ABSTRACT

Title of Dissertation: POSITIONING IN AN UPPER-LEVEL UNDERGRADUATE MATHEMATICS COURSE

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Dissertation directed by: Professor Daniel Chazan, Department of Teaching and Learning, Policy and Leadership

This study examined classroom interactions in an upper-level undergraduate mathematics course in order to investigate how they can be seen as positioning students in relation to mathematics. Students' experiences in undergraduate mathematics courses are often negative, yet few studies have focused their attention on what happens inside undergraduate mathematics classrooms, particularly for advanced-level courses in which proofs are the focus.

This study took place over the course of a semester in one section of an Introduction to Analysis course. Thirty-six of the 40 class sessions were observed and audio-recorded, and detailed field notes were taken. Additionally, selected students were interviewed at the beginning, middle, and end of the semester about their experiences, and the professor was interviewed at the end of the semester. These data were analyzed qualitatively to support the creation of a narrative description of patterns of interactions over the course of the semester. One

particular moment of mathematical disagreement between the professor and a student was examined closely to reveal the potential positioning of students in relation to mathematics. And patterns of commonly used phrases across the semester were analyzed as well, in order to reveal how the repeated use of language could potentially position students in relation to mathematics.

This analysis of classroom interactions suggested that the use of a traditional lecture format in an advanced mathematics class offers few opportunities for students to develop positive relationships with mathematics. Institutional constraints made it hard for the professor to shift away from a typical lecture format that efficiently covered the necessary content. But within this traditional lecture format, there is possibility for variation. The professor was able to establish a relatively comfortable classroom environment and to engage students in different kinds of mathematically meaningful classroom interactions. Within these interactions, different resources were available that could potentially position students as doers of mathematics, including storylines about mathematics as a logical system and about the classroom as a shared space of mathematical work.

POSITIONING IN AN UPPER-LEVEL UNDERGRADUATE MATHEMATICS COURSE

by

Elizabeth Fleming

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Advisory Committee:
Professor Daniel Chazan, Chair
Professor Tara Brown
Professor Patricia Campbell
Professor Andrew Elby
Professor KerryAnn O'Meara

Dedication

To my nieces and nephews

Budding mathematicians, one and all

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Table of Contents

Dedication ii

Acknowledgements iii

Table of Contents iv

List of Tables vii

List of Figures viii

CHAPTER 1: INTRODUCTION 1
- **My Story** 3
- **The Study** 7
  - Research Questions 7

CHAPTER 2: LITERATURE REVIEW 10
- **Students' Experiences in Undergraduate Mathematics Classes** 10
  - Instructional Practices 11
  - Faculty Attitudes and Accessibility 12
  - Classroom Culture 14
  - Experiences of Marginalized Students 15
- **Research in Undergraduate Mathematics Classrooms** 18
  - Communication of Mathematical Ideas 19
  - Characteristics of University Mathematics Teaching 19
  - Classroom Interactions and Student Positioning 21
- **Positioning Theory: A Conceptual Framework** 23
  - Positions 24
  - Storylines 25
  - Speech Acts 26
  - Positioning 27
  - Rationale for Choosing Positioning Theory 28
- **Research on K-12 Classroom Interactions and Positioning** 29
- **Summary of State of Research** 33

CHAPTER 3: METHODS 34
- **Description and Rationale for the Course Selected** 34
- **Description and Rationale for Data Collection** 37
  - Participants 37
  - Classroom Observations 38
  - Interviews 40
- **Description and Rationale for Data Analysis** 43
  - Stage 1: Working out the Affordances of the Data 45
  - Stage 2: Emergent Themes 51
  - Stage 2: Writing the Portrait of the Semester 55
  - Stage 3: Analysis of a Moment 56
  - Stage 3: Lexical Bundle Analysis 58
- **Methodological Stance** 59

CHAPTER 4: PORTRAIT OF THE SEMESTER 62
- **The Portrait** 62

August: Making the Familiar Strange (Week 1) ... 64
September: Shifting the Balance of Participation (Weeks 2, 3, and 4) ... 70
Into October: Playing with Authority (Week 5) ... 73
October: Disappointment and Redemption (Weeks 6 and 7) ... 76
October: Pushing on Boundaries (Week 8) ... 80
Into November: Feeling Comfortable (Weeks 9 and 10) ... 84
November: Moving Fast and Slow (Weeks 11, 12, and 13) ... 87
December: Too Much 'Magic' (Weeks 14 and 15) ... 93

CHAPTER 5: POSITIONING IN A MOMENT ... 99
Storylines about Mathematics ... 100
Storyline A: Teaching mathematics is explaining ... 101
Storyline B: Mathematics is an axiomatic system ... 103
Storyline C: The classroom is a shared space of mathematical work ... 104
Analyzing Positioning at Two Scales ... 106
One Moment in the Semester ... 107
Turn 1 ... 111
Turn 2 ... 113
Turns 3-5 ... 115
Turn 6 ... 115
Turn 7 ... 117
Turn 8 ... 118
Turns 9-10 ... 118
Turn 11 ... 119
Turn 12 ... 120
Turn 13-14 ... 120
Turn 15 ... 121
Turns 16-18 ... 122
Conclusion ... 122

CHAPTER 6: PATTERNS OF POSITIONING OVER TIME ... 126
Lexical Bundle Analysis ... 127
Mathematical relations bundles ... 131
Intention/prediction bundles ... 132
Checking in bundles ... 135
Desire bundles ... 136
Bundles spoken by students ... 139
Conclusion ... 141

CHAPTER 7: DISCUSSION ... 145
Conclusions ... 145
Speaking Back to Positioning Theory ... 150
The Affordances ... 150
The Challenges ... 153
Practical Implications ... 156
For Mathematicians ... 156
For Mathematics Departments ... 157
For Mathematics Teachers and Learners ... 160

Future Research 161

APPENDICES 164
Appendix A – Worked Example of Positioning Theory 164
Appendix B – Observation Protocol 166
Appendix C – Interview Participants 168
Appendix D – Interview Guiding Questions 169
Appendix E – Coding Table 174
Appendix F – Memo about Theme One 178
Appendix G – Analysis of Interaction 190

REFERENCES 194

List of Tables

Table 1. Heat map of interactions over the semester (excerpt).................................. 49

Table 2. Lexical bundles across nine transcripts 128

Table 3. Verbs preceding "*to be equal to*" 133

Table 4. Verbs following "*I want you to*" 137

List of Figures

Figure 1. Diagram of positioning (from Herbel-Eisenmann et al., 2015, p. 194)....... 27

Figure 2. Data collection timeline.. 43

Figure 3. Timeline of the semester (excerpt).. 46

Figure 4. Timeline of student participation (excerpt) 47

Figure 5. Instructional triangle (from Cohen, Raudenbush, & Ball, 2003, p. 124) 52

Figure 6. Themes in interactions across the weeks.. 62

Figure 7. The field axioms 66

Figure 8. Storylines and the instructional triangle 106

Figure 9. Categories of bundles organized by storyline 130

CHAPTER 1: INTRODUCTION

"Mathematics is the queen of the sciences" (Gauss, 1856)

"The Queen turned crimson with fury, and, after glaring at her for a moment like a wild beast, began screaming 'Off with her head!'" (Carroll, 1865/2000, p. 82)

Mathematics is powerful. Mathematics is an achievement of human rationality, a discipline practiced by a community of thinkers, and a valuable and productive tool. Mathematics can be empowering; as a way of thinking and exploring in the world, it can reveal beautiful and satisfying connections and insights. But, with great power comes great responsibility. Mathematics can also be deeply disempowering and alienating; it has a particular way of leaving some of us hurt and injured, walking around with hidden scars.

I have become more and more aware of this power of mathematics as I have become more involved in the field of mathematics education, and begun to identify myself as a mathematics educator—in part because it is unavoidable. The responses bubble out of people: "You study math? Oh, I hate math!"; "I've always been terrible at math"; "You know, I thought I was good at math, until…".

In particular, though, I have become aware of a set of stories existing around students' mathematical experiences in undergraduate courses. Here is one example, taken from a description of a pre-service teacher's mathematics identity:

> While always having a fondness for mathematics and receiving good grades in the subject area, Jan recalled one exception. She described an advanced-level mathematics class in college, of which she noted, 'No matter what I did, I

> could not get a success.' Jan's body language shifted as she talked about this experience and changed the topic of conversation quickly, noting that failing the course was something she tried not to think about. (Jones Frank, 2013, p. 94)

Over the past few years, I have accumulated numerous stories similar to this one, in that they describe an evidently painful undergraduate mathematics experience. I have heard these stories in talking with fellow mathematics educators, both professors and graduate students, with current and future mathematics teachers, with undergraduate mathematics majors, and even with graduate mathematics students. One thread common to many of these stories is a sense of rupture—of coming into an experience feeling intelligent, competent, and confident as a mathematics student, but leaving feeling insecure, intimidated, and doubtful of one's mathematical abilities; of coming in feeling excited about mathematics and full of questions, but leaving feeling demoralized and silenced. Consider this description from a mathematics education doctoral student about her experience taking an upper-level undergraduate mathematics course:

> I grew up loving mathematics, I was enrolled in a science and technology high school, and I went on [to the university] as an engineering student so I really didn't consider mathematics to be an obstacle for my success. It wasn't easy for me, but compared to most, I seemed to have a knack for it. ... Fast forward to Fall 1999 (my first semester in grad school), when I attempted to take an abstract algebra course by the advisement of members of the mathematics faculty, I remember the first day of classes thinking, 'What the hell?' I had no

> idea what in the world the professor was talking about and then worse off I had no idea what he was writing on the board. It all seemed so foreign. I felt ill. The semester got no better and after a handful of failed quizzes and tests, I opted to drop the course, with my confidence in mathematics totally bruised and shattered. I think I used to describe that semester as "traumatic" for me in terms of my mathematics learning (Faith, reflection). (Marshall, 2008, p. 107)

I personally experienced this sense of rupture in my own mathematics course-taking experiences, and find it important to share this story for several reasons. It provides an example of what I mean when I talk about these kinds of stories. It is a motivating force behind this study, and gives a sense of some of the personal questions that drive the study. And it is a lens that I bring to this work—in sharing it, I hope to acknowledge how it shapes my thinking, and to allow both the reader and myself to observe, understand, and even critique it as a lens. As elegantly explained in the methodology of portraiture:

> Paradoxically, the portraitist's reference to her own life story does not reduce the reader's trust—it enhances it. It does not distort the responsibility of the researcher and the authenticity of the work; it gives them clarity. A reader who knows where the portraitist is coming from can more comfortably enter the piece, scrutinize the data, and form independent interpretations. (Lawrence-Lightfoot & Davis, 1997, p. 96)

My Story

I started taking mathematics courses at the University of Maryland after a fairly unusual set of mathematical experiences at my undergraduate institution, St.

John's College. All students at St. John's are required to take four years of mathematics, and the courses are structured around engagement with classical mathematics texts from the Western tradition, beginning with Euclid's *Elements* and moving forward in time. In class students are expected to do mathematical demonstrations and to participate in discussions, driven by our own understanding and questions about these texts. Coming out of St. John's, I felt like a strong mathematical thinker with a rich grounding in the history of Western mathematics. I entered my first two upper-level mathematics courses at the University of Maryland feeling ready for the opportunity to delve into more formal and more modern mathematics—I was not particularly sure what abstract algebra or real analysis meant, but I was excited to find out.

My experience in abstract algebra had its ups and downs, but I emerged feeling that I had learned some beautiful mathematics and many of the norms for this community as well. I was able to figure out the path to being a successful student in the course and was able to see what kind of work and effort it would take on my part to get there. My experience in real analysis, however, felt like being knocked over by waves at the beach—I would think I had found the way up, only to get crashed into by another wave. These waves took many forms, from impossible-to-finish homework assignments to incomprehensible exam questions to lectures from the professor about how we were not studying hard enough. I left every exam in tears, and I failed every exam. Apparently so did the rest of the class; although my final grade in the course was in the 60s, somehow I emerged with a passing B. My goal in that course was survival. I wanted to get out of the course as unscathed as possible, with little to no regard for my own learning of the content.

At the same time as taking these two courses, I was taking a third course (with my current advisor), called Research in Mathematics for Mathematics Educators, which was an entirely different kind of mathematical experience. This experience provided an important contrast to the other two, while also being different from my courses at St. John's, because it provided an opportunity to play with mathematics and explore my own mathematical questions. Being in this course was especially significant, I think, because it reminded me of the sense of joy in mathematics, and of my own interest and engagement in mathematical thinking. I came to realize that the problematic experiences I was having in the other course were not about me, or about the mathematical ideas, but instead were about what was happening in that course.

Interestingly, after taking these courses, my subsequent experiences in upper-level undergraduate mathematics courses were mostly positive. I felt that I had learned the system, more or less, and was able to build on the tools and understandings developed in these prior course experiences. But perhaps most important was my realization that what was happening in the course *mattered*. Prior to this realization, I thought of these undergraduate mathematics courses as an individual challenge, in which I had to grapple with mathematical ideas and try to master them, where what happened in class was largely peripheral (for an example of this in the literature, see Weber, 2008). Instead, I came to realize that my mathematical experience was not solely a product of my one-on-one encounters with these mathematical ideas, but was in fact mediated and influenced by moments and interactions between the professor and students in the course. This realization gave me ground to stand on, a surer footing on which to meet the waves rather than get knocked over. If an instructor said we failed an exam because we did not study hard enough, I was able to see that moment as a choice made by the instructor, and critique it, rather than only feeling its weight of double failure: failure to master the mathematical ideas on the exam, and failure to work hard enough to conquer this challenge.

Once I began to observe and critique these moments and episodes of classroom interaction in my own experiences, I began to notice their influence everywhere in the stories I heard and collected about undergraduate mathematics experiences. It seemed so obvious—of course classroom interactions matter—and yet it also felt subtle, given that a typical undergraduate mathematics class is not usually seen as a highly interactive setting. I began to wonder: What is the nature of these moments of classroom interaction? Are there patterns; are there cues to which ones matter? *How* do these classroom interactions matter? How can I make visible the subtle (and sometimes not so subtle) influence they have on students' experiences and relationships with mathematics? And do all students experience these moments in the same way; are some of us knocked down while others ride the crest of the very same wave?

These are some of the questions and experiences that motivate this study. They are questions rooted in social interactions, in emotions, in what it feels like to be a mathematics student. Fundamentally they are about relationships with mathematics, and the experiences that shape them. It seems clear, both from the stories I have collected and from my own experiences, that what happens inside undergraduate classrooms can play a significant role in (re)shaping learners' relationships with and future participation in mathematics.

The Study

This study is motivated not only by these personal questions, but also by questions of interest to the field. As will be described in the review of the literature, studies of students' experiences in undergraduate mathematics courses point to particular problems such as poor instructional practices, faculty attitudes perceived as uncaring, and competitive and unwelcoming classroom climates (e.g., Seymour & Hewitt, 1997). But these studies rarely examine closely what takes place inside typical undergraduate mathematics classrooms, instead relying largely on student interviews and surveys. And very few studies have attempted to examine closely undergraduate mathematics classroom interactions in relation to student experiences, although similar research at the K-12 level has provided powerful insights into how students' experiences as instantiated in classroom interactions are linked to their relationships with mathematics (e.g., Bishop, 2012). With this study, I hope to extend these bodies of research, and examine how one understudied dimension of undergraduate students' experiences—classroom interactions—might play a role in students' relationships with mathematics. As argued by Nickerson and Bowers (2008), "the ways in which teachers and students interact can profoundly affect the attitudes students form as well as the content they learn" (p. 179).

Research Questions

This study took a qualitative approach to examining the classroom interactions in one upper-level undergraduate mathematics classroom, drawing on the conceptual framework of positioning theory (Harré & van Langenhove, 1999). Using field notes, transcripts, and interviews with selected students from over the course of the semester, the overarching question this descriptive study seeks to address is: How can classroom interactions in an undergraduate mathematics class be viewed as positioning students in relation to mathematics? This question is addressed through consideration of three main research questions:

1. What are patterns in the classroom interactions in an upper-level undergraduate mathematics class?
2. How does one moment in the semester position students in relation to mathematics?
3. How do frequently used phrases across the semester position students in relation to mathematics?

There are many levels of classroom interaction that can be studied: interactions between the professor and the students as a whole class, interactions between the professor and a particular student, interactions between students, interactions of a particular student with the course content. Thinking of this as an exploratory study, I started out open to all possibilities for classroom interactions; I did not have an empirical or theoretical grounding for choosing a particular level of interaction to focus on, given the relative lack of studies of undergraduate mathematics classroom interactions. It turned out that this class proceeded in a lecture-based format with virtually no public exchanges between students and with most interactions between the professor and students following a traditional Initiate-Respond-Evaluate (IRE) pattern, similar to many undergraduate mathematics classrooms (Nickerson & Bowers, 2008). Therefore, the primary classroom interactions studied were verbal exchanges between the professor and the students as a whole class and between the professor and a particular student.

In seeking to understand how these classroom interactions could be viewed as positioning students in relation to mathematics, it was critical to connect the moment-to-moment dynamics with larger storylines that structure and make sense of these dynamics. That is why I chose to use positioning theory as a conceptual framework to unpack features of the interactions such as the available positions, relevant storylines, and the corresponding interpretations of the words being said. More detailed definitions and descriptions of these terms (positions, storylines, positioning) and their relationships will be provided in the description of the conceptual framework at the end of Chapter 2.

