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Origin of Abelian gauge symmetries in heterotic/F-theory duality

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Abstract: We study aspects of heterotic/F-theory duality for compactifications with Abelian gauge symmetries. We consider F-theory on general Calabi-Yau manifolds with a rank one Mordell-Weil group of rational sections. By rigorously performing the stable degeneration limit in a class of toric models, we derive both the Calabi-Yau geometry as well as the spectral cover describing the vector bundle in the heterotic dual theory. We carefully investigate the spectral cover employing the group law on the elliptic curve in the heterotic theory. We find in explicit examples that there are three different classes of heterotic duals that have U(1) factors in their low energy effective theories: split spectral covers describing bundles with $\mathrm{S}\big(\mathrm{U}(m)\times\mathrm{U}(1)\big)$ structure group, spectral covers containing torsional sections that seem to give rise to bundles with $\mathrm{SU}(m)\times\mathbb{Z}_k$ structure group and bundles with purely non-Abelian structure groups having a centralizer in E_8 containing a U(1) factor. In the former two cases, it is required that the elliptic fibration on the heterotic side has a non-trivial Mordell-Weil group. While the number of geometrically massless U(1)'s is determined entirely by geometry on the F-theory side, on the heterotic side the correct number of U(1)'s is found by taking into account a Stückelberg mechanism in the lower-dimensional effective theory. In geometry, this corresponds to the condition that sections in the two half K3 surfaces that arise in the stable degeneration limit of F-theory can be glued together globally.

Keywords: F-Theory, String Duality, Superstrings and Heterotic Strings, M-Theory

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Contents

1 Introduction and summary of results

The study of effective theories of string theory in lower dimensions with minimal supersymmetry are both of conceptual and phenomenological relevance. Two very prominent avenues to their construction are Calabi-Yau compactifications of the $E_8 \times E_8$ heterotic string and of F-theory, respectively. The defining data of the two compactifications are seemingly very different. While a compactification to $10 - 2n$ dimensions is defined in the heterotic string by a complex n-dimensional Calabi-Yau manifold Z_n and a holomorphic, semi-stable vector bundle V [1, 2], in F-theory one needs to specify a complex $(n + 1)$-dimensional elliptically-fibered Calabi-Yau manifold X_{n+1} [3–5]. For an elliptic K3-fibered X_{n+1} and an elliptically fibered Z_n, however, both formulations of compactifications of string theory are physically equivalent. The defining data of both sides are related to each other by heterotic/F-theory duality [3–5]. Most notably, this duality allows making statements about the heterotic vector bundle V in terms of the controllable geometry of the Calabi-Yau manifold X_{n+1} on the F-theory side. Studying the structure of the heterotic vector bundle V is crucial for understanding the gauge theory sector of the resulting effective theories. In this note, we present key steps towards developing the geometrical duality map between heterotic and F-theory compactifications with Abelian gauge symmetries in their effective theories.

Since the advent of F-theory, the matching of gauge symmetry and the matter content in the effective theories has been studied in heterotic/F-theory duality [3–5]. Mathematically, the duality astonishingly allows to use the data of singular Calabi-Yau manifolds X_{n+1} in F-theory to efficiently construct vector bundles V on the heterotic side, which is typically very challenging. The duality can be precisely formulated in the so-called stable degeneration limit of X_{n+1} [6], in which its K3-fibration degenerates into two half K3-fibrations $X^{\pm}_{n+1}$,

$$X_{n+1} \;\rightarrow\; X^{+}_{n+1} \cup_{Z_n} X^{-}_{n+1}\,, \tag{1.1}$$

that intersect in the heterotic Calabi-Yau manifold, $X^{+}_{n+1} \cap X^{-}_{n+1} = Z_n$. It can be shown that $X^{\pm}_{n+1}$ naturally encode the heterotic vector bundle V on elliptically fibered Calabi-Yau manifolds Z_n [7]. The most concrete map between the data of X_{n+1} in stable degeneration and the heterotic side is realized if V is described by a spectral cover employing the Fourier-Mukai transform [7, 8] (for more details see e.g. [9] and references therein). Heterotic/F-theory duality has been systematically applied using toric geometry for the construction of vector bundles V with non-Abelian structure groups described both via spectral covers and half K3 fibrations, see e.g. [10, 11] for representative works. More recently, heterotic/F-theory duality has been used to study the geometric constraints on both sides of the duality in four-dimensional compactifications and to characterize the arising low-energy physics [12], see also [13]. Furthermore, computations of both vector bundle and M5-brane superpotentials could be performed by calculation of the F-theory superpotential using powerful techniques from mirror symmetry [14–16]. In addition, the heterotic/F-theory duality has been recently explored for studies of moduli-dependent prefactor of M5-instanton corrections to the superpotential in F-theory compactifications [17, 18]. The focus of all these works has been on vector bundles V with non-Abelian structure groups, see however [19, 20] for first works on aspects of heterotic/F-theory duality with U(1)'s.

In this work, we will apply the simple and unifying description on the F-theory side in terms of elliptically fibered Calabi-Yau manifolds X_{n+1} to study explicitly, using stable degeneration, the structure of spectral covers yielding heterotic vector bundles that give rise to U(1) gauge symmetry in the lower-dimensional effective theory, continuing the analysis explained in the 2010 talk [21].[1]

Abelian gauge symmetries are desired ingredients for controlling the phenomenology both of extensions of the standard model as well as of GUT theories. Recently, there has been tremendous progress on the construction of F-theory compactifications with Abelian gauge symmetries based on the improved understanding of elliptically fibered Calabi-Yau manifold X_{n+1} with higher rank Mordell-Weil group of rational sections, see the representative works [22–31]. In contrast, it has been long known that Abelian gauge symmetries in the heterotic theory can for example be constructed by considering a background bundle V with line bundle components [1]. The setup we are studying in this work is the duality map between the concrete and known geometry of the Calabi-Yau manifold X_{n+1} with a rank one Mordell-Weil group in [22] on the F-theory side and the data of the Calabi-Yau manifold Z_n and the vector bundle V defining the dual heterotic compactification. We will demonstrate, at the hand of a number of concrete examples, the utility of the F-theory Calabi-Yau manifold X_{n+1} for the construction of vector bundles with non-simply connected structure groups that arise naturally in this duality. In particular, the F-theory side will guide us to the physical interpretation of less familiar or novel structures in the heterotic vector bundle.

There are numerous key advancements in this direction presented in this work:

- We rigorously perform the stable degeneration limit of a class of F-theory Calabi-Yau manifolds X_{n+1} with U(1) Abelian gauge symmetry using toric geometry, applying and extending the techniques of [32]. We explicitly extract the data of the two half-K3 surfaces inside $X^{\pm}_{n+1}$, the spectral covers and the heterotic Calabi-Yau manifold Z_n. We point out the non-commutativity of the stable degeneration limit and birational

[1] We have recently learned that A. Braun and S. Schäfer-Nameki have been working on similar techniques.

maps, such as the one to the Weierstrass model. The stable degeneration limit we perform, which we denote as "toric stable degeneration", preserves the structure of the Mordell-Weil group of rational sections before and after the limit, which is, in contrast, obscured in the stable degeneration limit performed in the Weierstrass model. We apply our general techniques to Calabi-Yau manifolds with elliptic fiber in $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$, which yield one U(1) in F-theory [22].

- We illuminate the systematics in the mapping under heterotic/F-theory duality between F-theory with a Mordell-Weil group and heterotic vector bundles with non-simply connected structure groups leading to U(1)'s in their effective theories. We find that a single type of F-theory geometry X_{n+1} can be dual to a whole range of different phenomena in the heterotic string, at the hand of numerous concrete examples. We find three different classes of examples of how a U(1) gauge group is obtained in the heterotic string: one class of examples has a split spectral cover, which is a well-known ingredient for obtaining U(1) gauge groups in the heterotic literature starting with [33] and the F-theory literature, see e.g. [34–36]; another class of models have a spectral cover containing a torsional section of the heterotic Calabi-Yau manifold Z_n, where duality suggests that this should describe zero-size instantons of discrete holonomy, as considered in [37]; in a last set of examples, the U(1) arises as the commutant inside E_8 of vector bundles with purely non-Abelian structure groups. We analyze the emerging spectral covers by explicit computations in the group law on the elliptic curve in Z_n. In the first two classes of examples, it is crucial that the heterotic elliptic fibration Z_n exhibits rational sections, as also found in [38]. In addition, in certain examples, the U(1) is only visible in the half K3 fibration (and in Z_n), but not in the spectral cover.

- Whereas the number of massless U(1)'s on the F-theory side equals the Mordell-Weil rank of X_{n+1}, it is on the heterotic side a mixture of geometry and effective field theory effects: while the analysis of the spectral cover can be performed already in 8D, in 6D and lower dimensions U(1)'s can be lifted from the massless spectrum by a Stückelberg effect, i.e. gaugings of axions [1]. We understand explicitly in all three classes of examples how these gaugings arise and what is the remaining number of massless U(1) fields.

We note that although our analysis is performed in 8D and 6D, it is equally applicable also to heterotic/F-theory duality for compactifications to 4D.

This paper is organized in the following way: in section 2, we provide a brief review of the key points of heterotic/F-theory duality as well as a discussion of the new insights gained in this work into spectral covers and half K3-fibrations for vector bundles with non-simply connected structure groups. We review and discuss heterotic/F-theory duality in 8D and 6D, the spectral cover construction for SU(N) bundles, specializations thereof giving rise to U(1) factors in the heterotic string and the Stückelberg mechanism rendering certain U(1) gauge fields massive. Section 3 contains the toric description of a class of F-theory models X_{n+1} for which we describe a toric stable degeneration limit. We specialize to the toric fiber $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ and obtain the half K3-fibrations as well as the dual heterotic geometry and spectral cover polynomial. In section 4, we present selected examples of F-theory/heterotic dual compactifications. We illustrate the three different classes of examples with heterotic vector bundles of structure groups $\mathrm{S}\big(\mathrm{U}(n)\times\mathrm{U}(1)\big)$ and $\mathrm{S}\big(\mathrm{U}(n)\times\mathbb{Z}_k\big)$, as well as purely non-Abelian ones having a centralizer in E_8 with one $\mathrm{U}(1)$ factor. There we also illustrate the utility of the Stückelberg mechanism to correctly match the number of geometrically massless $\mathrm{U}(1)$'s on both sides of the duality. In section 5, we conclude and discuss possibilities for future works. This work has four appendices: we present the birational map of the quartic in $\mathbb{P}^{(1,1,2)}$ to Tate and Weierstrass form in appendix A; appendix B contains examples with no $\mathrm{U}(1)$ factor, consistently reproducing [4]; in appendix C we state the condition for the existence of two independent rational sections and appendix D illustrates explicitly the non-commutativity of the stable degeneration limit and the birational map to Weierstrass form.

2 Heterotic/F-theory duality and U(1)-factors

The aim of this section is two-fold: on the one hand, we review those aspects of heterotic/F-theory duality in 8D and 6D that are relevant for the analyses performed in this work. On the other hand, we point out subtleties and new insights into heterotic/F-theory duality with Abelian $\mathrm{U}(1)$ factors. In particular, we discuss in detail split spectral covers for heterotic vector bundles with non-simply connected gauge groups and the heterotic Stückelberg mechanism.

In section 2.1, we discuss the fundamental duality in 8D, the standard stable degeneration limit in Weierstrass form and the principal matching of gauge groups and moduli. There, we also discuss a subtlety in performing the stable degeneration limit of F-theory models with $\mathrm{U}(1)$ factors due to the non-commutativity of this limit with the map to the Weierstrass model. Section 2.2 contains a discussion of the spectral cover construction for $\mathrm{SU}(N)$ bundles as well as of split spectral covers giving rise to $\mathrm{S}\big(\mathrm{U}(N-1)\times\mathrm{U}(1)\big)$ and $\mathrm{S}\big(\mathrm{U}(N-1)\times\mathbb{Z}_k\big)$ bundles. In section 2.3 we briefly review heterotic/F-theory duality in 6D, before we discuss the Stückelberg effect in the effective theory of heterotic compactifications with $\mathrm{U}(1)$ bundles as well as the relation to gluing condition of rational sections in section 2.4.

In the review part, we mainly follow [7, 9, 39], to which we refer for further details.

2.1 Heterotic/F-theory duality in 8D

The basic statement of heterotic/F-theory duality is that the heterotic string (in the following, we always concentrate on the $\mathrm{E}_8\times\mathrm{E}_8$ string) compactified on a torus, which we denote by Z_1, is equivalent to F-theory compactified on an elliptically fibered K3 surface X_2. The first evidence is that the moduli spaces $\mathcal{M}$ of these two theories coincide and are parametrized by

$$\mathcal{M}=\mathrm{SO}(18,2,\mathbb{Z})\backslash\mathrm{SO}(18,2,\mathbb{R})/\big(\mathrm{SO}(18)\times\mathrm{SO}(2)\big)\times\mathbb{R}^+. \tag{2.1}$$

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From a heterotic perspective this is just the parametrization of the complex and Kähler structure of the torus Z_1 as well as of the 24 Wilson lines. On the F-theory side it corresponds to the moduli space of algebraic K3 surfaces X_2 with Picard number two. The last factor corresponds to the vacuum expectation value of the dilaton and the size of the base $\mathbb{P}^1$ of X_2, respectively.

Lower-dimensional dualities are obtained, applying the adiabatic argument [40], by fibering the eight-dimensional duality over a base manifold B_{n-1} of complex dimension $n-1$ that is common to both theories of the duality.

2.1.1 The standard stable degeneration limit

In order to match the moduli on both sides of the duality, the K3 surface X_2 has to undergo the so-called stable degeneration limit. In this limit it splits into two half K3 surfaces X_2^+, X_2^- as

$$X_2 \;\rightarrow\; X_2^+ \cup_{Z_1} X_2^- \,. \tag{2.2}$$

Each of these are an elliptic fibration $\pi_\pm : X_2^\pm \longrightarrow \mathbb{P}^1$ over a $\mathbb{P}^1$. These two $\mathbb{P}^1$'s intersect in precisely one point so that the two half K3 surfaces intersect in a common elliptic fiber which is identified with the heterotic elliptic curve, $X_2^+ \cap X_2^- = Z_1$. On the heterotic side, the stable degeneration limit corresponds to the large elliptic fiber limit of Z_1.

Matching the gauge groups. The F-theory gauge group is given by the singularities of the elliptic fibration of X_2, determining the non-Abelian part G, and its rational sections, which correspond to Abelian gauge fields [3, 5, 41]. In stable degeneration the non-Abelian gauge group of F-theory is distributed into the two half K3 surfaces $X_2^\pm$ and matched with the heterotic side as follows.

It is a well-known fact that the homology lattice of a half K3 surface $X_2^\pm$ is given in general by

$$H_2(X_2^\pm, \mathbb{Z}) = \Gamma_8 \oplus U \tag{2.3}$$

Here, U contains the classes of the elliptic fiber as well as of the zero section. Γ_8 equals the root lattice of E_8 and splits into a direct sum of two contributions [42]: the first contribution is given by the Mordell-Weil group of the rational elliptic surface while the second contribution is given by a sub-lattice which forms, for the half K3 surfaces $X_2^\pm$ at hand, the root-lattice of the part $G_\pm$ of the non-Abelian F-theory gauge group $G = G_+ \times G_-$ that is of ADE type. In the F-theory limit all fiber components are shrunken to zero size and the half K3 surface develops a singularity of type $G_\pm$. The possible ADE-singularities in the case of complex surfaces have been classified by Kodaira [43]. Thus, one can always read off the corresponding gauge group from the order of vanishings of f, g and Δ once the half K3 has been brought into affine Weierstrass normal form

$$y^2 = x^3 + fxz^4 + gz^6 , \qquad \Delta = 4f^3 + 27g^2 , \tag{2.4}$$

with f and g in $\mathcal{O}(4)$ and $\mathcal{O}(6)$ of $\mathbb{P}^1$, respectively. For convenience of the reader, we reproduce Kodaira's classification in table 1.

order (f)	order (g)	order (Δ)	singularity
$\geqslant 0$	$\geqslant 0$	0	none
0	0	n	A_{n-1}
$\geqslant 1$	1	2	none
1	$\geqslant 2$	3	A_1
$\geqslant 2$	2	4	A_2
2	$\geqslant 3$	$n+6$	D_{n+4}
$\geqslant 2$	3	$n+6$	D_{n+4}
$\geqslant 3$	4	8	E_6
3	$\geqslant 5$	9	E_7
$\geqslant 4$	5	10	E_8

Table 1. The Kodaira classification of singular fibers. Here f and g are the coefficients of the Weierstrass normal form, Δ is the discriminant as defined in (2.4) and order refers to their order of vanishing at a particular zero.

In contrast, the gauge group on the heterotic side is encoded in two vector bundles V_1, V_2 that generically carry the structure group E_8. Their respective commutants inside the two ten-dimensional E_8 gauge groups of the heterotic string are to be identified with the F-theory gauge group. As observed in [7], the moduli space of semi-stable E_8-bundles on an elliptic curve E corresponds to the complex structure moduli space of a half K3 surface S whose anti-canonical class is given by E. Furthermore, if S has an ADE singularity of type $\tilde{G}_\pm$ then the structure group of V_1, V_2 is reduced to the centralizer $H_\pm$ of $\tilde{G}_\pm$ within E_8, respectively. In heterotic/F-theory duality, a matching of the gauge group is then established by identifying $S \equiv X_2^\pm$ yielding $\tilde{G}_\pm \equiv G_\pm$.

Notice that the full eight-dimensional gauge group is given by $G \times U(1)^{16-\mathrm{rk}(G)} \times U(1)^4$. Here, the last factor accounts for the reduction of the metric and the Kalb Ramond B-field along the two one-cycles of the torus in the heterotic string. From the F-theory perspective, all U(1) factors arise from the reduction of the C_3 field along those 2-forms in the full K3 surface X_2 that are orthogonal to the zero section and the elliptic fiber. In particular, the $U(1)^{16-\mathrm{rk}(G)}$ arises from the generators of the Mordell-Weil group of the half K3 surfaces. For a derivation in Type IIB string theory, see the recent work [44].

Matching complex structure and bundle moduli. In this section, we discuss how the heterotic moduli can be recovered from the data of the F-theory K3 surface [4, 45]. Here we restrict the discussion to the moduli of the heterotic torus Z_1 and the vector bundle (i.e. Wilson line) moduli, ignoring the heterotic dilaton modulus.

So far, this discussion has been restricted to the case that the elliptic fibration of the K3 surface is described by a Weierstrass model. In this case, the standard stable degeneration procedure applies. Given the Weierstrass form (2.4) for X_2 with f, g sections of $\mathcal{O}(8)$ and $\mathcal{O}(12)$ on $\mathbb{P}^1$, respectively, we can expand these degree eight and twelve polynomials in the affine $\mathbb{P}^1$-coordinate u as

$$f = \sum_{i=0}^{8} f_i u^i, \qquad g = \sum_{i=0}^{12} g_i u^i. \tag{2.5}$$

Then, the two half K3 surfaces $X_2^{\pm}$ arising in the stable degeneration limit, given as the Weierstrass models

$$X_2^{\pm}: \qquad y^2 = x^3 + f^{\pm} z^4 + g^{\pm} z^6, \tag{2.6}$$

can be obtained from (2.5) by the split

$$f^+ = \sum_{i=0}^{4} f_i u^i, \quad f^- = \sum_{i=4}^{8} f_i u^i, \qquad g^+ = \sum_{i=0}^{6} g_i u^i, \quad g^- = \sum_{i=6}^{12} g_i u^i, \tag{2.7}$$

The "middle" polynomials f_4 and g_6 correspond to the heterotic elliptic curve, which then reads

$$Z_1: \qquad y^2 = x^3 + f_4 x z^4 + g_6 z^6, \tag{2.8}$$

while the "upper" and "lower" coefficients correspond to the moduli of the two E_8-bundles.

