Sliding Mode Control of Constrained Nonlinear Systems

Gian Paolo Incremona, Matteo Rubagotti, and Antonella Ferrara

Abstract—This technical note introduces the design of sliding mode control algorithms for nonlinear systems in the presence of hard inequality constraints on both control and state variables. Relying on general results on minimum-time higher-order sliding mode for unconstrained systems, a general order control law is formulated to robustly steer the state to the origin, while satisfying all the imposed constraints. Results on minimum-time convergence to the sliding manifold, as well as on the maximization of the domain of attraction, are analytically proved for the first-order and second-order sliding mode cases. A general result is presented regarding the domain of attraction in the general order case, while numerical results on the estimation of the domain of attraction and on minimum-time convergence are discussed for the third-order case, following a procedure applicable to a sliding mode of any order.

Index Terms—Sliding mode control, second order sliding mode, higher order sliding mode (HOSM), uncertain systems, constrained control.

I. INTRODUCTION

Input constraints are present in all practical control implementations, mainly in the form of saturations. In order to guarantee an acceptable performance of the controlled system in their presence, different solutions have been proposed, mainly in the field of anti-windup control (see, e.g., [1], [2]). In addition, in order to avoid failures or critical conditions of the controlled system, certain regions of the state space need to be avoided during the execution of many tasks. In some cases, a conservative tuning of the control laws can lead to the avoidance of these regions. On the other hand, control laws that directly consider the presence of state constraints can reduce conservativity and improve the overall system performance. The most well-known control methodology able to manage both input and state constraints is Model Predictive Control (MPC), for which the reader is referred to [3] and the references therein.

A control technique that naturally handles the presence of some classes of uncertainties and disturbances is Sliding Mode Control (SMC), in which a discontinuous control law steers the state onto a suitably-defined hyper-surface (the so-called *sliding manifold*), and, under suitable design conditions, makes the origin of the state space an asymptotically stable equilibrium point for the closed-loop system. The convergence to the sliding manifold is guaranteed in a finite-time interval if the control action is large enough to counteract the effect of the uncertain terms. After reaching the sliding manifold, the evolution of the state variables is insensitive to the so-called *matched disturbances*, i.e., those acting in the same channel as the control variable [4], [5]. The main drawback of SMC is the so-called *chattering* [6], [7], which is the high-frequency oscillatory motion around the sliding manifold due to the discontinuity of the control law. Higher Order Sliding Mode (HOSM) is a possible solution for chattering reduction (see, e.g., [8]–[17], and the references therein included). In particular, [16] proposes an algorithm that guarantees a time-optimal reaching of the sliding manifold *for arbitrary order* (i.e., dimension of the sliding manifold, see, e.g., [18]). Note that

This is the final version of the accepted paper submitted to IEEE Transactions on Automatic Control. G.P. Incremona is with the Dipartimento di Ingegneria Civile ed Architettura, University of Pavia, Via Ferrata 3, 27100 Pavia, Italy (e-mail: gianpaolo.incremona01@universitadipavia.it).

M. Rubagotti is with the Department of Engineering, University of Leicester, University Rd, Leicester LE1 7RH, United Kingdom (mr298@le.ac.uk).

A. Ferrara is with the Dipartimento di Ingegneria Industriale e dell'Informazione, University of Pavia, Via Ferrata 5, 27100 Pavia, Italy (e-mail: antonella.ferrara@unipv.it).

chattering reduction is not the only possible purpose of HOSM, since this latter allows the use of a relative degree greater than one between the discontinuous control input and the sliding variable.

In the first-order SMC formulation, input saturations are immediately satisfied if the control variable switches between values that are inside the imposed boundaries. When HOSM is used for chattering reduction, specific solutions have been proposed to achieve the satisfaction of saturation constraints (see, e.g., [19], [20] for the second-order case). On the other hand, the possible presence of state constraints is usually not taken into account in SMC formulations. Recently, few solutions have been proposed in order to merge SMC and MPC, and combine the constraints satisfaction property of MPC with the robustness of SMC [21]–[25]. As an alternative, in order to avoid the additional computational burden of MPC, the presence of state constraints has been directly inserted in the SMC law in [26]–[28] for the first-order sliding mode case, in [29]–[32] for the second order sliding mode case, and in [33] for third-order sliding mode with box constraints.

In this technical note, an HOSM control law of general order r is proposed, aimed at guaranteeing the minimum-time convergence of the state onto the sliding manifold, and, at the same time, satisfying the imposed hard inequality constraints on input and states, in the presence of matched disturbances. More specifically, the main contributions of the present work are the following: first of all, the formulation of a control law capable of solving an r-th order SMC problem for uncertain nonlinear affine systems with inequality constraints on both input and state variables (note that this was an open problem, to the best of the authors' knowledge); second, a procedure to select the sliding variable in order to satisfy the constraints, including two different sufficient conditions which provide guidance in the choice of the sliding variable; third, the proof of the minimum-time convergence with constraint satisfaction and maximization of the domain of attraction for the first and second-order cases (a preliminary result on this aspect, limited to the second-order case, was presented in [30], without proof of the minimum-time convergence); fourth, a general result on the domain of attraction for the r-th order case, along with the specific numerical study of the domain of attraction and of the minimum-time convergence for the third-order case. Note that preliminary results on the numerical evaluation of the domain of attraction for a third-order sliding mode controller (only for the case of box state constraints) were described in [33], and can be considered now as particular cases of the proposed general formulation.

This technical note is organized as follows. After formulating the regulation problem in Section II, Section III describes the design of the proposed control law. Theoretical results are proved in Section IV for the first and second-order cases, while more general results and possible developments are discussed in Section V. Conclusions are finally drawn in Section VI. For the sake of readability, most of the theoretical proofs are reported in the Appendix.

II. PROBLEM FORMULATION

In this technical note, we consider a class of uncertain nonlinear dynamical systems, defined by

$$\dot{x}(t) = \phi(x(t), t) + \gamma(x(t), t)u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\gamma : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are uncertain smooth vector fields, and the whole state vector is assumed to be available for feedback. The control objective is the regulation of the state x to the origin. Differently from classical SMC formulations, a set of (possibly mixed) hard input and state inequality constraints is introduced. More

specifically, it is required that

$$\bar{x} \in \bar{\mathcal{X}}, \quad \bar{x} \triangleq \begin{bmatrix} x' & u \end{bmatrix}' \in \mathbb{R}^{n+1}, \tag{2}$$

where $'$ denotes the transposition operator. $\bar{\mathcal{X}} \subseteq \mathbb{R}^{n+1}$ is a closed (possibly unbounded) set that includes the origin in its interior, and is formulated as a system of (possibly nonlinear) algebraic equations.