CHAPTER 2: LITERATURE REVIEW

In this chapter, I review literature on undergraduate and K-12 mathematics education in order to make clear how the proposed study fits into the landscape and has the potential to contribute new knowledge to the field. First, I review studies on students' experiences in undergraduate mathematics courses, which establish evidence of particular problems such as poor instructional practices, faculty attitudes perceived as uncaring, and competitive and unwelcoming classroom climates, along with the potential for improvement in these areas. Taken as a whole, this set of studies suggests the importance of students' classroom experiences, yet the studies rarely have data from within classrooms themselves. Next, I review the relatively small pool of studies that examine data from within undergraduate mathematics classrooms, which establishes that little is known about typical classroom interactions in relation to students' experiences. Finally, I introduce positioning theory as a conceptual framework and review studies from the K-12 literature on positioning and classroom interactions, in order to provide theoretical and empirical grounding for positioning theory as a helpful conceptual framework for considering these questions.

Students' Experiences in Undergraduate Mathematics Classes

There is a fairly substantial body of research on undergraduates' experiences in STEM. In surveying this literature, I attempted to identify studies that are relevant to students' experiences in undergraduate mathematics courses, which often meant including studies about introductory STEM courses or STEM course-taking in general. This research suggests a general consensus that what happens in the classroom matters for students. Three significant dimensions of students' overall experiences are the instructional practices of the professor, the attitude and accessibility of the professor, and the classroom culture.

Instructional Practices

The instructional practices of professors are perhaps the most significant influence on students' course experiences. In Seymour and Hewitt's (1997) seminal study of those who entered and stayed in SME [Science, Mathematics, and Engineering] majors and those who switched out, they found that "switchers and non-switchers were virtually unanimous in their view that no set of problems in SME majors was more in need of urgent, radical improvement than faculty pedagogy" (p. 165). In their study, they found that 90.2% of switchers raised concerns about poor teaching, with 36.1% of them identifying it as a significant factor in their decision to switch out of SME, and 73.7% of non-switchers complained about poor teaching, making it the most commonly-cited complaint among those who persisted in SME studies (p. 34). As an example of these kind of complaints, in a study of the mathematical experiences of mathematics education doctoral students, one participant critiqued, "it was pretty clear that they [the mathematics professors] either don't know how to teach or don't believe that there are other ways to teach other than standing at the board and lecture for an hour or two hours" (Marshall, 2008, p. 117). In a related finding, these "transmissionist" teaching practices have been found to be associated with a decline in student interest in further study of mathematics (Pampaka, Williams, Hutcheson, Davis, & Wake, 2012).

In contrast, "good teaching" of undergraduate mathematics courses has been shown to improve students' experiences. Three studies using student self-report data of pedagogical practices in Calculus I and introductory STEM courses identified the positive impact of fairly conventional pedagogical practices such as offering clear explanations, listening to and responding to student questions, and assigning appropriate homework and exams, as well as some more progressive practices such as having discussions or group work in class. An analysis of data from a large-scale study of Calculus I instruction in U.S. colleges and universities found that such good teaching practices were positively correlated with improved student attitudes toward mathematics (Sonnert, Sadler, Sadler, & Bressoud, 2015). A different analysis from the same large-scale study identified four factors that improved students' persistence in going on to take Calculus II: (1) showing students how to work specific problems, (2) preparing extra material for students, (3) requiring students to explain their thinking on exams, and (4) holding whole-class discussions (Ellis, Kelton, & Rasmussen, 2014). Similarly, a separate study of introductory STEM courses found that having discussions or group work during class and allowing space for student questions were associated with higher student engagement (Gasiewski, Eagan, Garcia, Hurtado, & Chang, 2012). These self-report findings are supported by a meta-analysis of 225 studies of undergraduate STEM courses, which revealed that students in "active learning" sections had significantly higher exam scores than students in traditional lecture-based sections (Freeman et al., 2014).

Faculty Attitudes and Accessibility

Faculty attitudes and accessibility are also a significant component of students' course experiences. One specific issue raised by students is that professors do not like or value teaching, because they are "just here for research" (Brown &

Rodd, 2004, p. 100). This claim is corroborated by a different analysis of the data from the large-scale Calculus I study, wherein 20% of professors at research universities expressed no interest or only a mild interest in teaching the course (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Students are aware of and care about these attitudes. As noted by one student in Seymour and Hewitt's (1997) study:

> Part of the problem with the math department is their attitude. I think they realize they're bad, but they really don't care. It's not their problem that their students are failing their courses. It's the students' problem. (p. 149)

Another concern is that students often perceive their professors to have no desire to interact with them. As one student explained:

> In the math class it seemed like the professor would just go up to the chalkboard and start doing problems. And when the bell rang, he'd set down the chalk, and he'd never turn around or say anything to the class. (Seymour & Hewitt, 1997, p. 147)

Even in Rodd's (2003) discussion of the potential power of mathematics lectures, she describes one lecturer who had seemingly effective pedagogical strategies, and yet "his disdain for the students comes through" (p. 18). She provides examples of his sharp answers to the only two student questions asked, and of the condescending language of some of his remarks, such as his comment, "'It might not be immediately obvious to you, but one is a multiple of the other' (of vectors (1, 3, -2) and (-2, -6, 4))" (p. 18).

Faculty accessibility can also affect student engagement; "if students perceive faculty to be uncaring, unengaged, or unavailable to help them succeed in learning, they may disengage from the course" (Gasiewski et al., 2012, p. 248). Going to office hours, e-mailing professors, and asking questions in class all require that students perceive their instructors as accessible, or are determined enough to seek assistance regardless (Marshall, 2008; Solomon, 2009). In general, it seems that students too often perceive instructors to be uninterested and disengaged participants in undergraduate mathematics teaching.

Classroom Culture

A third category relevant to students' experiences is the culture in the classroom. In general, the classroom culture of undergraduate mathematics courses is described as competitive, intimidating, and exclusive (Hill, Corbett, & St. Rose, 2010; Marshall, 2008; Seymour & Hewitt, 1997; Ward-Penny, Johnston-Wilder, & Lee, 2011). The common practice of grading on a curve, for example, can mean that students are reluctant to work together or help one another because one student's good grade can hurt other students' grades (Hill et al., 2010; Seymour & Hewitt, 1997; Ward-Penny et al., 2011). As one student depressingly commented, "now [that] I can't be the best, I just need to know that everyone else is as bad as me" (Ward-Penny et al., 2011, p. 24). Similarly, although not directly about mathematics classroom culture, Tobias' (1990) study of non-STEM college graduates auditing physics classes identified a particularly stark contrast between this competitive classroom culture and the environment they were familiar with from their humanities degrees. One participant explained, "Suddenly your classmates are your enemies" (p. 24).

One feature that is remarkably common in descriptions of undergraduate mathematics classroom culture is students' reluctance to ask or answer questions (Brown, McCrae, Rodd, & Wiliam, 2005; Marshall, 2008; Seymour & Hewitt, 1997; Ward-Penny et al., 2011; Yoon, Kensington-Miller, Sneddon, & Bartholemew, 2011). In discussing the tendency not to ask questions in these courses, one student commented, "Anyone who wants to ask questions should be able to ask questions but obviously everyone doesn't always feel like they can" (Yoon et al., 2011, p. 1115). Similarly, describing her experience in undergraduate mathematics courses, a mathematics education doctoral student explained, "[I] never asked a question- never felt comfortable doing so. People would ask questions I did not understand, so I just kept my mouth shut so as not to come across as stupid" (Marshall, 2008, p. 111). Given that there are relatively few opportunities for direct interaction in a typical lecture-based undergraduate mathematics course, this particular aspect of asking and answering questions can become very significant for students' experiences.

Experiences of Marginalized Students

This set of findings about the importance of instructional practices, faculty attitudes and accessibility, and classroom culture is corroborated by the body of studies about the experiences of women and students of color[1] in STEM. The

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[1] In referring to the "experiences of women and students of color," I do not mean to suggest that these groups have one monolithic set of experiences or are subject to the same oppressions. In strategically grouping them, I recognize that I am essentializing their experiences, as well as ignoring the experiences of other marginalized groups. By focusing in this brief review on the experiences of women of color, I seek to acknowledge the intersectionality of race and gender in students' experiences. Women of color are also an important group to focus on given the lack of parity in the awarding of bachelor's degrees in mathematics; in 2014, African American,

negative and positive impacts of each of these three dimensions seem to be magnified by the "outsider" or marginalized positioning of these groups of students. Without attempting to thoroughly review this large body of literature, I will describe a few findings related to these three dimensions from Ong, Wright, Espinosa, and Orfield's (2011) synthesis of 116 studies of the experiences of women of color in STEM.

The importance of teaching practices is evident in these studies, but largely as an assumption rather than a focus. Ong et al. (2011) explain that pedagogical approaches interact with faculty attitudes and classroom climates, "ultimately influencing the ways in which women of color approach the highly valued activity of classroom participation" (p. 183). For example, in Johnson's (2007) study of 16 women of color in introductory science courses, her participants sometimes interpreted teaching practices of posing and asking for questions in lectures as attempts to trap them or make them feel stupid, particularly when professors had not established any sort of rapport with them. Her participants almost universally reported feeling like they alone did not understand the material, perhaps a function of the competitive and isolating classroom climate, and therefore rarely answered or asked questions.

The significance of faculty relationships yielded a "mixed review," (Ong et al., 2011, p. 185). In some studies, students felt supported by their professors, particularly in helping them decide to major in mathematics (e.g., Ellington &

Hispanic or Latino, and Native American women earned only 7% of those degrees, while constituting 14% of all bachelor's degrees and 15% of the total U.S. population, while White women earned 28% of those degrees and made up 36% of all bachelor's degrees and 31% of the total U.S. population (NSF, 2017).

Frederick, 2010). But in many others, women of color did not feel supported by faculty, within or outside of the classroom. Returning to Johnson's (2007) study, for example, women of color felt "discouraged by—and unsatisfied with—faculty who focused their attention on relaying their subject matter of expertise rather than creating interpersonal connections with the students in their classrooms" (Ong et al., 2011, p. 185).

Classroom climate is perhaps the most significant component in this synthesis. The contrast between the generally supportive environments at historically Black colleges and universities (HBCUs) and the "chilly" climate and experiences of isolation at predominantly White institutions (PWIs) is particularly stark. At PWIs, women of color found themselves constantly questioning whether they "belonged" in these classrooms, often due to a lack of peers who looked like them and to the invisible work of having to prove themselves in the face of peers' and instructors' negative stereotypes and expectations of women of color in STEM. For example, one black female STEM major explained, "As far as being a woman, I don't think they expect too many women to be in that area; as far as being a black woman, they don't expect you to be there at all" (Varma, Prasad, & Kapur, 2006, p. 310, cited in Ong et al., 2011, p. 183).

Looking across this entire body of literature on students' experiences in STEM classrooms, there is clear and compelling evidence that classroom experiences are significant in shaping students' relationships with mathematics, particularly along the three inter-related dimensions of instructional practices, faculty attitudes, and classroom culture. However, this body of studies relies almost entirely on student interviews and student surveys as data sources, without collecting data within classrooms in particular. Given the problems and the promise identified in these studies related to classroom experiences, it is important to look more closely at studies that are focused within undergraduate mathematics classrooms themselves.

Research in Undergraduate Mathematics Classrooms

Overall, the body of studies that have collected observational data on what happens in undergraduate mathematics classrooms is relatively small. In 2010, Speer, Smith, and Horvath called attention to the lack of studies on day-to-day instructional practice in collegiate mathematics classrooms. Since then, there has been a relative increase in such studies; of particular interest for this study is the increase in research about teaching in traditional proof-based undergraduate mathematics courses (Paoletti et al., 2017). In surveying this literature on undergraduate mathematics classrooms, I focused on recent studies (since 2000) and on studies of traditional instruction rather than reform instruction, since that is the context for this study.

Taken as a whole, these studies of undergraduate mathematics classrooms do not tend to focus on students' experiences or relationships with mathematics. Instead the main foci are communication of mathematical ideas and characteristics of university mathematics teaching (as similarly categorized in a recent review of this literature, Biza et al., 2016); these studies will be surveyed in an overview fashion. Only five studies were identified as related to classroom interactions and positioning; these findings will be described in more detail, as this study hopes to build on this particular research.

Communication of Mathematical Ideas

The first set of studies focused on the communication of mathematical ideas and on student understanding. Unlike traditional studies of student understanding of mathematical concepts in undergraduate mathematics education, which often use written tests and cognitive interviews (e.g., Tall & Vinner, 1981), these studies were particularly attuned to what happened *inside* the classroom as impacting how students' understanding developed. Yu, Blair, and Dickinson (2006) found that a professor's careful use of metaphor helped to support students' intuitive geometric understandings. Other studies found that discrepancies or unacknowledged shifts in language use on the part of the instructor contributed to student confusion (Güçler, 2013) and devalued students' intuitive understandings (Stage, 2001). Lew, Fukawa-Connelly, Mejía-Ramos, and Weber (2016) found that students in a real analysis course did not understand the ideas that the professor identified as central to his lecture, perhaps due to his use of colloquial or informal language, such as "small" and "toolbox." Taken together, these studies point to the importance of attending closely to language use in lectures, given its potential impact on student understanding.

Characteristics of University Mathematics Teaching

Other studies have focused on characteristics of lectures in undergraduate mathematics classrooms; their findings illustrate a complexity in these teaching practices that goes against the common understanding of such lectures consisting "entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (Davis & Hersh, 1981, p. 151). Nickerson and Bowers (2008) identified two recurring interaction patterns in a case study of an expert professor's instruction of a course on functions for future teachers, both of which pushed toward conceptual understanding and were not the stereotypical Initiate-Respond-Evaluate pattern found in many classrooms. Mills (2014) and Fukawa-Connelly and Newton (2014) both examined lectures in order to characterize the use of examples in proofs-based courses, contrasting the stereotype that such courses only rely on definitions, theorems, and proofs without making use of examples.

Expanding upon these differences in lectures, two studies analyzed lectures in order to better understand components of the lecturing format (Bergsten, 2007; Weber, 2004). Weber's (2004) case study of a real analysis class identified three distinct lecture styles used by the professor depending on the content being covered, again suggesting that lectures are not as uniform as the "definition-theorem-proof" stereotype makes them out to be. Bergsten (2007) used a case study of an introductory calculus class to identify ten factors that define the quality of a mathematics lecture, which also highlighted the complexity and richness possible in such lectures.

In a slight contrast, two other studies have highlighted certain consistencies in undergraduate mathematics teaching. An international study of 50 mathematics instructors in seven countries claimed there is a pedagogical genre of "chalk talk" that has certain uniform features, such as instructors: verbalizing everything they write on the board, gesturing to the board to indicate relationships and highlight key points, referring to lecture notes, and using rhetorical questions (Artemeva & Fox, 2011). Similarly, Gerofsky (1999) analyzed four calculus classes and identified common features of the 'initial calculus lecture' genre, which shared many features with the language of persuasion. These two studies help identify particular features of classroom discourse and instructor moves that are likely to occur in the focal classroom of this study as well.

Classroom Interactions and Student Positioning

Finally, only five studies were identified as attending to classroom interactions as a particular object of study. These studies point to the importance of examining classroom interactions and also suggest important features of interactions to attend to in this study.

Kleinman (1995) looked at gendered patterns of communication in a calculus class at an Ivy League university. Through participant observation and semi-structured interviewing, she found that women used attenuated communication behaviors, such as directly or indirectly apologizing for asking a question, in contrast with men's more direct style. Additionally, women felt silenced in class by their male peers, as well as the professor. Kleinman's (1995) analysis of her observation notes supported this finding, in that women asked far fewer questions and were more likely to have their questions ignored or postponed, which never happened to men. These classroom findings support the findings from the literature on students' experiences reviewed earlier, in terms of highlighting the importance of question asking in lectures as a significant dimension of classroom interactions and students' experiences, particularly for women and other potentially marginalized students.

Mesa and Chang (2010) used linguistic tools to analyze positioning in transcripts of classroom interactions from two first-year undergraduate mathematics courses that both had high student engagement and participation. Their analysis revealed significant differences, in that one instructor's use of language indicated a much more authoritarian stance, potentially limiting students' agency, while the other instructor entertained alternative voices and options in his speech, in a way that allowed students more freedom. These findings affirm the importance of close attention to language use; at first glance the high student engagement in both classrooms seemed to suggest students were being positioned productively in relation to mathematics, until the linguistic analysis revealed important differences. As the authors argue, attending to language use in mathematics is critical because "how we use [language] conveys powerful messages that might exclude the students that we need to engage in the dialog" (p. 98).

The other three studies describe instructional practices in upper-level undergraduate courses that have the potential to position students as more or less meaningful participants in doing mathematics. Mills (2011) examined the pedagogical moves of three professors as they presented proofs, and found that all three professors expected students to contribute both factual information and key ideas in many of the proof presentations. Similarly, Fukawa-Connelly's (2012) case study of an abstract algebra course revealed that the professor invited students to participate in proof writing and presentation activities, positioning and supporting students as doers of proofs. Yet her questioning practices, while facilitating student participation, tended to "funnel" students toward the right answer and require only factual answers, suggesting a more limited positioning of students. The last study also focused on questioning practices, using a larger sample of 11 professors across three institutions, and found that instructor questions were a common occurrence but varied substantially by instructor (Paoletti et al., 2017). In general, they found that students' opportunities to engage in meaningful mathematical participation were limited, but some lecturers did provide students with "genuine" opportunities to participate (in one case by calling on students to elicit participation).