2.1.2 Stable degeneration with other elliptic fiber types

The focus of the present work are F-theory compactifications with one U(1) gauge group arising from elliptically fibered Calabi-Yau manifolds with two rational sections. These are naturally constructed using the fiber ambient space $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ [22]. More precisely, we will consider K3 surfaces given as sections χ of the anti-canonical bundle $-K_{\mathbb{P}^1\times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}}$ of $\mathbb{P}^1 \times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ reading

$$\chi = \sum_i s_i \chi^i. \tag{2.9}$$

Here s_i and χ^i are sections of the anti-canonical bundles $-K_{\mathbb{P}^1} = \mathcal{O}(2)$ and $-K_{\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}}$, respectively.

Then, analogously to the above construction, one can perform a stable degeneration limit for these hypersurfaces as well. However, it is crucial to note here that we can perform the stable degeneration limit in *two* possible ways, as shown in figure 1: one way is to first take the Weierstrass normal form W_χ (upper horizontal arrow) of the full $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$-model and then apply the split (2.7) to obtain two half K3 surfaces (right vertical arrow); a second way is to first perform stable degeneration (left vertical arrow), yielding two half K3 surfaces $\chi^{\pm}$ with elliptic fibers in $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$, and then compute their Weierstrass normal forms $W_\chi^{\pm}$ (lower horizontal arrow). It is important to realize, however, that these two possible paths in the diagram 1 do not commute, as explicitly shown in appendix D.

We propose and demonstrate in section 3 that the natural order to perform heterotic/F-theory duality for models with U(1) factors and different elliptic fiber types than the Weierstrass model is to first perform stable degeneration with the other fiber type (left vertical arrow) and then compute the Weierstrass model of the resulting half K3-fibrations (lower horizontal arrow) in order to analyze the physics of the model.

2.2 Constructing SU(N) bundles on elliptic curves and fibrations

While the description of the structure group of the vector bundle via half K3 surfaces as reviewed above is of high conceptual importance, it is in practice often easier to construct vector bundles with the desired structure group directly. In the following section, we review

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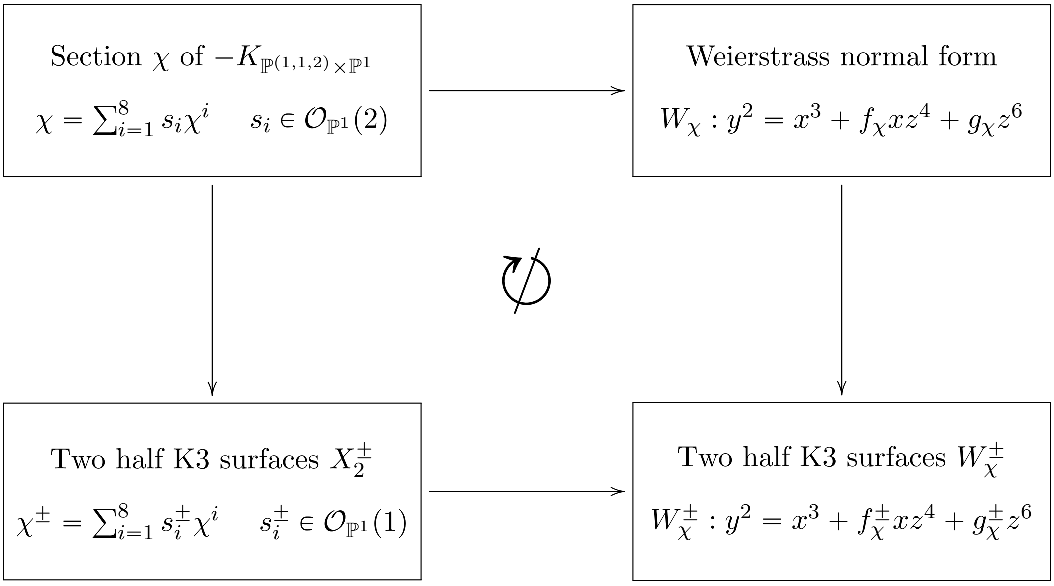

Figure 1. Computing the Weierstrass normal form (horizontal arrows) and taking the stable degeneration limit (vertical arrows) does not commute.

this construction for SU(N) bundles and specializations thereof which has been studied first in [46] and was further developed in [7, 8, 47].

In this section E always denotes an elliptic curve with a marked point p. The curve is defined over a general field K, which does not necessarily have to be algebraically closed. It is well-known that an elliptic curve with a point p has a representation in the Weierstrass normal form (2.4), where p reads $[x : y : z] = [1 : 1 : 0]$. In general, a degree zero line bundle $\mathcal{L} \longrightarrow E$, i.e. a U(1)-bundle, takes the form

$$\mathcal{L} = \mathcal{O}(q) \otimes \mathcal{O}(p)^{-1} = \mathcal{O}(q - p)\,, \tag{2.10}$$

where q denotes another arbitrary rational point on E (note that over $K = \mathbb{C}$ every point is rational). Furthermore, we note that there is a bijective map ϕ from the elliptic curve E onto its Picard group of degree zero which is defined by

$$\phi : E \longrightarrow \mathrm{Pic}^0(E)\,, \qquad q \mapsto q - p\,. \tag{2.11}$$

In particular, this extends to an isomorphism from the space of line bundles onto $\mathrm{Pic}^0(E)$, defined by $\mathrm{div}(\mathcal{L}) = q - p$. To be more precise, the divisor map 'div' is to be applied to a meromorphic section[2] of $\mathcal{L}$. For later purposes, we also recall that the addition law in $\mathrm{Pic}^0(E)$ can be identified with the group law on E, which we denote by $\boxplus$, via this isomorphism.

[2]This map is independent of the section chosen.

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A semi-stable SU(N) vector bundle of degree zero V is then given as the sum[3] of N holomorphic line bundles $\mathcal{L}_i$, i.e. we have $V = \oplus_{i=1}^N \mathcal{L}_i = \oplus_{i=1}^N \mathcal{O}(q_i - p)$, such that the determinant of V is trivial. The latter implies that

$$\otimes_{i=1}^N \mathcal{O}(q_i - p) = \mathcal{O} \qquad \Leftrightarrow \qquad \boxplus_{i=1}^N q_i = 0\,. \tag{2.12}$$

An SU(N) vector bundle is therefore determined by the choice of N points on E that sum up to zero. Any such N-tupel is determined by a projectively unique element of $H^0\big(E, \mathcal{O}(Np)\big)$, i.e. a function with N zeros and a pole of order N at p. Thus, the moduli space of SU(N) vector bundles is given by

$$\mathcal{M}_{\mathrm{SU}(N)} = \mathbb{P}H^0\big(E, \mathcal{O}(Np)\big)\,. \tag{2.13}$$

In the affine Weierstrass form of E, given by (2.4) with z set to one, the coordinates x, y have a pole of order two and three at p, respectively. Accordingly, any element of $\mathbb{P}H^0\big(E, \mathcal{O}(Np)\big)$ enjoys an expansion

$$w = c_0 + c_1 x + c_2 y + c_3 x^2 + \ldots + \begin{cases} c_N x^{\frac{N}{2}} & \text{if } N \text{ is even}\,, \\ c_N x^{\frac{N-3}{2}} y & \text{if } N \text{ is odd}\,, \end{cases} \tag{2.14}$$

with $c_i \in K$. The section w is called the spectral cover polynomial and has N common points with E, called the *spectral cover*, which define the desired SU(N) bundle. Counting parameters of (2.14), one is lead to the conclusion that

$$\mathcal{M}_{\mathrm{SU}(N)} = \mathbb{P}^{N-1}. \tag{2.15}$$

Finally, a comment on rational versus non-rational points is in order. Generically, p is the only point on E over a general field K. However, in such a situation, it is possible to mark N points in a rational way by the polynomial $w = 0$ which give rise to an $SU(N)$ bundle in the way just described. Nevertheless, under the circumstances that there are additional rational points on E and the spectral cover polynomial $w = 0$ specializes appropriately, the structure group reduces in a certain way, as discussed next.

2.2.1 Vector bundles with reduced structure groups

As described in the previous section, the choice of N points on E describes an SU(N) bundle. If we consider just an elliptic curve E over $\mathbb{C}$, which is the geometry relevant for the construction of heterotic compactifications to 8D, the spectral cover (2.14) can be factorized completely. This corresponds to the 16 possible Wilson lines on T^2.

In contrast, if we consider an elliptic curve over a function field, as it arises in elliptic fibrations Z_n of E over a base B_{n-1} used for lower-dimensional heterotic compactifications, the N points are the zeros of (2.14), which defines an N-section of the fibration. In non-generic situations, where subsets of the N sheets of this N-section are well-defined globally,

[3]If two or more points coincide, the situation is a bit more subtle [47, 48]. In this case the bundle is given by $\oplus_{i=1}^N \mathcal{O}(q_i - p) I_{r_i}$, where r_i denotes the multiplicity of the point q_i and I_r is inductively defined by the extension sequence $0 \longrightarrow \mathcal{O} \longrightarrow I_{r-1} \longrightarrow \mathcal{O} \longrightarrow 0$. However, one usually only considers bundles up to S-equivalence which identifies I_r with $\mathcal{O}^{\oplus r}$.

i.e. are monodromy invariant, the structure group of the vector bundle is reduced. For example, a separation into two sets of k and l sheets (with $k + l = N$), respectively, results in the structure group $\mathrm{S}\big(\mathrm{U}(k) \times \mathrm{U}(l)\big)$. The spectral cover defined by (2.14) is called "split" and defines a reducible variety inside Z_n, see e.g. [33–36]. In the most extreme case, one could have $k = 1$ and $l = N - 1$. In this case, the elliptic fibration of Z_n has to necessarily have another well-defined section in addition to the section induced by the rational point p: it is the one marked by the component of the spectral cover $w = 0$ with just one sheet [38]. Thus, the fiber E has a rational point, which we denote by q and one can, as discussed above, define a U(1) line bundle $\mathcal{L}$ via (2.10). As this fiberwise well-defined line bundle is also well-defined globally, it will induce a line bundle on Z_n, whose first Chern class is given, up to vertical components, by the difference of the sections induced by q and p, cf. [49]. The structure group H of the vector bundle is in this case given by

$$H = \mathrm{S}\big(\mathrm{U}(N-1) \times \mathrm{U}(1)\big)\,. \tag{2.16}$$

We will see later that this situation will be relevant situation for the construction of U(1) gauge groups in the heterotic string.

We emphasize that for a U(1)-bundle alone there is no spectral cover polynomial (2.14) that would be able to detect this additional rational point. This is due to the fact that there is no function that has only one zero on an elliptic curve E . However, if the rational point is accompanied by further points, rational or non-rational points over the field K, it can very well be seen by the spectral cover. For instance, one could construct a spectral cover from q and $-q$, which would describe a bundle of structure group $\mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(1)\big)$.

Finally, it needs to be discussed what interpretation should be given to the case that the rational point q on the curve E happens to be torsion of order k. In this case the structure group H reduces further to $\mathrm{S}\big(\mathrm{U}(N) \times \mathbb{Z}_k\big)$. To argue for this, we invoke again a fiberwise argument. The fiber at a generic point in B_{n-1} admits a line bundle $\mathcal{L} = \mathcal{O}(q-p)$ with the property that $\mathcal{L}^k = \mathcal{O}$. This is clear as the transition functions g_{ij} will be subject to $g_{ij}^k = 1$ in Čech cohomology as k times the Poincaré dual of its first Chern class is trivial. However, this is just the statement that the fiberwise structure group of $\mathcal{L}$ is contained in $\mathbb{Z}_k$. Employing that p and q are globally well-defined sections then suggests that this argument also holds on Z_n.

2.3 Heterotic/F-theory duality in 6D

Six-dimensional heterotic/F-theory duality arises by fibering the eight-dimensional duality over a common base $B_1 = \mathbb{P}^1$, employing the adiabatic argument [40]. Thus, the heterotic string gets compactified on an elliptically fibered K3 surface Z_2 while F-theory is compactified on an elliptically fibered Calabi-Yau threefold X_3 over a Hirzebruch surface $\mathbb{F}_n$. Our presentation will be brief and focused on the later applications in this work. For a more detailed discussion we refer to the classical reference [4, 5, 7] or the reviews [9, 39].

On the F-theory side, the non-Abelian gauge content originates from the codimension one singularities of the elliptic fibration $\pi : X_3 \to \mathbb{F}_n$. The singularity is generically of type G', which gets broken down to $G \subset G'$ by monodromies corresponding to outer

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automorphisms of the Dynkin diagram of G' [41]. The resulting gauge symmetry is encoded in the order of vanishing of the coefficients a_0, a_1, a_2, a_3, a_4, a_6 in the Tate form of the elliptic fibration

$$y^2 + a_1xy + a_2 = x^3 + a_3x^2 + a_4x + a_6\,. \tag{2.17}$$

In addition, we introduce the *Tate vector* $\vec{t}_X$ which encodes the orders of vanishing of the coefficients a_i along the divisor defined by the local coordinate X:

$$\vec{t}_X = \big(\mathrm{ord}_X(a_0), \mathrm{ord}_X(a_1), \mathrm{ord}_X(a_2), \mathrm{ord}_X(a_3), \mathrm{ord}_X(a_4), \mathrm{ord}_X(a_6), \mathrm{ord}_X(\Delta)\big)\,. \tag{2.18}$$

The results of the analysis of singularities, known as Tate's algorithm, are summarized in table 2 [41, 50], see, however, [51] for subtleties.

On the heterotic side, the gauge theory content is encoded in a vector bundle V where the following discussion restricts itself to the case of $\mathrm{SU}(N)$ bundles. The six-dimensional bundle is defined in terms of two pieces of data, the spectral cover curve C as well as a line bundle $\mathcal{N}$ which is defined on C. Here, the spectral curve C is the 6D analog of the points defined by the section of $\mathbb{P}H^0\big(E, \mathcal{O}(np)\big)$ which has been discussed in 2.2. In 6D, the elliptic curve $Z_1 \cong E$ gets promoted to an elliptic fibration, which can again be described by a Weierstrass form (2.4) with coordinates x, y, z being sections of $\mathcal{L}^2$, $\mathcal{L}^2$, $\mathcal{O}$, respectively, for $\mathcal{L} = K^{-1}_{\mathbb{P}^1} = \mathcal{O}(-2)$ and coefficients f, g being in $\mathcal{L}^4$, $\mathcal{L}^6$, respectively. Accordingly, the coefficients c_i entering the spectral cover (2.14) are now sections of $\mathcal{M} \otimes \mathcal{L}^{-i}$, $\mathcal{M}$ being an arbitrary line bundle on $\mathbb{P}^1$ and C is defined as the zero locus of the section of (2.14). Thus, C defines an N-sheeted ramified covering of $\mathbb{P}^1$, i.e. a Riemann surface. The spectral cover C defines the isomorphism class of a semi-stable vector bundle above each fiber. The line bundle $\mathcal{N}$ describes the possibility to twist the vector bundle without changing its isomorphism class. It is usually fixed, up to a twisting class γ, by the condition $c_1(V) = 0$ for an $\mathrm{SU}(N)$ bundle, see [7] for more details.

2.4 Massless U(1)-factors in heterotic/F-theory duality

As previously discussed, the perturbative heterotic gauge group is obtained by commuting the structure group H of the vector bundle V within the two E_8-bundles. We propose three possibilities, how $\mathrm{U}(1)$ gauge groups can arise from this perspective:

- H contains a $\mathrm{U}(1)$ factor, i.e. it is of the form $H = H_1 \times \mathrm{U}(1)$, or $\mathrm{S}\big(\mathrm{U}(M) \times \mathrm{U}(1)\big)$,
- H contains a discrete piece, i.e. a part taking values in $\mathbb{Z}_k$,
- or H is non-Abelian and is embedded such that its centralizer in E_8 necessarily contains a $\mathrm{U}(1)$-symmetry.

The construction of a vector bundle for these three different cases employing spectral covers has been discussed in section 2.2.

In general, we emphasize that $\mathrm{U}(1)$-factor which arises from a split spectral cover is usually massive due to a Stückelberg mass term which is induced by the first Chern class of the $\mathrm{U}(1)$ background bundle, as we review next. However, if the $\mathrm{U}(1)$ term originates from a

Type	Group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	$\{e\}$	0	0	0	0	0	0
I_1	$\{e\}$	0	0	1	1	1	1
I_2	SU(2)	0	0	1	1	2	2
I_3	SU(3)	0	1	1	2	3	3
$I_{2k}, k \geqslant 2$	Sp(k)	0	0	k	k	$2k$	$2k$
$I_{2k+1}, k \geqslant 1$	Sp(k)	0	0	$k+1$	$k+1$	$2k+1$	$2k+1$
$I_n, n \geqslant 4$	SU(n)	0	1	$\left[\frac{n}{2}\right]$	$\left[\frac{n+1}{2}\right]$	n	n
II	$\{e\}$	1	1	1	1	1	2
III	SU(2)	1	1	1	1	2	3
IV	Sp(1)	1	1	1	2	2	4
IV	SU(3)	1	1	1	2	3	4
I_0^*	G_2	1	1	2	2	3	6
I_0^*	Spin(7)	1	1	2	2	4	6
I_0^*	Spin(8)	1	1	2	2	4	6
I_1^*	Spin(9)	1	1	2	3	4	7
I_1^*	Spin(10)	1	1	2	3	5	7
I_2^*	Spin(11)	1	1	3	3	5	8
I_2^*	Spin(12)	1	1	3	3	5	8
I_{2k-3}^*, $k \geqslant 3$	SO($4k+1$)	1	1	k	$k+1$	$2k$	$2k+3$
I_{2k-3}^*, $k \geqslant 3$	SO($4k+2$)	1	1	k	$k+1$	$2k+1$	$2k+3$
I_{2k-2}^*, $k \geqslant 3$	SO($4k+3$)	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
I_{2k-2}^*, $k \geqslant 3$	SO($4k+4$)	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
IV*	F_4	1	2	2	3	4	8
IV*	E_6	1	2	2	3	5	8
III*	E_7	1	2	3	3	5	9
II*	E_8	1	2	3	4	5	10
non-min	–	1	2	3	4	6	12

Table 2. Results from Tate's algorithm.

background bundle with non-Abelian structure group there is tautologically no U(1) background factor which could produce a mass term and therefore the six-dimensional U(1) field is expected to be massless. Finally, we propose, for consistency with heterotic/F-theory duality, that a six-dimensional torsional section gives rise to a point-like instanton with discrete holonomy, as introduced in [37]. Indeed, we will show in several examples in section 4 that all three cases naturally appear in heterotic duals of F-theory compactifications with one U(1) and that a matching of the corresponding gauge groups is only possible if the arising spectral covers are interpreted as suggested here.