III. The proposed HOSM control law

A. Definition of the augmented system

The control law can either be defined as a discontinuous control law (classical SMC), or as the result of an m-fold time integration of a discontinuous signal $w(t)$ (HOSM aimed at chattering reduction). The case $m > 0$, m being an integer number, is considered first. An integrator-chain dynamics is added to the system, starting from $x_{n+1} \triangleq u$, as

$$\begin{cases} \dot{x}_{n+1}(t) = x_{n+2}(t) \\ \dot{x}_{n+2}(t) = x_{n+3}(t) \\ \quad \vdots \\ \dot{x}_{n+m}(t) = w(t). \end{cases} \tag{3}$$

Note that the chain of integrators will be an element of the closed loop control system for which the stability results proved in Sections IV and V hold.

In order to provide bounds on the derivatives of the actual control variable u, and recalling that the constraints on $u = x_{n+1}$ are already imposed in (2), a set of box constraints is defined, as

$$w \in [-\alpha, \alpha], \ x_{n+i} \in [-\beta_i, \beta_i], \ i = 2, \dots, m, \tag{4}$$

$\alpha, \beta_i > 0$ being fixed parameters. The overall augmented system dynamics, with state $x_a \triangleq \begin{bmatrix} x_1 & \dots & x_{n+m} \end{bmatrix}' \in \mathbb{R}^{n+m}$ is now represented by

$$\dot{x}_a(t) = \Phi(x_a(t), t) + \Gamma(x_a(t), t)w(t) \tag{5}$$

where $\Phi : \mathbb{R}^{n+m+1} \to \mathbb{R}^{n+m}$ and $\Gamma : \mathbb{R}^{n+m+1} \to \mathbb{R}^{n+m}$ are smooth vector fields immediately obtainable from (1) and (3). For the sake of compactness, the sets of disjoint constraints in (2) and (4) can be merged as

$$x_a \in \mathcal{X}_a, \ w \in [-\alpha, \alpha], \tag{6}$$

with implicit definition of $\mathcal{X}_a$.

Remark 1: As m increases, the smoothness of the control signal u also increases [12], together with the number of states of system (5), with a consequent increase in the complexity of the controller. On the other hand, the chattering amplitude does not necessarily decrease as m increases [34]. Thus, there is not a "best value" for m in general: we propose an approach valid for any value of m, and leave the decision on its value, for each specific application, to the designer of the control law. □

Remark 2: The analysis of chattering is outside the scope of this paper. Yet, it is an interesting topic that the reader may deepen making reference to [6], [34]–[37]. Indeed, in systems with chains of integrators and certain dynamics of lag type in series, chattering may exist or not depending on the type of the discontinuous control law (ideal relay or relay with hysteresis) and the effect of parasitic dynamics. □

In the particular case when u is directly defined as the discontinuous control variable (i.e., $m = 0$ and therefore $w = u$) the constraints in (2) are required to be formulated as disjoint input and state constraints, and precisely

$$\bar{\mathcal{X}} = \{(x, u) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathcal{X}, u \in [-\alpha, \alpha]\}, \tag{7}$$

where $\mathcal{X}$ is, in general, a closed set, while $\alpha > 0$ is the same fixed parameter as in (4), which defines the maximum amplitude of the control variable. In order to consider (5) as the general formulation, the case $m = 0$ can be formulated as a particular case of (5), with $x_a = x$, $\Phi(\cdot, \cdot) = \phi(\cdot, \cdot)$, $\Gamma(\cdot, \cdot) = \gamma(\cdot, \cdot)$, and $\mathcal{X}_a = \mathcal{X}$.

B. Definition of the sliding manifold

After defining the augmented system (5) and the related constraint sets in (6), the next step consists of defining a suitable output variable $\sigma_1(t) \in \mathbb{R}$, as a (in general, nonlinear) function of x_a. Following the standard design procedure of HOSM control [10], a vector of time derivatives of σ_1 is defined as

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_r \end{bmatrix}' \triangleq \begin{bmatrix} \sigma_1 & \dot{\sigma}_1 & \dots & \sigma_1^{(r-1)} \end{bmatrix}' \in \mathbb{R}^r \tag{8}$$

where $r \in [0, n+m]$ is the well defined, uniform and time-invariant relative degree of the system (assuming w as input and σ_1 as output).

In the $(n+m)$-dimensional space defined by the components of x_a, the manifold $\Sigma \triangleq \{x_a : \sigma(x_a) = 0\}$ is referred to as *sliding manifold*. The control variable will be defined in order to ensure the finite-time convergence of x_a on the sliding manifold, which (assuming a correct definition of σ) will in turn imply the asymptotic convergence of x_a to zero. With reference to [38, Theorem 13.1], a diffeomorphism $\Omega : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m-r} \times \mathbb{R}^r$ is defined, such that

$$(\zeta, \sigma) = \Omega(x_a) \tag{9}$$

where $\zeta \in \mathbb{R}^{n+m-r}$ is the internal state vector. The diffeomorphism allows one to transform system (5) into the normal form

$$\dot{\zeta}(t) = \psi(\zeta(t), \sigma(t), t) \tag{10a}$$

$$\dot{\sigma}_i(t) = \sigma_{i+1}(t), \ i = 1, \dots, r - 1 \tag{10b}$$

$$\dot{\sigma}_r(t) = f(\zeta(t), \sigma(t), t) + g(\zeta(t), \sigma(t), t)w(t) \tag{10c}$$

with $\psi : \mathbb{R}^{n+m+1} \to \mathbb{R}^{n+m-r}$, $f : \mathbb{R}^{n+m+1} \to \mathbb{R}$, $g : \mathbb{R}^{n+m+1} \to \mathbb{R}$. The dynamics of system (5) includes uncertain terms, as specified when defining the vector fields ϕ and γ in (1). As a consequence, ψ, f and g in (10) also contain uncertain terms, but some information about them is available, as stated in the following assumption.

Assumption 1: There exist positive constants F, G_1, G_2, such that

$$|f(\zeta(t), \sigma(t), t)| \le F, \tag{11}$$

$$0 < G_1 \le g(\zeta(t), \sigma(t), t) \le G_2, \tag{12}$$

$$\alpha_r \triangleq G_1 \alpha - F > 0. \tag{13}$$

Moreover, the following two hypotheses are valid for the internal dynamics (10a). Firstly, for all initial conditions $(\zeta(0), \sigma(0)) \in \mathbb{R}^{n+m}$, for all realizations of the uncertain terms satisfying (11)-(12), the internal dynamics (10a) does not present finite time escape phenomena. Secondly, the zero dynamics

$$\dot{\zeta}(t) = \psi(\zeta(t), 0, t). \tag{14}$$

is globally asymptotically stable. □

These are typical assumptions in SMC: (11)-(12) require the boundedness of the uncertain terms, (13) ensures that the control amplitude is large enough to counteract their effect, the assumption on the internal dynamics and on the zero dynamics (14) imply that, during the reaching phase the internal states remain bounded, while, once σ has been steered to zero, ζ will also converge to zero asymptotically. By means of the diffeomorphism $\Omega(x_a)$ in (9), the set $\mathcal{X}_a$ can be in general mapped into a new set $\mathcal{S}$, i.e.,

$$x_a \in \mathcal{X}_a \iff (\zeta, \sigma) \in \mathcal{S} \tag{15}$$

which will be used to enforce the constraints in the coordinate system defined by (10). The presence of these constraints requires an additional hypothesis.