In summary, this small pool of studies focused within undergraduate mathematics classrooms points to the significance of the understudied dimension of classroom interactions. Studies of "traditional" instruction in such classrooms suggest that there is greater variation in lectures than might be expected, that language use in lectures can be significant for student understanding and engagement, and that interaction patterns around student participation such as asking and answering questions are especially significant dynamics. Looking across all of the undergraduate research that has been reviewed, classroom interactions are clearly a significant component of students' experiences, but one that is only beginning to receive purposeful attention in the literature.

I turn now to the conceptual framework of positioning theory that I will use to study classroom interactions and introduce it using theoretical literature. I then review studies from the K-12 literature that foreground the importance of attending to classroom interactions, establishing empirical evidence for positioning theory as a helpful lens to understand students' relationships with mathematics.

Positioning Theory: A Conceptual Framework

Positioning theory comes from the field of social psychology, and takes moments and episodes of everyday language and discourse as its object of study. It offers a conceptual and methodological framework for analyzing social interactions, in the form of attention to three basic features:

i. Positions—the moral positions of the participants and the rights and duties they have to say certain things;

ii. Storylines—the conversational history and the sequence of things already said; and

iii. Speech acts—the actual sayings with their power to shape certain aspects of the social world. (adapted from Harré & van Langenhove, 1999, p. 6)

This triad of positions, storylines, and speech acts is the fundamental conceptual apparatus that positioning theory offers for analyzing interactions.

Positions

A position is a collection of attributes and beliefs that orient participants in an interaction in relation to one another. A position can be ascribed to a person as a result of their discursive actions, and it can shape the discursive actions available to the person to whom it is ascribed. Positions are relational, "in that for one to be positioned as powerful others must be positioned as powerless" (Harré & van Langenhove, 1999, p. 2). And positions are often reciprocally defined; "One can position oneself or be positioned as e.g., powerful or powerless, confident or apologetic, dominant or submissive, definitive or tentative, authorized or unauthorized, and so on" (p. 17).

Positions are also associated with relational "rights" and "duties," where rights are "what you (or they) must do for me" and duties are "what I must do for you (or them)" (Harré, 2012, p. 197). For example, a fundamental right is the "right to speak," which is associated with some positions (e.g., teacher or parent) more strongly than others, and has a corresponding "duty to listen" (e.g., for students or children). As pointed out by Tan and Moghaddam (1999) in their discussion of intergroup relations, "In many situations, only a few claim the right and/or are socially ascribed the right to speak and be heard" (p. 184).

Although positions are ascribed to participants, they are never permanent, but rather exist on a spectrum from fleeting to long-term, where long-term positions are the most like "roles" (Davies & Harré, 1999, p. 39; Harré, 2012). The strength with which a position is taken up by or forced upon a participant determines its effect as an orienting device. For example:

> Once having taken up a particular position as one's own, a person inevitably sees the world from the vantage point of that position and in terms of the particular images, metaphors, storylines and concepts which are made relevant within the particular discursive practice in which they are positioned. (Davies & Harré, 1999, p. 35)

Storylines

Storylines and positions are closely intertwined; "With every position goes a story line" (Harré & Slocum, 2003, p. 106). Storylines are the narratives that people use to navigate and interpret episodes of interaction. They can be organized around "events, characters and moral dilemmas" and they can draw on "cultural stereotypes such as nurse/patient, conductor/orchestra, mother/son" (Davies & Harré, 1999, p. 37). These storylines are relevant to interactions because "the words the speaker chooses inevitably contain images and metaphors which both assume and invoke *the* *ways of being* that the participants take themselves to be involved in," whether or not the speaker is deliberately aware of it (p. 38, emphasis mine).

Storylines can exist at many levels, from cultural discourses (e.g., men are dominant and women are submissive) to classic stories (e.g., David and Goliath) to local occasions (e.g., inside jokes). Storylines can be explicit or implicit in an interaction. For example, "explicit storylines are exemplified in the playing out of structures like ceremonies, rule-bound games, or routines in church" (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015, p. 188). And different participants in episodes of interaction can draw on different storylines, resulting in different interpretations of the episode; "It is important to remember that these cultural resources may be understood differently by different people" (Davies & Harré, 1999, p. 37).

Speech Acts

The third component, speech acts, is about the meaning of the words spoken in an interaction, which is shaped by the positions and storyline(s) that a particular participant takes as relevant. Positioning theory is careful to distinguish between the literal words spoken and the force and meaning that the words are taken to have, calling them speech *actions* and speech *acts*, respectively. As one example of this distinction, consider the phrase or speech action, "I'm sorry we're late." The speech act, or social meaning of the speech action, can be taken as a genuine apology, as a pro-forma statement, or even as an accusation, depending on the positions and storylines of the particular episode. "This way of thinking about speech acts allows for there to be multiple speech acts accomplished in any one saying and for any speech act hearing to remain essentially defeasible" (Davies & Harré, 1999, p. 34).

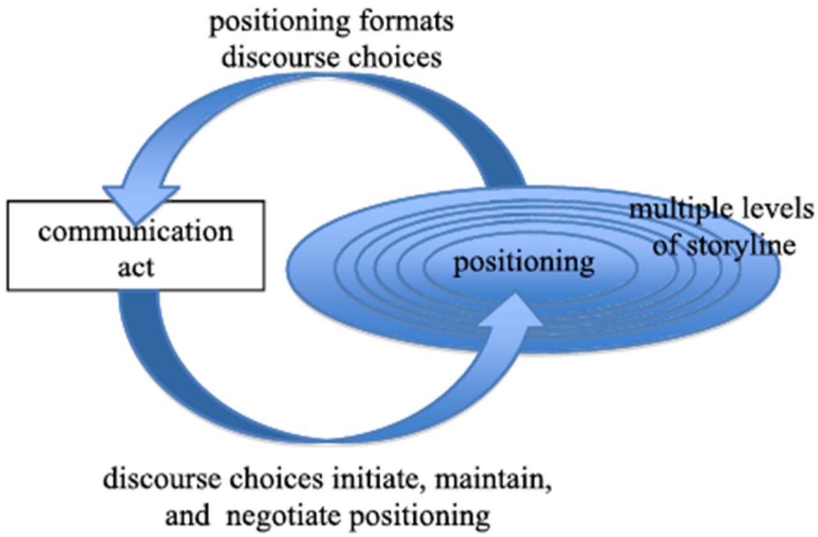

Figure 1. Diagram of positioning (from Herbel-Eisenmann et al., 2015, p. 194)

Positioning

Positioning pulls together these three components into a dynamic, discursive process (Figure 1); "positioning is the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced storylines" (Davies & Harré, 1999, p. 37). Positioning involves the transformation of speech actions into speech acts through the interpretive and negotiated context of particular storylines and their corresponding positions. To summarize these relationships, in other words:

> …we have shown how both the social act performed by the uttering of those words and the effect that action has is a function of the narratives employed by each speaker as well as the particular positions that each speaker perceives the other speaker to be taking up. (Davies & Harré, 1999, p. 48)

These relations are also made clear with reference to a worked example from Davies and Harré (1999), which analyzes a short episode of interaction into its features of speech acts, positions, and storylines (Appendix A).

This study focused in particular on students' positioning in relation to mathematics, which is slightly different from the focus of many of these definitions and of the worked example on positioning between people. By positioning in relation to mathematics, I am referring specifically to interactions that involve storylines about mathematics, and to how students are positioned within these storylines—for example, as incapable of understanding mathematics, as receivers and reproducers of mathematics, and/or as doers of mathematics (as will be described in more detail at the start of Chapter 5).

Rationale for Choosing Positioning Theory

Positioning theory offers an important lens for understanding educational issues, particularly the study of classroom interactions. In attending to different dimensions and scales of interactions, positioning theory offers insight into day-to-day dynamics as well as larger social and cultural forces.

> A powerful aspect of the use of positioning theory as an analytical tool is that not only persons and their identities both individual and social, but also societal issues on a cultural level can be tackled with the same conceptual apparatus. (Harré & van Langenhove, 1999, p. 12)

Not only does positioning theory help in unpacking different dimensions of our experiences of interactions, it also holds some promise for improving these experiences. In the first place, there is a power in noticing patterns in positioning at all:

> Revealing the subtle patterns of the distributions of rights to speak and act in certain ways can open up the possibility of their transformation. At the same time, analysis of patterns of meaningful actions in terms of story lines can bring to light previously unnoticed presumptions about what is going on in an episode. (Harré & Slocum, 2003, p. 102)

In other words, noticing patterns in our interactions is a way to reveal assumptions and to imagine new possibilities. As many have pointed out, it is only when we point to patterns and give them names that we become free to play with them in our thoughts and use them as tools to transform the world around us (e.g., Vygotsky's semiotic mediation).

Research on K-12 Classroom Interactions and Positioning

In addition to these theoretical arguments, empirical research on classroom interactions in K-12 instruction using positioning theory shows the importance of this lens and the potential insights it offers into the development of students' relationship with mathematics. Researchers' attention to positioning in K-12 classrooms has grown dramatically in the past decade, including in the field of mathematics education (Herbel-Eisenmann et al., 2015). The variety of approaches taken to studying positioning in mathematics classrooms help make the case that it is has much to offer in terms of better understanding students' experiences and relationships with mathematics in the classroom.

Within the body of studies in mathematics education, I identified two general approaches to studying positioning in classroom interactions. One very common approach is to track the positioning of one student within a particular event or sequence of events, with findings that address the relation of classroom interactions to identity development.[2] Many of these studies take the dynamics of group work as their focus, attending to the small-scale negotiations of identity within a particular group of students and its impact on those students' opportunities for mathematical engagement and learning (Esmonde & Langer-Osuna, 2013; Langer-Osuna, 2011; 2015; Pinnow & Chval, 2015; Wood, 2013). For example, Wood (2013) analyzed the small group interactions in one lesson between three 4th-grade students and their teacher and identified three micro-identities assigned/enacted by one of the students, two of which (mathematical student and mathematical explainer) positioned him as mathematically capable and one of which (menial worker) positioned him as minimally engaged with mathematics. Similarly, Langer-Osuna (2011) found that a female 9th-grader was positioned as "bossy" and slowly excluded from the group work with her three male classmates. Other studies take a longer-term view, and look across multiple events to analyze the ways that in-the-moment positioning can "stick" and lead to more stable identities, such as not being competent at math (Anderson, 2009; Bishop, 2012) or being mathematically proficient (Yamakawa, Forman, &

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[2] Identity and mathematics identity, in particular, are constructs defined in many ways. Here is one definition that captures a general sense of mathematics identity: "the ideas, often tacit, one has about who he or she is with respect to the subject of mathematics and its corresponding activities. Note that this definition includes a person's ways of talking, acting, and being and the ways in which others position one with respect to mathematics" (Bishop, 2012, p. 39).

Ansell, 2009). Taken together, these studies demonstrate the importance of classroom interactions in terms of their potential impact on students' mathematics identities, both in-the-moment and over time.

The second approach focuses less on individual identities and more on larger patterns of positioning in classroom interactions. Some focus on patterns of positioning across multiple groups within one classroom, with the goal of understanding how these patterns establish or inhibit opportunities to learn (DeJarnette & González, 2015; Esmonde, 2009). For example, in looking at three high school mathematics classes, Esmonde (2009) found that when one student was positioned as the group expert, it often closed off opportunities for learning for students positioned as novices, unless there was a student positioned as facilitator who helped the "novice" students in the group take advantage of the expert's understanding. Another study focused on patterns of positioning in whole group instruction in an mathematics after-school program for 4^{th} and 5^{th} graders, and found that instructional moves such as revoicing were used to deliberately position English language learner students as important contributors to the group's mathematical activity (Turner, Domínguez, Maldonado, and Empson, 2013). Finally, one set of studies has used critical discourse analysis to examine a corpus of 148 high school mathematics classroom observations, connecting the use of particular words or phrases with possibilities for agency and dialogue (Herbel-Eisenmann & Wagner, 2010; Herbel-Eisenmann, Wagner, & Cortes, 2010; Wagner & Herbel-Eisenmann, 2008). For example, Wagner and Herbel-Eisenmann (2008) showed that students' frustrations with the word "just" could perhaps be explained by the fact that it was used by teachers in ways that often closed off further discourse and positioned the teacher as the sole source of authority (as in "Just multiply straight across"). These studies emphasize the importance of attending to patterns in instructional moves and in discourse choices, because these patterns can provide insight into positioning as a larger classroom dynamic that has the potential to influence many students' relationships with mathematics.

In looking across these studies on the use of positioning in K-12 mathematics classrooms, one critique I offer, particularly of the first approach, is that their conclusions sometimes feel limited. Learning about the trajectories of one student or one group of students is only powerful insofar as it enables us to better understand, shift, or support trajectories for other students, which requires some understanding of the larger social forces that contribute to such trajectories. Yet very few of these studies take up positioning theory as a complete conceptual framework; many attend to positions, but not as explicitly to speech acts or storylines (Herbel-Eisenmann et al. (2015). Therefore their conclusions cannot speak to the nature and impact of storylines as larger narratives and forces shaping classroom interactions and trajectories of positioning. More explicit attention to all three features of the theory and to storylines, in particular, would be an important contribution, which shaped my own application of the theory.

In summary, positioning theory offers a framework for bringing to light underlying patterns in classroom interactions and thereby holds out the possibility of helping to identify and possibly shift some storylines for students in undergraduate mathematics classrooms. These theoretical strengths as well as the empirical evidence from the review of the literature led me to choose positioning theory as the conceptual framework for this study.

Summary of State of Research

We do not know very much about the in-class dynamics of students' experiences in undergraduate mathematics classrooms, but what we do know suggests that classroom experiences are significant in shaping students' relationships with mathematics. Research in undergraduate mathematics classrooms is only just beginning to focus on the patterns and dynamics of interactions, while research from K-12 mathematics classrooms suggests that studying classroom interactions is vital in discerning how students come to see themselves as mathematics learners and doers. The use of positioning theory as a conceptual framework can inform our understanding of both in-the-moment dynamics as well as larger sociocultural storylines that shape students' experiences of classroom interactions. That is why this descriptive study seeks to address the overarching question: How can classroom interactions in an undergraduate mathematics class be viewed as positioning students in relation to mathematics? Having reviewed the literature and described the conceptual framework that situate and motivate the study, it is time to turn to the methods for how the study was conducted.

CHAPTER 3: METHODS

Data collection occurred over the course of a semester in an upper-level undergraduate course, *Introduction to Analysis*[3], at a mid-Atlantic research university. The primary data source was field notes, supplemented by interviews with students and with the professor, audio recordings of class sessions, and transcripts of particular classes and particular interactions. I analyzed these data in order to construct descriptions of interactions and interpretations of positioning that address the three research questions:

1. What are patterns in the classroom interactions in an upper-level undergraduate mathematics class?
2. How does one moment in the semester position students in relation to mathematics?
3. How do frequently used phrases across the semester position students in relation to mathematics?

In this chapter, I provide a brief overview of the context of the course and reasons for selecting this course, explain the data collection and data analysis process, and describe my methodological stance.

Description and Rationale for the Course Selected

Introduction to Analysis is a fairly traditional lecture-based course, comparable in many ways to the other upper-level courses in the mathematics department at this mid-Atlantic research university. Instruction typically proceeds in the standard "Definition-Theorem-Proof" format, where the instructor introduces and

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[3] This course title is a pseudonym.

defines new content, states an important theorem related to this content, and then works through the proof of this theorem (Davis & Hersh, 1981; Weber, 2004). Students are expected to master proof-related material specific to the course content (in this case, Real Analysis), and these expectations are assessed through weekly homework and on midterm and final exams. Similar to the other upper-level courses, the required prerequisites include the Calculus sequence (up through multivariable) and Linear Algebra, as well as an Introduction to Proof course.

The textbook for this course is Fitzpatrick's (2006) *Advanced Calculus*, and the course covers roughly:

Chapter 1. Tools for analysis;

Chapter 2. Convergent sequences;

Chapter 3. Continuous functions;

Chapter 4. Differentiation;

Chapter 5. Elementary functions as solutions of differential equations;

Chapter 6. Integration: Two fundamental theorems;

Chapter 8. Approximation by Taylor polynomials; and

Chapter 9. Sequences and series of functions.

Portions of these later chapters and often all of Chapter 5 are sometimes skipped. There is a subsequent course, *Introduction to Analysis II,* that is not required but that students who intend to go to graduate school in mathematics are strongly advised to take, which covers the remaining chapters in the textbook beginning with Chapter 10. This course, as well as the general goal of preparing potential students for graduate school, means that instructors of *Introduction to Analysis* are expected to get through all of this content in the 15 weeks of the semester.

Introduction to Analysis is different from other courses in this department in that it is the only upper-level course that is required for all mathematics majors, regardless of concentration (Traditional, Education, Applied, and Statistics). Between 20 to 30 percent of students typically get a D, an F, or withdraw from the course, which means they have to re-take the course or change their major. Although the department anticipates that most students will be juniors, it is possible for a student to have satisfied the prerequisites and to take this course as a freshman or sophomore. There are often seniors in the course as well, re-taking it after withdrawing or failing to get a C, or squeezing in the requirement at the end. Thus, a wide variety of students with varying mathematical experiences and backgrounds are represented in the course.