2.4.1 The heterotic Stückelberg mechanism

In six and lower dimensions, it is well-known that a geometric Stückelberg effect can render a U(1) gauge field massive [1]. To identify the mass term of the six- (or lower-) dimensional U(1), one considers the modified ten-dimensional kinetic term of the Kalb-Ramond field B_2 which reads, up to some irrelevant proportionality constant, as

$$\mathcal{L}_{\text{kin}}^{10d} = H \wedge \star_{10d} H\,, \qquad H = dB_2 - \frac{\alpha'}{4}\big(\omega_{3Y}(A) - \omega_{3L}(\Omega)\big)\,. \tag{2.19}$$

Here, $\star_{10d}$ is the ten-dimensional Hodge-star and ω_{3Y}, ω_{3L} denote the Chern-Simons terms of the gauge field and the spin connection, respectively. The physical effect we want to discuss here arises from the former one, which is given explicitly by

$$\omega_{3Y} = \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{2.20}$$

Now, we perform a dimensional reduction of the kinetic term (2.19) in the background of a U(1) vector bundle on the heterotic compactification manifold Z_n, ignoring possible additional non-Abelian vector bundles for simplicity. On such a background, we can expand the ten-dimensional field strength $F^{10D}_{\text{U}(1)}$ of the U(1) gauge field as

$$F^{10D}_{\text{U}(1)} = F_{\text{U}(1)} + \mathcal{F} = F_{\text{U}(1)} + k_\alpha \omega^\alpha. \tag{2.21}$$

Here $\mathcal{F} = \frac{1}{2\pi i} c_1(\mathcal{L})$ is the background field strength, i.e. the first Chern class $c_1(\mathcal{L})$ of the corresponding U(1) line bundle $\mathcal{L}$, and $F_{\text{U}(1)}$ is the lower-dimensional gauge field. We have also introduced a basis ω^α, $\alpha = 1, \ldots, b_2(Z_n)$, of harmonic two-forms in $H^{(2)}(Z_n)$, where $b_2(Z_n)$ is the second Betti number of Z_n, along which we have expanded $\mathcal{F}$ into the flux quanta k_α. We also expand the ten-dimensional Kalb-Ramond field as

$$B_2 = b_2 + \rho_\alpha \omega^\alpha, \tag{2.22}$$

where b_2 is a lower-dimensional two-form and ρ_α are lower-dimensional axionic scalars . We readily insert this reduction ansatz into the ten-dimensional field strength H in (2.19), where we only take into account the gauge part, to arrive, dropping unimportant prefactors, at the lower-dimensional kinetic term for the axions ρ_α of the form

$$\mathcal{L}_{\text{Stück.}} = G^{\alpha\beta}(d\rho_\alpha + k_\alpha A_{\text{U}(1)}) \wedge \star(d\rho_\beta + k_\beta A_{\text{U}(1)})\,. \tag{2.23}$$

Here we introduced the kinetic metric

$$G^{\alpha\beta} = \int_{Z_n} \omega^\alpha \wedge \star\omega^\beta. \tag{2.24}$$

It is clear from (2.23) that a single U(1) gauge field will be massive if we have a non-trivial $c_1(\mathcal{L}) \neq 0$. However, we note that in the presence of multiple massive U(1) gauge fields, appropriate linear combinations of them in the kernel of the mass matrix can remain massless U(1) fields. A computation similar to the one above has appeared in e.g. [33], where also the case of multiple U(1)'s is systematically discussed.

2.4.2 U(1)-factors from gluing conditions in half K3-fibrations

We conclude this section by discussing the connection between the previous field theoretic considerations that lead to a massive U(1) via the Stückelberg action (2.23) on the heterotic side and geometric glueing conditions of the sections of half K3 surfaces to global sections of the two half K3-fibrations $X_n^\pm$ that arise in stable degeneration as well as of the full Calabi-Yau manifold X_n. We illustrate this in 6D for concreteness, i.e. for F-theory on a Calabi-Yau threefold X_3 and the heterotic string on a K3 surface Z_2, although the arguments hold more generally.

It is well known that the number of U(1) factors in F-theory is given by the rank of the Mordell-Weil group, i.e. by the number of independent global rational sections of the elliptic fibration X_3 in addition to the zero section. As discussed in section 2.1, a half K3 surface with ADE singularity of rank r has an $(8-r)$-dimensional Mordell-Weil group. Promoting the half K3 surface to a fibration of half K3 surfaces over the base $\mathbb{P}^1$, such as the threefolds $X_3^\pm$, these sections need not necessarily give rise to sections of the arising three-dimensional elliptic fibrations. Considering the half K3 surfaces arising in the stable degeneration limit of F-theory, there are those sections which also give rise to sections of e.g. the full half K3 fibration X_3^+. These sections will induce a U(1)-factor on the heterotic side which is embedded into one E_8-bundle and which is generically massive with a mass arising via the Stückelberg action (2.23). If there is also a globally well-defined section of the other half K3 fibration X_3^- and this section glues with the section in the first half K3 fibration X_3^+, then there is a linear combination of U(1)'s that remains massless in the Stückelberg mechanism on the heterotic side. This is clear from the F-theory perspective, as these two sections can then be glued along the heterotic two-fold Z_2 to a section of the full Calabi-Yau threefold X_3, i.e. give rise to an element in its Mordell-Weil group and a massless U(1).

3 Dual geometries with toric stable degeneration

In this section, we describe a toric method in order to study the stable degeneration limit of an elliptically fibered K3 surface. This stable degeneration limit will be at the heart of the analysis of the examples of heterotic/F-theory dual geometries in section 4. In a first step in section 3.1, we construct an elliptically fibered K3 surface. Afterwards in section 3.2, we fiber this K3 surface over another $\mathbb{P}^1$ which is used to investigate the splitting of the K3 surface into two rational elliptic surfaces, as discussed in section 3.3. In the concluding section 3.4, we prove that the surfaces arising in the stable degeneration of the K3 surface indeed define rational elliptic surfaces, i.e. half K3 surfaces.

3.1 Constructing an elliptically fibered K3 surface

We start by constructing a three-dimensional reflexive polytope Δ_3° given as the convex hull of vertices that are the rows of the following matrix:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & 0 & x_1 \\ \vdots & & 0 & x_i \\ a_n & b_n & 0 & x_n \\ 0 & 0 & 1 & U \\ 0 & 0 & -1 & V \end{array}\right). \tag{3.1}$$

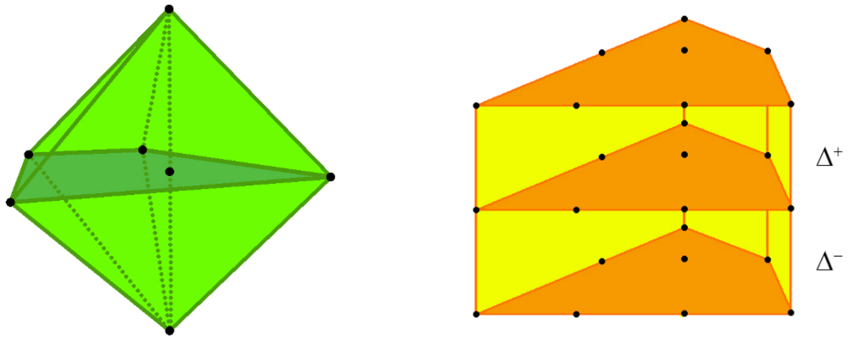

Figure 2. On the left we show the reflexive polytope Δ_3°, while its dual Δ_3 is shown on the right. In this example, the ambient space for the elliptic fiber, specified by Δ_2°, is given by $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$.

Here $(a_i\, b_i)$ denote the points of a two-dimensional reflexive polytope Δ_2°, which will specify the geometry of the elliptic fiber E. It is embedded into Δ_3° in the xy-plane, see the first picture in figure 2. The last column contains the homogeneous coordinate associated to a given vertex. We label the rays of the two-dimensional polytope counter-clockwise by the coordinates $x_1, \dots x_n$. In addition, we assign the coordinates U, V to the points $(0\,0\,1)$ and $(0\,0\,-1)$ which correspond to the rays of the fan of the $\mathbb{P}^1$-base. We use the shorthand notation $\mathbb{P}^1_{[U:V]}$ to indicate its homogeneous coordinates. Finally, we use the notation ρ_H for the ray with corresponding homogeneous coordinate H. We denote by Σ_3 the natural simplicial fan associated to Δ_3° and denote the corresponding toric variety over the fan of Δ° as $\mathbb{P}_{\Sigma_3}$. Provided a fine triangulation of the polytope Δ_3° has been chosen, the toric ambient space $\mathbb{P}_{\Sigma_3}$ will be Gorenstein and terminal.

A general section χ of the anti-canonical bundle $\mathcal{O}_{\mathbb{P}_{\Sigma_3}}(-K_{\mathbb{P}_{\Sigma_3}})$ defines a smooth elliptically fibered K3 surface X_2. The ambient space of its elliptic fiber E is the toric variety $\mathbb{P}_{\Sigma_2}$ that is constructed from the fan Σ_2 of the polytope Δ_2° induced by Σ_3. As the toric fibration of Σ_2 over $\Sigma_{\mathbb{P}^1}$ is direct, the section χ takes the form

$$\chi = s_i \eta^i \quad \text{for} \quad s_i = s_i^0 U^2 + s_i^1 UV + s_i^2 V^2 . \tag{3.2}$$

Here η^i are the sections of the anti-canonical bundle of $\mathcal{O}_{\mathbb{P}_{\Sigma_2}}(-K_{\mathbb{P}_{\Sigma_2}})$, i.e. the range of the index i is given by the number of integral points in Δ_2, and s_i^k, $k = 1,2,3$, are constants. Note that, for a very general[4] X_2, the dimension $h^{(1,1)}(X)$, of the cohomology group $H^{(1,1)}(X_2, \mathbb{C})$ can be computed combinatorially from the pair of reflexive polyhedra Δ_3, Δ_3° by a generalization of the Batyrev's formula [52]:

$$h^{(1,1)}(X) = l(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ} l^*(\Gamma^\circ) + \sum_{\Theta^\circ} l^*(\Theta^\circ) l^*(\hat{\Theta}^\circ) \,. \tag{3.3}$$

Here $l(\Delta)$ ($l^*(\Delta)$) denote the number of (inner) points of the n-dimensional polytope Δ. In addition, Γ (Γ°) denote the codimension one faces of Δ (Δ°), while Θ denotes a codimension two face with $\hat{\Theta}$ being its dual.

[4]A point is very general if it lies outside a countable union of closed subschemes of positive codimension.

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3.2 Constructing K3 fibrations

As a next step, we fiber this ambient space over a second $\mathbb{P}^1_{[\lambda_1,\lambda_2]}$ with homogeneous coordinates λ_1, λ_2. The following construction is such that the generic fiber consists of a smooth K3 surface X_2 over a generic point of $\mathbb{P}^1_{[\lambda_1,\lambda_2]}$ and a split fiber, i.e. a splitting into two half K3 surfaces, over a distinguished point of $\mathbb{P}^1_{[\lambda_1,\lambda_2]}$, as explained below.

The four-dimensional polytope which describes this construction is given by

$$\Delta_4 = \left\{(m_1,m_2,m_3,m_4)\in\mathbb{Z}^4 \,|\, (m_1,m_2,m_3)\in\Delta_3\,,\ -1\leqslant m_4\leqslant 1\,, \begin{cases} m_4\geqslant -1 & \text{if } m_3\leqslant 0\,, \\ m_4\geqslant m_3-1 & \text{if } m_3\geqslant 0\,. \end{cases}\right\}. \tag{3.4}$$

Here, Δ_3 denotes the dual polytope of Δ_3°, cf. the second picture in figure 2. The faces of Δ_4 are given by the (intersection of the) hyperplanes

$$m_4 = 1\,, \quad m_4 = -1\,, \quad m_4 = -1 + m_3\,, \qquad m_3 = -1\,, \quad m_3 = 1\,, \qquad \sum_{j=0}^{2} a_i^j m_j = 1\,, \tag{3.5}$$

where the last expression is given by the defining hyperplanes of Δ_2, the dual of Δ_2°. We denote by Σ_4 the fan associated to the dual polytope Δ_4° of Δ_4. In particular, the normal vectors of the facets of Δ_4 give the rays of Σ_4. To be explicit, the rays of Σ_4 are given by the rows of the matrix

$$\left(\begin{array}{cccc|c} a_1 & b_1 & 0 & 0 & x_1 \\ \vdots & & 0 & 0 & x_i \\ a_n & b_n & 0 & 0 & x_n \\ 0 & 0 & 1 & 0 & U \\ 0 & 0 & 0 & 1 & \lambda_1 \\ 0 & 0 & -1 & 1 & \mu \\ 0 & 0 & -1 & 0 & V \\ 0 & 0 & 0 & -1 & \lambda_2 \end{array}\right). \tag{3.6}$$

We note that the coordinates assigned to its rays as displayed in (3.6) transform as follows under the $\mathbb{C}^*$-actions

$$(U : \lambda_1 : \mu : V : \lambda_2) \sim (a^{-1}U : ab^{-1}\lambda_1 : a^{-1}bc^{-1}\mu : b^{-1}cV : c^{-1}\lambda_2) \tag{3.7}$$

with $a, b, c \in \mathbb{C}^*$.

In analogy to the discussion in the previous section, a section χ_4 of the anti-canonical bundle $-K_{\mathbb{P}_{\Sigma_4}}$ of the toric variety $\mathbb{P}_{\Sigma_4}$ defines a three-dimensional smooth Calabi-Yau manifold $\mathfrak{X}$. In particular, the Calabi-Yau constraint (3.2) generalizes as

$$\chi_4 = s_i \eta^i, \tag{3.8}$$

where the η^i are given as before and the coefficients s_i now read

$$\begin{aligned} s_i(U,V,\lambda_1,\lambda_2,\mu) = {} & s_i^1\lambda_1\lambda_2U^2 + s_i^2\lambda_1^2\mu U^2 + s_i^3\lambda_2^2UV + s_i^4\lambda_1\lambda_2\mu UV + s_i^5\lambda_1^2\mu^2UV \\ & + s_i^6\lambda_2^2\mu V^2 + s_i^7\lambda_1\lambda_2\mu^2V^2 + s_i^8\lambda_1^2\mu^3V^2 \end{aligned} \tag{3.9}$$

with constants $s_i^j \in \mathbb{C}$.

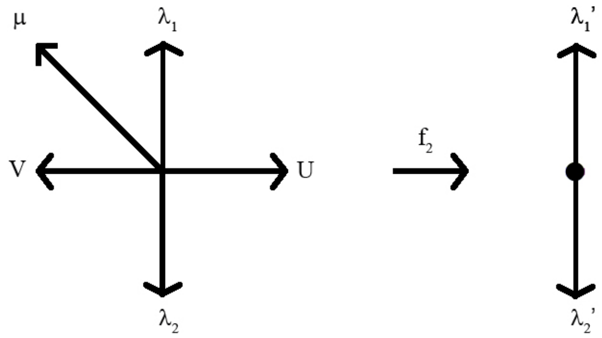


Figure 3. The toric morphism f_2.

We proceed by observing that the projection on the last two columns in (3.6) yields the polytope $\Delta^\circ_{dP_2}$ of the toric variety dP_2, cf. figure 3. Denoting the fan of $\Delta^\circ_{dP_2}$ by Σ_{dP_2} this projection gives rise to a toric map

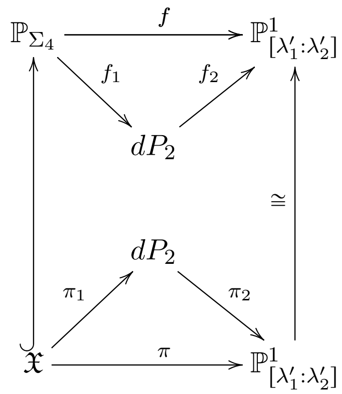
$$f_1: \ \Sigma_4 \longrightarrow \Sigma_{dP_2} \, . \tag{3.10}$$

In addition, dP_2 is fibered over the $\mathbb{P}^1_{[\lambda'_1:\lambda'_2]}$ as can be seen by projecting onto the fourth column of Δ_4, cf. figure 3, i.e. there is a toric map

$$f_2: \ \Sigma_{dP_2} \longrightarrow \Sigma_{\mathbb{P}^1} \, , \tag{3.11}$$

where $\Sigma_{\mathbb{P}^1}$ is the fan of $\mathbb{P}^1_{[\lambda'_1:\lambda'_2]}$. Note that this $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^1_{[\lambda_1:\lambda_2]}$. We denote the composition map of the two by $f = f_2 \circ f_1$.

In summary, we have the following diagram of toric morphisms and induced maps on $\mathfrak{X}$:



Here we denote the toric maps f_1, f_2, f and their induced morphisms of toric varieties by the same symbol, respectively. Note that for a generic point, the fiber of π is given by a smooth K3 surface X_2.

In order to prepare for the discussion of the stable degeneration limit, we proceed by discussing the fibration map in more detail. For this purpose, we note the correspondence of facets and rays as displayed in table 3. The dual Δ_{dP_2} of $\Delta^\circ_{dP_2}$ with associated monomials is shown in figure 4. These monomials are the global sections of K_{dP_2} and are constructed

ray	facet	constraint
ρ_{λ_1}	$m_4 = -1$	$s_{\lambda_1} = s_i^3 U + s_i^6 \mu$
ρ_μ	$m_4 = m_3 - 1$	$s_\mu = s_i^1 \lambda_1 + s_i^3 V$
ρ_{λ_2}	$m_4 = 1$	$s_{\lambda_2} = s_i^2 U^2 + s_i^5 UV + s_i^8 V^2$

Table 3. The correspondence between the rays of $\Delta^\circ_{dP_2}$ and the facets of Δ_{dP_2}. The last column displays the global sections that embed the associated divisor into $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. The coefficients on the right-hand side refer to equation (3.9).

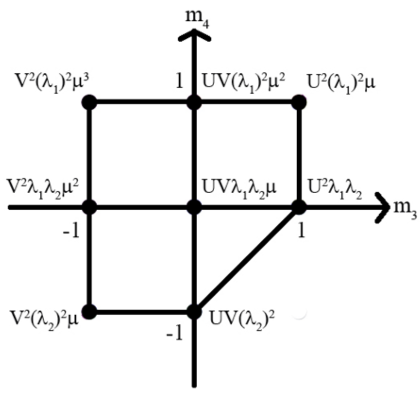


Figure 4. The dual polytope Δ_{dP_2} and the associated monomials.

according to [53]

$$\chi_{dP_2} = \sum_{P \in \Delta_{dP_2}} \prod_{P* \in \Delta^*_{dP_2}} a_P x_{P*}^{\langle P, P* \rangle + 1}. \tag{3.12}$$

Here x_{P*} denotes the coordinate which is associated to the corresponding ray of the toric diagram and a_P are constants. By the correspondence between cones of $\Delta^\circ_{dP_2}$ and vertices of Δ_{dP_2} the vertex corresponding to the monomial $V^2\lambda_2^2\mu$ is dual to the cone spanned by the rays ρ_U and ρ_{λ_1}. We denote the coordinates associated to the two rays of $\mathbb{P}^1_{[\lambda_1:\lambda_2]}$ inside $\Delta^0_{dP_2}$ appearing in (3.11) by $\rho_{\lambda'_1}$ and $\rho_{\lambda'_2}$. Note that $f_2^{-1}(\rho_{\lambda'_1}) = \{\rho_{\lambda_1}, \rho_\mu\}$, while $f_2^{-1}(\rho_{\lambda'_2}) = \{\rho_{\lambda_2}\}$.

3.3 The toric stable degeneration limit

In the following, we aim to show that the general fiber of the map π gives rise to a smooth K3 surface while the pre-image of the point $[\lambda'_1 : \lambda'_2] = [1 : 0]$ gives rise to a degeneration into two half K3 surfaces $X_2^\pm$ that intersect in the elliptic fiber Z_1 over the point of intersection of the two $\mathbb{P}^1$ which are the respective bases of their elliptic fibrations.