Assumption 2: The diffeomorphism $\Omega(x_a)$ is defined such that $\mathcal{S} = \mathcal{S}(\sigma) \subset \mathbb{R}^r$, i.e., the expression of the constraint set does not depend on the internal state ζ. $\square$

Assumption 2 is required by the fact that, in SMC, the control law is defined as a function of σ. As a consequence, it would not be possible to enforce constraint satisfaction on the internal state ζ during the reaching phase. For this reason, the constraints have to be expressed in the r-dimensional space defined by σ.

An explicit expression for $\Omega(x_a)$ is not available in general. However, in the following we provide two different sufficient conditions, for which $\Omega(x_a)$ can be immediately defined. We also show an example in which neither of these conditions are satisfied, but nonetheless $\Omega(x_a)$ can be easily found. For the general case, the solution to this problem has to be considered by the designer of the control system as a specific step of the design process.

C. Design of the sliding manifold

The definition of $\Omega(x_a)$ (or, equivalently, of σ_1) such that σ satisfies Assumption 2 requires, in general, some attention. Guidelines are provided in the following. Let us start by describing two particular cases in which Assumption 2 can be easily satisfied.

Lemma 1: Consider a given augmented dynamics (5) with associated constraints (6). If σ is defined such that $r = n + m$, then Assumption 2 is satisfied. $\square$

Proof: Since $r = n + m$, $\Omega(x_a)$ defines a mapping from x_a to σ, and no internal state ζ exists. Therefore, each element of x_a can be expressed as a function of the elements of σ. Substituting the corresponding expressions of σ in the formulation of $\mathcal{X}_a$, the set $\mathcal{S}$ is obtained, which is a function of σ only. ∎

Example 1: Consider the augmented system given by

$$\dot{x}_1(t) = \sin(x_2(t)) + u(t)$$
$$\dot{x}_2(t) = u(t)$$

with constraint set $\mathcal{X}_a = \{(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1\}$. Choosing $\sigma_1 = x_1 - x_2$, we obtain a vector $\sigma \in \mathbb{R}^2$, the dynamics of which is defined by

$$\dot{\sigma}_1(t) = \sigma_2(t)$$
$$\dot{\sigma}_2(t) = \cos(x_2(t))w(t)$$

with

$$\sigma = \Omega(x_a) = \begin{bmatrix} x_1 - x_2 \\ \sin(x_2(t)) \end{bmatrix} \quad x_a = \Omega^{-1}(\sigma) = \begin{bmatrix} \sigma_1 + \sin^{-1}(\sigma_2) \\ \sin^{-1}(\sigma_2) \end{bmatrix}.$$

The constraint set is defined as $\mathcal{S} = \{(\sigma_1, \sigma_2) : |\sigma_1 + \sin^{-1}(\sigma_2)| \leq 1, |\sigma_2| \leq \sin(1)\}$, which is a function of σ only. $\square$

Lemma 2: Assume that the state vector $x_a \in \mathbb{R}^{n+m}$ of the augmented system is composed of two subvectors, as $x_a = \begin{bmatrix} x_b' & x_c' \end{bmatrix}'$, with $x_b \in \mathbb{R}^{n+m-p}$, $x_c \in \mathbb{R}^p$, $p \in \{1, 2, \ldots, n - m\}$. As a consequence, (5) takes the form

$$\dot{x}_b(t) = \Phi_b(x_b(t), x_c(t), t) \tag{16}$$
$$\dot{x}_c(t) = \Phi_c(x_c(t), t) + \Gamma_c(x_c(t), t)w(t) \tag{17}$$

with implicit definition of Φ_b, Φ_c, and Γ_c. In addition, assume that the constraint set $\mathcal{X}_a$ is a function of x_c only. Then, if $\sigma = \sigma(x_c)$ with $r = p$, Assumption 2 is satisfied. $\square$

Proof: Since $r = p$, the same development of the proof of Lemma 1 can be followed, for subsystem (17). $\Omega^{-1}(\sigma)$ will map σ to x_c, and the evolution of the internal state variables ζ will not influence the dynamics of σ. Again, by substituting the expression of σ into the expression of $\mathcal{X}_a$, the set $\mathcal{S}$ will be obtained as a function of σ only. ∎

Example 2: Consider as augmented system

$$\dot{x}_1(t) = -x_1(t)e^{-x_1(t)} + x_2^5(t)$$
$$\dot{x}_2(t) = -x_2^3(t) + w(t)$$

with constraint set $\mathcal{X}_a = \{(x_1, x_2) : x_2 \in [-1, 3]\}$. In this case, the state vector can be divided into two scalar components $x_b = x_1$ and $x_c = x_2$.

Choosing $\zeta = x_1$ and $\sigma_1 = x_2$, we obtain a scalar $\sigma \in \mathbb{R}$, and the dynamics of the system in form (10) is described by

$$\dot{\zeta}(t) = -\zeta(t)e^{-\zeta(t)} + \sigma_1^5(t)$$
$$\dot{\sigma}_1(t) = -\sigma_1^3(t) + w(t)$$

with

$$\begin{bmatrix} \zeta \\ \sigma_1 \end{bmatrix} = \Omega(x_a) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_a = \Omega^{-1}(\zeta, \sigma) = \begin{bmatrix} \zeta \\ \sigma_1 \end{bmatrix}.$$

The constraint set is defined as $\mathcal{S} = \{\sigma_1 : \sigma_1 \in [-1, 3]\}$, which is a function of σ only. $\square$

In the general case, Assumption 2 is not straightforwardly verified so that a careful choice of σ_1 must be made for each case. A simple example is shown next.

Example 3: The augmented system dynamics is given by

$$\dot{x}_1(t) = -x_1(t) + x_2(t) + w(t)$$
$$\dot{x}_2(t) = -x_2(t) + w(t)$$

with associated constraint set $\mathcal{X}_a = \{(x_1, x_2) : |x_1 + x_2| \leq 1\}$. Since $\mathcal{X}_a$ depends on both state variables, it is not possible to use the result in Lemma 2. The constraint set $\mathcal{S}$ has to be a function of σ only, so we choose $\sigma_1 = x_1 + x_2$. The internal state can be chosen as $\zeta = x_1 - x_2$.