This course is viewed by students, instructors, and the department as the "make-or-break" class for mathematics majors, both because students have to pass the course to complete the major, and because of the demanding nature of the content and the pace. As one student described it, "I heard that it is terrible. I heard that it's very hard and very challenging. But the general consensus is, after you've taken it, it's like you're like a veteran math student." *Introduction to Analysis* is clearly an intense course in terms of expectations and demands on students, while also attracting the widest possible array of mathematics majors, making it a particularly compelling choice of context for a study of students' positioning in relation to mathematics.

Description and Rationale for Data Collection

The primary data source was field notes from all class sessions over the course of the fall semester, supplemented by audio recordings of these class sessions. Using the audio recordings, I transcribed several episodes of classroom interaction during the semester (to share in interviews with students), and then transcribed nine entire classes after data collection, along with all interviews. I also interviewed a small pool of students at the beginning, middle, and end of the semester, as well as the professor after the semester was over.

Participants

On the first day of class, I obtained informed consent from students in the course for my doing observations, audio recordings, and writing field notes about their interactions. All of the students consented, except for one student who was not old enough to, resulting in 26 student participants in observations. I did not take field notes about the one student who did not consent, or transcribe his speech from any audio recordings.

I also solicited participants for interviews on the first day of class; 12 students ended up participating in interviews at some point over the course of the semester. More details about the selection of the interview participants are presented below. I only collected detailed data, including demographic information, from the interview participants (see Appendix C). All students are described in this dissertation using pseudonyms; students I interviewed chose and/or approved their pseudonym, while for the other students I chose their pseudonym.

In addition to the students, I also obtained informed consent from the professor of the course to do observations, audio record, and write field notes over the course of the semester, as well as to do an interview at the end of the semester.

Classroom Observations

The section I observed met three days a week (MWF) for 50 minutes at a time. I attended class every day, except the three days of in-class exams and one day when I was sick, for a total of 39 sessions. The professor was absent and had substitutes for one week, leaving 36 sessions that were the main focus of subsequent analysis. I recorded each class using two audio recording devices, one placed on the table at the front of the room and one placed at my seat in the back right corner of the classroom. The second recording served as a back-up, in cases when the first recorder did not work and in cases when the students' talk was not audible from the primary recorder. As soon as possible after each class session, I uploaded the recordings to my computer and external hard drive.

Because of the relatively high stakes associated with the class, and my relative unfamiliarity to the professor and students, I decided to audio record the class only, rather than video record. One significant feature of class that was not captured by audio is the instructor writing on the chalkboard; however the board work was rarely a direct referent point in classroom interactions, but rather served as a record of formal proofs work that was usually quite close to the work in the textbook. Whenever the board work became the focus of classroom interaction or deviated significantly from the textbook, I kept track of it in my field notes.

During class, I took field notes by hand in a notebook, which allowed me certain flexibility (e.g., being able to sketch graphs drawn on the board) and also fit into the classroom environment better, as all students took notes by hand and not on laptops. In these notes I tracked the time, the gist of the lecture at that time, any particular phrases the professor used that stood out, any interactions with students that occurred, as well as other general notes (e.g., who was taking notes and who was not). Given the use of audio recordings rather than video, the field notes were very important for capturing non-verbal features of interactions, such as gestures and expressions. At the start of the semester I assigned each student an abbreviation consisting of a letter and a number (e.g., M7) so that I could track each student's participation in my field notes in a de-identified manner. Using this form of shorthand, I was able to capture essentially all of the student-instructor interactions in the classroom, which proved very helpful and important for later data analysis. For example, it helped me to identify which student was talking when I transcribed audio recordings of interactions.

Within 48 hours of each class session, I documented the field notes from my notebook in an observation protocol (Appendix B) in order to capture consistent information across different class sessions and to organize and focus my field notes. The information about the date, class make-up, and seating chart were primarily for record-keeping purposes, such as keeping track of who came to class and any patterns in who participated. I constructed a summary timeline of the key events that established roughly how long was spent on different content during that class (e.g., how many textbook sections were covered, how many theorems proved) that was helpful for describing the overall pacing in the course. The bottom half of the observation protocol contained the actual field notes, and was intended to be fairly open-ended in order to allow for details and in-the-moment descriptions of particular events. I modified the original observation protocol after the first week of class to add a section for keeping track of my own thoughts and comments, which was useful as a way to explicitly document my researcher perspective while keeping it separate from the more observational account of what happened. The final "of note" column was a way to focus my attention on capturing particular types of interactions that have been noted in the literature as potentially significant - when the professor asks a direct question to the students, when the students ask a question to the professor, and when students make a comment to the professor (e.g., Kleinman, 1995; Yoon et al., 2011) – as well as a star for an overall significant-seeming interaction.

Every few weeks, episodes that were marked as significant in the observation protocol were logged with a brief description that included the day and time of the event, the participants, and a summary of the event. This log of event summaries was used to select significant episodes to discuss with students in interviews, as well as later in the analytic process to help identify patterns across significant events.

Interviews

On the first day of class, I gave students consent forms that explained the project and asked them to indicate their willingness to be written about in field notes, and to indicate their willingness to participate in these interviews (with compensation of $15 per interview, plus another $15 for doing all three). I had anticipated that few students would be willing to be interviewed; instead, 19 of the 27 students consented.

As originally planned if more than five students volunteered, my primary thought was to represent a wide variety of experiences, such as a student who often participated in class and a student who never talked in class. I made categories of participation (Often, Sometimes, Once, Not at all) at the end of the second week, in order to select at least one student from each category. Within those categories, I selected students on the basis of diversity related to gender/race, as well as convenience (having student emails) and different experiences (having a friend in the class, being in the education concentration). These participation categories, the list of students selected, and the interviews students did participate in are all included in Appendix C.

For the first interview I asked a total of 10 students, and nine of them agreed. These interviews took place between the third and sixth weeks of class (9/14 to 10/8); all of them took place before the first exam. For the second interview I asked eight of the participants from the first interview (the ninth student had dropped the class) and added two participants. These two new participants had significant increases in their participation and significant events they were involved in between the first and second interviews; they were also chosen on a somewhat opportunistic basis of running into them outside the classroom. These interviews took place between the tenth and thirteenth weeks of class (11/2 to 11/25); all of them took place before Thanksgiving break. For the final interview I asked all 11 of the participants so far (including the student who dropped the class) and added one participant. This student was unique in his participation in that he was the only student who never participated or was called on for the entire semester; also, other students discussed him in particular in their interviews, suggesting his insights on the class might be important for understanding the class dynamics. These interviews took place during the final week of the course and exams week (12/7 to 12/18); all but two took place before the final exam.

Each interview was audio recorded and followed a semi-structured protocol (Appendix D). The first interview focused on collecting information about the student, his or her expectations for the course, and his or her initial experiences in the course, and took about 10-20 minutes. This information provided important context for understanding and describing students' perspectives. The second interview focused on asking the student to describe and respond to audio clips from the course that I identified (using the log of significant events described earlier). Each student responded to one clip that was common across all participants (a speech by the professor from after the first exam), to one clip in which they participated, and to one clip in which they were not a participant, with the idea of getting an "insider" and "outsider" perspective on each of these moments (for more detail on the clips, see Appendix D). This interview was the longest, taking between 30-90 minutes. The third interview asked the students to reflect on their experiences in the course, and took between 30-60 minutes. This information allowed for context and comparison with the initial interview.

In addition to interviews with the students, I also conducted one interview with the professor at the end of the semester, on December 21st, once the course was finished and grades were submitted. The interview lasted one hour and generally covered the following three topics: the professor's goals for the course lectures; his goals for student participation; and his goals for homework and exams. The professor also discussed how this year's course went compared to the previous time he had taught it, as well as his plans for the course in the coming semesters.

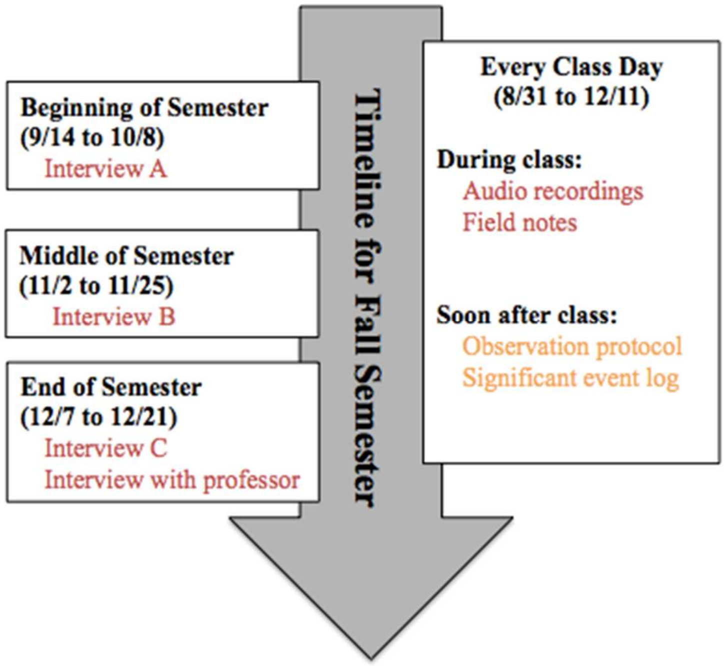

Figure 2. Data collection timeline

The overall data collection process can be seen in the timeline (Figure 2). The observations occurred throughout the semester, with the interviews occurring at three time frames across the semester. The observations involved both raw data collection in the form of audio recordings and field notes, as well as a first analytic pass (noted in orange) when completing the observation protocol for each lesson and the significant event log.

Description and Rationale for Data Analysis

Data analysis proceeded in three main stages. In the first stage, I worked with the different types of data I had collected to determine their affordances, and how these affordances might serve to illuminate directions to take in addressing my research questions. In this stage, I organized and analyzed my data in the following ways: (1) Coded interactions and created timelines of the semester from my field notes; (2) Refined codes and re-coded field notes for types of interactions; (3) Created tables and memos about interactions over the semester; (4) Transcribed all interviews with students; and (5) Categorized significant events from the significant event log and from student interviews.

In the second stage, I worked to address the first research question by writing a narrative description of interactions over the semester. To write this portrait, I first identified "emergent themes" in the interactions, drawing heavily on the analysis from the first stage, and wrote memos detailing these themes. Then I proceeded to outline and write a narrative "portrait" of the semester, organized around those emergent themes. I member checked this portrait with the professor.

In the third stage, I worked to address the second and third research questions and describe positioning in interactions using two different approaches. First, I identified a significant moment in the semester, which I then closely analyzed using tools from discourse analysis and using positioning theory as a conceptual framework. I member checked this analysis with the focal student from this moment. Second, I transcribed nine lessons from across the semester, which I then analyzed using MAXQDA to identify commonly occurring four-word phrases, or lexical bundles. I categorized and analyzed these bundles, again using tools from discourse analysis and using positioning theory as a conceptual framework.

The following sections will describe each stage of analysis in greater detail.

Stage 1: Working out the Affordances of the Data

> The ultimate power of field research lies in the researcher's emerging map of what is happening and why. So any method that will force more differentiation and integration of that map, while remaining flexible, is a good idea. Coding, working through iterative cycles of induction and deduction to power the analysis, can accomplish these goals. (Miles & Huberman, 1994, p. 65)

Given the large quantity of data collected, I spent a lot of time at the start getting familiar with my data. I began by going through my field notes in order to make a timeline of the semester that I thought would help orient me to the general context of each class session (Figure 3). It included the content covered (sections of the textbook and key ideas), assessments and review days, number of students in class that day (male/female), a rough count of the number of instructor questions/student questions/student comments, and a holistic assessment of participation that day as high/medium/low. This latter part, in particular, was unsatisfying – my rough counts felt unreliable and my assessment of high/medium/low participation felt extremely subjective. My next step was to create an expanded timeline of the semester that included categories of student participation and identified each student with a color (Figure 4). The categories of student participation developed in vivo as I went through the field notes chronologically; the resulting codes (from top to bottom in the timeline) were: Answered a factual question; Asked a question; Called on by the professor; Participated in a proof; and Non-mathematics related participation. The very bottom of the timeline tracked the students who participated that day, with a box around the first time a student participated for the semester.

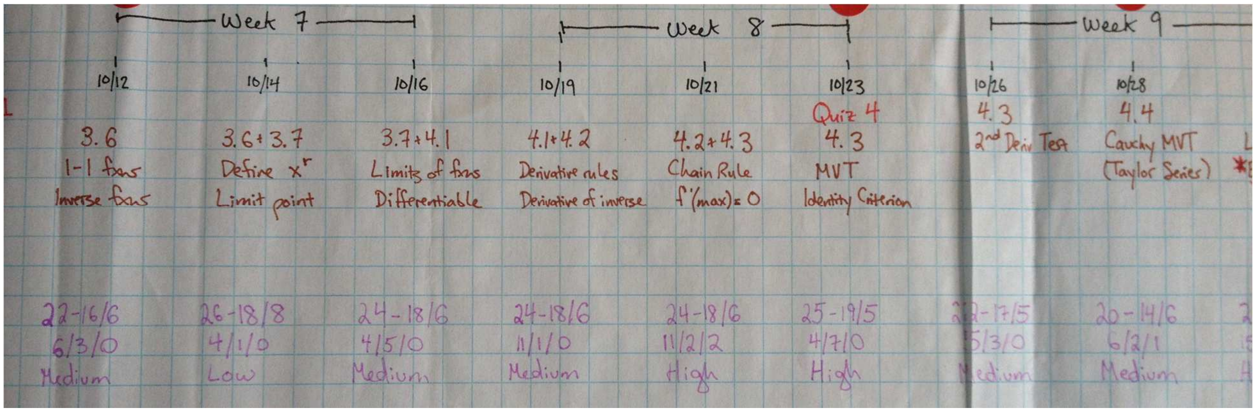

Figure 3. Timeline of the semester (excerpt)

Figure 4. Timeline of student participation (excerpt)

Both timelines were useful as ways to immerse myself in the field notes; the second timeline in particular was a way to begin to develop codes for the kinds of student-professor interactions that occurred over the semester, which were necessary for identifying both patterns and unusual or important events.

In studying these timelines, I began to develop preliminary hypotheses about the data – for example, the beginning of the semester seemed notable for how often the professor called on students to contribute to proofs; the middle of the semester had the most participation in general; and the end of the semester had comparatively little participation. In order to refine and test these hypotheses, I found myself wanting to compare counts of codes across different days; this task was challenging when working from my hand-made timelines, and it also raised questions for me about the distinctions between the codes I had created. Therefore, I decided to re-code the field notes using a qualitative data analysis program (MAXQDA), which would help me track and tabulate the codes much more quickly, and which provided an easy way to pull all instances of one code and create and organize memos describing the different codes.

Using MAXQDA I coded all 36 days of field notes using codes for each student and codes for the kind of interaction that occurred. These interaction codes continued to be refined throughout the data analysis process; the final five resulting high-level codes for student interaction were: Answered a question; Contributed to a proof; Checked understanding; Called on by the professor; and Responded to another student. Within each of these high-level codes (except the last one) were sub-codes; for example, the category of "Called on by the professor" had two distinct sub-codes, one for when the professor called on a student to contribute to a proof, and one for when the professor called on a student to check that they were following or to see if they had questions. For a complete coding table, see Appendix E.

Using these codes I made a variety of different tables representing interactions over the semester. For example, to explore my earlier hypothesis about changes over the semester, I created one table that looked at the three categories of answered a question, contributed to a proof, and asked a question, then took the sum of these codes per day and sorted the days into Low participation (4 to 10), Medium participation (11 to 21), and High participation (21 to 28) using color-coding (see

Table 1). I then wrote a brief memo on the basis of this table, describing patterns that I noticed in the different codes as well as overall patterns in amount of participation.

Table 1. Heat map of interactions over the semester (excerpt)

Class number	Date of class	Answered question	Contributed to proof	Asked question	SUM
1	8/31	13	0	0	13
2	9/2	3	6	7	16
3	9/4	1	7	7	15
4	9/9	1	11	12	24
5	9/11	1	7	0	8
6	9/14	3	8	1	12
7	9/16	5	7	2	14
8	9/18	7	4	5	16

My original plan was to identify one representative lesson as well as significant event(s) in order to describe those interactions in detail, but based on these initial analyses of the data, my emphasis instead became on describing interactions over the course of the semester. This direction felt more productive and more true to the data – the idea of a "representative" lesson was much harder to capture than I had originally predicted (I had expected there to be minimal interactions, more on the order of 5-10 interactions per class).

In addition to working with the field notes as described above, I also transcribed all of the student interviews. After transcribing these interviews, I made a list of common themes with quotes from the student interviews and shared it with the professor because he had expressed interest in the student feedback from these interviews. These themes were about the professor (general praise, makes students comfortable, critique); participation (calling on students – positive, calling on students – mixed); homework (learned a lot, want more guidance, want solutions, work alone, using the internet); and the textbook (learning from textbook vs. lecture, reading textbook before lecture). Although these themes were not focused on classroom interactions in particular, I do think that identifying them was helpful for immersing myself in the interview data.

The other analytic work I did with the student interviews was to pull out significant events mentioned by students in any of the three interviews, each of which included a prompt about any moments they'd noticed in class or that stood out to them. (I pulled out these clips by importing the interviews into MAXQDA, coding the moments with a "significant events" code, and then exporting all coded segments to an Excel file). I combined these coded segments with the events identified in the significant event log I created throughout the semester, which resulted in a total of 76 segments; I then organized all of these into categories. There were four events that stood out as unique: the professor's speech after the first exam, his comment after the second exam, his asking students to turn and talk to a partner, and his discussion of a confusing problem during an exam review day. The remaining 72 segments were categorized into ten groups, including Challenges (when a student expressed a different understanding from the professor) and Checking in interactions (when the professor had an extended interaction with a student who was not understanding). In the same way that coding types of interactions in the field notes was helpful for identifying patterns in interactions over the semester, creating these categories was helpful for identifying patterns in significant events over the semester and for identifying events that were unique.