Let us first consider the toric variety $f_2^{-1}(\lambda'_2 = 0)$ corresponding to the pre-image in $\Delta^0_{dP_2}$ of $\rho_{\lambda'_2}$. It is given by the star of ρ_{λ_2} in $\Delta^0_{dP_2}$ which is just the generic fiber of f_2. Indeed, if $\lambda_2 = 0$, the coordinates μ and λ_1 are non-vanishing due to the Stanley-Reisner ideal. Two of the scaling relations (3.7) can be used in order to eliminate the latter two variables while the remaining (linear combination) endows the coordinates U, V with the well-known scaling relations of $\mathbb{P}^1_{[U,V]}$. In addition, the monomials associated to the vertices

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of the dual facet of ρ_{λ_2} give rise to the following sections

$$s_{\lambda_2} := s_i^2 U^2 + s_i^5 UV + s_i^8 V^2, \tag{3.13}$$

as follows from (3.9) by setting $\lambda_2 = 0$. These provide precisely the global sections of $O_{\mathbb{P}^1}(2)$ that are needed for the Veronese embedding, i.e. the embedding of $\mathbb{P}^1_{[U,V]}$ into $\mathbb{P}^2$ as a conic

$$[U : V] \longmapsto [U^2 : UV : V^2]\,. \tag{3.14}$$

In contrast, the preimage of $\rho_{\lambda_1'}$ consists of the two divisors $\lambda_1 = 0$ and $\mu = 0$. In this case the Stanley-Reisner ideal forbids the vanishing of the coordinates V, λ_2 and U, λ_2 respectively. Taking again into account the scaling relations (3.7), one observes that the pre-image of the divisor $\lambda_1' = 0$ consists of two $\mathbb{P}^1$'s that are given by

$$D_{\lambda_1} = [U : 0 : \mu : 1 : 1]\,, \qquad D_\mu = [1 : \lambda_1 : 0 : V : 1]\,. \tag{3.15}$$

These intersect in precisely one point given by $[1 : 0 : 0 : 1 : 1]$. One identifies the dual facets of ρ_{λ_1} and ρ_μ as $m_4 = -1$ and $m_4 = m_3 - 1$. In this case the global sections are given by

$$s_{\lambda_1} := s_i^3 U + s_i^6 \mu\,, \qquad s_\mu := s_i^1 \lambda_1 + s_i^3 V\,, \tag{3.16}$$

as follows again from (3.9). This induces in this case only the trivial embedding via the identity map. Note that the union of the two divisors D_{λ_1} and D_μ is given by a degenerate conic

$$z_1 z_3 = z_2^2 \lambda_1 \mu\,, \quad \text{with} \quad (V^2\mu, UV, U^2\lambda_1) \mapsto [z_1 : z_2 : z_3] \in \mathbb{P}^2, \tag{3.17}$$

which splits as just observed into the two lines $z_1 = 0$, $z_3 = 0$ at $\lambda_1 = 0$ and $\mu = 0$.

A similar reasoning applies to the pre-image of $\rho_{\lambda_1'}$ under the composite map f. As noted above, we have $f^{-1}(\rho_{\lambda_1'}) = \{\rho_\mu, \rho_{\lambda_1}\}$, which implies that the pre-image is given by the two divisors $\mathbb{P}_{\Sigma_3^+} = \{\mu = 0\}$ and $\mathbb{P}_{\Sigma_3^-} = \{\lambda_1 = 0\}$ in $\mathbb{P}_{\Sigma_4}$. They are obtained as the star of ρ_μ and ρ_{λ_1} in Δ_4^0, respectively, with their fans $\Sigma_3^\pm$ induced by Σ_4. The corresponding respective dual facets are given by the three-dimensional facets $m_4 = -1$ and $m_4 = m_3 - 1$ in Δ_4. In addition, this gives rise to a splitting of Δ_3 as

$$\Delta_3^+ = \big\{(m_1, m_2, m_3) \in \mathbb{Z}^3 \mid (m_1, m_2) \in \Delta_2\,,\ m_3 \in \{0, 1\}\big\}\,, \tag{3.18}$$
$$\Delta_3^- = \big\{(m_1, m_2, m_3) \in \mathbb{Z}^3 \mid (m_1, m_2) \in \Delta_2\,,\ m_3 \in \{-1, 0\}\big\}\,, \tag{3.19}$$

which is also referred to as the top and bottom splitting [10], cf. figure 2. Thus, the section of the anti-canonical bundle $\mathcal{O}(-K_{\mathbb{P}_{\Sigma_3}})$ in (3.2) in the limit becomes the sum of

$$\begin{aligned} \chi_{X_2^+} &:= s_i^+ \eta^i, \quad \text{with} \quad s_i^+ = s^{+0}_{\ i} V + s^{+1}_{\ i} \lambda_1\,, \\ \chi_{X_2^-} &:= s_i^- \eta^i, \quad \text{with} \quad s_i^- = s^{-0}_{\ i} U + s^{-1}_{\ i} \mu\,, \end{aligned} \tag{3.20}$$

so that we can define the two surfaces $X_2^\pm$ as

$$X_2^+ = X_2|_{\mathbb{P}_{\Sigma_3^+}} = \{\chi = \mu = 0\}\,, \qquad X_2^- = X_2|_{\mathbb{P}_{\Sigma_3^-}} = \{\chi = \lambda_1 = 0\}\,. \tag{3.21}$$

As we will prove in the next subsection, X_2^+ and X_2^- are two rational elliptic surfaces (half K3 surfaces).

In contrast, the pre-image of $\rho_{\lambda_2'}$ is given by the whole three-dimensional fan Σ_3 as it is also for a generic point in $\mathbb{P}^1_{[\lambda_1:\lambda_2]}$. To justify the latter statement inspect the fiber above the origin 0 of the fan $\Sigma_{\mathbb{P}^1}$ corresponding to a generic point in $\mathbb{P}^1_{[\lambda_1:\lambda_2]}$.

Finally, we remark that the two rational elliptic surfaces X_2^+, X_2^- that arise at the loci $\{\mu = 0\}$ and $\{\mu = 0\}$, respectively, are independent of the K3 surface which appears over the locus $\{\lambda_2 = 0\}$. In the following, we explain how the half K3 surfaces can be obtained from the data of the K3 surface X_2 directly. As the notation used so far is rather heavy, which is unfortunately necessary, we introduce a slightly easier notation that will be used in the discussion of explicit examples in section 4. We rewrite a general hypersurface constraint as

$$\chi = s_i \eta^i, \qquad s_i = s_{i1}U^2 + s_{i2}UV + s_{i3}V^2, \tag{3.22}$$

which requires, depending on the situation at hand, the following identifications between the coefficients of (3.22) and of (3.9):

$$\begin{aligned} & s_{i1} \equiv s_i^2\,, \qquad s_{i2} \equiv s_i^5\,, \qquad s_{i3} \equiv s_i^8\,, \\ \text{or} \quad & s_{i1} \equiv s_i^1\,, \qquad s_{i2} \equiv s_i^3\,, \qquad s_{i3} \equiv s_i^6\,. \end{aligned} \tag{3.23}$$

However, it is crucial to note that the pair of coordinates U, V is only suited to describe the base $\mathbb{P}^1$ of the K3 surface X_2, while the base coordinates $\mathbb{P}^1$'s of X_2^+ and X_2^- are given by λ_1, V and U, μ, respectively.

3.4 Computing the canonical classes of the half K3 surfaces $X_2^\pm$

In this subsection, we discuss how the half K3 surfaces $X_2^\pm$ can be re-discovered in the toric stable degeneration limit. Note that the two components $\mathbb{P}_{\Sigma_3^+}$ and $\mathbb{P}_{\Sigma_3^-}$ of the degenerate fiber, as divisors in $\mathbb{P}_{\Sigma_4}$, should equal the generic fiber $\mathbb{P}_{\Sigma_3}$:

$$\mathbb{P}_{\Sigma_3^+} + \mathbb{P}_{\Sigma_3^-} \cong \mathbb{P}_{\Sigma_3}\,. \tag{3.24}$$

In addition, we have

$$\mathbb{P}_{\Sigma_3} \cdot \mathbb{P}_{\Sigma_3^\pm} = 0 \tag{3.25}$$

as the generic fiber can be moved away from the locus $\lambda_1' = 0$, cf. figure 3. This allows us to compute the canonical bundle of $\mathbb{P}_{\Sigma_3^\pm}$ using adjunction in $\mathbb{P}_{\Sigma_4}$ as

$$K_{\mathbb{P}^\pm_{\Sigma_3}} = \left(K_{\mathbb{P}_{\Sigma_4}} \otimes \mathcal{O}_{\mathbb{P}_{\Sigma_4}}(\mathbb{P}_{\Sigma_3^\pm})\right)\big|_{\mathbb{P}^\pm_{\Sigma_3}} = K_{\mathbb{P}_{\Sigma_4}}\big|_{\mathbb{P}^\pm_{\Sigma_3}} \otimes \mathcal{O}_{\mathbb{P}^\pm_{\Sigma_3}}(-\mathbb{P}_{\Sigma_3^\pm} \cdot \mathbb{P}_{\Sigma_3^\mp})\,, \tag{3.26}$$

where we used (3.24) and (3.25). Note that the divisor corresponding to the last term equals the class of the ambient space $\mathbb{P}_{\Sigma_2}$ of elliptic fiber of X_2, i.e. $\mathbb{P}_{\Sigma_3^+} \cdot \mathbb{P}_{\Sigma_3^-} = \mathbb{P}_{\Sigma_2}$. Making one more time use of the adjunction formula, one finally arrives at

$$\begin{aligned} K_{X_2^\pm} &= \left(K_{\mathbb{P}_{\Sigma_3^\pm}} \otimes \mathcal{O}_{\mathbb{P}_{\Sigma_3^\pm}}(X_2^\pm)\right)\big|_{X_2^\pm} = \left(K_{\mathbb{P}_{\Sigma_4}}\big|_{\mathbb{P}^\pm_{\Sigma_3}} \otimes \mathcal{O}_{\mathbb{P}^\pm_{\Sigma_3}}(-\mathbb{P}_{\Sigma_2}) \otimes \mathcal{O}_{\mathbb{P}_{\Sigma_3^\pm}}(X_2^\pm)\right)\big|_{X_2^\pm} \\ &= \mathcal{O}_{\mathbb{P}^\pm_{\Sigma_3}}(-\mathbb{P}_{\Sigma_2})\big|_{X^\pm} = \mathcal{O}_{X_2^\pm}(-\mathcal{E})\,, \end{aligned}$$

where we used (3.26) in the second equality and $K_{\mathbb{P}_{\Sigma_4}}|_{\mathbb{P}^{\pm}_{\Sigma_3}} = \mathcal{O}_{\mathbb{P}_{\Sigma_3^{\pm}}}(X_2^{\pm})$. Thus, the anti-canonical class of $X_2^{\pm}$ is given by that of the elliptic fiber E which leads to the conclusion that $X_2^{\pm}$ is indeed a rational elliptic surface.

4 Examples of heterotic/F-theory duals with U(1)'s

In the section, we use the tools of section 3 to construct explicit elliptically fibered Calabi-Yau two- and threefolds whose stable degeneration limit is well under control. Our geometries have generically two sections, which give rise to a U(1)-factor in the corresponding F-theory compactification. Performing the toric symplectic cut allows us to explicitly track these sections through the stable degeneration limit and to make non-trivial statements about the vector bundle data on the heterotic side in which the U(1)-factor in the effective theory is encoded. Finally, after having performed the stable degeneration limit as discussed in section 3.3, we split the resulting half K3 surfaces into the spectral cover polynomial and the constraint for the heterotic elliptic curve. Then, we determine the common solutions of the latter two constraints which encode the data of a (split) spectral cover. The general geometries we consider as well as the procedure we apply is discussed in section 4.1. Despite the fact that we do not determine the embedding of the structure group into E_8 directly, we are able to match the spectral cover with the resulting gauge group in all cases. In particular, we consider three different classes of examples. In subsection 4.2 we investigate a number of examples whose heterotic dual gives rise to a split spectral cover. This class of examples has generically one U(1) factor embedded into both E_8-bundles of which only a linear combination is massless. The next class of examples considered in subsection 4.3 displays torsional points in its spectral cover. There is one example with a U(1)-factor on the F-theory side which is found to be only embedded into one E_8-bundle while the other E_8-bundle is kept intact. Finally, in the last subsection 4.4 we consider an example where the structure group reads $SU(2) \times SU(3)$. However, we argue that it is embedded in such a way that its centralizer necessarily contains a U(1) factor.

4.1 The geometrical set-up: toric hypersurfaces in $\mathbb{P}^1 \times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$

For convenience, we recall the three-dimensional polyhedron Δ_3° for the resolved toric ambient space $\mathbb{P}_{\Sigma_3} = \mathbb{P}^1 \times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$. It is given by the points

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & x_1 \\ -1 & -1 & 0 & x_2 \\ 1 & 0 & 0 & x_3 \\ 0 & 1 & 0 & x_4 \\ -1 & 0 & 0 & x_5 \\ 0 & 0 & 1 & U \\ 0 & 0 & -1 & V \end{array}\right). \tag{4.1}$$

Here, $x_1, \ldots x_5$ are homogeneous coordinates on the resolved variety $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$, while U, V denote the two homogeneous coordinates of $\mathbb{P}^1$. In particular, x_5 resolves the A_1-singularity of the space $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$.

A generic section of the anti-canonical bundle of the ambient space $\mathbb{P}_{\Sigma_3}$ takes the form

$$\begin{aligned}\chi := & s_1 x_1^4 x_4^3 x_5^2 + s_2 x_1^3 x_2 x_4^2 x_5^2 + s_3 x_1^2 x_2^2 x_4 x_5 + s_4 x_1 x_2^3 x_5^2 + s_5 x_1^2 x_3 x_4^2 x_5 \\ & + s_6 x_1 x_2 x_3 x_4 x_5 + s_7 x_2^2 x_3 x_5 + s_8 x_3^2 x_4 = 0\,,\end{aligned} \tag{4.2}$$

where the coefficients s_i are homogeneous quadratic polynomials in U, V. An elliptically fibered K3 surface is defined by $X_2 = \{\chi = 0\}$. As can be seen for example in its Weierstrass form, the K3 surface generically has a Kodaira fiber of type I_2 at the locus $s_8 = 0$. It is resolved by the divisor $\{x_5 = 0\} \cap X_2$ as mentioned above.[5] In addition, X_2 generically has a Mordell-Weil group of rank one. A choice of zero section S_0 and generator of the Mordell-Weil group S_1 are given by

$$S_0 = X_2 \cap \{x_1 = 0\}\,, \qquad S_1 = X_2 \cap \{x_4 = 0\}\,. \tag{4.3}$$

Explicitly, their coordinates read

$$S_0 = \begin{cases} [0:1:1:s_7:-s_8] & \text{generically}\,, \\ [0:1:1:0:1] & \text{if } s_7 = 0\,, \\ [0:1:1:1:0] & \text{if } s_8 = 0\,, \end{cases} \qquad S_1 = \begin{cases} [s_7:1:-s_4:0:1] & \text{generically}\,, \\ [0:1:1:0:1] & \text{if } s_7 = 0\,, \\ [1:1:0:0:1] & \text{if } s_4 = 0\,. \end{cases} \tag{4.4}$$

Here we distinguished the special cases with $s_7 = 0$ and $s_8 = 0$, respectively, from the generic situation. Using the fact that the generic K3 surface X_2 has $h^{(1,1)} = 5$ [52], we, hence, conclude that the full F-theory gauge group G_{X_2} is[6]

$$G_{X_2} = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,. \tag{4.5}$$

We note that if $s_7 = 0$, one observes that the two sections coincide, as was also employed in [54]. That the converse is true is shown in appendix C. This is expected as the vanishing of s_7 can be interpreted as a change of the toric fibre ambient space from $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ to $\mathbb{P}^{(1,2,3)}$, which has a purely non-Abelian gauge group [30].

4.1.1 Engineering gauge symmetry: specialized sections of $-K_{\mathbb{P}^1 \times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}}$

In order to construct examples with higher rank gauge groups, we tune the coefficients of χ further. To be concrete, every s_i in (4.2) takes the form

$$s_i = s_{i1} U^2 + s_{i2} UV + s_{i3} V^2 \tag{4.6}$$

and a specialization corresponds to the identical vanishing of some s_{ij}. This specialization of coefficients implies that Δ, the dual polyhedron of $\mathbb{P}^1 \times \mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$, gets replaced by the Newton polytope $\Delta_{\text{spec.}}$ of the specialized constraint, compare also figure 5. As a technical side-remark we note that we strictly speaking refer with $\Delta_{\text{spec.}}$ to the convex

[5] As the details of the resolution are not important, we can set $x_5 = 1$ in most computations performed here.

[6] We note that $s_8 = 0$ has two solutions on $\mathbb{P}^1$. If we consider higher dimensional bases of the elliptic fibration, we will just have one SU(2) factor as $s_8 = 0$ is in general an irreducible divisor.

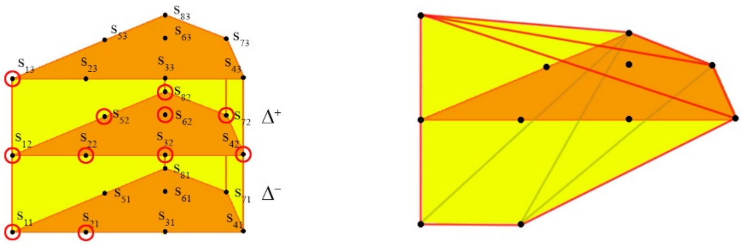

Figure 5. This figure illustrates a specialization of the coefficients of the hypersurface $\chi = 0$ such that the resulting gauge group is enhanced to $\mathrm{E}_7 \times \mathrm{E}_6 \times \mathrm{U}(1)$, see also the discussion in section 4.2.2. In the left picture, the non-vanishing coefficients are marked by a circle in the polytope Δ_3. In the right figure the new polytope, i.e. the Newton polytope of the specialized constraint $\chi = 0$, is shown.

hull of the points defined by the non-vanishing monomials in (4.6) respectively (4.2). As a consequence, also Δ° changes to the dual of $\Delta^\circ_{\text{spec.}}$. Thus, we have secretly changed the toric ambient space by this specialization of coefficients. It is crucial to note that only those specializations are allowed whose corresponding specialized polyhedron $\Delta_{\text{spec.}}$ is reflexive. The latter condition is equivalent to the associated toric variety being Gorenstein and Fano. Set-ups which are not reflexive can in particular have ill-defined stable-degeneration limits within our formalism.

In order to determine the gauge group of this specialized hypersuface, we need to transform $\chi = 0$ into its corresponding Tate or Weierstrass normal form. For convenience, we provide the Weierstrass as well as the Tate form of the most general hypersurface in appendix A.

4.1.2 Stable degeneration and the spectral cover polynomial

As a next step, we show how the K3 surface X_2 defined via (4.2) can be decomposed into the two half K3 surfaces $X_2^\pm$ and the heterotic elliptic curve as well as the two spectral cover polynomials, respectively. First, we write the Calabi-Yau hypersurface equation (4.3) for X_2 as

$$\chi = p^+(x_i, s_{j1})U^2 + p_0(x_i, s_{j2})UV + p^-(x_i, s_{j3})V^2, \tag{4.7}$$

for appropriate polynomials p^+, p_0 and p^- depending on the fiber coordinates x_i. By the results of the previous section 3.3, the K3 surface X_2 in the semistable degeneration limit can be described by the half K3 surfaces $X_2^\pm$. Practically, X_2^+ arises by setting the term V^2 in (4.7) to zero and factoring out U and vice versa, X^- is obtained by setting U^2 to zero and factoring out V. In conclusion, the defining equations for $X_2^\pm$ are given as

$$X_2^+ : \quad p^+(x_i, s_{j1})U + p_0(x_i, s_{j2})V = 0\,, \qquad X_2^- : \quad p^-(x_i, s_{j3})V + p_0(x_i, s_{j2})U = 0\,. \tag{4.8}$$

However, it is crucial to note that this is just a shortcut to obtain the defining equations for $X_2^\pm$ and that U, V are not the right coordinates for the base $\mathbb{P}^1$ of the two rational elliptic surfaces $X_2^\pm$. As explained in more detail in section (4.7), X^+ is coordinized by V, λ_1 and X^- is coordinized by U, μ. Thus, the correct form of the defining constraint for

X_2^+ is given by

$$X_2^+ : \quad p^+(x_i, s_{j1})\lambda_1 + p_0(x_i, s_{j2})V = 0 \,, \qquad X_2^- : \quad p^-(x_i, s_{j3})\mu + p_0(x_i, s_{j2})U = 0 \,. \tag{4.9}$$

It follows that generically the two linearly independent sections (4.4) of the K3 become independent sections in the half K3s, which we denote, by abuse of notation, by the same symbols. They intersect along the common (heterotic) elliptic curve. This is a novel property of our toric degeneration.