As in the previous example, we obtain a scalar $\sigma \in \mathbb{R}$, and the dynamics of the system in form (10) is

$$\dot{\zeta}(t) = -\frac{3}{2}\zeta(t) + \frac{1}{2}\sigma_1(t)$$
$$\dot{\sigma}_1(t) = \frac{1}{2}\zeta(t) + \frac{1}{2}\sigma_1(t) + 2w(t)$$

with

$$\begin{bmatrix} \zeta \\ \sigma_1 \end{bmatrix} = \Omega(x_a) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}, x_a = \Omega^{-1}(\zeta, \sigma) = \frac{1}{2}\begin{bmatrix} \zeta + \sigma_1 \\ \zeta - \sigma_1 \end{bmatrix}.$$

The constraint set is defined as $\mathcal{S} = \{\sigma_1 : |\sigma_1| \leq 1\}$, which is a function of σ only. $\square$

D. Definition of the control law

The control law $w(t)$ has to be defined as a discontinuous function of σ. In [16, Sec. III.C], an HOSM control law for a system in form (10b)-(10c) was proposed (i.e., for arbitrary value of r), which guaranteed the reaching of the sliding manifold in minimum time, for the worst-case realization of the uncertain terms (i.e., $f(\zeta(t), \sigma(t), t) \equiv -F \cdot \text{sgn}(w(t))$ and $g(\zeta(t), \sigma(t), t) \equiv G_1$). In this worst-case formulation, α_r in (13) can be interpreted as a *reduced control amplitude* (however, notice that α_r and α have different units of measure). We denote the control law proposed in [16], which *did not take into account the presence of constraints*, as $w_{in}(\sigma) \triangleq (-1)^{r+1} \cdot \alpha \cdot c(\sigma)$, where function $c(\sigma) : \mathbb{R}^r \to \{-1, +1\}$ is determined as reported in [16]. In general, one obtains an $(r-1)$-dimensional manifold in the r-dimensional space generated by vector σ, described by the nonlinear equation $s(\sigma) = 0$. This manifold,

which includes the origin, divides the σ-space into two half-spaces. By imposing a constant value of c (either -1 or $+1$) in each half-space, the control law can be written as

$$w_{in}(\sigma) = -\alpha \cdot \text{sgn}(s). \qquad (18)$$

As mentioned in [16], the complexity of the expression of $s(\sigma)$ grows very fast with the order r, and the derivation of efficient methods (numerical or exact-algebraic) in order to define it, is still an open problem. A method for determining the expression of $s(\sigma)$ for generic order is proposed in [39]. For relatively low-dimensional cases, the analytical expression of $s(\sigma)$ is given by

$$\sigma \in \mathbb{R}^1 \Rightarrow s(\sigma) = \sigma_1 \qquad (19)$$

$$\sigma \in \mathbb{R}^2 \Rightarrow s(\sigma) = \sigma_1 + \frac{\sigma_2|\sigma_2|}{2\alpha_r} \qquad (20)$$

$$\sigma \in \mathbb{R}^3 \Rightarrow s(\sigma) = \sigma_1 + \frac{\sigma_3^3}{3\alpha_r^2} + \text{sgn}\left(\sigma_2 + \frac{\sigma_3^2 \, \text{sgn}(\sigma_3)}{2\alpha_r}\right) \times$$

$$\times \left[\frac{1}{\sqrt{\alpha_r}}\left(\text{sgn}\left(\sigma_2 + \frac{\sigma_3^2 \, \text{sgn}(\sigma_3)}{2\alpha_r}\right)\sigma_2 + \frac{\sigma_3^2}{2\alpha_r}\right)^{\frac{3}{2}} + \frac{\sigma_2\sigma_3}{\alpha_r}\right]. \qquad (21)$$

To be precise, in the problem defined in [16], the expression of w_{in} is found also in the different sub-cases when $\sigma = 0$, which would make the formulation of the control law more complicated. However, as pointed out also in [16], being $s(\sigma)$ a null-measure set, the definition of the control law for $\sigma = 0$ has no practical relevance. For this reason, the value of $w_{in}(\sigma)$ when $s(\sigma) = 0$ will not be explicitly defined, and, when needed in the proofs, will be determined as the Filippov solution of the given discontinuous vector field at σ (see [40]).

In order to take the presence of constraints into account, the control law proposed in this technical note coincides with $w_{in}(\sigma)$ when $\sigma \in \mathcal{S}$, while for $\sigma \notin \mathcal{S}$ is defined as $w_{out}(\sigma) \triangleq -\alpha \cdot \text{sgn}(\sigma_r)$. Overall,

$$w(\sigma) = \begin{cases} -\alpha \cdot \text{sgn}(s(\sigma)), & \sigma \in \mathcal{S} \\ -\alpha \cdot \text{sgn}(\sigma_r), & \sigma \notin \mathcal{S}. \end{cases} \qquad (22)$$

which is an extremely simple law, yet capable of solving a rather complex nonlinear constrained control problem for uncertain systems.

After the introduction of the proposed control law (22), three interesting issues regarding the closed-loop system are addressed in the following. The first is the determination of the domain of attraction $\mathcal{S}_I$ of the origin for the closed-loop system, i.e., all the initial conditions $\sigma(0)$ for which $\sigma(t)$ converges to the origin in a finite time *without violating the constraints*. The second is the problem of determining if the obtained domain of attraction can be enlarged by using a different control law $w(t)$, with the same system dynamics, the same bounds on the uncertain terms and on the control amplitude, and the same state constraints. The third issue is related to investigating if $w(\sigma)$ in (22) still leads to minimum-time convergence to $\sigma = 0$ for the constrained case, as it was in [16] for the unconstrained case. In the next section, the cases $r = 1$ and $r = 2$ with box constraints on σ will be considered. For such cases, formal results will be presented. The general case will be tackled in Section V.

Remark 3: In many practical applications, SMC laws are implemented using a finite sampling time, or one of the so-called pseudo-sliding techniques (typically, employing a saturation function instead of the sign function, or low-pass filtering the discontinuous control signal). All of these techniques typically lead to the convergence to a boundary layer of the sliding manifold, rather than on the sliding manifold itself [41]. If such techniques are directly applied to approximate the proposed control law, the imposed constraints can be slightly violated, as the state evolves on a boundary layer around them. The typical solution consists in defining more conservative

constraint set $\tilde{\mathcal{S}} \subset \mathcal{S}$, so as to prevent the original constraints from being violated. $\square$

IV. ANALYTICAL RESULTS ON FIRST AND SECOND-ORDER SLIDING MODES WITH BOX CONSTRAINTS

In this section, the convergence in a finite (minimum) time of the sliding variable associated with the proposed closed-loop control system is discussed. Note that this result directly implies asymptotic stability of the origin of the closed-loop system since, by assumption, the zero dynamics (14) of system (1) transformed via the diffeomorphism $\Omega(x_a)$ is globally asymptotically stable.

A. First-Order Sliding Mode Control

The formulation of the control law (22) for $r = 1$, using (19), becomes

$$w(\sigma) = -\alpha \, \text{sgn}(\sigma_1) \qquad (23)$$

which coincides with the "unconstrained" control law w_{in} in (18). The constraint set in this simple case can only take the form

$$\mathcal{S} = \{\sigma_1 : \sigma_1 \in [\underline{\sigma}_1, \overline{\sigma}_1]\}. \qquad (24)$$

with $\underline{\sigma}_1 < 0$ and $\overline{\sigma}_1 > 0$. Then, the following result can be proved.