Stage 2: Emergent Themes

> Memoing helps the analyst move easily from empirical data to a conceptual level, refining and expanding codes further, developing key categories and showing their relationships, and building towards a more integrated understanding of events, processes, and interactions. (Miles & Huberman, 1994, p. 158-159)

Looking across these categories of significant events as well as the kinds of interactions I coded in the field notes, I began to identify "emergent themes" that seemed to organize different threads across classroom interactions. I developed these themes both inductively, by sorting the codes for interactions and the categories of significant events into larger groupings, and deductively, by listing out threads I had noticed or hypothesized about and then supporting them with evidence (several codes and categories were included in more than one theme). I ended up with three themes - one about patterns in the professor's classroom interactions and how they were structured around the goal of students learning how to do proofs, one about patterns in students' participation in the classroom and how they potentially related to students' positioning, and one about constraints on classroom interactions in the form of assessments and time pressure.

Each of these three themes captures an important dimension to consider in classroom interactions, as supported by their parallels to the three dimensions of the instructional triangle (Figure 5). The first theme about learning to do proofs sits along the content dimension—focusing on how the interactions drive and are driven by this goal related to the content. The second theme about student participation sits along the student dimension—focusing on the interactions that students participate in and how this positions them in relation to the content, the professor, and one another. And the third theme about constraints on the course sits along the teacher dimension—focusing on the professor's obligations to the content and students, within this particular departmental and institutional environment.

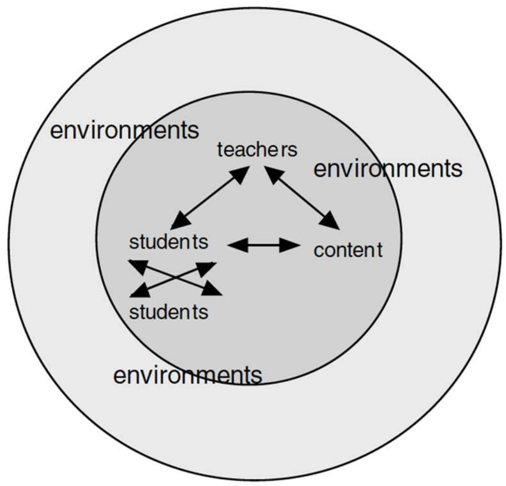

Figure 5. Instructional triangle (from Cohen, Raudenbush, & Ball, 2003, p. 124)

In developing these themes, I decided to construct memos that would flesh out the patterns of interactions, claims about these patterns, and evidence to support them. The resulting documents were rather extensive; the memo for the first theme is included as an example (Appendix F). Writing these memos involved going back-and-forth between the memo and various data sources in a fairly fluid process. Every time I wrote a claim, I tried to return to the data: to find support for that claim, to flesh out the claim, to check for counter-examples, and so forth. To give an example, at first I wrote "the professor often asked information-gathering questions" in the first memo about the professor's classroom interactions. I then asked, "Well, how often is often?" and went into MAXQDA and pulled all the field note segments coded as

"Answering question," and then used the software to create a table that showed the frequency of this code across every field note document in chronological order. Using this table, I could see that "often" was in fact every day of field notes, and I could then provide details on the range of how often (between 1 and 13 times a class). Additionally, returning to the data forced me to be very careful in the language of my claim; what I had coded was not in fact every time the professor asked an information-gathering question, but instead every time a student answered one of those questions. I shifted the final language of my claim to reflect the precision and the frequency: "the students answered the professor's information-gathering questions every day of class (ranging from 1 to 13 answers a class)."

This fluid back-and-forth was the principal kind of interaction between the earlier analytic work and the memos. But on occasion, writing the memos influenced the original analytic work as well. For example, as I went to describe the kinds of questions students asked during class, I was looking through all the coded segments to see if there were any patterns and to try to find helpful examples for distinguishing between two of the codes. These initial two codes were "questions about the proof" and "challenge/extension questions" and had been created based on my experiences sitting in this class and similar classes. As I looked through all the questions, though, it felt very hard to pin down this distinction, and the two codes were unsatisfying. I decided to re-code all of the student questions in these two codes from the ground up, which resulted in four specific categories - similar to the two original ones but with much clearer specifications. In this way the work of describing led me back to the work of coding. Having spent the time re-coding was extremely influential and helpful for then returning to the memo and describing students' questions and patterns in them.

The other important point to mention is the role of the interview data. For the first memo about patterns in the professor's interactions, I often found myself wanting to make claims about the professor's intentions, and for this reason I went and transcribed my interview with the professor in order to look for confirming and dis-confirming evidence about his intentions. After transcribing his interview I went through and pulled out all quotes that seemed relevant to any of the three themes, and inserted them in each of the memos. These quotes were particularly relevant for the first memo and I often incorporated them into claims and/or refined my claims on the basis of them.

The student interview data came into play in a similar way for the second memo about patterns in student participation, in that I often found myself wanting to make claims about students' experiences or perceptions of particular interactions or kinds of interactions. To justify these claims, I proceeded in a manner similar to the back-and-forth fluidity section described earlier. When I found myself typing a claim about students' perceptions, or wondering about support for a particular interpretation, I would return to the student interviews and use my familiarity with them to locate particular quotes. This process was by no means exhaustive, but I did not find that it needed to be; the quotes often acted as existence proofs that a student could interpret it this way, rather than as a claim about all or most students. This fluid process of returning to the interview data as needed felt very helpful as a way to use these secondary data to expand, refine, and sometimes challenge (i.e., sometimes I would misremember the force of a student's interview comment) my thinking about student participation.

Stage 2: Writing the Portrait of the Semester

In order to address the first research question, I constructed a narrative about classroom interactions that I call a portrait of the semester (Chapter 4), as inspired by portraiture (Lawrence-Lightfoot & Davis, 1997). The goal of this portrait is to allow the reader to enter this classroom and experience the patterns and significant moments of classroom interactions over time. I decided that a portrait was the most appropriate choice for responding to the research question and describing classroom interactions because I wanted to synthesize these themes and present them in the time and space in which they occurred. In other words, the professor's goal of students learning to do proofs (the first theme) did not occur separately from the students' choices to participate (the second theme), nor did it occur separately from the physical space and arrangements of this classroom, or from the time of year and the point in the semester. As described in the methodological tradition of portraiture, "the narrative is always embedded in a particular context, including physical settings, cultural rituals, norms, and values, and historical periods. The context is rich in cues about how the actors or subjects negotiate and understand their experience" (Lawrence-Lightfoot & Davis, 1997, p. 12).

Writing this portrait was a process of synthesizing much of the previous analytic work, and presenting these patterns of interaction in an engaging and authentic form for the reader. In order to write it, I had to select the particular events to describe, which relied heavily on the "emergent themes" memos. I went through the memos and selected events, which I then arranged in a chronological outline from which to begin writing. My rationale for selecting an event was that it either needed to be important for elucidating one of the themes, or an important event for understanding those particular weeks of the semester. After selecting an event to include, I then tried to write a rich, thick description of it, including selected quotes that gave voice to the instructor and students, and trying to develop the central themes without being heavy-handed. The following considerations from portraiture served as a useful guide in crafting the portrait overall:

> The portraitist attends to *resonance,* which designates particular stories and convergent themes as pertinent parts of the whole. The portraitist attends as well to *coherence*, through which the various parts gain meaning from their relationship to each other. And ultimately, the portraitist attends to *necessity* or the indispensability of any designated part to the aesthetic whole. (Lawrence-Lightfoot & Davis, 1997, p. 274)

After a complete draft of the portrait was written, I sent it to the professor as a form of member checking; he said it matched his intentions and was a revealing portrait of the semester.

Stage 3: Analysis of a Moment

In order to analyze positioning in these interactions, I took two approaches. To address the second research question, I identified a significant moment in the semester and consider positioning within that interaction. To address the third research question, I examined patterns of commonly used phrases across the semester and the positioning within these patterns.

I chose to examine an interaction in which a student expressed a different understanding from the professor, one of the categories of significant events. These types of interactions are likely to be illuminating about positioning because they represent a tension between the professor and student. The student's understanding was accepted as correct over the professor's only three times (out of 24 total); I selected the first of these three to analyze because it happened earliest in the semester and thus had the most potential for novelty/tone setting.

To analyze this interaction from a positioning theory lens, I attempted to map each speech turn onto a set of possible storylines, positions, and speech acts, using discourse analytic tools to provide evidence for these. In particular, I printed out the transcript of this interaction and went through the interaction twice: once attending to voice, and once attending to temporal structure (as guided by Pimm's (2004) categories of features of discourse). Analysis of voice involved identifying patterns and shifts in personal pronoun usage (e.g., I, you, we), as well as the use of deictics (e.g., it, they, them). Analysis of temporal structure involved identifying patterns and shifts in tense usage (e.g., present, past, future) and the use of connectives (e.g., hence, therefore, but).

After analyzing these particular features of discourse, I then mapped each speech turn onto possible positioning triads of speech actions – positions – storylines (as in the example in Appendix A), using the discourse features as evidence for these interpretations. As I sketched out possible positioning triads, I attended in particular to the different storylines, and ended up with three storylines in particular that were relevant and offered distinct interpretations of how student(s) were positioned in relation to mathematics. These storylines became central to how I discussed my interpretations of positioning in this interaction, and are described in detail in Chapter 5. In writing up these interpretations, I also relied on evidence from my interview with the focal student in which I asked him about this particular event. After completing the write-up, I shared it with the focal student as a form of member checking; he expressed general agreement with the analysis and different interpretations offered of the interactions.

Stage 3: Lexical Bundle Analysis

The second approach to analyzing positioning in interactions was lexical bundle analysis, which I modeled closely after the work in Herbel-Eisenmann, Wagner, and Cortes (2010). This approach complements the first, in analyzing particular repeated four-word phrases to better understand how broader patterns of interactions can position students in relation to mathematics.

In order to identify commonly occurring four-word phrases, or lexical bundles, across the semester, I transcribed nine class sessions (25% of the 36 audio recordings). I divided the semester into three time periods of roughly equal length (four to five weeks of audio) using exams as the boundary points (as significant shifts in terms of both content and potential participation), and then selected three days to transcribe from each time period.

Using this set of nine transcripts, I used the “word combinations” feature in MAXQDA to identify the most frequently occurring four-word bundles. I decided that the bundles had to occur in at least five of the nine transcripts in order to be included in the analysis, to establish a certain consistency of use, and that the bundles had to appear at least 15 times (in the ~64,000 total words), in order to qualify as "frequently" used. (More details about these choices are described in Chapter 6).

Once the bundles were identified, I then proceeded to categorize them. I began by using categories already developed in similar studies of lexical bundles in classrooms (Biber, 2006; Herbel-Eisenmann et al., 2010; Herbel-Eisenmann & Wagner, 2010) but since most of the bundles were unique to this study, I ended up developing two new categories and refining two of the previously established ones, drawing again on tools from discourse analysis to help identify and describe similarities (and differences) in voice and temporal structure across the bundles. I then sorted these categories of bundles according to potential interpretations of them as positioning students through the three storylines identified earlier. The resulting chapter both describes the categories of bundles and offers interpretations of how they could be seen as positioning students in relation to mathematics.

Methodological Stance

My stance as a researcher has been influenced by a variety of methodological traditions. While not strictly adhering to one in particular, I provide here a brief account of the influence of two of these, discourse analysis and portraiture, on my orientation toward this research.

From the tradition of discourse analysis, I have taken a deep interest and concern about the nature and importance of words and language. In studying classroom interactions, I chose to attend to the use of words in particular, because they simultaneously illuminate and construct the perceived reality of an interaction. As Gee (2005) explains:

> Language has a magical property: when we speak or write, we design what we have to say to fit the situation in which we are communicating. But, at the same time, how we speak or write creates that very situation. It seems, then, that we fit our language to a situation that our language, in turn, helps to create in the first place. (p. 10)

In this way, it was evident to me that the words spoken in a classroom merit close attention because they tell us about the classroom as a particular space, with a particular history of interactions that constrains the possibilities for future interactions. Words do not need to be consciously deliberated on or intentionally chosen in order to do this social work.

> Sometimes participants accomplish action that they do not intend, and sometimes they are unaware of social actions that they demonstrably orient to but do not consciously understand. In many cases, discourse analysis reveals mechanisms of social action that participants use but do not consciously recognize. (Wortham & Reyes, 2015, p. 11)

In a quite different but equally important vein, from the tradition of portraiture I have taken a perspective of looking for the "good" in the data. My personal experiences in upper-level courses such as this one might suggest that I would be very focused on documenting and describing negative interactions in the course and places of failure. "But the relentless scrutiny of failure has many unfortunate and distorting results […] we begin to get a view of our social world that magnifies what is wrong and neglects evidence of promise and potential" (Lawrence-Lightfoot & Davis, 1997, p. 9). In other words, I have taken from portraiture an understanding that it is not beneficial to document only the shadows, the negative moments, the absence of healthy interactions that I might be inclined to expect. Instead, I intended to look for features of classroom interaction that are "good and healthy" and not to "discover the sources of failure" (p. 9).

In summary, drawing on the qualitative traditions of discourse analysis and portraiture, my methodological stance was one of attending to the power of words as conveyors and constructors of social realities, and looking at data with a critical but well-intentioned eye.

Conclusion

In spending one semester in a section of *Introduction to Analysis*, an upper-level undergraduate mathematics course, I took observational field notes, collected audio recordings of the lessons, and conducted interviews with students and the professor. These data were transformed through several stages of analysis into answers to my research questions. I developed a portrait of patterns of interaction over the semester (Chapter 4), in order to address the first research question. I developed interpretations of how a particular interaction (Chapter 5) and commonly used four-word phrases (Chapter 6) can be seen as positioning students in relation to mathematics, addressing the second and third research questions, respectively. In the end, these descriptions and interpretations allowed me to speculate about how classroom interactions can be viewed as positioning students in relation to mathematics (Chapter 7), and thereby address the overarching question of this study.

CHAPTER 4: PORTRAIT OF THE SEMESTER

The following portrait illustrates classroom interactions over the course of the fall semester, in order to address the first research question: What are patterns in the classroom interactions in an upper-level undergraduate mathematics class? Patterns in classroom interactions are described over the course of weeks of the semester; certain patterns are established, others change over time, and moments that break from patterns are noted as well. The following timeline (Figure 6) summarizes the themes for interactions across the weeks:

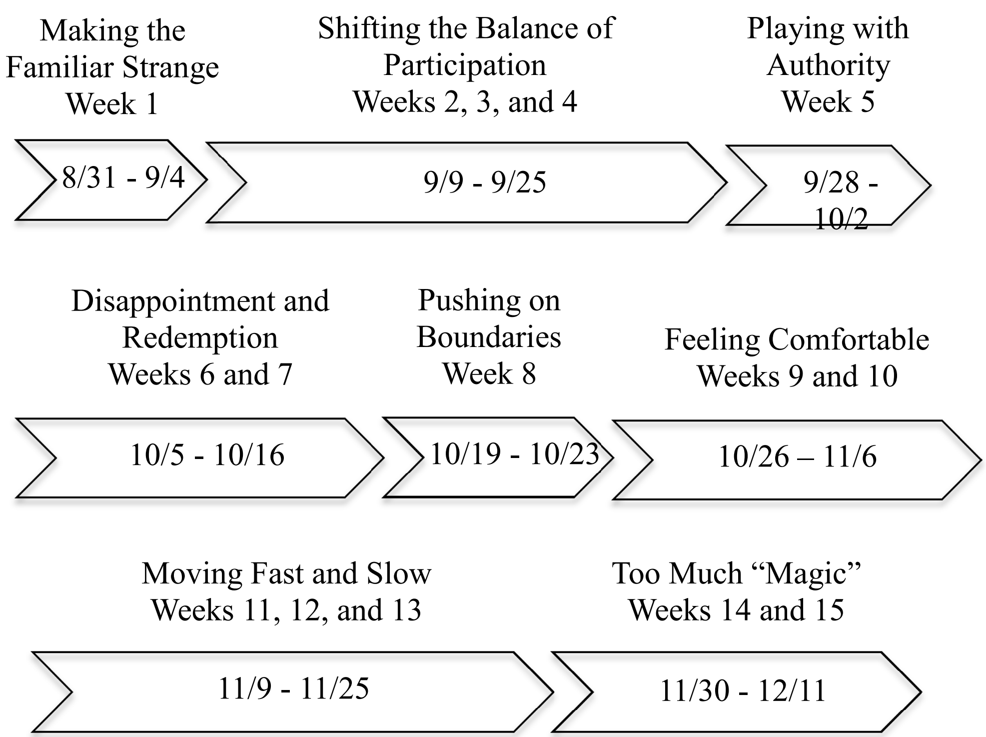

Figure 6. Themes in interactions across the weeks

The Portrait

The Mathematics building sits near the front of campus, on the edge of the maze of inter-connected Physics, Chemistry, and Engineering buildings. Stately columns and a semi-circular staircase frame the main entrance to the brick building, facing out to a fountain surrounded by benches. Walking through the tall doors leads to an open two-story dome, pristine and polished and empty except for a ring of chairs around the edge. A large staircase near the entrance leads up to research centers, professor and graduate student offices.