In addition, the heterotic elliptic curve is given as $p_0(x_i, s_j) = 0$ while the data of the two background bundles are given by the spectral cover polynomials $p^+(x_i, s_{j1}) = 0$ and $p^-(x_i, s_{j3}) = 0$. The structure group of the two heterotic bundles is then determined by the common solutions of $p_0(x_i, s_j) = 0$ with $p^\pm(x_i, s_{j1/3}) = 0$ using the results and techniques from section 2.2.

4.1.3 Promotion to elliptically fibered threefolds

Eventually, we are interested in examples of six-dimensional heterotic/F-theory duality. In order to promote the K3 surfaces X_2 constructed above to elliptically fibered threefolds we promote the coefficients s_{ij}, defined in (4.6), to sections of a line bundle of another $\mathbb{P}^1$ with homogeneous coordinates R, T. The base of the previously considered K3 surface and the new $\mathbb{P}^1$ form a Hirzebruch surface $\mathbb{F}_n$. At this point, we only consider base geometries which are Fano and restrict our discussion to $\mathbb{F}_0$ and $\mathbb{F}_1$ for simplicity, avoiding additional singularities in the heterotic elliptic fibration. For these two geometries, the explicit form of the s_{ij} reads

$$s_{ij} = s_{ij1}R^2 + s_{ij2}RT + s_{ij3}T^2, \tag{4.10}$$

for $\mathbb{F}_0$ and

$$s_i = s_{i11}R + s_{i12}T + s_{i21}R^2 + s_{i22}RT + s_{i23}T^2 + s_{i31}R^3 + s_{i32}R^2T + s_{i33}RT^2 + s_{i34}T^3, \tag{4.11}$$

for the geometry $\mathbb{F}_1$.

Next, we observe that the explicit expression of the discriminant of the heterotic Calabi-Yau manifold Z_n, which is given by $p_0 = 0$, contains a factor of s_{82}^2. While this is certainly not a problem in 8D, as s_{82} is just a constant there, it gives rise to an SU(2)-singularity at co-dimension one in the heterotic K3 surface Z_2. This can be cured by a resolution of this singularity through an exceptional divisor E, which is the analog of x_5 in (4.1). In particular, the solutions to the spectral cover constraint will pass through the singular point in the fiber. Thus, one expects that the spectral cover curve will pick up contributions from the class E in general. A similar situation has been analyzed in [55] where it has been argued that the introduction of this exceptional divisor will not change the structure of the spectral cover as an N-sheeted branched cover of the base except for a finite number of points where it wraps a whole new fiber component over the base. As discussed in [55], this introduces more freedom in the construction of the heterotic vector bundle V. As this work focuses on the mapping of U(1)-factors under the heterotic/F-theory duality, we only concentrate on the generic structure of the spectral cover and leave the resolution of this singularity as well as an exploration of the freedom in the construction of V to future works.

4.2 U(1)’s arising from U(1) factors in the heterotic structure group

In this section, we consider examples that have an additional rational section in the dual heterotic geometry. We consider K3 surfaces in F-theory, which are given as hypersurfaces in $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)} \times \mathbb{P}^1$ with appropriately specialized coefficients generating a corresponding gauge symmetry. Elliptic K3 fibered Calabi-Yau threefolds are constructed straightforwardly as described in section 4.1.3. Thus, our following discussion will be equally valid in 6D, although, in order to avoid confusion, we present our geometric discussions in 8D. Having this in mind we, therefore, drop here in the rest of this work the subscripts on all considered manifolds X_{n+1}, $X^{\pm}_{n+1}$ and Z_n, respectively. In the following, we discuss the main geometric properties of the Calabi-Yau manifold X, demonstrate heterotic/F-theory duality and relations among different examples by a chain of Higgsings.

We begin by a summary of key results and by setting some notation. As we will see, all considered examples have the same heterotic Calabi-Yau manifold Z in common. It is given by the most generic section of the anti-canonical bundle in $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ reading

$$Z: \quad s_{12}x_1^4 + s_{22}x_1^3x_2 + s_{32}x_1^2x_2^2 + s_{42}x_1x_2^3 + s_{52}x_1^2x_3 + s_{62}x_1x_2x_3 + s_{72}x_2^2x_3 + s_{82}x_3^2 = 0\,. \tag{4.12}$$

Here and in the following, we refer to the $\mathbb{P}^{(1,1,2)}$ coordinates only and work with the limit $x_4 \to 1, x_5 \to 1$.

The examples considered in the following only differ among each other by the spectral covers, i.e. by the choice of the coefficients s_{i1} and s_{i3} in (3.22), which will be different in each case. Generically, all examples will have a U(1)-factor embedded into the structure groups of both heterotic vector bundles V_1, V_2. Thus, the maximal non-Abelian gauge group determining any chain of Higgsings is given by $\mathrm{E}_7 \times \mathrm{E}_7$. For later reference we also note the Weierstrass normal form of (4.12) is given by

$$\begin{aligned}
W_Z: \quad y^2 = x^3 &+ \Big(-\frac{1}{48}s_{62}^4 + \frac{1}{6}s_{52}s_{62}^2s_{72} - \frac{1}{3}s_{52}^2s_{72}^2 - \frac{1}{2}s_{42}s_{52}s_{62}s_{82} + \frac{1}{6}s_{32}s_{62}^2s_{82} \\
&\qquad + \frac{1}{3}s_{32}s_{52}s_{72}s_{82} - \frac{1}{2}s_{22}s_{62}s_{72}s_{82} + s_{21}s_{72}^2s_{82} - \frac{1}{3}s_{32}^2s_{82}^2 + s_{22}s_{42}s_{82}^2 \Big)x \\
&+ \Big(\frac{1}{864}s_{62}^6 - \frac{1}{72}s_{52}s_{62}^4s_{72} + \frac{1}{18}s_{52}^2s_{62}^2s_{72}^2 - \frac{2}{27}s_{52}^3s_{72}^3 + \frac{1}{24}s_{42}s_{52}s_{62}^3s_{82} \\
&\quad - \frac{1}{72}s_{32}s_{62}^4s_{82} - \frac{1}{6}s_{42}s_{52}^2s_{62}s_{72}s_{82} + \frac{1}{36}s_{32}s_{52}s_{62}^2s_{72}s_{82} + \frac{1}{24}s_{22}s_{62}^3s_{72}s_{82} \\
&\quad + \frac{1}{9}s_{32}s_{52}^2s_{72}^2s_{82} - \frac{1}{6}s_{22}s_{52}s_{62}s_{72}^2s_{82} - \frac{1}{12}s_{21}s_{62}^2s_{72}^2s_{82} + \frac{1}{3}s_{21}s_{52}s_{72}^3s_{82} \\
&\quad + \frac{1}{4}s_{42}^2s_{52}^2s_{82}^2 - \frac{1}{6}s_{32}s_{42}s_{52}s_{62}s_{82}^2 + \frac{1}{18}s_{32}^2s_{62}^2s_{82}^2 - \frac{1}{12}s_{22}s_{42}s_{62}^2s_{82}^2 \\
&\quad + \frac{1}{9}s_{32}^2s_{52}s_{72}s_{82}^2 - \frac{1}{6}s_{22}s_{42}s_{52}s_{72}s_{82}^2 - \frac{1}{6}s_{22}s_{32}s_{62}s_{72}s_{82}^2 + s_{21}s_{42}s_{62}s_{72}s_{82}^2 \\
&\quad + \frac{1}{4}s_{22}^2s_{72}^2s_{82}^2 - \frac{2}{3}s_{21}s_{32}s_{72}^2s_{82}^2 - \frac{2}{27}s_{32}^3s_{82}^3 + \frac{1}{3}s_{22}s_{32}s_{42}s_{82}^3 - s_{21}s_{42}^2s_{82}^3 \Big)\,.
\end{aligned} \tag{4.13}$$

In the stable degeneration limit the heterotic geometry Z generically inherits two sections S_0^Z, S_1^Z from the elliptically fibered K3 defined in (4.2). These are the zero section as well as the generator of the U(1) and are in Weierstrass normal form explictly given as

$$S_0^Z = [1:1:0]\,, \tag{4.14}$$

$$S_1^Z = \Big[\frac{1}{12}(s_{62}^2 s_{72}^2 - 4s_{52}s_{72}^3 - 12s_{42}s_{62}s_{72}s_{82} + 8s_{32}s_{72}^2 s_{82} + 12s_{42}^2 s_{82}^2) : \frac{1}{2}s_{82}(s_{42}s_{62}^2 s_{72}^2 - s_{42}s_{52}s_{72}^3 - s_{32}s_{62}s_{72}^3 + s_{22}s_{72}^4 - 3s_{42}^2 s_{62}s_{72}s_{82} + 2s_{32}s_{42}s_{72}^2 s_{82} + 2s_{42}^3 s_{82}^2) : s_7\Big]\,. \tag{4.15}$$

Here, the first section S_0^Z is the point at infinity, while the second section S_1^Z can be seen also in the affine chart. We note that S_0^Z can be obtained by a simple coordinate transformation[7] from S_0 defined in (4.3), while S_1^Z needs to be constructed using the procedure of Deligne applied in [22].

4.2.1 Structure group U(1) × U(1): E$_7$ × E$_7$ × U(1) gauge symmetry

We start with a model which has a heterotic vector bundle of structure group U(1) × U(1). Upon commutation within the group E$_8$ × E$_8$, the centralizer is given as E$_7$ × U(1) × E$_7$ × U(1). On the heterotic side, the two U(1) factors acquire a mass term so that only a linear combination of them is massless. This matches the F-theory gauge group given by E$_7$ × E$_7$ × U(1).

Our example is specified by the following non-vanishing coefficients:

Coefficient	X	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{11}\lambda_1 + s_{12}V$
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	$s_{72}UV$	$s_{72}U$	$s_{72}V$
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

Here, the first columns denotes the coefficient in the Calabi-Yau constraint (3.2), the second column indices the chosen specialization and the third as well as fourth column contain the resulting coefficient in the half K3 fibrations $X^\pm$, respectively.

Using the identities (3.16) and (3.21), we readily write down the defining equations for the half K3 surfaces $X_2^\pm$ obtained via stable degeneration explicitly. They read

$$\begin{aligned} X^+ :\quad & (s_{11}\lambda_1 + s_{12}V)x_1^4 + s_{22}Vx_1^3x_2 + s_{32}Vx_1^2x_2^2 + s_{42}Vx_1x_2^3 \\ & + s_{52}Vx_1^2x_3 + s_{62}Vx_1x_2x_3 + s_{72}Vx_2^2x_3 + s_{82}Vx_3^2 = 0\,, \\ X^- :\quad & (s_{12}U + s_{13}\mu)x_1^4 + s_{22}Ux_1^3x_2 + s_{32}Ux_1^2x_2^2 + s_{42}Ux_1x_2^3 \\ & + s_{52}Ux_1^2x_3 + s_{62}Ux_1x_2x_3 + s_{72}Ux_2^2x_3 + s_{82}Ux_3^2 = 0\,. \end{aligned} \tag{4.16}$$

[7] To be more precise, we refer in this case to (4.3) as a section of the heterotic geometry.

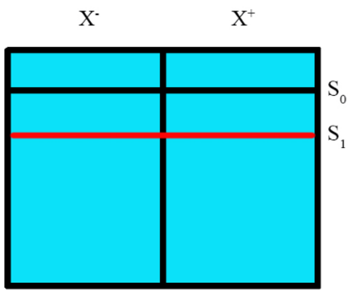

Figure 6. This figure shows the stable degeneration limit of a K3 surface which has $E_7 \times E_7 \times U(1)$ gauge symmetry. There are the two half K3 surfaces, X^+ and X^- which have both an E_7 singularity and intersect in a common elliptic curve. Both have two sections, S_0 and S_1 which meet in the common elliptic curve. Thus, there are two global sections in the full K3 surface and therefore a U(1) factor.

By explicitly evaluating the Tate coefficients (A.3), one obtains the following orders of vanishing for the Tate vector at the loci $U = 0$ and $V = 0$ for the full K3 surface,

$$\vec{t}_U = \vec{t}_V = (1, 2, 3, 3, 5, 9)\,, \tag{4.17}$$

which reveal two E_7 singularities. Also, the two half K3 surfaces inherit an E_7 singularity each, which are located at $U = 0$, $V = 0$, respectively. Thus, the non-Abelian part of the gauge group is given by $E_7 \times E_7$. Both half K3 surfaces have two rational sections given by $S_0^{X^\pm} = [0 : 1 : 0]$ and $S_1^{X^+} = [0 : s_{82}V : -s_{72}s_{82}V^2]$ and $S_1^{X^-} = [0 : s_{82}U : -s_{72}s_{82}U^2]$, respectively. In the intersection point of the two half K3's given by $[U : \lambda_1 : \mu : V : \lambda_2] = [1 : 0 : 0 : 1 : 1]$, the sections $S_0^{X^\pm}$ from both half K3's intersect and meet each other, and similarly the sections $S_1^{X^\pm}$ from both half K3's intersect and meet each other, cf. figure 6. Thus, the six-dimensional gauge group contains a U(1) factor.

However, if one evaluates the spectral cover, as described in section 4.1.2, one obtains[8]

$$p_+ = s_{11}x_1^4\,, \qquad p_- = s_{13}x_1^4\,. \tag{4.18}$$

which is mapped by use of the transformations (A.9) onto

$$p_+^W = s_{11}z^4, \qquad p_-^W = s_{13}z^4. \tag{4.19}$$

These expressions gives rise to a constant spectral cover in affine Weierstrass coordinates x, y defined by $z = 1$. However, on an elliptic curve there does not exist any function which has exactly one zero at a single point, in this case S_1.[9] Nevertheless, one can use the two points S_0^Z and S_1^Z on the heterotic elliptic curve in order to construct the bundle $\mathcal{L} = \mathcal{O}(S_1^Z - S_0^Z)$ fiberwise, which is symmetrically embedded into both E_8-bundles. As argued in [49], this bundle promotes to a bundle $\mathcal{L}^{6D}$ in 6D whose first Chern class is

[8] Here, and in the following we set $x_4 \to 1$, $x_5 \to 1$ for convenience.

[9] However, note that the homogeneous expression x_1^4 vanishes at the loci of S_0^Z and of S_1^Z. Indeed, the common solutions of $\lim_{x_4\to 1}\lim_{x_5\to 1}\chi$ defined in (4.2) with $x_1 = 0$ are given by $[x_1 : x_2 : x_3] = [0 : 1 : 0]$ or $[0 : s_8 : -s_7 s_8]$.

given by the difference of the two sections $c_1(\mathcal{L}^{6D}) = \sigma_{S_1^Z} - \sigma_{S_0^Z}$, up to fiber contributions. Thus, the heterotic gauge group is given by $E_7 \times E_7 \times U(1) \times U(1)$. Due to the background bundle $\mathcal{L}^{6D}$, these two U(1)'s seem both massive according to the Stückelberg mechanism discussed in section 2.4. However, due to the symmetric embedding into both E_8's their sum remains massless. Thus, one obtains a perfect match with the F-theory gauge group.

We conclude with the remark that one can interpret this model also as a Higgsing of a model with $E_8 \times E_8$ gauge symmetry as presented in the appendix B.1. Here, the Higgsing corresponds to a geometrical transition from the ambient space geometry $\mathbb{P}^{(1:2:3)}$ to $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ where the vacuum expectation value of the Higgs corresponds to the non-vanishing coefficient s_{72}.

4.2.2 Structure group $\mathbf{U(1) \times S\big(U(2) \times U(1)\big)}$: $\mathbf{E_7 \times E_6 \times U(1)}$ gauge symmetry

As a next step, we investigate an example which has $E_7 \times E_6 \times U(1)$ gauge symmetry. On the heterotic side we find an $U(1) \times S\big(U(2) \times U(1)\big)$ structure group which directly matches the non-Abelian gauge group and gives rise to one massless as well as one massive U(1). The model is specified by the following non-vanishing coefficients:

Coefficient	X	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{11}\lambda_1 + s_{12}V$
s_2	$s_{21}U^2 + s_{22}UV$	$s_{22}U$	$s_{21}\lambda_1 + s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	$s_{72}UV$	$s_{72}U$	$s_{72}V$
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

The evaluation of the order of vanishing of the Tate coefficients is summarized in the two Tate vectors

$$\vec{t}_V = (1,2,2,3,5,8) \qquad \vec{t}_U = (1,2,3,3,5,9)\,. \tag{4.20}$$

It signal one E_6 singularity at $V = 0$ and one E_7 singularity at $U = 0$. The E_7 singularity is inherited by the half K3 surface X^- while the E_6 singularity is contained in the half K3 surface X^+ after stable degeneration.

Next, we turn to the heterotic side. Here, the analysis of sections and U(1) symmetries from the perspective of the gluing condition is completely analogous to the geometry with $E_7 \times E_7 \times U(1)$ gauge symmetry discussed in the previous section 4.2.1. The situation at hand is summarized in figure 7. However, there is a crucial difference in the evaluation of the spectral cover which we discuss next.

The corresponding split of the two half K3 surfaces into a spectral cover polynomial and the heterotic elliptic curve results in

$$p^+ = s_{11}x_1^4 + s_{21}x_1^3x_2\,, \qquad p^- = s_{13}x_1^4\,. \tag{4.21}$$

Again, in order to evaluate the spectral cover information, one needs to transform both constraints into Weierstrass normal form. p^- is again just a constant and its interpretation

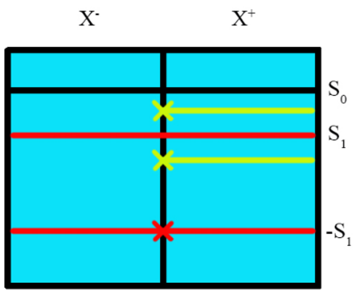

Figure 7. The interpretation of this figure is similar to figure 6. The additional structure arises from two sections shown in yellow which form together with $\boxminus S_1$ the zeros of the spectral cover.

is along the lines of the previous example in section 4.2.1. However, in the case of p^+ something non-trivial happens. Its transform into Weierstrass coordinates reads explicitly

$$\begin{aligned} p_W^+ = {}& (s_{21}s_{62}^3 s_{72} - 4s_{21}s_{52}s_{62}s_{72}^2 - 2s_{11}s_{62}^2 s_{72}^2 + 8s_{11}s_{52}s_{72}^3 - 2s_{21}s_{42}s_{62}^2 s_{82} \\ & - 4s_{21}s_{42}s_{52}s_{72}s_{82} - 4s_{21}s_{32}s_{62}s_{72}s_{82} + 24s_{11}s_{42}s_{62}s_{72}s_{82} + 12s_{21}s_{22}s_{72}^2 s_{82} \\ & - 16s_{11}s_{32}s_{72}^2 s_{82} + 8s_{21}s_{32}s_{42}s_{82}^2 - 24s_{11}s_{42}^2 s_{82}^2 - 12s_{21}s_{62}s_{72}x \\ & + 24s_{11}s_{72}^2 x + 24s_{21}s_{42}s_{82}x + 24s_{21}s_{72}y)/\big(2(-s_{62}^2 s_{72}^2 + 4s_{52}s_{72}^3 \\ & + 12s_{42}s_{62}s_{72}s_{82} - 8s_{32}s_{72}^2 s_{82} - 12s_{42}^2 s_{82}^2 + 12s_{72}^2 x)\big) \,. \end{aligned} \tag{4.22}$$

In contrast to the well-known case of the spectral cover in the $\mathbb{P}^{(1,2,3)}$-model which takes only poles at infinity, one observes that the denominator of (4.22) has two zeros at S_1^Z and at $\boxminus S_1^Z$, the negative of S_1^Z in the Mordell-Weil group of Z. In addition, the numerator has zeros at two irrational points Q_1, Q_2 and at $\boxminus S_1$. Finally, there is a pole of order one at S_0^Z. Here, S_0^Z and S_1^Z refer to the two sections (4.14), (4.15). Thus, the divisor of p_W^+ is given by

$$\operatorname{div}(p_W^+) = Q_1 + Q_2 - S_1 - S_0 \,. \tag{4.23}$$

Clearly, in order to promote the points defined by the spectral cover polynomial in 8D to a curve in 6D, the current form of p_W^- is not suitable due to its non-trivial denominator. However, one observes that the polynomial given by the numerator of p_W^+ gives rise to the divisor

$$\operatorname{div}\big(\text{Numerator}(p_W^+)\big) = Q_1 + Q_2 + \boxminus S_1 - 3S_0 \tag{4.24}$$

which is, however, linearly equivalent[10] to the divisor (4.23). Consequently, a spectral cover, valid also for the construction of lower-dimensional compactifications, is defined by the numerator of (4.22).