Theorem 1: Given system (10) with $r = 1$, assuming that Assumption 1 holds, if the control law (23) is applied, then σ_1 is steered to the origin with domain of attraction $\mathcal{S}_I$ coinciding with $\mathcal{S}$ in (24). The set $\mathcal{S}_I$ is the maximum obtainable domain of attraction for the given set of constraints, and the convergence takes place in minimum time for the worst possible realization of the disturbance terms (i.e., $f(\zeta(t), \sigma_1(t), t) \equiv -F \cdot \text{sgn}(w(t))$ and $g(\zeta(t), \sigma_1(t), t) \equiv G_1$). $\square$

Proof: See Appendix A. $\blacksquare$

B. Second-Order Sliding Mode Control

In the case $r = 2$, the second-order SMC law, according to (20) and (22), is defined as

$$w(\sigma) = \begin{cases} -\alpha \, \text{sgn}\left(\sigma_1 + \frac{\sigma_2|\sigma_2|}{2\alpha_r}\right) & \text{if } (\sigma_1, \sigma_2) \in \mathcal{S} \\ -\alpha \, \text{sgn}(\sigma_2) & \text{if } (\sigma_1, \sigma_2) \notin \mathcal{S} \end{cases} \qquad (25)$$

while the box constraints can be expressed as

$$\mathcal{S} = \{\sigma \in \mathbb{R}^2 : \sigma_1 \in [\underline{\sigma}_1, \overline{\sigma}_1], \, \sigma_2 \in [\underline{\sigma}_2, \overline{\sigma}_2]\} \qquad (26)$$

with $\underline{\sigma}_1, \underline{\sigma}_2 < 0$ and $\overline{\sigma}_1, \overline{\sigma}_2 > 0$. A graphical representation of the switching regions is reported in Figure 1.

Theorem 2: Given system (10) with $r = 2$, assuming that Assumption 1 holds, if the control law (25) is applied, then σ is steered to the origin with domain of attraction

$$\mathcal{S}_I \triangleq \mathcal{S} \setminus \{\mathcal{M}_0 \cup \mathcal{M}_1\} \qquad (27)$$

where

$$\mathcal{M}_0 \triangleq \left\{(\sigma_1, \sigma_2) : \sigma_1 > -\frac{\sigma_2|\sigma_2|}{2\alpha_r} + \overline{\sigma}_1, \, \sigma_2 > 0\right\}, \qquad (28)$$

$$\mathcal{M}_1 \triangleq \left\{(\sigma_1, \sigma_2) : \sigma_1 < -\frac{\sigma_2|\sigma_2|}{2\alpha_r} + \underline{\sigma}_1, \, \sigma_2 < 0\right\}. \qquad (29)$$

The set $\mathcal{S}_I$ is the maximum obtainable domain of attraction for the given set of constraints, and the convergence takes place in minimum time for the worst possible realization of the disturbance terms (i.e., $f(\zeta(t), \sigma(t), t) \equiv -F \cdot \text{sgn}(w(t))$ and $g(\zeta(t), \sigma(t), t) \equiv G_1$). $\square$
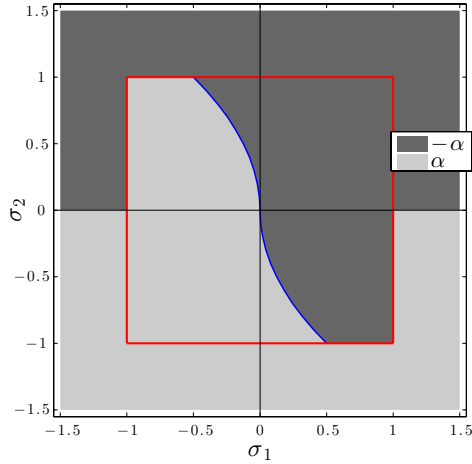
Proof: See Appendix B. $\blacksquare$

Figure 1. Graphical representation of the switching regions and box constraints in the second order sliding mode case.

Figure 2. Numerical estimate of the domain of attraction $\mathcal{S}_I$ for Example 4: (18) with $\alpha = 1$ (a), (25) with $\alpha = 1$ (b), (18) with $\alpha = 5$ (c), and (25) with $\alpha = 5$ (d).

V. GENERAL CASE: HIGHER ORDER SLIDING MODES WITH ARBITRARY CONSTRAINTS

In Section IV, the properties of the proposed scheme have been analyzed for constrained first-order and second-order sliding modes with box constraints. These represent very common and relevant cases (for instance, a mechanical system with constraints on position and velocity). Yet, sometimes one has to design higher-order sliding mode controllers and/or to deal with more general constraints. We provide now general considerations and results for the higher-order case with arbitrary constraints on σ.

In the cases analyzed in Section IV, it has been proved that the obtained domain of attraction is maximized for the given set of constraints. The following result can be proved for the general case.

Lemma 3: Given system (10) with generic r, assuming that Assumption 1 holds, if the control law (22) is applied, then the domain of attraction $\mathcal{S}_I$, for the given set of constraints, includes the domain of attraction $\mathcal{S}'_I$ obtained by applying the unconstrained control law (18). □

Proof: The result immediately follows by observing that, if $\sigma(0) \in \mathcal{S}'_I$, then $w(0) = w_{in}(0)$. Being $\mathcal{S}'_I$ by definition a robust positively invariant (RPI) set [42, Def. 4.3] when applying (18), the time evolution of σ will coincide with that obtained by applying (18). This implies that $\mathcal{S}'_I \subseteq \mathcal{S}_I$. ∎

Example 4: In order to show the domain of attraction in a rather general case, we consider the third-order SMC, for which the dynamics of the sliding variable is defined as

$$\dot{\sigma}_1(t) = \sigma_2(t)$$
$$\dot{\sigma}_2(t) = \sigma_3(t)$$
$$\dot{\sigma}_3(t) = w(t).$$

Note that, in order to be able to make comparisons, the uncertain terms have been fixed as $f(\zeta(t), \sigma(t), t) \equiv 0$ and $g(\zeta(t), \sigma(t), t) \equiv 1$, which implies $\alpha = \alpha_r$. The set of constraints is defined as $\|\sigma\|_2 \leq 1$, which is a sphere centered at the origin. By fixing $\alpha = 1$, we numerically obtain the domains of attraction shown in Figures 2a (63% of the sphere volume) and 2b (76% of the sphere volume) by applying the unconstrained control law (18) and the new proposed control law (22), respectively. Increasing the control amplitude to $\alpha = 5$, the corresponding domains of attraction are shown in Figures 2c and 2d, again for (18) and (22), respectively. The corresponding
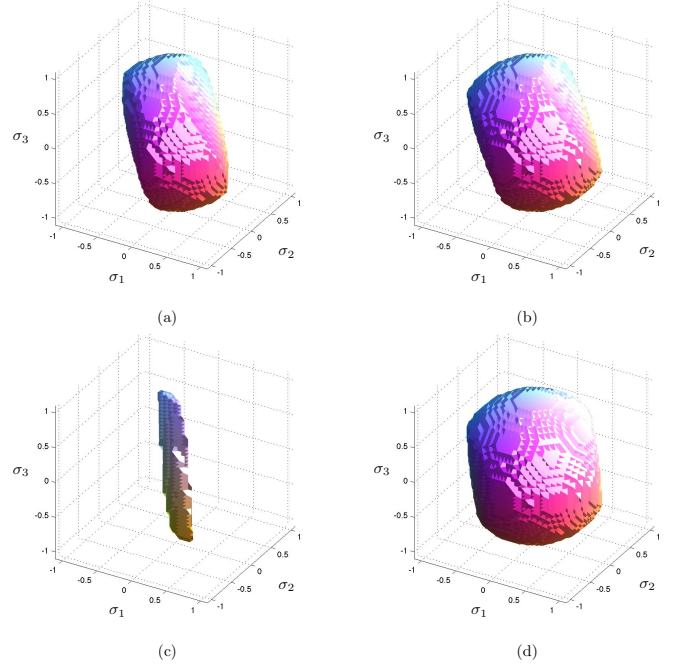
domains of attraction amount at 3% and 86% of the sphere volume, respectively. Two observations are in place:

- The domain of attraction using the constrained control law (22) is not reduced (actually, it is enlarged) with respect to that obtained by using (18), as expected from Lemma 3.
- The difference between the two domains of attraction seems to widen as α increases: this is due to the ability of the proposed control law (22) to impose a state trajectory that, when possible, moves on the boundary of $\mathcal{S}$ before moving to its interior. □

The result regarding minimum-time convergence also in the presence of constraints, formally proved in the previous section for lower-order cases, becomes analytically intractable for higher sliding mode orders. However, in order to show that the proposed method might be achieving the minimum-time convergence for the general case, we show the following example.

Example 5: Consider again the system of Example 4, this time with constraints defined as $\|\sigma\|_2 \leq 0.1$. In order to test the minimum-time converge properties of the controller, we set $\alpha = \alpha_r = 0.5$. Considering an initial condition $\sigma = [0.05 \ 0.05 \ 0]'$, the control law (22) is simulated with MATLAB using the Ode4 solver with a fixed discretization interval of 10^{-3} s. With the same initial condition and the same constraints, a minimum-time constrained optimization problem is solved numerically by using CVX [43], by implementing a bisection routine which runs a feasibility problem at every step, for a fixed time interval. In Figures 3-4 it is possible to compare the time evolution of the two control laws, and the trajectories of the two state vectors. From 0 to about 0.6 s, both control laws are equal to $-\alpha$. The two state trajectories are indistinguishable, and they both approach the boundary of the sphere. At about 0.6 s, the boundary is reached: the proposed control law switches between $-\alpha$ and $+\alpha$, while the numerical solver determines a solution that is continuous between the boundaries. The two state trajectories slide along the surface of the sphere, and are again indistinguishable. After 2 s, in both cases the control law keeps the value $+\alpha$, while at about 3.3

s it switches to $-\alpha$ (with a quick but continuous transition in the numerical case). The two state trajectories are still indistinguishable, and converge to the origin in about 4 s.

□

A similar procedure has been repeated for different systems and different sets of constraints, always obtaining indistinguishable state trajectories, and the same time interval for convergence to the origin. This can lead us to the conjecture that the proposed control law achieves minimum-time convergence in the general r-order case. The formal proof of such a result can be a topic for further research.
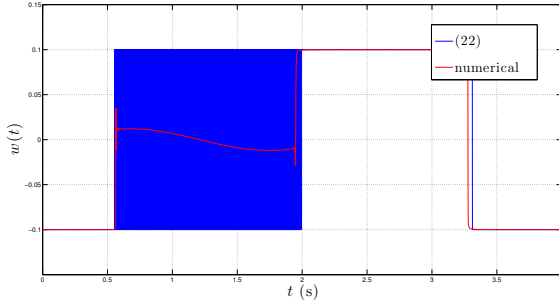

Figure 3. Comparison between the time evolutions of the proposed control law (22) and of the numerical minimum-time control law, for Example 5.
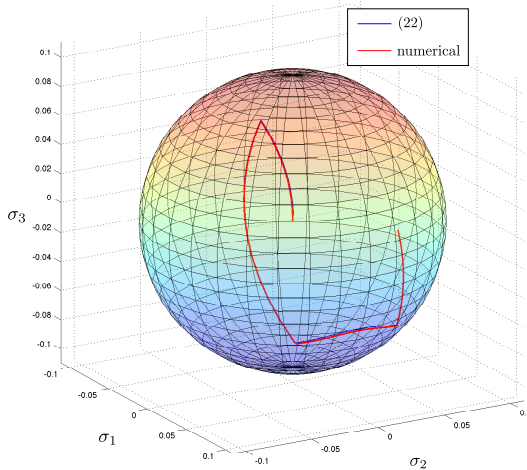

Figure 4. Comparison between the state trajectories using the proposed control law (22) and the numerical minimum-time control law, for Example 5.

VI. CONCLUSIONS

This technical note has presented a new approach to design HOSM control laws for nonlinear uncertain systems with arbitrary relative degree subject to input and state constraints. The presence of constraints, while being a topic of paramount importance in practical applications, has not been dealt with extensively in the SMC literature up to now. The proposed control laws, apart from maintaining the system state and the control variable always within the admissible domain, are aimed at achieving the minimum-time convergence on the sliding manifold, and the maximization of the domain of attraction. Analytical proofs together with numerical results have been reported in order to show the potential of the proposed method.

APPENDIX A
PROOF OF THEOREM 1

Assume that $\sigma_1(0) \in \mathcal{S}$, i.e., $\sigma_1(0) \in [\underline{\sigma}_1, \overline{\sigma}_1]$. Specifically, from Assumption 1 it follows that

$$0 < \sigma_1(t) \leq \overline{\sigma}_1 \Rightarrow \dot{\sigma}_1(t) \leq F - G_1\alpha = -\alpha_r < 0, \quad (30)$$

$$\underline{\sigma}_1 \leq \sigma_1(t) < 0 \Rightarrow \dot{\sigma}_1(t) \geq -F + G_1\alpha = \alpha_r > 0. \quad (31)$$

As a consequence, $\sigma_1(t)$ is steered to zero in finite time without leaving $\mathcal{S}$ for all $\sigma_1(0) \in \mathcal{S}_I \equiv \mathcal{S}$. Being $\mathcal{S}$ the set of state constraints, a larger set $\mathcal{S}_I$ cannot be obtained, which proves the maximization of the domain of attraction. Also, being signals $w(\sigma(t))$ and $w_{in}(\sigma(t))$ coinciding for all t, the controller solves a minimum-time problem for the worst-case disturbance, as proved in [16].