To get to the undergraduate classrooms, I enter a side entrance instead and head down a wide flight of stairs to the basement of the building. The hallway is white and bare except for an assortment of colorful photos of animals, landscapes, and some campus scenes with dates from the early 2000s and the professor's initials in the corners. There are backless wooden benches at various points along the hallway for students to sit on, as they wait for classes to begin and end. Walking by identical rooms, I look closely at the arrowed signs – 300-310 to the right, 200 and below to the left. Coming up I see the undergraduate math lounge with a large glass window that features a drawing of a duck and a math joke. As I walk by, I peek inside and see a few students working at the table, and one sleeping in an armchair; the door needs a four-digit code to get inside. Just past the lounge is a bulletin board advertising internship opportunities, math talks, and many offers of math tutoring services with phone number slips to tear off along the bottom.

I walk into the classroom, which matches all the ones I passed, except that it is wide rather than long. The front of the room has two long chalkboards and an empty table with a lectern underneath it; there is an overhead projector stuck in the far left corner. The room is tightly packed with forty-some small wooden desks in eight rows; although there are also two chalkboards on the side walls, the desks make them inaccessible. As I slip back to a seat in the back corner of the room, I step over students' bags in the narrow aisle, and hold my bag to my side to avoid bumping anyone. The fluorescent lighting hums quietly; the only window in the room is on the back wall and at feet-level, providing little natural light.

Students slowly and silently filter in over the next ten minutes before class begins; there are no greetings or conversations as most students look at their phones or take out their notebooks. The professor arrives a few minutes before nine and quietly says "Good morning," then distributes the one-page syllabus to each row. He is dressed nicely, wearing a blue plaid short-sleeve button up shirt tucked into khakis with a belt. While waiting a few more minutes, he stands quietly at the front of the room with a hand in his pocket, shifting his weight from foot to foot. The only noises in the room are the 27 students unzipping and re-zipping backpacks and bags to take out notebooks, and the creaks of the chairs as they shift in their seats.

"Good morning, and welcome to Intro to Analysis. Right?" With a few polite laughs, the class has begun. In the first five minutes the professor covers all the logistics of the course – from exams, homework, and office hours to the pronunciation of his name and adjusting to his handwriting and his accent. "But besides that, let's try to have fun. This is a fun class, it's challenging, but I think if you work hard, you will be able to enjoy it." And with that, we dive in to the content. "All right, so, ready?"

August: Making the Familiar Strange (Week 1)

The content of the first few days is called "Preliminaries," and consists of an overview of sets, functions, and the axioms for the real numbers. The language and notation are fairly technical; the point of these preliminaries is to establish and formalize a shared set of properties “that one has always assumed” (Fitzpatrick, 2006, p. 2). Much of the second day is taken up with the professor writing the following axioms on the chalkboard and providing brief explanations of them (Figure 7). Preliminaries like these are presented at the start of most upper-level math courses; I find them almost soothing in their familiarity, although I remember feeling bewildered when I first encountered them, and later bored when every course spent more than one day introducing them as though they were brand new.

The Field Axioms

Commutativity of Addition: For all real numbers a and b,

$$a + b = b + a.$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

The Additive Identity: There is a real number, denoted by 0, such that

$$0 + a = a + 0 = a \qquad \text{for all real numbers } a.$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0.$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba.$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc).$$

The Multiplicative Identity: There is a real number, denoted by 1, such that

$$1a = a1 = a \qquad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

The Nontriviality Assumption:

$$1 \neq 0.$$

The Field Axioms are simply a record of the properties that one has always assumed about the addition and multiplication of real numbers.

Figure 7. The field axioms

Students are not just expected to recognize these properties as familiar, but also to be able to use and apply them as tools in doing proofs. On the second day, after reviewing all the above axioms, the professor writes up two problems that he tells students to try to prove on their own outside of class. The first is that any number multiplied by zero is equal to zero; the second is that if two numbers multiply to equal zero then one of them must be zero. After writing them up on the chalkboard, the professor pauses and re-considers having students prove them on their own, commenting, “Maybe we should try to do this quickly, to sort of see, I mean these are simp- trivial, but how do you use the rules that we just did to prove these two statements? Let’s try the first one [...] So, can somebody help me do this one?” After a long fifteen seconds of silence, a student in the front corner, Oliver [4], throws out “Zero is *a* minus *a*, we could start with that. Maybe?” The professor responds, “So let’s try it,” and after a few more exchanges with Oliver, the professor then writes up the complete proof using a slightly different approach. This writing up of the proof takes about three minutes, after which the professor asks “Is that clear for everyone?” and transitions into the next topic.

These are the patterns of the class at the beginning – the expectation that students can apply the given tools to complete a proof, the professor’s invitation asking for students to help him with a proof, the taking up of a student idea while also adjusting it, and the general silence from most students in the class. Within that silence flows an undercurrent of unspoken student questions and confusion about the strangeness of both the goal – “why do you have to prove [that zero times a number is zero], that's ridiculous” – and the process of completing this proof – “I didn't really get the proof at all […] that was the point where I was just like ooh proofs. I'm lost still” (Anne – September interview). The professor takes time over these first few weeks to continue to establish these expectations that students should become comfortable doing proofs, no matter how familiar or obvious-seeming the idea, and

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[4] All student names are pseudonyms.

doing so publicly; different students are more and less comfortable adapting to these expectations.

About halfway into the third day of class, the professor is working to establish the Completeness Axiom, which is stated in the textbook as follows: "Suppose that S is a nonempty set of real numbers that is bounded above. Then, among the set of upper bounds for S there is a smallest, or least, upper bound" (Fitzpatrick, 2006, p. 8). This axiom is quite different from the familiar properties of the field axioms; as the textbook explains, "At first glance, it is not at all apparent that the Completeness Axiom will help our development of mathematical analysis" (p. 9). In order to illustrate that this axiom is in fact meaningful and useful, the professor has written up an application problem, asking students to prove that the set of real numbers $\{s^2 < 2\}$ has a least upper bound. The square root of 2 is the clear candidate for the least upper bound, as it is the smallest you can get without being in the set. But the professor then makes the following point, tapping on the front table for emphasis: "Now remember, so far I have not defined what the square root is, so you cannot use anything about square root to prove this problem." But this feels confusing; if the clear answer is square root of 2, how can you avoid using anything about square roots? I note my confusion in my field notes, wondering if students are similarly confused, particularly given that this commentary about not using square roots is not discussed in the textbook's treatment of this problem at all.

The professor then opens up the problem to the class: "So how do we prove this? How can we find an upper bound for this set? *Just* an upper bound for this set." Nico, a soft-spoken student with glasses sitting a few desks in front of me, offers an idea phrased as a question: "Do we just um find a number greater than 0 that does not belong in S? So, for example, 2?" The professor takes a beat before responding; he agrees that 2 is an upper bound, but reiterates the issue of how to prove that's true if they're not allowed to take the square root.

Now Ryan jumps in, sitting two rows directly to the left of Nico. He is bald, bearded, and tall, with a deep voice; he seems older than the typical undergraduate although he is not. Ryan suggests: "You could just square 2, and say, 2^2 is 4, which is greater than 2, so it must be an upper bound for S." The professor starts to write up this idea on the board, pauses to ask for Ryan's name, and then summarizes his contribution: "Ryan suggests to sort of say that [s^2] is less than 2^2 which is 4. Right?" This is the first time the professor has asked for a student's name; he will continue to do so over the next few weeks, in order to attribute ideas to students as well as to call on students to participate.

Damien and another student begin to speak at the same time, both of the opinion that Ryan's suggestion should complete the proof. Damien is already established as an active participant in the class by this third day; he is well dressed, bearded, and 27 years old with a fairly thick French accent. When the professor pushes back that the proof is not quite done because they do not yet have an upper bound, Damien interrupts, "- And by definition," and the professor interrupts right back, "- What's the definition?" This point about not having defined the square root yet, and therefore not being able to use it, is clearly as counterintuitive for the students as it is for me; the professor knows it and pushes that point. Speaking quickly and animatedly, he asks: "You guys with me? Here I have an upper bound of

s^2, but not on s itself. If I could take square root then I'll be done, but I'm not allowed to take square root yet, so what do I do?" The silent pause, although only a few seconds long, suggests the students have no more ideas to offer, and the professor finishes the proof himself. The students attend very closely, peering towards the far right corner board where the professor is working. It is perhaps the most attentive they have been as a whole class so far; I wonder if it is because it is the first time the professor is writing up steps that are not in the textbook. Damien quietly talks through a few steps at the same time as the professor writes them on the board; other students are nodding their heads as they follow along. The professor finishes and asks, "Is that clear?" and Damien gives him an affirmative "Mhm." The class continues, and the professor goes on to formally define the square root, and finish this introductory section of the textbook.

In this proof, the familiar square root was undefined and "not allowed" to be used, while the strangely-phrased Completeness Axiom and idea of a least upper bound were indispensable as the tools that establish the existence of square roots. In many ways, the professor's goal in this proofs-based course is to make the familiar strange and the strange familiar, especially in these first few weeks.

September: Shifting the Balance of Participation (Weeks 2, 3, and 4)

At the start of the second week, the professor introduces the Archimedean Property, which states that given any positive natural number, there exists an integer that is larger than it. This property hardly feels necessary to prove; it is the intuition behind the childhood game of "I can count to a hundred" "Oh yeah? Well, I can count to a hundred and one!" And yet it is not part of the set of axioms, and can be proved from them. Its simplicity affords the professor the opportunity to discuss some approaches to proofs, such as how to prove two statements are equivalent, and how to prove a statement by contradiction. If you assume the opposite of the statement is true, that would mean there is a largest natural number, or an upper bound on the set of natural numbers. The proof by contradiction, as the professor explains, "means that you have to at some point get something that's sort of contradicting the fact that you've already established. So where do we get the contradiction from?"

After a six second pause, Arielle, a petite student who wants to be a high school math teacher, offers her idea of the contradiction: "Saying that N is bounded above, because the natural numbers aren't bounded above." The professor points out that, while true, they haven't yet proved that the natural numbers aren't bounded, which makes Arielle blush and wryly say, "Just kidding." The professor responds in a comforting tone that proving this Archimedean property will also prove that the natural numbers aren't bounded above; the two ideas go hand-in-hand. The proof then continues on, with several students offering ideas, one of which the professor builds on to complete the proof. There is a feeling of excitement and energy in the offering of these ideas, of genuine student participation and contribution, on the part of both the students and the professor. Indeed, 11 of the 24 students in attendance participate in a substantial mathematical interaction during this class.

In the middle of writing up these proof steps, the professor pauses to ask Arielle, "By the way, what's your name?" which feels out of the blue and random. A few minutes later, though, after completing the proof, the reason becomes clear; he uses her name to return to her earlier comment: "As Arielle mentioned, now we have established that the natural numbers don't have a least upper bound, right? Because of this [Archimedean property]. Given any large number, you can always find an integer larger than it. Given another large number, you can always find an integer larger than it. So the natural numbers cannot have a least upper bound, it's not bounded above. Make sense to everyone?"

Although we see the professor striving to be positive and encouraging of Arielle's participation, this interaction is the only time she voluntarily participates in class. For the remainder of the semester, Arielle only participates when called on, and then only to ask questions. As she explains in her interview after the semester is over: "I don't participate because I'm not confident in this class and like I don't want to say something and just be wrong and it's just like, oh. Like I- especially if it's something that's so simple that like I should definitely know. So I just try to avoid it at all costs" (December interview). The consequences of being publicly wrong, no matter how it is received by the professor, seem very significant; it is as though the weight of all previous bad experiences in math class sits above the balance of participation, ready to drop down and add its weight to what may be a small moment, tipping the balance towards discouragement and ensuring an end to future risk-taking.

The professor's invitations to the students to contribute to proofs continue over these weeks; he asks for students' names, attributes ideas and proofs to them, and at the start of the third week of class he begins to cold-call on students who have not yet participated, saying, "Okay so there are a couple of people who have never spoken in class, so what I'll do is I'll go row by row and choose someone who has never spoken." Although this prospect is alarming and frightening to many of those quiet students, it has the potential to be a proud and validating experience when it goes well. “I'd been thinking oh I shouldn't talk in this class, I don't know what I'm doing (Laughing) But then, he called on me the first time and it was right, and it might not've been totally right, like there was more to go with it, but still super cool” (Anne, September interview). It is also worth noting that the professor learning students’ names and using them in class is not typical, even for a small upper-level math course. As one student commented with a laugh: “It's also neat that he actually knows all of our names, cause most professors don't” (Nate, November interview). By the end of the third week, two-thirds of the 27 students have participated in at least one substantive mathematical interaction.

The fourth week interrupts these nascent routines because the professor goes out of town. Each day of this week is a different substitute, a professor and then two graduate students. There is almost no student participation on these days; on Friday only one student speaks (to suggest a notation correction). This contrast highlights for me just how novel and tenuous this thread of student participation is for an upper-level mathematics course; something different is coming to exist within this class.

Into October: Playing with Authority (Week 5)

The professor begins the fifth week by going over some logistics—the first exam has been postponed a week, to give them time to finish the chapter on continuity—while completing his daily routine of setting up the chalkboard. First he takes chalk from the holder next to the board and distributes pieces along the two boards at the front of the room. “So more or less we'll have like a quiz every week from now on except the week during which we have exam, okay?” His back to the students, he then draws a vertical line down the middle of each of the two boards, creating four areas for writing. “I mean I could also change my mind and give a quiz during exam week, so, it depends, if you guys want me to, I can- I can always do the quiz every day if you want to.” A student snorts, the professor now turns back to face the students, a smile on his face and in his voice, as he rustles through his notes on the front table. “Would that be okay? Doing a quiz every lecture? During every lecture? Five-minute quiz if you want to, I mean we can play with that if you want.” Students shake their heads, one or two say no. “Nah? Okay. Usually I take the opposite of what you say for- for my answer, so if you say no, I'll say yes.” With barely a pause after this disorienting statement, the professor turns back to the board, “Okay 3.3, the Intermediate Value Theorem,” and begins writing.

This is not the first time the professor has joked about when to give quizzes or how he will interpret the opposite of what students say, nor will it be the last. “Yeah he does that a lot. Like, oh if you guys say no then I'm gonna go with yes. And I'm like okay, like I don't know if I should just say yes from here on out or? I like never know what to do” (Arielle, October interview). It feel as though he is attempting to play with his authority, perhaps to soften it, to acknowledge that it feels silly to him as well, to establish camaraderie with the students around being trapped in this system of grading. But the joke is hard to sell because the fact remains that he does have that authority and can in fact decide to give a quiz whenever he likes. Offering the students a voice and then flipping the interpretation to be the opposite of what they say feels more like a betrayal than if he openly admitted that his is the only voice that counts.

During the next class, the professor is much more successful in playing with his authority, though admittedly around the lower-stakes topic of class participation rather than quizzes and grading. He has introduced the idea of uniform continuity, and is now presenting a theorem on its relationship to the more familiar notion of continuity. Having written up the theorem, the professor steps away from the board, seeming also to step away from the theorem and closer to the students. "So we've been at this thing for maybe three or four weeks now, so at this point my hope is that you start sort of guessing how we start, at least how we get started on the proof of a statement you have no idea about." He explicitly acknowledges the strangeness and the discomfort of what he is pushing students to do: "This is- this is different from the homework, I want to take you on the spot, and try to see if you can help me- guide me through what are the different steps in trying to establish this theorem." In a quiet, slower voice he invites them to develop their intuitions, to try out being a mathematician with him. The incongruity of asking them to step into what is typically seen as the professor's role is not lost on him, "So I don't know what we need to use yet- well okay I do," or on the students, who respond with genuine laughter. He is asking both him and them to pretend that he does not know what he knows, a vulnerable place for both parties to be in. "But if I were you in your position, what rings a bell? What- what is it that I- I need to- I have to use?" And this vulnerable place is one that is hard to stay in, as with a moment's pause, he then pushes them toward the bigger picture he knows is there. "There is a word that we used here," he says as he underlines the words "closed" and "bounded" in the statement of the theorem on the board. “Do you remember anything about closed bounded intervals before?”

The work on this proof takes them through the next 25 minutes of the 50-minute class; the professor continues to emphasize the importance of developing their intuitions and their feeling for the proof. Although several students offer ideas, none come up with the key approach of proof by contradiction. This opening speech is the most explicit account the professor will offer over the semester of his expectations for students learning to do proofs, and his rationale for asking them to participate in proofs during class.

These two instances of the professor joking with the students about daily quizzes and encouraging them to prove a theorem are similar; the professor invites students to contribute ideas as though they had equal authority with him. These attempts to play with his institutional role reveal there are some constraints that cannot be escaped (grades; department guidelines), while others perhaps can be shifted (classroom norms for participation).

October: Disappointment and Redemption (Weeks 6 and 7)

Coming up to the first exam at the end of week six, then, the professor has laid out expectations, challenged students to meet expectations, played with logistical expectations, and invited students to participate in this strange process of proof. The expectations for the exam are the highest stakes, and also the least clear for students. The Monday of exam week, the professor asks students if they’re ready, and few of them respond. He makes a joke about giving the exam that day, which is received with anxious laughter and a comment of “That’s not funny.” Arielle has raised her hand and when called on, says in a nervous tone, “I just don't know what to expect on it. Like I feel like you could literally ask anything.” The professor explains that they will have a review day on Wednesday, and that he’ll be available for extra office hours throughout the week. But in our interview on Wednesday after the review, Arielle reiterates her concern: “it's very nerve-wracking when it comes time for exams, because I don't know what he can ask” (October interview). She was not the only one; when I asked Jamie how she felt before the exam, she said “scared […] definitely scared,” and explained, “I don't really know how to study for it” (October interview).