Thus, the three zeros Q_1, Q_2 and $\boxminus S_1$ form, following section 2.2, a split SU(3) spectral cover, i.e. an $\mathrm{S}\big(\mathrm{U}(2)\times\mathrm{U}(1)\big)$ spectral cover. All three points extend as sections into the half K3 surface X^+, cf. figure 7. Two of these sections are linearly independent and are in

[10]To see this, one notices that the element $-S_1 + S_0$ in $\mathrm{Pic}^0(E)$ is equivalent to $-S_1 + S_0 + f$ where f is defined as $x - x_{S_1}$ on E with x_{S_1} denoting the x-coordinate of S_1. It holds that $\operatorname{div}(f) = S_1 + \boxminus S_1 - 2S_0$. Thus, $-S_1 + S_0$ maps to $\boxminus S_1$ on E under the map (2.11).

8D the generators of the rank two Mordell-Weil group corresponding to a rational elliptic surface with an E_6 singularity. However, due to monodromies of Q_1 and Q_2 only $\boxminus S_1$ survives in 6D as a rational section.

In conclusion, this spectral cover gives rise to an $S\big(U(2) \times U(1)\big)$ background bundle which is embedded into the E_8 factor corresponding to X^+. The centralizer of this is given by $E_6 \times U(1)$. The latter factor seems again massive due to the U(1) background bundle. However, this U(1) forms together with the seemingly massive U(1) of the half K3 surface X^- a massless linear combination. In conclusion, there is a perfect match with the F-theory analysis of the low energy gauge group. Analogously to the previous case in section 4.2.1, this model can be understood as arising by Higgsing the non-Abelian model B.2 with gauge symmetry $E_8 \times E_7$. Here, a (massive) U(1) factor is embedded minimally into both factors. Again, the vacuum expectation value of the Higgs corresponds to the coefficient s_{72}. In addition, we can view this model also as arising by a Higgsing process from a compactification with $E_7 \times E_7 \times U(1)$ gauge group where a vacuum expectation value of the Higgs corresponds to s_{21}.

4.2.3 Structure group $\mathbf{U(1) \times \big(SU(2) \times SU(2) \times U(1)\big)}$: $\mathbf{E_7 \times SO(9) \times U(1)}$ gauge symmetry

The final example in this chain of Higgsings is given by a model with $E_7 \times SO(9) \times U(1)$ gauge symmetry. On the heterotic side we find an $U(1) \times \big(SU(2) \times SU(2) \times U(1)\big)$ structure group which matches the non-Abelian gauge content. Also in this case we find one massless as well as one massive U(1) on the heterotic side.

As before, we define the model by the following choice of coefficients in X:

Coefficient	X	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{11}\lambda_1 + s_{12}V$
s_2	$s_{21}U^2 + s_{22}UV$	$s_{22}U$	$s_{21}\lambda_1 + s_{22}V$
s_3	$s_{31}U^2 + s_{32}UV$	$s_{32}U$	$s_{31}\lambda_1 + s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	$s_{72}UV$	$s_{72}U$	$s_{72}V$
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

Once again we begin the analysis on the F-theory side with the evaluation of the order of vanishing of the Tate coefficients. We obtain the Tate vectors

$$\vec{t}_V = (1, 1, 2, 3, 4, 7)\,, \qquad \vec{t}_U = (1, 2, 3, 3, 5, 9)\,, \tag{4.25}$$

which signal one SO(9) singularity at $V = 0$ and one E_7 singularity at $U = 0$, each of which being inherited by one half K3 surface.

For the analysis of the heterotic side, we split the two half K3 surfaces into a spectral cover polynomial and the heterotic elliptic curve. We obtain

$$p^+ = s_{11}x_1^4 + s_{21}x_1^3x_2 + s_{31}x_1^2x_2^2\,, \qquad p^- = s_{11}x_1^4\,, \tag{4.26}$$

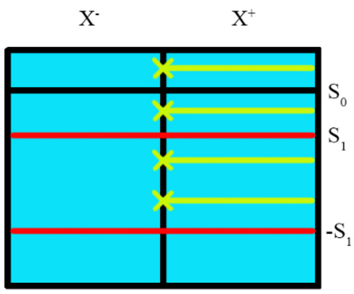

Figure 8. The half K3 surface X^- only exhibits the section S^1 in addition to the zero section. In contrast, X^+ gives rise to a spectral cover polynomial that has two pairs of irrational solutions Q_1, Q_2, R_1, R_2 that sum up to S_1^Z each.

from which we see that p^- is again a trivial spectral cover. Again, in order to evaluate the non-trivial spectral cover p^+, one needs to transform both constraints into Weierstrass normal form. The interpretation of p^+ is as in the previous cases. We again obtain a Weierstrass form p_W^+ with a denominator. The explicit expression is rather lengthy and can be provided upon request. Its divisor is given by

$$\mathrm{div}(p_W^+) = Q_1 + Q_2 + R_1 + R_2 - 2S_1 - 2S_0\,, \tag{4.27}$$

Here Q_1, Q_2 and R_1, R_2 are two pairs of irrational points which obey $Q_1 \boxplus Q_2 \boxminus S_1 = 0$ and $R_1 \boxplus R_2 \boxminus S_1 = 0$. The divisor of p_W^+ is again equivalent to the divisor of its numerator reading

$$\mathrm{div}\left(\mathrm{Numerator}(p_W^+)\right) = Q_1 + Q_2 + R_1 + R_2 + 2 \boxminus S_1 - 6S_0\,. \tag{4.28}$$

By a similar token as before, we thus drop the denominator and just work with the numerator of p_W^+.

All the points appearing here extend to sections of the half K3 surface X^+. However, while Q_1, Q_2, R_1, R_2 extend to rational sections of the half K3 surface they do not lift to rational sections of the rational elliptic threefold. Altogether, we obtain as in the previous examples two rational sections in both half K3 surfaces which glue to global sections and therefore give rise to a U(1) factor. Besides that the spectral cover is split and describes a vector bundle with structure group $\mathrm{S}\big(\mathrm{U}(2)\times\mathrm{U}(1)\big)\times\mathrm{S}\big(\mathrm{U}(2)\times\mathrm{U}(1)\big)$, where the U(1) part in both factors needs to be identified. This is due to the fact that in both cases the same point, $\boxminus S_1^Z$, splits off. Thus, the spectral cover is isomorphic to $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ whose centralizer[11] within E_8 is given by $\mathrm{SO}(9)\times\mathrm{U}(1)$. Thus, we obtain again two seemingly massive U(1)'s which give rise to one massless linear combination.

This model can be understood by a Higgsing mechanism. Either it can be viewed as arising from the non-Abelian model in section B.3 with $\mathrm{E}_8\times\mathrm{SO}(11)$ gauge symmetry, by giving a vacuum expectation value to a Higgs corresponding to s_{72}, or from the previous example in section 4.2.2, by giving a vacuum expectation value to a Higgs associated to s_{31}.

[11] We employ here the breaking $\mathrm{E}_8 \longrightarrow \mathrm{SO}(9)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$.

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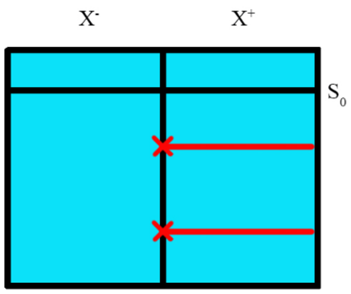

Figure 9. The half K3 surface X^- has only one section $S_0^{X^-}$ which merges with the section $S_0^{X^+}$ from the other half K3 surface X^+. X^+ has in addition also the section S^{X^+} which does not merge with a section of X^-. Thus, there is no U(1)-factor on the F-theory side.

4.2.4 Example with only one massive U(1): $\mathbf{S}\big(\mathbf{U(1)} \times \mathbf{U(1)}\big)$ structure group

Finally, we conclude the list of examples with a model which has only one U(1)-bundle embedded into one of its E_8 factors while the other E_8 stays untouched. Accordingly there is only one massive U(1) symmetry. On the F-theory side we obtain an $E_8 \times E_6 \times SU(2)$ gauge symmetry which matches the findings on the heterotic side.

The model is defined by the following specialization of the coefficients in the constraint (4.2):

Coefficient	X	X^-	X^+
s_1	$s_{13}V^2$	$s_{13}\mu$	0
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{41}U^2 + s_{42}UV$	$s_{42}U$	$s_{42}V + s_{41}\lambda_1$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	0	0	0
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

First of all, we note that the coefficient s_7 vanishes identically. Thus, we have changed the ambient space of the fiber from $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$ to $\mathbb{P}^{(1,2,3)}$. Therefore, we do not expect to see another section besides the zero section on the F-theory side and therefore no U(1), cf. appendix C.

First, we determine the gauge group on the F-theory side. As before, we evaluate the Tate coefficients along the singular fibers which are in the case at hand located at $U = 0$, $V = 0$ and $s_{41}U + s_{42}V = 0$. One obtains the Tate vectors

$$\vec{t}_U = (1,2,3,4,5,10)\,, \qquad \vec{t}_V = (1,2,2,3,5,8)\,, \qquad \vec{t}_{s_{41}U+s_{42}V} = (0,0,1,1,2,2)\,. \tag{4.29}$$

Clearly, these signal an $E_8 \times E_6 \times SU(2)$ gauge group in F-theory. Also, after the stable degeneration limit, one obtains one half K3 surface X^- with an E_8 singularity and one, X^+, with an $E_6 \times SU(2)$ singularity.

For the further analysis we remark that there is the zero section $S_0 = [0:1:0]$ in the K3 surface only. As in the previous discussion, we refer to the $\mathbb{P}^{(1,1,2)}$ coordinates

$[x_1 : x_2 : x_3]$ only, i.e. we work in the limit $x_4 \to 1$, $x_5 \to 1$. For the two half K3s one finds that X^- has only a zero section. In contrast, one observes the sections[12] $S^{X^+} = [1:0:0]$ and $\boxminus S^{X^+} = [s_{82}V : 0 : -s_{52}s_{82}V^2]$ in the other half K3 surface X^+. However, these sections do not glue with another section of X^- and therefore do not give rise to a U(1) symmetry from the F-theory perspective. However, from the heterotic perspective they should give rise to a *massive* U(1) which upon commutation within E_8 leaves an $E_6 \times SU(2)$ gauge symmetry.

This result is in agreement with the spectral cover analysis. One evaluates the spectral cover polynomials as

$$p^- = s_{13}x_1^4 \qquad p^+ = s_{41}x_1x_2^3 \,. \tag{4.30}$$

As observed already before, the Weierstrass transform p_W^- of p^- does not have any common solution with the heterotic elliptic curve and therefore the E_8-symmetry does not get broken. For the half K3 surface X^+, the common solutions to p_W^+ and the heterotic elliptic curve are given in Weierstrass coordinates $[x : y : z]$ as

$$\begin{aligned} S_W^Z &= \left[\frac{1}{12}(s_{62}^2 - 4s_{32}s_{82}) : -\frac{1}{2}s_{42}s_{52}s_{82} : 1\right], \\ \boxminus S_W^Z &= \left[\frac{1}{12}(s_{62}^2 - 4s_{32}s_{82}) : \frac{1}{2}s_{42}s_{52}s_{82} : 1\right]. \end{aligned} \tag{4.31}$$

Here, S_W^Z and $\boxminus S_W^Z$ denote the intersections of S^{X^+} and $\boxminus S^{X^+}$ with the heterotic geometry Z respectively, in Weierstrass coordinates. Thus, we observe a split spectral cover pointing towards the structure group $S\big(U(1) \times U(1)\big)$. Using the breaking $E_8 \longrightarrow E_6 \times SU(2) \times U(1)$, this spectral cover matches with the observed gauge group. The U(1) is decoupled from the massless spectrum via the Stückelberg effect of section 2.4.

4.3 Split spectral covers with torsional points

In the following, we discuss examples which exhibit a torsional section in their spectral covers. As mentioned before, heterotic/F-theory duality suggests that the structure group of the heterotic vector bundle should contain a discrete part.

4.3.1 Structure group $\mathbb{Z}_2$: $E_8 \times E_7 \times SU(2)$ gauge symmetry

We consider a model which arises by the following specialization of coefficients in (4.2):

Coefficient	X	X^-	X^+
s_1	$s_{13}V^2$	$s_{13}\mu$	0
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{41}U^2 + s_{42}UV$	$s_{42}U$	$s_{42}V + s_{41}\lambda_1$
s_5	0	0	0
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	0	0	0
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

[12]Clearly, as the rank of the Mordell Weil group of X^+ is positive, there are in fact infinitely many sections.

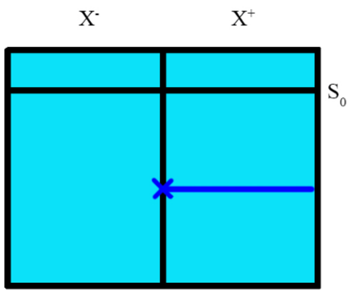

Figure 10. The stable degeneration limit of a K3 surface with $E_8 \times \big(E_7 \times SU(2)\big)/\mathbb{Z}_2$. The half K3 surface X^- has trivial Mordell-Weil group, while the half K3 surface X^+ has a torsional Mordell-Weil group $\mathbb{Z}_2$.

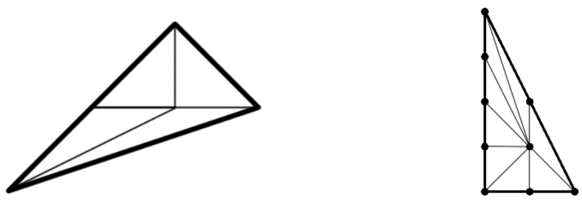

Figure 11. The left picture shows the specialized two-dimensional polytope Δ_2 corresponding to the monomials parameterizing the elliptic fiber of the half K3 surface X^+. The right figure shows its dual, Δ_2°, which specifies the ambient space of the elliptic fiber of X^+.

We start the analysis with the gauge group on the F-theory side first. There are three singular loci of the fibration at $U = 0$, $V = 0$ and $s_{41}U + s_{42}V = 0$. The evaluation of the Tate coefficients reveals the Tate vectors

$$\vec{t}_U = (1, 2, \infty, 4, 5, 10)\,, \qquad t_V = (1, 2, \infty, 3, 5, 9)\,, \qquad \vec{t}_{s_{41}U+s_{42}V} = (0, 0, \infty, 1, 2, 2)\,. \tag{4.32}$$

Thus, there are an E_8 singularity as well as an E_7 and an SU(2) singularity. The E_8 singularity is inherited by the half K3 surface X^- while X^+ gets endowed with an E_7 and an SU(2) singularity.

As a next step, we observe that there is only one section given by $[1 : 0 : 0]$ in the half K3 surface X^- and two sections given by $[x_1 : x_2 : x_3] = [0 : 1 : 0]$ and $[x_1 : x_2 : x_3] = [1 : 0 : 0]$ in the half K3 surface X^+. Here, we work again in the limit $x_4 = x_5 = 1$. In contrast, the full K3 surface has only one section namely the point at infinity. Moreover, a transformation into Weierstrass coordinates shows that the generic section $S_1^{X^+}$ has specialized into a torsional section of order two as can be checked using the results of [37]. This is expected, as the centralizer of the gauge algebra[13] $E_7 \times SU(2)$ within E_8 is given by $\mathbb{Z}_2$, which is also expected from the general analysis of [56]. In contrast, the full K3 surface X does not have a torsional section of order two [57].

[13]To be precise, E_8 only contains the group $\big(E_7 \times SU(2)\big)/\mathbb{Z}_2$ as a subgroup.

Finally, we turn towards the analysis of the gauge group from the heterotic side. Here, the spectral cover is given by

$$p^- = s_{13}x_1^4 \,, \qquad p^+ = s_{41}x_1x_2^3 \,. \tag{4.33}$$

After transformation to Weierstrass normal coordinates p_W^- is given by a constant which has no common solution with the elliptic curve. In contrast, the transformed quantity p_W^+ gives rise to the point

$$[x:y:z] = \left[\frac{1}{3}\left(\frac{s_{62}^2}{4} - s_{32}s_{82}\right) : 0 : 1\right]. \tag{4.34}$$

which is a torsion point of order two. In other words we see that the spectral cover is just given by a torsional point.

In [37] it has been suggested that an F-theory compactification with a torsional section in an elliptically fibered Calabi-Yau manifold and its stable degeneration limit should be dual to pointlike instantons with discrete holonomy on the heterotic side. Due to the similarity to the considered example, we propose that the spectral cover p_W^+ is to be interpreted as describing such a pointlike instanton with discrete holonomy. In addition, as pointed out above, the matching of gauge symmetry on both sides of the duality only works if the spectral cover p_W^+ is interpreted in this way. It would be exciting to confirm this proposal further by a more detailed analysis of the spectral cover, computation of the heterotic tadpole, or an analysis of codimension two singularities in F-theory.

4.3.2 Structure group $\mathrm{S}(\mathrm{U}(2) \times \mathbb{Z}_2)$: $\mathrm{E}_8 \times \mathrm{E}_6 \times \mathrm{U}(1)$ gauge symmetry

In this section we present another example whose spectral cover polynomial contains a torsional point and leads to an $\mathrm{E}_8 \times \mathrm{E}_6 \times \mathrm{U}(1)$ gauge symmetry. As one E_8 factor is left intact, the U(1) factor needs to be embedded solely into one E_8 bundle.

The starting point of our analysis is the following specialization of coefficients in (4.2):

Coefficient	X	X^-	X^+
s_1	$s_{13}V^2$	$s_{13}\mu$	0
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{41}U^2 + s_{42}UV$	$s_{42}U$	$s_{42}V + s_{41}\lambda_1$
s_5	0	0	0
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	$s_{71}U^2$	0	$s_{71}\lambda_1$
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

As in the previous cases, we compute the orders of vanishing of the Tate coefficients in order to determine the gauge group on the F-theory side. The computed Tate vectors signal an E_8 symmetry at $U = 0$ and an E_6 symmetry at $V = 0$. As a next step, we investigate the rational sections of X. As the coefficient s_7 does not vanish for the full K3 surface, there are the two generic sections S_0, S_1 realized in this model. However, the half K3 surface X^- only has the zero section $S_0^{X^-}$. In contrast, the half K3 surface X^+

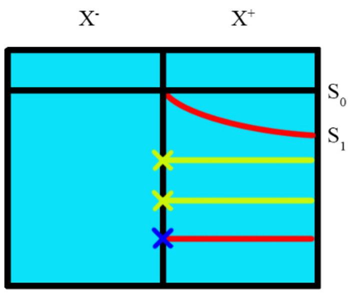

Figure 12. The half K3 surface X^- exhibits only the zero section, while the half K3 surface X^+ has also the section $S_1^{X^+}$ which merges with the section $S_0^{X^-}$ along the heterotic geometry. Thus there are two independent sections in the full K3 surface giving rise to a U(1) gauge group factor. In addition, the inverse of $S_1^{X^+}$ becomes a torsion point of order two when hitting the heterotic geometry.

has two linear independent sections given by $S_0^{X^+}$, $S_1^{X^+}$, which lift to rational sections of the elliptic threefold. Due to the theory of rational elliptic surfaces, there is another linear independent section of X^+ in 8D which nevertheless does not lift to a rational section of the rational elliptic threefold. The two linear independent rational sections unify in the heterotic elliptic curve and continue as one section into the other half K3 surface, see figure 12. This behavior of rational sections explains the origin of the U(1)-factor from the gluing condition discussed in section 2.4.2.

