B(K)$ is directly opposite the center of K.

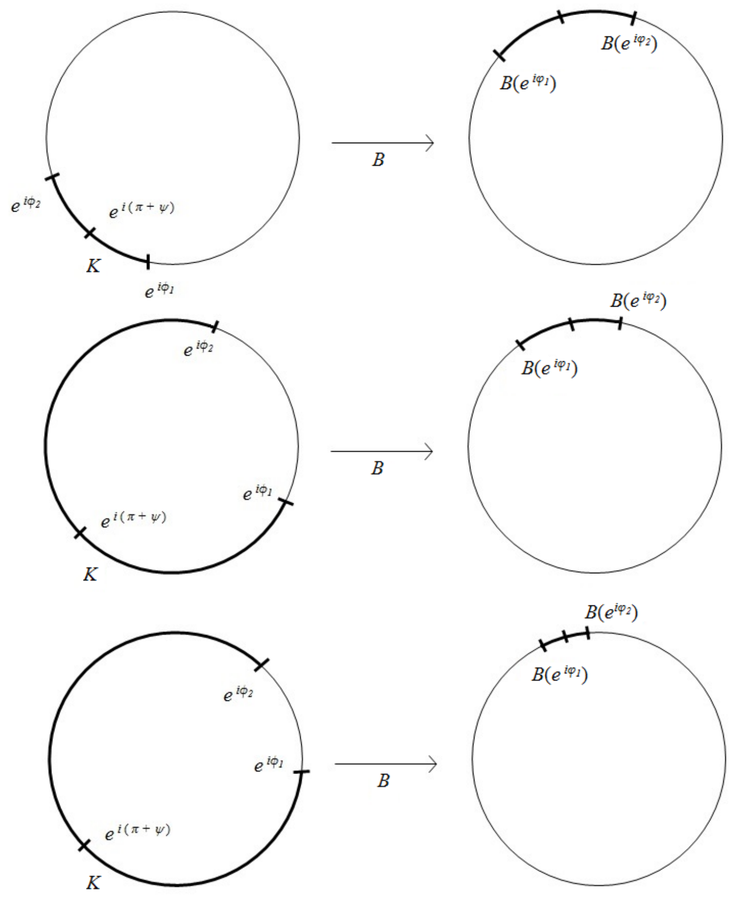

Figure 5.5: Diagram showing the three situations that arise in Theorem 5.0.3, the elliptic case ($s < s_0$) where there are no fixed points on B in K (top), the parabolic case ($s = s_0$) where $B(e^{i\phi_2}) = e^{i\phi_2}$ (middle), and the hyperbolic case ($s > s_0$) where $B(K)$ is strictly contained in K (bottom).

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