The exam happens on Friday. The following Wednesday, the professor walks in carrying a white canvas bag on his shoulder and silently takes it off, takes the graded exams out of the bag, and begins walking around the room, returning them to individual students. Except for the professor occasionally calling out a student’s name to give them back their exam, the room is silent. Students are bent over their desks, scanning their graded blue books as soon as they are returned. Arielle looks straight down and begins rubbing her eyes. Griffin, wearing his over-sized hoodie and slip-on sandals as always, exclaims “What!” under his breath as he pushes his tousled hair off his forehead. Jamie, a blonde-haired preppy student, frowns at her friend Michael, whose hands are resting on the brim of his backwards baseball cap with his elbows out and his head tilted toward the ceiling. The process of returning exams takes about ten minutes; although the room is usually quiet, today distress and frustration feel palpable in the silence. The professor finally breaks in: “All right, um, so I sent you the statistics. Um, I don't- I don't think it looks good.” At the time I do not know the statistics; I find out after class how poorly the first exam went for most students – the median score was 24 out of 50 and the mean was 26.

The professor gives a 10-minute speech reviewing the five problems on the exam and the students' overall issues for each. Speaking quickly, and going to the board to illustrate his point several times, the professor explains, "But uh I mean uh I was a little bit disappointed because there were a few problems that I felt like we've actually done essentially in class." The speech feels like a combination of a review of common mistakes, a re-stating of expectations of how to do proofs, and a defense of the exam he wrote as being appropriate. His rapid speech dramatically slows as he gets to the fifth and final problem, and with halting pauses and a sad tone in his voice he says, "And uh, and number five, um. Number five, um, is not straightforward, that's- you have to think a little bit on number five." But then he picks up the pace and tone again, "But it's not that bad either…" and launches into an outline of the problem, that lasts for eight of these ten minutes. By the time he gets to this fifth problem, it almost feels as though he cannot help but spend time explaining it in detail, trying to get the students to see his logic and follow along with him.

The speech ends with the professor reminding students to come to his office hours, "if I can help I'll just be happy to talk to you." He is concerned they're not taking advantage of all the resources available to them, relying instead on their good grades on the homework (graded by a TA). "You're doing well on the homework, but the first exam, I wasn't too happy about," he concludes, which seems to me to match with and explain the students' obvious unhappiness as well. Both professor and students seem to feel the weight of this exam result, in the practical effect it has on their grades and chances to pass the course, and in the relational effect this disappointment has on the classroom culture. Indeed, the professor keeps trying to move past the exam speech but then returning to the subject to provide one more comment—his tests are conceptual, not calculation-based like previous calculus classes. When he finally does move to new content, only three students are willing participants –Nico, Patrick, and Damien – all of whom are consistent in participating almost every class.

Amidst all this disappointment, however, there is a glimmer of something different. Before launching into his speech about the exam problems, the professor made an offer that I'd never heard before in an upper-level math class: "I feel like maybe I should give you a chance to get some points back on some of the problems." Together with the students, he sorts out the logistical details for how to make up a maximum of ten points by re-doing their choice of two of the five problems. This opportunity was unexpected to all students interviewed afterwards, and independently referred to as "redemption" by two students. A third student explained, "it sort of like gave us another chance to like show what we actually know and what we can- what we're capable of" (Rohit, November interview).

On the whole, students had mixed responses to the first exam, ranging from agreement with the professor that the exam was fair and feeling that they had let him down to serious qualms about the integrity and appropriateness of both the exam and the speech. Many students said it helped them to buckle down and apply the effort the class clearly needed: to read the relevant textbook section before coming to class, to work on homework with others, to go to office hours more regularly. And all of them improved their scores with the make-up points, although not always relative to the overall curve; the class median improved from a 24 to a 37[5].

The weeks following the first exam are some of the most comfortable of the semester, in terms of the overall atmosphere in the class, the familiarity of the material on derivatives, and the kinds of student participation that arise. The expectations have been set and the norms established, the stakes are high but the first exam is over, and there is time and space for students to play with ideas and even to challenge the professor. In the beginning part of the semester, the professor seemed to be coaxing the students into participating, inviting them to try to think like a mathematician. In this middle period, it feels as though a few students are willing to take him up on these offers, and push on the boundaries of the content as well as the classroom norms.

October: Pushing on Boundaries (Week 8)

During the eighth week, the content covered includes the chain rule, a familiar property from earlier calculus courses for computing the derivative of the composition of two functions. A classic application of this theorem is to calculate the derivative of x^r where r is a rational number. Having sketched out the idea, the professor doesn't write up all the steps of this proof, instead saying, "I'm not going to finish this proof, I hope it's clear for everyone. Okay, can I skip it?" He gets one sentence into the next topic when Oliver, a student who often comes late and rarely

[5] This change may also have been impacted by two students dropping the course at this point; assuming they had low scores on the exam, then their dropping would also improve the median.

takes notes, interrupts him to ask a question: “Can we show this for irrationals as well? Where *r* is irrational?”

The professor pauses and turns around to face Oliver, in the front row on the right half of the room; this is an interruption from normally scheduled programming. “That's a very good point. So. Very good point.” Smiling, he then turns the question back to Oliver, asking him if it’s possible. Oliver slouches back in his chair and mutters, “Maybe if you’re smart enough…I don’t know, maybe if you think about it long enough.” The professor then launches into an extended conversation about Oliver’s question, taking up ten minutes to discuss how irrational numbers are defined, using pi as a specific example.

At various points in the conversation students contribute tentative ideas of how to approach the question of defining pi or x^r, but make little progress; there is a sense that this question is truly open-ended and unknown for the whole class. Indeed, the final take-away from the professor is that their typical strategies from earlier in the class cannot be applied here, that this situation requires something novel. Silently he writes up $x^\pi = e^{\pi * ln(x)}$ and then rhetorically asks if it is true, which gets a few unintended laughs from students. He summarizes, “So when you have irrational number, you have to go to the definition of exponential [e^x]. And technically we have not introduced the notion of exponential function.”

Oliver’s question is clearly an unusual one, pushing the boundaries of the content beyond what has been introduced in class or the textbook so far. It has a very different flavor to it, matching the feel of many of the questions the professor asks to motivate the introduction of new content, rather than a more typical student question that is intended to master the content already presented. Oliver's question also stood out strongly to at least a few students. Nico explained, "I was like oh that's- like a clever question. How come I couldn't think about that?" (November interview), while Sara said, "I would never even think of something like that" (December interview). The comparative language used by both Nico, a consistent participant who did well in the class, and Sara, a low confidence student who seemed to struggle, is especially interesting to me. While Oliver's question is exciting in terms of pushing the boundaries of the content, it seems to also push the boundaries of student participation and make some students feel inadequate or less than.

The following day, another boundary-pushing question arises. Toward the end of class, Patrick asks a seemingly innocuous question about whether they used the lemma the professor just wrote up to prove Rolle's Theorem at the start of class. The professor responds immediately and uncharacteristically tersely, "Absolutely not. We never used it." His response is somewhat jarring; usually when a student isn't sure and asks a question, the professor is patient and encouraging. But in fact this question represents a significant challenge to the mathematics at hand, because it could imply circular reasoning.

Also surprising, Patrick persists with his question, despite the professor's response. The professor interrupts him, sketching up a graph on the blackboard on the far left side of the room that has never been used before, close to Patrick's seat in the front row. Patrick jumps in again to point out exactly where he sees the lemma being used in the line of reasoning, and the professor is genuinely surprised, "Oh! I- I sort of … I see." Patrick drives his point home, "And you used this [lemma]. Didn't you?" and the professor gracefully concedes, "Okay I see what you're saying now, yes I used it. But, I can prove it!" There is genuine laughter around the room, including from the professor and Patrick, as the professor thanks Patrick. Patrick jokes with a bright red face, "Just trying to hold you accountable," to which the professor responds, "Absolutely. I like it." He then continues on to clarify how Patrick's point is technically accurate but that the larger concern of circular reasoning is avoidable.

This challenge to the professor, especially with Patrick's persistence after being flatly contradicted, is a particularly interesting incident. As Arielle explained when I asked her about this moment, "every time one of the students questions the professor I'm always like, oh my gosh, that's something like I would never do," explaining further that "you're always taught like don't question authority" (November interview). But at the same time, Patrick's challenge was appreciated by students, in contrast with Oliver's question from the day before. Patrick's question seemed "productive" and not a tangent into unfamiliar content (Arielle – November interview). And Patrick's ideas are generally heard as clear and accessible, rather than "taking everything to the highest level" like Oliver does (Jamie – November interview). These two interactions around unexpected questions provide a sense of the comfort of at least some students to push boundaries with the professor and the material at this point in the semester, but also of the potential for discomfort it creates for other students.

Into November: Feeling Comfortable (Weeks 9 and 10)

At the end of week nine, the professor walks in a few minutes late and explains that he did not have time to write a quiz before class so they're off the hook. Luke, a student who often struggles in class, asks him to say it again, exclaiming happily "Are you serious!" to general laughter from the class; the professor jokes that maybe students came to his before-class office hours on purpose to keep him from writing a quiz. He then writes up a problem that he gave to them at the end of last class, and explains that he wants everyone to at least be able to say the main idea they need to use to prove it. After a 20-second pause a few students start muttering, and Michael, who rarely contributes, offers the Mean Value Theorem. "Does everyone thinks the same thing?" the professor asks, and then moves across the room from left to right, pointing to students and asking what they used, getting a few different responses but mostly the Mean Value Theorem. He jokes, "Math is not a democracy, but I'll go with the majority now," and finishes the problem with some suggestions from students along the way. He then spends about five minutes on the new content for the day, a section in the textbook discussing Leibniz notation.

After the last class, the professor posted a set of "extra" problems on the course website, which he asked students to work on but which are separate from the homework and won't be turned in or graded. This is a fairly unusual practice, and it's even more surprising to me that he is willing to spend class time reviewing the problems rather than covering new material. "So hopefully you guys have thought about each one of them. So maybe I can get somebody to come to the board to do the problem instead of- but I want volunteers, or I'll choose volunteers." This suggestion of a student coming to the board is one he has made before, but it has never happened; then, as now, the professor does not follow up on the idea. Instead he borrows a paper with the problems from Anne in the front row, and moves directly into writing up a problem on the board. He asks if everyone had a chance to try the problems, and to raise their hands if they haven't tried it, which gets some laughs as only Damien and Becky raise their hands. "Is that a trick?" he asks, and he calls on Becky, who has never spoken all semester; she blushes and explains that no, she actually didn't see the problems. He checks in with Nate, who says he tried it, and then with Luke, who gives no audible response, and then asks the whole class, "Because you tried it, can we give it as a quiz?" which brings the room back to silence. After a beat, he decides, "Let's not do that because it's going to be unfair to people who haven't tried" and then returns to his typical questioning pattern, "So what's the key word here again?"

The problem proceeds fairly normally, with Damien and Nico contributing ideas and answering small questions about steps in the proof. The professor then adds two new questions onto this problem and gives the students about four minutes to work on it; most are diligently and silently writing in their notebooks. Griffin, who never takes notes or has a notebook out, is sitting in the front row and appearing to stare into space; the professor asks him if he wants to help with the problem. Griffin responds slowly, "I don't…I don't think you can do it," which surprises the professor, so Griffin repeats it, still somewhat hesitantly. The professor does not directly disagree, instead saying, "Maybe…because, I thought you could do it. Then try to convince me that you cannot." So Griffin reaches into his bag to pull out a notebook and sketch a counterexample. While he does that, Anne jumps in and explains how she started the problem; the professor uses her idea to more formally write up the first problem, then turns back to Griffin and asks if he can see now that he's wrong. Griffin flatly says no, and the professor laughs, "Oh, that will be challenging for me then, okay." At this point several other students start jumping in to challenge the professor, going back and forth with one another and with the professor. The professor sketches up a graph on the front board, and in the subsequent discussion the students are finally able to convince him that the problem is not possible as written and needs another condition. There is a general feeling of amusement in the air; the professor acknowledges that they "got him on that one," and then jokes that this is how professors come up with problems.

Similarly to the earlier interaction when Patrick challenged the professor and was right, the students seem to have a positive takeaway from this interaction. As Nate explained with a laugh, "It's better when he's wrong…cause people are more confident in pointing it out," adding, "When he's wrong then it's like, I don't know, more exciting somehow. More people were trying to fill in the steps here I guess … It was sort of exciting when you know you have somebody who's like, he's really good at this stuff and messes up" (November interview).

This day encapsulates many of the expectations from the professor, even the ones that are not realized – that the students are trying problems on their own just to learn, that they would be willing to come to the board and explain their thoughts, that they can solve new problems in class on the spot. It also provides a sense of the students' comfort with the material, the professor, and one another at this point in the semester. This day has the most participation from students of any in the course, with 13 of the 23 students present that day offering meaningful mathematical contributions, and with students responding directly to one another's ideas, which only happened four other times in the whole semester.

Although their nervousness certainly builds up again the following week, the tenth week of the semester and the week of the second exam, there is still an overall sense in the class atmosphere that students are not as overwhelmed and perhaps even having some fun. Worth considering, though, is the possibility that this atmosphere of excitement and comfort can be excluding in its own way, and possibly even more so than an atmosphere of shared silence and dread, because the students who still feel uncomfortable or confused may feel all the more like they don't belong.

November: Moving Fast and Slow (Weeks 11, 12, and 13)

The Monday after the second exam is the final drop date for students; two students dropped after the first exam, and three more students withdraw now, leaving 22 students in the class. The statistics for the second exam are better than the first but not great: the median is 33 (out of 50), the maximum is 50, and the minimum is 19. The professor waits until the end of class that Monday to return the exams, and then talks for two minutes about whether or not students should consider dropping the course. "I said at the beginning of the semester that I don't curve. And that I curve down, so I'll bring your grade down if you do too well." The students look up from reviewing their graded blue books with confused, anxious expressions. "Okay. I'm just kidding, sorry," he says with a laugh, and then goes on to explain: "what I mean by I curve down is- I'll probably bring down the cut-offs to assign the letter grade."

The last time he taught the class a 55% became a C minus; without committing to that curve, he suggests they can use that benchmark as an indicator of whether or not to drop the course today.

He treats the topic of grading the students here, as earlier in the class, with a joking attitude that seems to bewilder the students; his efforts to bridge the distance of this power disparity seem only to exacerbate it. He wants the students to trust him, to trust the relationship they have built over the semester, that of course he would not lower their grades. What he may not recognize is that from other professors that statement would not be completely ridiculous, and that curving in general is an unclear and frightening tool. From my experience in similar courses, it is common for professors to say at the start of the semester that they don't curve, even though at the end of the semester almost all of them do. Particularly in upper-level courses where there are not common exams or common graders across different sections of the course, curves seem to occur in completely idiosyncratic and opaque ways.

After this second exam, the semester begins to feel like a blur. The content coverage significantly speeds up; the first ten weeks of the semester were spent on the first four chapters of the textbook, while these last five weeks are spent on the next four chapters. The Friday after the second exam is the first day on integration, a new chapter but familiar territory. The professor has introduced the notion of the lower and upper Darboux sums, both with formal definitions and informal sketches; these sums are essentially dividing the area under a curve into shorter and taller rectangles and adding up those areas to put bounds on the area under the curve. He then writes the first lemma for the chapter on the board, commenting as he writes, "So these are easy statement, and uh, I'm not going to prove them, you guys will tell me how to prove them very quickly." This is the start of a trend in these later weeks of the semester, of the professor skipping proofs and asking students to work at an increased tempo. But when he finishes writing up the statement and asks, "Can we all prove this statement easily, in one or two lines?" the class sits in silence for a lengthy 18 seconds. In a more worried tone, he continues, "So these are a few statements that I-I do want to make sure that everybody can actually sort of try to- try to prove." The professor then suggests that each student write down their proof in one or two lines and then turn to talk to a neighbor and compare ideas. This approach takes me entirely by surprise; I have never experienced it in an upper-level mathematics course before.

After five seconds of silence, Nico is the first to embrace the idea, turning to Griffin on his left and saying cheerily, "Hi neighbor!" Other students need more explicit encouragement; "Come on, come on" the professor says as he walks around the room and directs specific students to work together and talk to one another. Slowly the room fills with pockets of conversation, while the professor circulates and asks students if they've got it. All the students he asks say yes, except Arielle who says she's stuck. She has not been talking with anyone, so he directs her to talk to Sara in front of her or Griffin to her right, and then moves to check in with another student.

This episode is brief, less than three minutes long, at which point the professor summarizes the lemma again and then asks them if they've got it. He says that whoever hasn't gotten it will come to the board and asks for a show of hands of who didn't get it, which generates the silence typical to these comments. Since no one volunteers, he begins calling on students to come help him. First he calls on Oliver, who is in the front right corner eating a sandwich. "You came late, so should I choose you? You're still eating though, so you cannot talk." Then he asks Arielle if she will help him, but she vehemently shakes her head no, and there is general laughter. "Who wants to help me?" He calls on a third student, who sits in silence for some seconds, until Patrick jumps in with what he sees as the key idea. The professor wants them to notice something else first; Ryan starts to offer an idea but the professor continues his explanation. Over the next several minutes, the professor summarizes the proof and writes it up himself. He does return to Arielle and lead her through a few yes-or-no, what's the definition questions that she hesitantly and very quietly answers.