As a further step, we investigate how this U(1) factor is reflected in the spectral cover on the heterotic side. The spectral cover polynomials computed by stable degeneration read

$$p^- = s_{13}x_1^4\,, \qquad p^+ = s_{41}x_1x_2^3 + s_{71}x_2^2x_3\,. \tag{4.35}$$

The interpretation of p^- is as in all the other cases just a trivial spectral cover. The common solution to p^+ and the heterotic Calabi-Yau manifold Z is given by a pair of irrational points R_1, R_2 as well as a further point T_t which has in Weierstrass normal form coordinates

$$T_t = \left[\frac{1}{3}\left(\frac{1}{4}s_{62}^2 - s_{32}s_{82}\right) : 0 : 1\right]. \tag{4.36}$$

Thus, it is a torsion point of order two. However, it does not extend as a full torsional section into the half K3 surface X^+. The corresponding section is rather the inverse of S_1.

Again we see that the split spectral cover p^+ contains a torsional section. Let us comment on the interpretation of this for the structure group of the heterotic vector bundle. Heterotic/F-theory duality implies that the low-energy effective theory contains a massless U(1)-symmetry. However, as we have seen in section 2.4, a U(1) background bundle in the heterotic theory has a non-trivial field strength and thus a non-vanishing first Chern class, which would yield a massive U(1) in the effective field theory. Thus, we can not interpret the torsional component T_t to the spectral cover as a U(1) background bundle. By the arguments of section 2.2 and the similarity to the setups considered in [37], it is tempting to identify this torsional component T_t as a pointlike heterotic instanton with discrete holonomy. In order to justify this statement, it would be necessary to compute the first Chern class of a heterotic line bundle that is defined in terms of components to the cameral cover given by rational sections of the half K3 fibrations arising in stable degeneration. In [49], it has been argued that the first Chern class is given, up to vertical components, by the difference of the rational section and the zero section. If the first Chern class would be completed into the Shioda map of the rational section, which we conjecture to be the case, it would be zero precisely for a torsional section [58]. Consequently, the U(1) in the commutant of E_8 would remain massless as the gauging in (2.23) would be absent. It would be exciting to confirm this conjecture by working out the missing vertical part in the formula for the first Chern class of a U(1) vector bundle.

4.4 U(1) factors arising from purely non-Abelian structure groups

In this final section, we present an example in which the heterotic vector bundle has only purely non-Abelian structure group, while the F-theory gauge group analysis clearly signals a U(1) factor.

As in the previous cases, we start by specifying the specialization of the coefficients in the defining hypersurface equation for X:

Coefficient	X	X^-	X^+
s_1	$s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{12}V$
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	$s_{71}U^2$	0	$s_{71}\lambda_1$
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

We determine the gauge symmetry of the F-theory side by analysis of the Tate coefficients. We obtain the Tate vectors

$$\vec{t}_U = (1,2,3,4,5,10) \qquad \vec{t}_V = (1,1,2,2,4,6)\,, \tag{4.37}$$

which reveal an E_8 singularity at $U = 0$ and an SO(7) singularity at $V = 0$. We note that it is not directly possible to distinguish an SO(7) singularity from an SO(8) singularity using the Tate table 2 only. To confirm that the type of singularity is indeed SO(7) we have to investigate the monodromy cover [59] which is for an I_0^* fiber given by

$$A: \quad \psi^3 + \left(\frac{f}{v^2}\bigg|_{v=0}\right)\psi + \left(\frac{g}{v^3}\bigg|_{v=0}\right) = 0\,. \tag{4.38}$$

Here, v is the affine coordinate V/U and f, g are the Weierstrass coefficients. An I_0^* fiber is SO(7) if the monodromy cover A factors into a quadratic and a linear constraint, which is indeed the case for the example at hand.

The stable degeneration limit yields two half K3 surfaces, X^+ and X^-, cf. figure 13. There only exists the zero section in X^-. In contrast, X^+ has two independent sections which are given by $S_0^{X^+}$ and $S_1^{X^+}$ and lift to rational sections of the rational elliptic

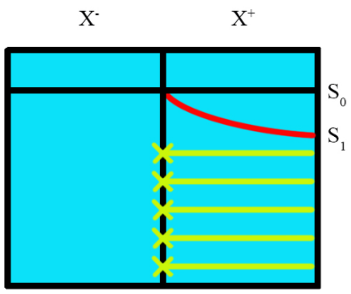

Figure 13. The half K3 surface X^- exhibits only the zero section, while the half K3 surface X^+ has also the section $S_1^{X^+}$ which merges with the section $S_0^{X^-}$ in the heterotic geometry. Thus, there are two independent sections in the full K3 surface giving rise to a U(1) gauge group factor.

threefold.[14] As in the previously considered case in section 4.3.2, $S_1^{X^+}$ unifies with $S_0^{X^-}$ on the heterotic elliptic curve. Thus, there are two global sections in the full K3 surface and therefore a U(1) factor in the F-theory compactification.

Turning towards the discussion of the heterotic gauge bundles, one finds that the U(1) factor is encoded in the data of the spectral cover polynomial as follows. We observe that the spectral covers following X^+ and X^-, respectively, are given by

$$p^- = s_{13} x_1^4 \, , \qquad p^+ = s_{71} x_2^2 x_3 \, . \tag{4.39}$$

The intersection of its Weierstrass transform p_W^+ with the heterotic elliptic curve gives five irrational points R_1, R_2, T_1, T_2, T_3 with $R_1 \oplus R_2 = 0$ and $T_1 \oplus T_2 \oplus T_3 = 0$. Thus we have a heterotic vector bundle with SU(2) × SU(3) structure group. As the spectral cover p^+ has one free parameter only, namely s_{71}, this model does not seem to have any moduli.

As our understanding of the precise embedding of the structure group into E_8 is limited, we have checked all possible ways to embed the group SO(7) × SU(2) × SU(3) into E_8. Independently of the chosen embedding, there is always a U(1) in all possible breakings. Thus, we are led to conclude that the centralizer of SU(2) × SU(3) necessarily produces a U(1) factor which matches with the F-theoric analysis.

5 Conclusions and future directions

In this paper we have presented a first explicit analysis of the origin of Abelian gauge symmetries for string theory compactifications within the duality between the $E_8 \times E_8$ heterotic string and F-theory. Here we summarize the framework of the analysis, highlight the key advancements, and conclude with future directions.

Framework. We have focused on F-theory compactifcations with a rank one Mordell-Weil group of rational sections both for compactifications to 8D and 6D. We have systematically studied a broad class F-theory compactifications on elliptically fibered Calabi-Yau $(n+1)$-folds (with $n = 1, 2$, respectively) with rational sections and rigorously performed the

[14] On the level of the rational elliptic surface X_2^+, there are additional linear independent sections which however do not lift into the six-dimensional geometry.

stable degeneration limit to dual heterotic compactifications on elliptically fibered Calabi-Yau n-folds. All considered examples are toric hypersurfaces and the stable degeneration limit is performed as a toric symplectic cut.

The key aspects of the analysis are the following:

- We have carefully investigated the solutions of the spectral cover polynomial and the hypersurface for the heterotic elliptically fibered Calabi-Yau manifold. We have used the group law of the elliptic curve in Weierstrass normal form in order to determine the structure group of the heterotic background bundle.
- We have analyzed the origin of the resulting gauge group. In 6D this involves incorporation of the massive U(1) gauge symmetries, due to the heterotic Stückelberg mechanism, that are not visible in F-theory.

Key results. While the F-theory side provides a unifying treatment of Abelian gauge symmetries, as encoded in the Mordell-Weil group of elliptically fibered Calabi-Yau $(n+1)$-folds, a detailed analysis of a broad classes of toric F-theory compactifications has resulted in the proposal of three different classes of heterotic duals that give rise to U(1) gauge group factors:

- Split spectral covers describing bundles with $\mathrm{S}\big(\mathrm{U}(m) \times \mathrm{U}(1)\big)$ structure group. Examples of this type have been discussed in section 4.2.
- Spectral covers containing torsional sections giving rise to bundles with $\mathrm{SU}(m) \times \mathbb{Z}_k$ structure group. Classes of examples with this structure group have been presented in section 4.3.
- The appearance of bundles with structure groups of the type $\mathrm{SU}(m) \times \mathrm{SU}(l)$ whose commutants inside E_8 contain a U(1)-factor. Explicit examples of this form can be found in section 4.4.

Future directions. While the work presents a pioneering effort, addressing comprehensively the origin of Abelian gauge group factors in heterotic/F-theory duality for a class of compactifications, the analysis provides a stage for further studies, both by extending the systematics of the analysis and by further detailed studies of the dual heterotic geometry and vector bundle data.

- It would be important to extend the studies to examples within larger classes of pairs of dual toric varieties as well as of more general elliptically fibered Calabi-Yau manifolds, respectively. In particular, this would allow to account for studies of dual geometries with broader classes of complex structure moduli spaces, and thus for an analysis of more general spectral covers of dual heterotic vector bundles. In 6D our analysis has been limited to a specific elliptically fibered Calabi-Yau $(n + 1)$-fold, which has resulted in constrained appearances of non-Abelian gauge symmetries and additional U(1)'s. In particular, it would be illuminating to elaborate on the stable degeneration limit for general toric fibrations of two-dimensional polyhedra over $\mathbb{P}^1$ in 8D and, in addition, over Hirzebruch surfaces in 6D.

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- It would be interesting to have the tools to study the spectral cover directly in the $Bl_1\mathbb{P}^{(1,1,2)}$ model or more generally for fiber geometries which are given by the sixteen two-dimensioal reflexive polyhedra. This would require in particular a notion of the group law for these representations of elliptic curves.

- The study of the properties of the spectral cover was primarily confined to the derivation of the resulting gauge symmetries and the structure groups of the heterotic vector bundles. Further analysis of the spectral cover in compactifications to 6D (and extensions to 4D) is needed; it should shed light on the further spectral cover data, which enter Chern classes, anomaly cancellation and matter spectrum calculations. This study is complicated by the resolution of singularities of the heterotic geometry that may have to be performed, resulting in spectral covers, which are not finite [55].

- Our analysis has been primarily constrained to studies of Abelian gauge symmetries in the language of a perturbative heterotic dual. Although we have encountered spectral covers which seem to describe small instantons, i.e. non-perturbative M5-branes, with discrete holonomy, we have not systematically analyzed their effect. In F-theory, M5-branes are visible as non-minimal singularities which occur at co-dimension two loci that have to be blown up. It would be interesting to thoroughly perform this geometric analysis. We expect in addition rich structures of Abelian gauge symmetry factors in F-theory whose heterotic duals are due to other types of non-perturbative M5-branes. Furthermore, it would be interesting to study the geometric transitions between F-theory geometries with different numbers of tensor multiplets, whose discussion is again related to this resolution process.

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A Weierstrass and Tate form of the hypersurface χ

In this appendix, we summarize the Weierstrass normal form as well as the Tate coefficients of the χ model. For convenience, we recall the most general form of the hypersurface χ

which reads in the limit $x_4 \to 1$, $x_5 \to 1$

$$\chi := s_1 x_1^4 + s_2 x_1^3 x_2 + s_3 x_1^2 x_2^2 + s_4 x_1 x_2^3 + s_5 x_1^2 x_3 + s_6 x_1 x_2 x_3 + s_7 x_2^2 x_3 + s_8 x_3^2 = 0\,, \quad s_i \in \mathcal{O}_{\mathbb{P}^1}(2)\,. \tag{A.1}$$

This can be brought in the so-called Tate form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 + a_4 x + a_6\,. \tag{A.2}$$

The Tate coefficients are explicitly given as [30]

$$\begin{aligned}
a_1 &= s_6\,,\\
a_2 &= -s_5 s_7 - s_3 s_8\,,\\
a_3 &= -s_4 s_5 s_8 - s_2 s_7 s_8\,,\\
a_4 &= s_3 s_5 s_7 s_8 + s_1 s_7^2 s_8 + s_2 s_4 s_8^2\,,\\
a_6 &= -s_1 s_3 s_7^2 s_8^2 - s_1 s_4^2 s_8^3 + s_4 s_7(-s_2 s_5 s_8^2 + s_1 s_6 s_8^2)\,.
\end{aligned} \tag{A.3}$$

In addition, it is useful, to introduce the quantities

$$\begin{aligned}
b_2 &= a_1^2 + 4a_2\,,\\
b_4 &= a_1 a_3 + 2a_4\,,\\
b_6 &= a_3^2 + 4a_6\,.
\end{aligned} \tag{A.4}$$

The Weierstrass normal form of χ^{sing} reads

$$\begin{aligned}
f &= \Big(-\frac{1}{48}s_{62}^4 + \frac{1}{6}s_{52}s_{62}^2 s_{72} - \frac{1}{3}s_{52}^2 s_{72}^2 - \frac{1}{2}s_{42}s_{52}s_{62}s_{82} + \frac{1}{6}s_{32}s_{62}^2 s_{82}\\
&\quad + \frac{1}{3}s_{32}s_{52}s_{72}s_{82} - \frac{1}{2}s_{22}s_{62}s_{72}s_{82} + s_{21}s_{72}^2 s_{82} - \frac{1}{3}s_{32}^2 s_{82}^2 + s_{22}s_{42}s_{82}^2\Big)\,.\\
g &= \Big(\frac{1}{864}s_{62}^6 - \frac{1}{72}s_{52}s_{62}^4 s_{72} + \frac{1}{18}s_{52}^2 s_{62}^2 s_{72}^2 - \frac{2}{27}s_{52}^3 s_{72}^3 + \frac{1}{24}s_{42}s_{52}s_{62}^3 s_{82}\\
&\quad - \frac{1}{72}s_{32}s_{62}^4 s_{82} - \frac{1}{6}s_{42}s_{52}^2 s_{62}s_{72}s_{82} + \frac{1}{36}s_{32}s_{52}s_{62}^2 s_{72}s_{82} + \frac{1}{24}s_{22}s_{62}^3 s_{72}s_{82}\\
&\quad + \frac{1}{9}s_{32}s_{52}^2 s_{72}^2 s_{82} - \frac{1}{6}s_{22}s_{52}s_{62}s_{72}^2 s_{82} - \frac{1}{12}s_{21}s_{62}^2 s_{72}^2 s_{82} + \frac{1}{3}s_{21}s_{52}s_{72}^3 s_{82}\\
&\quad + \frac{1}{4}s_{42}^2 s_{52}^2 s_{82}^2 - \frac{1}{6}s_{32}s_{42}s_{52}s_{62}s_{82}^2 + \frac{1}{18}s_{32}^2 s_{62}^2 s_{82}^2 - \frac{1}{12}s_{22}s_{42}s_{62}^2 s_{82}^2\\
&\quad + \frac{1}{9}s_{32}^2 s_{52}s_{72}s_{82}^2 - \frac{1}{6}s_{22}s_{42}s_{52}s_{72}s_{82}^2 - \frac{1}{6}s_{22}s_{32}s_{62}s_{72}s_{82}^2 + s_{21}s_{42}s_{62}s_{72}s_{82}^2\\
&\quad + \frac{1}{4}s_{22}^2 s_{72}^2 s_{82}^2 - \frac{2}{3}s_{21}s_{32}s_{72}^2 s_{82}^2 - \frac{2}{27}s_{32}^3 s_{82}^3 + \frac{1}{3}s_{22}s_{32}s_{42}s_{82}^3 - s_{21}s_{42}^2 s_{82}^3\Big)\,.
\end{aligned} \tag{A.5}$$

In particular, the discriminant reads

$$\Delta = 4f^3 + 27g^2 = \frac{1}{48}s_{82}^2(\dots)\,. \tag{A.6}$$

where the expression in the bracket denotes a generic polynomial.

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A.1 The map to Weierstrass normal form

In this subsection we discuss the bi-rational map of (A.1) to Weierstrass normal form. As a first step, we transform (4.2) into the form

$$\tilde{s}_1 x_1^4 + \tilde{s}_2 x_1^3 x_2 + \tilde{s}_3 x_1^2 x_2^2 + \tilde{s}_4 x_1 x_2^3 + s_7 x_2^2 x_3 + x_3^2 = 0 \,. \tag{A.7}$$

Here, we have introduced the new quantities

$$\tilde{s}_1 = -\frac{1}{4}s_5^2 + s_0 s_8 \,, \quad \tilde{s}_2 = -\frac{1}{2}s_5 s_6 + s_1 s_8 \,, \quad \tilde{s}_3 = -\frac{1}{4}s_6^2 - \frac{1}{2}s_5 s_7 + s_3 s_8 \,, \quad \tilde{s}_4 = -\frac{1}{2}s_6 s_7 + s_4 s_8 \tag{A.8}$$

Next, one uses the transformations provided in [22]

$$\begin{aligned}
x_1 &\longmapsto z \\
x_2 &\longmapsto \frac{6 s_7 y + 6\tilde{s}_4 x z + 2\tilde{s}_3 \tilde{s}_4 z^3 + 3\tilde{s}_2 s_7^2 z^3}{2(3 s_7^2 x - 3\tilde{s}_4^2 z^2 - 2\tilde{s}_3 s_7^2 z^2)} \\
x_3 &\longmapsto (108 s_7^3 x^3 - 108 s_7^3 y^2 - 108\tilde{s}_4 s_7^2 x y z - 216\tilde{s}_4^2 s_7 x^2 z^2 - 108\tilde{s}_3 s_7^3 x^2 z^2 - 108\tilde{s}_4^3 y z^3 \\
&\quad - 144\tilde{s}_3\tilde{s}_4 s_7^2 y z^3 - 108\tilde{s}_2 s_7^4 y z^3 - 36\tilde{s}_3\tilde{s}_4^2 s_7 x z^4 - 54\tilde{s}_2\tilde{s}_4 s_7^3 x z^4 + 12\tilde{s}_3^2\tilde{s}_4^2 s_7 z^6 \\
&\quad - 54\tilde{s}_2\tilde{s}_4^3 s_7 z^6 + 16\tilde{s}_3^3 s_7^3 z^6 - 72\tilde{s}_2\tilde{s}_3\tilde{s}_4 s_7^3 z^6 - 27\tilde{s}_2^2 s_7^5 z^6)/ \\
&\quad 12(3 s_7^2 x - 3\tilde{s}_4^2 z^2 - 2\tilde{s}_3 s_7^2 z^2)^2
\end{aligned} \tag{A.9}$$

in order to finally bring (A.7) into Weierstrass normal form in $\mathbb{P}^{(1,2,3)}$. We also note that the transformations (A.9) simplify in the case $s_7 = 0$, in particular their denominators loose their dependence on x, y.

B Spectral cover examples with no U(1)

For convenience and to demonstrate how our formalism works in a well-understood situation, we analyze several examples with pure non-Abelian gauge content only. These are related to the examples 4.2.1, 4.2.2 and 4.2.3 by a Higgsing process which gives s_{72} a vacuum expectation value.

B.1 Trivial structure group: $E_8 \times E_8$ gauge symmetry

As described in the section 4.1, we can obtain examples with higher rank gauge symmetry by specializing the coefficients of χ. Aiming for a model with $E_8 \times E_8$ gauge symmetry, one obtains the following coefficients.

Coefficient	K3	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{12}V + s_{11}\lambda_1$
s_2	$s_{22}UV$	$s_{22}U$	$s_{22}V$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	0	0	0
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

Here the second row displays the coefficients of the full K3 surface while the coefficients of the two half K3 surfaces are displayed in row three and four. In particular, one notices that the coefficient s_7 is missing which means that one is passing from the toric ambient space $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)} \times \mathbb{P}^1$ to the ambient space $\mathbb{P}^{(1,2,3)} \times \mathbb{P}^1$. Clearly, a generic section of the anti-canonical bundle of $\mathbb{P}^{(1,2,3)}$ does not have a second rational section, so there is also no reason to expect any U(1).