APPENDIX B
PROOF OF THEOREM 2

Define the external perimeter $\partial\mathcal{S}_I$ of $\mathcal{S}_I$ as the union of the segments AB, CD, EF, GH, BC, DE, FG, and HA (see Figure 5). The extreme points (e.g., A and B in AB) are not considered as part of the segment for the first four ones, while they are taken into account for the other segments. Figure 5 shows a realization of the invariant set $\mathcal{S}_I$ in the simple case $\overline{\sigma}_1 = \overline{\sigma}_2 = 1$, $\underline{\sigma}_1 = \underline{\sigma}_2 = -1$, $\alpha_r = 1$. The white region is $\mathcal{S}_I$, while the black regions represent the set $\{\mathcal{M}_0 \cup \mathcal{M}_1\}$. The so-called 'switching line' of equation

$$\sigma_1 = -\frac{\sigma_2|\sigma_2|}{2\alpha_r} \quad (32)$$

is also shown as a solid blue line.

A. Positive invariance of $\mathcal{S}_I$

As a preliminary result, it will be shown that $\mathcal{S}_I$ is a robust positively invariant (RPI) set [42, Def. 4.3] for the closed-loop system, which is proved by checking that for each $\sigma \in \partial\mathcal{S}_I$, the vector field $\dot{\sigma} = [\dot{\sigma}_1, \dot{\sigma}_2]'$ never points outside $\mathcal{S}_I$ [42, Theorem 4.10].

Case 1 ($\sigma \in HA \lor \sigma \in DE$): Assume $\sigma \in DE$ so that $\dot{\sigma} = [\sigma_2, f - g\alpha]'$. Notice that, from this point and for all the remainder of this proof, the dependency of f and g from their arguments will be omitted for the sake of readability. Consider that $\sigma \in DE$ implies $\sigma_2 < 0$, while $-F - G_2\alpha \leq f - g\alpha \leq F - G_1\alpha < 0$. Then, the
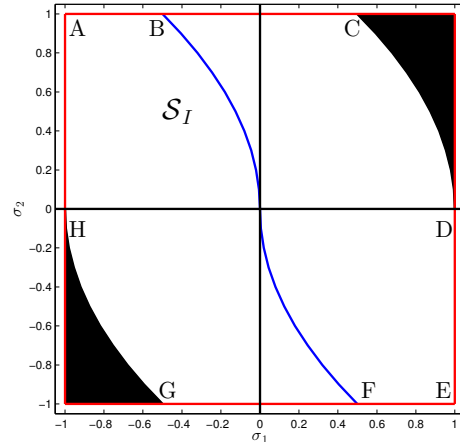

Figure 5. Graphical representation of a possible invariant region $\mathcal{S}_I$ in the second order case.

vector field is pointing down-left, that is towards the interior of $\mathcal{S}_I$. Analogous considerations can be done if $\sigma \in HA$, where the vector field is always pointing up-right.

Case 2 ($\sigma \in BC \ \vee \ \sigma \in FG$): Assume $\sigma \in FG \setminus \{G\}$ so that $\dot{\sigma} = [\sigma_2, \ f + g\alpha]'$, with $\sigma_2 = \underline{\sigma}_2 < 0$ and $0 < -F + G_1\alpha \leq f + g\alpha \leq F + G_2\alpha$. Then, the vector field $\dot{\sigma}$ is always pointing up-left, which means towards the interior of $\mathcal{S}_I$. Analogous considerations can be done if $\sigma \in BC$, where the vector field is always pointing down-right.

Case 3 ($\sigma \in CD \ \vee \ \sigma \in GH$): Assume $\sigma \in GH$ so that $\dot{\sigma} = [\sigma_2, \ f + g\alpha]'$. One has always that $\sigma_2 < 0$ is on this segment, while $0 < -F + G_1\alpha = \alpha_r \leq f + g\alpha \leq F + G_2\alpha$. It is easy to notice that, since all the points on this segment verify $\sigma_1 = -\frac{\sigma_2|\sigma_2|}{2\alpha_r} + \underline{\sigma}_1$, $\dot{\sigma}$ can be at most tangent to the line but never points outside. The same considerations can be stated for $\sigma \in CD$.

Case 4 ($\sigma \in AB \ \vee \ \sigma \in EF$): Assume $\sigma \in AB$. The control law is discontinuous on AB (see Figure 1). The vector field is therefore generated as the Filippov solution (see, e.g., [4], [5]) of the state space equations (10b)-(10c) for the second-order case. More precisely, $\dot{\sigma}$ belongs to the convex hull of $\dot{\sigma}^+ = [\sigma_2, f - g\alpha]'$ and $\dot{\sigma}^- = [\sigma_2, f + g\alpha]'$. The solution is obtained as $\dot{\sigma} = \mu\dot{\sigma}^+ + (1 - \mu)\dot{\sigma}^-$. Finding μ from condition $\nabla\sigma_2 \cdot \dot{\sigma} = 0$, with $\nabla\sigma_2 = [0, \ 1]'$, one has that

$$\dot{\sigma} = \frac{\nabla\sigma_2 \cdot \dot{\sigma}^-}{\nabla\sigma_2 \cdot (\dot{\sigma}^- - \dot{\sigma}^+)}\dot{\sigma}^+ - \frac{\nabla\sigma_2 \cdot \dot{\sigma}^+}{\nabla\sigma_2 \cdot (\dot{\sigma}^- - \dot{\sigma}^+)}\dot{\sigma}^-$$
$$= \frac{f + g\alpha}{2g\alpha}\dot{\sigma}^+ - \frac{f - g\alpha}{2g\alpha}\dot{\sigma}^- = [\sigma_2, \ 0]'$$

which is always tangent to the segment AB and pointing left. An analogous procedure has been used to analyze the case $\sigma \in EF$.

We can therefore conclude that $\mathcal{S}_I$ is an RPI set for the considered closed-loop system.

B. Finite-time convergence to the origin

The convergence property will be proved in three parts, showing that, for any initial condition $\sigma(0) \in \mathcal{S}_I$, $\sigma(t)$ reaches the origin in a finite time.

Case 1: Assume that one has $\sigma(0) \in \mathcal{S}_I \ \cap \ \{\sigma : \sigma_1 > -\sigma_2|\sigma_2|/2\alpha_r, \ \sigma_2 > \underline{\sigma}_2\}$. The vector field in this case is $\dot{\sigma}(0) = [\sigma_2(0), \ f - g\alpha]'$, being $w(\sigma(0)) = -\alpha$, and it is possible to state that $f - g\alpha < 0$. As a consequence, for all the considered $\sigma(0)$, $\dot{\sigma}(0)$ has a vertical component which is always strictly negative. When $f - g\alpha < -\alpha_r$, $\sigma(t)$ will reach in finite time either EF, or the switching line $\sigma_1 = -\frac{\sigma_2|\sigma_2|}{2\alpha_r}$. An analogous proof is obtained for $\sigma(0) \in \mathcal{S}_I \ \cap \ \{\sigma : \sigma_1 < -\sigma_2|\sigma_2|/2\alpha_r, \ \sigma_2 < \overline{\sigma}_2\}$: in that case, the sliding variable will reach in a finite time the segment AB, or the switching line.