We proceed by analyzing the F-theory gauge group. The analysis of the Tate vectors reveals that

$$\vec{t}_U = \vec{t}_V = (1,2,3,4,5,10) \tag{B.1}$$

and thus there is an $E_8 \times E_8$ gauge symmetry. After the stable degeneration limit, both half K3 surfaces X^+ and X^- obtain one E_8 singularity each.

Finally, we turn to the heterotic side. The splitting of the two half K3's into the heterotic elliptic curve and the spectral cover contributions reveals that

$$p^+ = s_{11}x_1^4\,, \qquad p^- = s_{13}x_1^4\,. \tag{B.2}$$

After transforming these expressions into the affine Weierstrass coordinates x, y, one obtains

$$p_W^+ = s_{11}\,, \qquad p_W^- = s_{13} \tag{B.3}$$

In both cases, one obtains an SU(1) spectral cover. However, the centralizer of the identity in E_8 is E_8 and one obtains a perfect match with the F-theory calculation.

B.2 Structure group SU(1) × SU(2): $\mathbf{E_8 \times E_7}$ gauge symmetry

We consider the following model which is specified by the following coefficients in (4.2).

Coefficient	K3	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{12}V + s_{11}\lambda_1$
s_2	$s_{21}U^2 + s_{22}UV$	$s_{22}U$	$s_{22}V + s_{21}\lambda_1$
s_3	$s_{32}UV$	$s_{32}U$	$s_{32}V$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	0	0	0
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

This time, we obtain the following Tate vectors

$$\vec{t}_V = (1,2,3,3,5,9)\,, \qquad \vec{t}_U = (1,2,3,4,5,10) \tag{B.4}$$

which signal an E_7 singularity at $V=0$ as well as an E_8 singularity at $U=0$. The latter one is inherited by the half K3 surface X^- while the former one moves into X^+.

The spectral cover is in this case given by

$$p^+ = x_1^3(s_{11}x_1 + s_{21}x_2)\,, \qquad p^- = s_{11}x_1^4\,. \tag{B.5}$$

We only comment on the non-trivial spectral cover. After applying the transformation (A.9), it reads

$$p_W^+ = c_0 + c_1 x \tag{B.6}$$

which defines an SU(2) spectral cover and is precisely what is expected. Explicitly, the c_i's read

$$c_0 = s_{21}s_{62}^2 - 4s_{21}s_{32}s_{82} + 12s_{11}s_{42}s_{82} \qquad c_1 = -s_{21} \tag{B.7}$$

Note that the c_i are indeed proportional to s_{11}, s_{21} which define the spectral cover. Thus, we obtain an SU(2) spectral cover in the case of X^+ and a trivial structure group for the case of X^-. In conclusion, there is a perfect match with the F-theory analysis.

B.3 Example with gauge group $\mathbf{E_8} \times \mathbf{SO(11)}$

We consider the following model which is specified by the following coefficients in (4.2).

Coefficient	K3	X^-	X^+
s_1	$s_{11}U^2 + s_{12}UV + s_{13}V^2$	$s_{12}U + s_{13}\mu$	$s_{12}V + s_{11}\lambda_1$
s_2	$s_{21}U^2 + s_{22}UV$	$s_{22}U$	$s_{22}V + s_{21}\lambda_1$
s_3	$s_{31}U^2 + s_{32}UV$	$s_{32}U$	$s_{32}V + s_{31}\lambda_1$
s_4	$s_{42}UV$	$s_{42}U$	$s_{42}V$
s_5	$s_{52}UV$	$s_{52}U$	$s_{52}V$
s_6	$s_{62}UV$	$s_{62}U$	$s_{62}V$
s_7	0	0	0
s_8	$s_{82}UV$	$s_{82}U$	$s_{82}V$

This time, we obtain the following Tate vectors

$$\vec{t}_V = (1,1,3,3,5,8)\,, \qquad \vec{t}_U = (1,2,3,4,5,10) \tag{B.8}$$

which signal an SO(11) singularity at $V = 0$ as well as an E_8 singularity at $U = 0$. The former one is inherited by the half K3 surface X^+ while the latter one moves into X^-.

The spectral cover is in this case given by

$$p^+ = x_1^2(s_{11}x_1^2 + s_{21}x_1x_2 + s_{31}x_2^2)\,, \qquad p^- = s_{11}x_1^4\,. \tag{B.9}$$

We only comment on the non-trivial spectral cover. After applying the transformation (A.9), it reads

$$p_W^+ = c_0 + c_1 x + c_2 x^2 \tag{B.10}$$

which defines an $Sp(2) \cong SO(5)$ spectral cover[15] [7] and is precisely what is expected. Thus, we obtain an Sp(2) spectral cover in the case of X^+ and a trivial structure group for the case of X^-. The commutant of SO(5) within E_8 is given by SO(11).

[15]Sometimes, SP(N) is denoted by SP($2N$).

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C Tuned models without rational sections

In this appendix we reproduce [22, 54] the following

Lemma C.1. *The two sections denoted by* $x_1 = 0$ *and* $x_4 = 0$ *in* (4.3) *merge into a single section if and only if* $s_7 = 0$ *in* (4.2)*. Furthermore, the single section is given by* $[x_1 : x_2 : x_3 : x_4 : x_5] = [0 : 1 : 1 : 0 : 1]$.

Proof. Suppose the two sections $x_1 = 0$ and $x_4 = 0$ merge into a single section. Then this single section obeys both $x_1 = 0$ and $x_4 = 0$, everywhere. Thus the Stanley-Reisner ideal requires $x_2 \neq 0$, $x_3 \neq 0$ and $x_5 \neq 0$ everywhere. Making use of the skaling relations (3.7) of the resolved space $\mathrm{Bl}_1\mathbb{P}^{(1,1,2)}$, one obtains that this section is indeed given by $[x_1 : x_2 : x_3 : x_4 : x_5] = [0 : 1 : 1 : 0 : 1]$.

Suppose now that $s_7 = 0$. Setting x_1 in (4.2) to zero, results in the equation $s_8 x_3^2 x_4 = 0$. As $x_3 \neq 0$ due to the Stanley Reisner ideal, x_4 has to vanish as well resulting in the merging of the two sections. Similarly, $x_4 = 0$ requires that $s_4 x_1 x_2^3 x_5^2 = 0$. The Stanley Reisner ideal requires x_2 and x_5 to be non-vanishing. Thus, there is also in this case only one section given by $[x_1 : x_2 : x_3 : x_4 : x_5] = [0 : 1 : 1 : 0 : 1]$. □

D Non-commutativity of the semi-stable degeneration limit and the map to Weierstrass form

We illustrate the non-commutativity of the diagram 1 using the above example with gauge group $\mathrm{E}_7 \times \mathrm{SO}(9) \times \mathrm{U}(1)$. To be precise, on the top left corner of the diagram, the section χ of $-K_{\mathrm{Bl}_1\mathbb{P}^{(1,1,2)} \times \mathbb{P}^1}$ is given by

$$\begin{aligned} \chi: \quad & s_1 x_1^4 + s_2 x_1^3 x_2 + s_3 x_1^2 x_2^2 + s_4 x_1 x_2^3 + s_5 x_1^2 x_3 + s_6 x_1 x_2 x_3 + s_7 x_2^2 x_3 + s_8 x_3^2 = 0\,, \\ \text{where} \quad & s_1 = s_{11} U^2 + s_{12} UV + s_{13} V^2, \quad s_i = s_{i1} U^2 + s_{i2} UV \text{ for } 2 \leqslant i \leqslant 3\,, \\ & s_i = s_{i2} UV \text{ for } 4 \leqslant i \leqslant 8\,. \end{aligned} \tag{D.1}$$

Under the stable degeneration limit, denoted by the left map in the diagram 1, χ is split into $\chi^{\pm}$, which are in turn defined by

$$\begin{aligned} \chi^{\pm}: \quad & s_1^{\pm} x_1^4 + s_2^{\pm} x_1^3 x_2 + s_3^{\pm} x_1^2 x_2^2 + s_4^{\pm} x_1 x_2^3 + s_5^{\pm} x_1^2 x_3 + s_6^{\pm} x_1 x_2 x_3 + s_7^{\pm} x_2^2 x_3 + s_8^{\pm} x_3^2 = 0\,, \\ \text{where} \quad & s_1^{+} = s_{12} U + s_{13} \mu\,, \quad s_1^{-} = s_{11} \lambda_1 + s_{12} V, \\ & s_i^{+} = s_{i2} U \text{ and } s_i^{-} = s_{i1} \lambda_1 + s_{i2} V \text{ for } 2 \leqslant i \leqslant 3\,, \\ & s_i^{+} = s_{i2} U \text{ and } s_i^{-} = s_{i2} V \text{ for } 4 \leqslant i \leqslant 8\,. \end{aligned} \tag{D.2}$$

We further map $\chi^{\pm}$, under the bottom map of the diagram 1, into their respective Weierstrass forms

$$W_{\chi}^{\pm} : y^2 = x^3 + f_{\chi}^{\pm} x z^4 + g_{\chi}^{\pm} z^6. \tag{D.3}$$

We can show that $W_{\chi}^{\pm}$ obtained in this way is different compared to $W_{\chi}^{\prime\pm}$ obtained by taking the other route in diagram 1, namely start from χ on the top left corner of the diagram, first map χ into its Weierstrass form W_χ using the map on top of 1, and then use the map on the right of 1 to split W_χ into

$$W'^{\pm}_\chi : y^2 = x^3 + f'^{\pm}_\chi x z^4 + g'^{\pm}_\chi z^6. \tag{D.4}$$

Indeed,

$$W^+_\chi \neq W'^+_\chi, \qquad W^-_\chi \neq W'^-_\chi. \tag{D.5}$$

To be precise,

$$f^{\pm}_\chi = f'^{\pm}_\chi \quad \text{but} \quad g^{\pm}_\chi \neq g'^{\pm}_\chi, \quad g^+_\chi - g'^+_\chi = \frac{2}{3} U^6 s_{13} s_{31} s_{72}^2 s_{82}^2, \quad g^-_\chi - g'^-_\chi = \frac{2}{3} V^6 s_{13} s_{31} s_{72}^2 s_{82}^2. \tag{D.6}$$

References

[1] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Volume 2: Loop amplitudes, anomalies and phenomenology*, Cambridge University Press, Cambridge U.K. (2012).

[2] J. Polchinski, *String theory. Volume 1: An introduction to the bosonic string. Volume 2: Superstring theory and beyond*, Cambridge University Press, Cambridge U.K. (1998).

[3] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022] [INSPIRE].

[4] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds (II)*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161] [INSPIRE].

[5] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds (I)*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114] [INSPIRE].

[6] P.S. Aspinwall and D.R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys.* **B 503** (1997) 533 [hep-th/9705104] [INSPIRE].

[7] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [hep-th/9701162] [INSPIRE].

[8] R.Y. Donagi, *Principal bundles on elliptic fibrations*, *Asian J. Math.* **1** (1997) 214 [alg-geom/9702002] [INSPIRE].

[9] B. Andreas, *$N = 1$ heterotic/F theory duality*, *Fortschr. Phys.* **47** (1999) 587 [hep-th/9808159] [INSPIRE].

[10] P. Candelas and A. Font, *Duality between the webs of heterotic and type-II vacua*, *Nucl. Phys.* **B 511** (1998) 295 [hep-th/9603170] [INSPIRE].

[11] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, *Adv. Theor. Math. Phys.* **2** (1999) 1307 [hep-th/9811217] [INSPIRE].

[12] L.B. Anderson and W. Taylor, *Geometric constraints in dual F-theory and heterotic string compactifications*, *JHEP* **08** (2014) 025 [arXiv:1405.2074] [INSPIRE].

[13] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New aspects of heterotic–F-theory duality*, *Nucl. Phys.* **B 806** (2009) 224 [arXiv:0805.1057] [INSPIRE].

[14] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Computing brane and flux superpotentials in F-theory compactifications*, *JHEP* **04** (2010) 015 [arXiv:0909.2025] [INSPIRE].

[15] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Five-brane superpotentials and heterotic/F-theory duality*, *Nucl. Phys.* **B 838** (2010) 458 [arXiv:0912.3250] [INSPIRE].

[16] H. Jockers, P. Mayr and J. Walcher, *On $\mathcal{N} = 1$ 4d effective couplings for F-theory and heterotic vacua*, *Adv. Theor. Math. Phys.* **14** (2010) 1433 [arXiv:0912.3265] [INSPIRE].

[17] M. Cvetič, I. García Etxebarria and J. Halverson, *Three looks at instantons in F-theory — new insights from anomaly inflow, string junctions and heterotic duality*, *JHEP* **11** (2011) 101 [arXiv:1107.2388] [INSPIRE].

[18] M. Cvetič, R. Donagi, J. Halverson and J. Marsano, *On seven-brane dependent instanton prefactors in F-theory*, *JHEP* **11** (2012) 004 [arXiv:1209.4906] [INSPIRE].

[19] G. Aldazabal, A. Font, L.E. Ibáñez and A.M. Uranga, *New branches of string compactifications and their F-theory duals*, *Nucl. Phys.* **B 492** (1997) 119 [hep-th/9607121] [INSPIRE].

[20] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional noncritical strings*, hep-th/9607139 [INSPIRE].

[21] A. Grassi, *Toric Weierstrass models*, talk given at the *Workshop on Elliptic fibrations and F-theory*, Kavli I.P.M.U., Tokyo Japan (2010), http://www.ipmu.jp/node/552.

[22] D.R. Morrison and D.S. Park, *F-theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds*, *JHEP* **10** (2012) 128 [arXiv:1208.2695] [INSPIRE].

[23] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *Elliptic fibrations for* $SU(5) \times U(1) \times U(1)$ *F-theory vacua*, *Phys. Rev.* **D 88** (2013) 046005 [arXiv:1303.5054] [INSPIRE].

[24] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple* $U(1)$*-factors: constructing elliptic fibrations with rational sections*, *JHEP* **06** (2013) 067 [arXiv:1303.6970] [INSPIRE].

[25] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral four-dimensional F-theory compactifications with* $SU(5)$ *and multiple* $U(1)$*-factors*, *JHEP* **04** (2014) 010 [arXiv:1306.3987] [INSPIRE].

[26] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, $SU(5)$ *tops with multiple* $U(1)$*s in F-theory*, *Nucl. Phys.* **B 882** (2014) 1 [arXiv:1307.2902] [INSPIRE].

[27] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple* $U(1)$*-factors: addendum*, *JHEP* **12** (2013) 056 [arXiv:1307.6425] [INSPIRE].

[28] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with* $U(1) \times U(1) \times U(1)$ *gauge symmetry*, *JHEP* **03** (2014) 021 [arXiv:1310.0463] [INSPIRE].

[29] V. Braun, T.W. Grimm and J. Keitel, *Complete intersection fibers in F-theory*, *JHEP* **03** (2015) 125 [arXiv:1411.2615] [INSPIRE].

[30] D. Klevers, D.K. Mayorga Pena, P.-K. Oehlmann, H. Piragua and J. Reuter, *F-theory on all toric hypersurface fibrations and its Higgs branches*, *JHEP* **01** (2015) 142 [arXiv:1408.4808] [INSPIRE].

JHEP04(2016)041

[31] M. Cvetič, D. Klevers, H. Piragua and W. Taylor, *General* $U(1) \times U(1)$ *F-theory compactifications and beyond: geometry of unHiggsings and novel matter structure*, JHEP **11** (2015) 204 [arXiv:1507.05954] [INSPIRE].

[32] A. Grassi and V. Perduca, *Weierstrass models of elliptic toric K3 hypersurfaces and symplectic cuts*, Adv. Theor. Math. Phys. **17** (2013) 741 [arXiv:1201.0930] [INSPIRE].

[33] R. Blumenhagen, G. Honecker and T. Weigand, *Loop-corrected compactifications of the heterotic string with line bundles*, JHEP **06** (2005) 020 [hep-th/0504232] [INSPIRE].

[34] J. Marsano, N. Saulina and S. Schäfer-Nameki, *Monodromies, fluxes and compact three-generation F-theory GUTs*, JHEP **08** (2009) 046 [arXiv:0906.4672] [INSPIRE].

[35] R. Blumenhagen, T.W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, Nucl. Phys. **B 829** (2010) 325 [arXiv:0908.1784] [INSPIRE].

[36] T.W. Grimm and T. Weigand, *On Abelian gauge symmetries and proton decay in global F-theory GUTs*, Phys. Rev. **D 82** (2010) 086009 [arXiv:1006.0226] [INSPIRE].

[37] P.S. Aspinwall and D.R. Morrison, *Nonsimply connected gauge groups and rational points on elliptic curves*, JHEP **07** (1998) 012 [hep-th/9805206] [INSPIRE].

[38] K.-S. Choi and H. Hayashi, $U(n)$ *spectral covers from decomposition*, JHEP **06** (2012) 009 [arXiv:1203.3812] [INSPIRE].

[39] P.S. Aspinwall, *K3 surfaces and string duality*, in *Fields, strings and duality. Proceedings, Summer School, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI'96*, Boulder U.S.A., 2–28 Jun 1996, pp. 421–540 [hep-th/9611137] [INSPIRE].

[40] C. Vafa and E. Witten, *Dual string pairs with* $N = 1$ *and* $N = 2$ *supersymmetry in four-dimensions*, Nucl. Phys. Proc. Suppl. **46** (1996) 225 [hep-th/9507050] [INSPIRE].

[41] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, Nucl. Phys. **B 481** (1996) 215 [hep-th/9605200] [INSPIRE].

[42] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, JHEP **04** (1998) 019 [hep-th/9802194] [INSPIRE].

[43] K. Kodaira, *On compact analytic surfaces II*, *Ann. Math.* **77** (1963) 563.

[44] M.R. Douglas, D.S. Park and C. Schnell, *The Cremmer-Scherk mechanism in F-theory compactifications on K3 manifolds*, JHEP **05** (2014) 135 [arXiv:1403.1595] [INSPIRE].

[45] R. Donagi, *ICMP lecture on heterotic/F-theory duality*, in *Mathematical physics. Proceedings, 12th International Congress, ICMP'97*, Brisbane Australia, 13–19 Jul 1997 [hep-th/9802093] [INSPIRE].

[46] M.F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **s3-7** (1957) 414.

[47] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9709029 [INSPIRE].

[48] J.W. Morgan, *Holomorphic bundles over elliptic manifolds*, lecture given at the *School on Algebraic Geometry*, Trieste Italy, 26 Jul–13 Aug 1999.

[49] P.S. Aspinwall, *An analysis of fluxes by duality*, hep-th/0504036 [INSPIRE].

[50] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in *Modular functions of one variable IV*, Lect. Notes Math. **476** (1975) 33.

[51] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **08** (2011) 094 [arXiv:1106.3854] [INSPIRE].

[52] U. Bruzzo and A. Grassi, *Picard group of hypersurfaces in toric 3-folds*, *Int. J. Math.* **23** (2012) 1250028 [arXiv:1011.1003].

[53] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *J. Alg. Geom.* **3** (1994) 493 [alg-geom/9310003] [INSPIRE].

[54] D.R. Morrison and W. Taylor, *Sections, multisections and* U(1) *fields in F-theory*, arXiv:1404.1527 [INSPIRE].

[55] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard models from heterotic M-theory*, *Adv. Theor. Math. Phys.* **5** (2002) 93 [hep-th/9912208] [INSPIRE].

[56] T. Shioda and K. Oguiso, *The Mordell-Weil lattice of a rational elliptic surface*, *Comment. Math. Univ. St. Paul* **40** (1991) 83.

[57] I. Shimada, *On elliptic K3 surfaces*, *Michigan Math. J.* **47** (2000) 423.

[58] C. Mayrhofer, D.R. Morrison, O. Till and T. Weigand, *Mordell-Weil torsion and the global structure of gauge groups in F-theory*, *JHEP* **10** (2014) 016 [arXiv:1405.3656] [INSPIRE].

[59] A. Grassi and D.R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, *Commun. Num. Theor. Phys.* **6** (2012) 51 [arXiv:1109.0042] [INSPIRE].