Case 2: Assume that $\sigma(0) \in EF$. As proved in Case 4 in Appendix B-A, the trajectory of the system is kept on the line $\sigma_2 = \underline{\sigma}_2$. Since $\dot{\sigma}$ has a strictly negative horizontal component during this time interval, point F (which is on the switching line $\sigma_1 = -\frac{\sigma_2|\sigma_2|}{2\alpha_r}$) is reached in finite time. Analogous considerations hold for $\sigma(0) \in AB$.

Case 3: Assume that $\sigma(0) \in \{\sigma : \sigma_1 = -\sigma_2|\sigma_2|/2\alpha_r, \ \sigma_2 < 0\}$. Since the control law is discontinuous on the switching line, we need to use the Filippov solution, with $\nabla\left(\sigma_1 - \frac{\sigma_2^2}{2\alpha_r}\right) = \left[1, -\frac{\sigma_2}{\alpha_r}\right]'$ and

$\sigma_2 < 0$, obtaining

$$\dot{\sigma} = \frac{\nabla\left(\sigma_1 - \frac{\sigma_2^2}{2\alpha_r}\right) \cdot \dot{\sigma}^-}{\nabla\left(\sigma_1 - \frac{\sigma_2^2}{2\alpha_r}\right) \cdot (\dot{\sigma}^- - \dot{\sigma}^+)}\dot{\sigma}^+$$
$$- \frac{\nabla\left(\sigma_1 - \frac{\sigma_2^2}{2\alpha_r}\right) \cdot \dot{\sigma}^+}{\nabla\left(\sigma_1 - \frac{\sigma_2^2}{2\alpha_r}\right) \cdot (\dot{\sigma}^- - \dot{\sigma}^+)}\dot{\sigma}^-$$
$$= \frac{f + g\alpha - \alpha_r}{2g\alpha}\dot{\sigma}^+ - \frac{f - g\alpha - \alpha_r}{2g\alpha}\dot{\sigma}^- = [\sigma_2, \ \alpha_r]'$$

which is always tangent to the switching curve so that the state moves towards the origin, and converges to it in a finite time. The same holds for $\sigma(0) \in \{\sigma : \sigma_1 = -\sigma_2|\sigma_2|/2\alpha_r, \ \sigma_2 > 0$, for which $\dot{\sigma} = [\sigma_2, \ -\alpha_r]'$. Combining the two obtained vector fields we can also obtain that $\dot{\sigma} = [0, \ 0]'$ for $\sigma = 0$.

We have therefore proved that $\mathcal{S}_I$ is a domain of attraction for the origin, with finite-time convergence.

C. Maximal region of attraction

Assume that $\sigma(0) \in \mathcal{M}_0$. Considering that $\sigma_1(0), \sigma_2(0) > 0$, then σ_1 will continue to increase in time until $\sigma_2 > 0$. The quickest way to make σ_2 decrease is to use the control variable $w(\sigma) = -\alpha$. In this case, the system will move on a parabolic arc, the equation of which, in the worst case, taking into account the uncertain terms, is

$$\sigma_1 = -\frac{\sigma_2|\sigma_2|}{2\alpha_r} + \overline{\sigma}_1 + \varepsilon \qquad (33)$$

with $\varepsilon > 0$. It is immediate to see that this arc intersects the σ_1-axis outside $\mathcal{S}$. One can easily see that any other realization of the control variable will also lead to the same outcome. As a consequence, $\mathcal{M}_0$ cannot be part of the region of attraction for any realization of the control variable, given the constraints imposed on w, and on σ. Analogous considerations hold if $\sigma(0) \in \mathcal{M}_1$. In conclusion $\mathcal{S}_I$ is the largest achievable region of attraction.

D. Minimum-time convergence

The proof of the minimum-time convergence to the origin of the space $\{\sigma_1, \sigma_2\}$ follows from [44, Chapter 8] and [45]. Considering the worst-case realization of the disturbance terms, it is possible to express the system dynamics as that obtained by applying the control input (25) to the double integrator plant

$$\begin{array}{rcl} \dot{\sigma}_1(t) &=& \sigma_2(t) \\ \dot{\sigma}_2(t) &=& w_r(t). \end{array} \qquad (34)$$

in which $w_r = (\alpha_r/\alpha)w$ implicitly takes into account the effect of the disturbance terms. Let $[0, \ 0]' = [\sigma_1(t_c), \ \sigma_2(t_c)]'$, where t_c is the time needed to reach the origin from the given initial condition for a given control law. Following the approach discussed in [45], given a linear system subject to convex state constraints and strongly convex control constraints, if a *covariant function* $\psi(t)$ and functions $\eta(t)$ and $\phi(t)$ can be found such that

$$\dot{\phi}(t) = \psi(t) + \xi(t)\eta(t) \qquad (35)$$

for some time-optimal control law in the absence of state constraints $w_{nc}(\sigma)$, and for any $0 \leq t \leq t_c$, then the obtained evolution of the sliding variable $\sigma(t)$ is the one and only minimal time path and

$$w(\sigma) = \begin{cases} w_{nc}(\sigma) & \text{if } \underline{\sigma}_2 < \sigma_2 < \overline{\sigma}_2 \\ 0 & \text{if } \sigma_2 = \underline{\sigma}_2 \text{ or } \sigma_2 = \overline{\sigma}_2 \end{cases} \qquad (36)$$

is the one and the only minimal time control connecting points $(\sigma_1(0), \sigma_2(0))$ and $(0, 0)$. In our case, the defined set of state

constraints in (26) is convex, while the input constraint set $[-\alpha_r, \ \alpha_r]$ is strongly convex. Also, relying on the linear system (34), it is possible to define $\psi(t) \equiv 0$, $\xi(t) = |\sigma_2(t)| + \sigma_2(t)w(t)/\alpha_r$, $\eta(t) = \text{sgn}(\sigma_2(t))$ if $\sigma(t) \notin \mathcal{S}$, and $\eta(t) \equiv 0$ if $\sigma(t) \in \mathcal{S}$. Moreover, it is known that the time-optimal control law in the absence of state constraints would be the bang-bang control law

$$w_{nc}(\sigma) = -\alpha_r \, \text{sgn} \left(\sigma_1 + \frac{\sigma_2|\sigma_2|}{2\alpha_r} \right). \tag{37}$$

Condition (35) is therefore satisfied, as $\dot{\phi}(t) = \psi(t) + \xi(t)\eta(t) = \text{sgn}(\sigma_2(t)) \left(|\sigma_2(t)| + \sigma_2(t)\frac{w_{nc}(\sigma(t))}{\alpha_r} \right)$, from which it follows that $\phi(t) = \sigma_1(t) + \frac{\sigma_2(t)|\sigma_2(t)|}{2\alpha_r}$. Given the definition of $\mathcal{S}$ in (26), the effect of the application of (25) to the uncertain system (10b)-(10c) would be the same of applying (36) to system (34), with w_{nc} defined in (37). This proves that the proposed control law (25) drives σ to the origin in minimum time, for the worst-case realization of the disturbance terms.

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