This interaction of turning and talking to a partner, while surprising, is generally well received by students as a new kind of interaction. They seem to appreciate the opportunity to learn from one another; "it was probably good to get us working with other people that have the exact same prior knowledge that we do […] if our partner gets it and we don't, we're going to learn something" (Nate, November interview). And the lower stakes of talking to a partner are also helpful; "It was nicer to like talk with a student than like having him just call on you, that's like terrifying for me" (Rohit, November interview). But there are also issues that arise: Arielle had no one to work with and felt stuck; Jamie did not understand her partner's idea and ended up feeling more confused (November interview); Rohit and his partner had trouble articulating their thinking: "Like that was literally what we said, like okay, yeah, okay" (November interview). As Nate summarized, "I'm a bit anti-social and didn't know her [his partner], so that was slightly awkward" (November interview). Learning how to talk to one another is a skill that is not developed much in this course; overcoming the awkwardness would seem possible if this interaction happened more often rather than just this once.

This surprising move by the professor offers a moment of slowing down, a moment of something quite different in way of student interaction, before returning to the fast pace and the standard space of professor-led lecture. After this day, student participation drops precipitously, as the professor continues to work through the material in fast-moving lectures, skipping over proofs and telling them to follow in the textbook. His goal is clear; he wants to get through the chapter on integration before Thanksgiving, so that students have the break to study for the third exam.

There is one other surprising moment that interrupts this breakneck pace; the Monday before Thanksgiving break a student does come to the board for the first and only time in the semester. The professor has written a theorem about integration on the board, and asks students how they can use the lemma from last class to prove it. Getting no response, he returns to his typical question about what key fact needs to be established, in this case in order to prove that this function is integrable. James, an athletic student with a buzz-cut and a quiet voice, is sitting in the front row on the left side of the room as usual, and after a brief pause he begins offering an explanation involving the upper and lower Darboux sums. Although correct, he is stumbling through the words, and he interrupts his own explanation, blushing and saying, "I can't say it very well." The professor responds, half disappointed but also half teasing, “Oh, well. Then we have a problem. You have to be able to say it very well.” After a moment’s pause, James smiles and gives a little shrug, offering, “I could write it.” The professor double-checks, “You can write it?” and then hands James a piece of chalk, “Okay why don’t you help me out?” James goes to the board and silently writes, in a very small script, the definition he was trying to say out loud, against a backdrop of coughs and sniffles from the professor and students. After 30 seconds of James writing, the professor jumps in, “That’s all right, yup, that’s exactly what we need to do!” He then continues the proof, as James sits down, adding some writing before and after what James has written up, as though he is giving James’s work the final stamp of approval.

The entire interaction only lasts about two minutes, and then class returns to business as usual. None of the students seem to react or take it with surprise—there are no whispers or changes in expressions or posture—although I think it is pretty exciting. I had never before seen a student write on the chalkboard in an instructor-led college mathematics class (only in TA sections). When I checked students’ reactions after the fact in interviews, none of them found it especially memorable, although they all agreed that it never happens in their other upper-level classes. They remembered rooting for James to do well at the board because it’s intimidating, and thinking it was “kinda cool” but in character for the professor to give him the opportunity to “redeem” himself (Nate, December interview). As Michael summarized, “it was like oh that's kinda interesting, but that seems like kinda how the professor is” (December interview).

This moment feels like a confirmation, then, of what the professor has been promising—that he really wants to and will give students the chalk and transfer some of that power into their hands. It is not a regular occurrence or an expected responsibility of the students; it requires a special combination of circumstances such as the particular student being willing, the particular content being familiar rather than new, possibly even the particular fact that James is sitting in the front row so that the physical distance between him and the professor and the board is smaller. The fact that it does happen is an important instantiation of the professor's overall philosophy of wanting students to take ownership of the ideas; these are not empty words. Both James writing on the board and the previous interaction of having students turn and talk to one another are confirmations of these intentions. Yet each interaction only happens once; they are moments of ebb and slowness amid the fast-rushing tide of the end of the semester, which pushes the professor and students relentlessly onward.

December: Too Much 'Magic' (Weeks 14 and 15)

After Thanksgiving break there are only six more classes, including the third exam that Friday. As the professor welcomes students back, there is a faint feeling of desperation in the air, as though everyone is hanging on and just trying to make it through to the end. But there is much to do; the professor explains the plan first thing on that Monday back – to go through three sections of the chapter on Taylor polynomials that Monday, review for the third exam Wednesday, take the third exam Friday. And then cover three sections of the next chapter on convergent series on Monday and Wednesday, and spend the final Friday discussing the construction of a very counterintuitive function that is continuous everywhere but differentiable nowhere. This leaves no time for reviewing for the final exam, so he proposes an additional meeting on the Sunday of finals week, to go through some previous final exams and the practice final together.

The actual pace does not match this planned one; there end up being only five minutes to review for the third exam on Wednesday, and so the following week he does not make it to the construction of the Weierstrass function, choosing instead to review for the final exam on the final Friday. Not only does he skip sections of the textbook in order to hit the highlights of these last two chapters, he also skips proofs within those sections, choosing to write up the statements of theorems and verbally outline their proofs rather than formally write up the steps.

When he does present proofs, there is a feeling of "magic" to it that was not there earlier in the semester. On Wednesday of the final week, he is writing up a proof about uniform convergence of sequences, which requires "playing the game" of choosing the right epsilon so that a sequence will stay within that epsilon bound of the limit. Playing this game typically requires working backwards in the proof, starting from what you want to be true at the end and then deliberately picking the epsilon that will make it work. Rather than playing this game with the class and working through the proof backward, as he typically did earlier in the semester, during this proof he simply states the result. "And you play the game with epsilon over two, f_n minus f_m will be less than f_n minus f plus f_m minus f. You play that game and then you get this." The letters whiz by as he says them out loud, easy to lose track of unless you already know what he should say. The understanding of where the epsilon over two comes from is similarly hard to access unless you have already done the proof. “It's just like the professor, magician, makes it work. And like, I followed everything you said, but I can't reproduce it” (Patrick, December interview).

The following and final class the professor reviews a similar proof involving epsilons, on a problem taken from the third exam. Damien helps provide the steps involving the triangle inequality and epsilon during class, but after class he notes, “When he applied the definition of uniform convergence, he said f_n minus f was less than epsilon over three - it's not up until the end that we knew why we had to have- why epsilon over three makes sense. But at first he had it, so he already knew why it was gonna be like this” (December interview). Again there is a feeling of “magic” to the proof, where the professor knows the ending and can make the rabbit appear, without being explicit about how he got it into the hat in the first place. Damien tries to put himself in the professor’s shoes, acknowledging that it takes more time to show the steps that get there, but reiterates his concern: “The students don't know why it was epsilon over three, they don't know why you have to think about the triangle inequality, and how you to split it in a very particular way to get the answer. And that's really where it's important to know why he did it, so you can do any type of problem related, even if it's- the problem is tweaked a little bit, you will be able to do it” (December interview).

Going along with the “magic” in the proofs is a certain “magic” being expected in students’ participation. On Wednesday the professor introduces the idea of a uniformly Cauchy sequence of functions and asks students how it should be defined. “So the sequence f_n from D to R is said to be uniformly Cauchy on D if -

can anyone give me the definition?" Nico gives several attempts but none are quite right, the professor shakes his head at each one. On the one hand, this question seems like a reasonable conceptual question, extending the idea of uniformly Cauchy from a sequence to a sequence of functions. On the other hand, given how quickly the content is being covered (sequences of functions were first introduced on Monday, only one class previous), the opportunity for processing this new information has been limited. "These last few weeks have actually been very rushed, so it's kinda hard to fully understand everything as much as I did back with like derivatives and sequences" (Nico, December interview).

Right after this definition, the professor asks students if they can guess what comes next because it is parallel to what happened after they first introduced Cauchy sequences. When he writes up the theorem, he expects it to be easy: "So by now I think we've seen enough of these proofs that, even if I don't prove it, and even if I sort of tried to quiz you right now, as a pop quiz can you prove this statement, my hope is everybody should be able to prove it. Is that a fair statement? That even without looking in the book, without looking anywhere, you should be able to attempt to prove this statement." And yet when he asks students how they'd approach the proof, he is met with silence, a tentative response from James (that is wrong), and then more silence. The professor expresses his own frustration, half reprimanding and half pleading, "I need you guys to get a feel for this, please!" It feels as though there is an obvious mismatch now between the professor's expectations of the students and where they are able to meet him. "I feel like I can get everything he's gone over so far if I think about it for like a really long time. But I can't think about it like that," Nate said as he snapped his fingers (December interview). “He's expecting a little bit more of that magic stuff.”

Perhaps the other pressing reality explaining this mismatch is that at this point in the semester the students are simply overwhelmed and not able to spend as much time preparing and focusing on this class. Whereas earlier in the semester it felt like some students were not answering because they chose not to, it now feels that many students are not even sure where to begin. “Oh my gosh, no one's prepared anymore! Yeah I feel like there was a solid group of people before who would like read the textbook before and then like came prepared and kinda knew where we were going. I don't think anybody is there right now” (Anne, December interview). Or as Jamie explained, “But yeah, definitely like haven't had the time to really focus on it. So like then, if I don't review the notes from last class I'm even more lost the next time, so it's like this just downward spiral” (December interview). And one more for good measure, “I think the combination of all my other classes being very um, they're also rushing, and they have all these projects and stuff, and then, adding this class which is supposed to be very hard, rushing that I think has been a little overwhelming. But, it is what it is” (Nico, December interview). While it is perhaps tempting to dismiss these as excuses, the feeling of end-of-semester exhaustion does permeate the classroom atmosphere, in the increasing emptiness of the desks as students are absent, the stooped postures and drooping eyelids, the lack of note taking, and the bouts of coughing and sneezing that serve as the backdrop to these last few weeks.

The final day of class is spent reviewing for the final exam. The professor writes up selected problems from the practice exam, pitching the problems out to the students as usual and evaluating ideas as students offer them. Both the professor and the students know it is perhaps their last chance to make clear what theorems have to be understood, what assumptions can be used, and what content will be covered. In an effort to extend their time together, the professor spends the last two minutes of class coordinating plans for a final study session, eventually deciding to send out an electronic scheduling poll. I have never seen a professor offer to schedule an extra class session like this; it seems like a generous gift of his time, and one that he is trying to make accessible to as many students as possible. And then, with those details set, class is finished. “All right, thank you very much, and I'll see you for the final.”

Walking out of the class for the last time, out of the rows of desks, into the hallway crowded with students, back up the staircase and into the cold winter sunlight, I feel a sense of accomplishment mixed with a deep sense of relief. I try to capture the students’ faces in my mind as I watch them walking out of the Mathematics building and down various sidewalk pathways. I wonder what they are walking away feeling: crushing anxiety about the final exam; worried about what the curve will be, fretfully calculating and re-calculating possible grade scenarios; agitation about whether or not they will have to re-take the course next spring. Or perhaps they are walking away with a feeling of triumph; a deeper picture of the mathematical landscape and some “Aha” moments about calculus; an appreciation for the novel opportunities this experience had to offer. It seems certain to be a different mixture for each student.

CHAPTER 5: POSITIONING IN A MOMENT

Having captured some patterns and important moments in classroom interaction through the portrait of the semester, it is now time to turn to the second and third research questions about how classroom interactions position students in relation to mathematics. In order to make sense of these questions, it is helpful to consider what possible answers could look like – for example, that a particular interaction positions a student as a 'math genius,' or that a pattern of interactions positions all of the students as receivers and reproducers of mathematics rather than as mathematics doers. So the question of positioning in relation to mathematics can bring up notions of access, agency, ability, and authority, but is focused on in-the-moment relationships to mathematics rather than longer-term or more stable mathematics identities. It is also important to recognize that by mathematics I mean something slightly different from mathematics as a body of knowledge or content and from mathematics as a discipline. Instead, I am treating mathematics as a discourse (Gee, 2005), or a particular way of speaking and interacting in the world (for similar stances, see Sfard, 2002; Rotman, 1993, p. 68). In this way, students can be positioned as "apprentices" who are learning this discourse, or "receivers and reproducers" of this discourse, or as "mathematics doers" who actively take up and participate in this discourse, or even as excluded entirely from this discourse.

Positioning theory focuses on the triad of speech acts, positions, and storylines as a way of making sense of interactions. One of the commitments of this dissertation is to try to conscientiously use this triadic framework, in part because there is a tendency in mathematics education literature to refer to positioning theory and to discuss positions without making explicit use of all three components (see Chapter 2 and Herbel-Eisenmann et al., 2015). In making this commitment to fully applying this framework, my hope is to be able to speak to the affordances and limitations of this theory as a conceptual framework, as well as to the larger repertoire of positions and storylines in undergraduate mathematics classrooms. The following section will spell out the storylines about mathematics and their associated positions that I found to be most relevant within this classroom.

Storylines about Mathematics

Storylines are the narratives that people use to navigate and interpret episodes of interaction, as described in Chapter 2. Storylines about mathematics are not the same as epistemological beliefs about mathematics; they can look more like stereotypes or caricatures and can feel over-simplified compared to our more complex belief systems. They are fundamentally cultural creations that we can use to make sense of our interactions with and related to mathematics, in classrooms and outside of them (depending on the scale of the storyline). With every storyline, there are also associated positions. In the descriptions below, I describe the “rights” and “duties” associated with different positions as a way to flesh out these positions, where rights are “what you (or they) must do for me” and duties are “what I must do for you (or them)” (Harré, 2012, p. 197).

I identified the following three storylines as the ones that were the most relevant and consistently occurring in the interactions in the classroom (see Chapter 7 for more detail). Other storylines would certainly emerge from the student interview data, or from the perspective of another researcher; these were the focal ones I identified at the level of what happened in the classroom in the shared public space of discourse, and that I found supported in the literature and in conversations with others. It is important to note that the three storylines are independent of one another. They do not represent variations of one dimension or a continuum of approaches to teaching and learning mathematics. They can overlap, to a certain extent, and can be relevant to multiple participants in an interaction at the same time. I argue at the conclusion of the following descriptions (Figure 8) that a different aspect of the experience of being in a mathematics classroom is central to each storyline and gives each one its particular character.

Storyline A: Teaching mathematics is explaining

In considering students' positioning in relation to mathematics, I anticipated that the teacher would be largely in control of the discourse with the students as relatively passive receivers and reproducers. This vision is common to lectures in general:

> "In [the lecture-based paradigm], knowledge, by definition, consists of matter dispensed or delivered by an instructor. The chief agent in the process is the teacher who delivers knowledge; students are viewed as passive vessels, ingesting knowledge for recall on tests" (Barr & Tagg, 1995, p. 21)
>
> "…the students' role in lectures is relatively passive. They sit listening; their activity usually consists of selecting information from what is said, possibly translating it into their own words or some form of shorthand, and then writing it down" (Bligh, 2000, p. 9)

And this vision is particularly prevalent in undergraduate mathematics courses, given the general consensus that lectures are the instructional norm (Mills, 2011).

Taken together, these ideas suggest the existence of a storyline about teaching mathematics, in particular, which says that teaching mathematics is about explaining the body of mathematical knowledge (Storyline A). This storyline exists at the timescale of an educational system[6] – it has existed at least for the past hundred years or so as an educational model where mathematics teachers lecture and students listen and practice (Cajori, 1890/1974; Hiebert et al., 2003). In this storyline the teacher is the "sage on the stage," who has access to this body of knowledge by virtue of his or her expertise and intelligence, and who has both the right and the duty to communicate it to the students. The students in turn are a relatively passive audience, and have few rights associated with this position; they do have the duty to follow along and to be able to (re)produce evidence of understanding this body of knowledge.

However, as I sat in the classroom as an observer and as I analyzed the data afterwards, I found that, while this storyline was certainly present, there were alternative possibilities in terms of how students were positioned in classroom interactions. As captured in the themes of the portrait, something different was happening at times in this classroom, something that invited or allowed (some)

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[6] As pointed out by Herbel-Eisenmann et al. (2015), "identifying scales allow[s] researchers to explore the relationships among positions and storylines, make stronger connections between and among articles using positioning theory, and be more precise about the foci of their studies" (p. 12). Building on their recommendation, I situate the three focal storylines within Herbel-Eisenmann et al.'s (2015) adapted version of Lemke's (2000) timescales for education and related processes.

students to have different relationships with mathematics. In order to make sense of these other possibilities, I turned to two other storylines about mathematics.

Storyline B: Mathematics is an axiomatic system

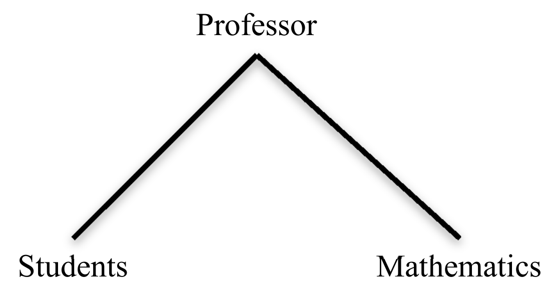
One of these alternate storylines (Storyline B) is focused on mathematics itself rather than teaching mathematics, and says that mathematics is an axiomatic system, in which first principles are decided upon and then theorems are derived through a sequence of logical and consistent steps. This storyline has a rich history as an approach to mathematics, from the "embodied formalism" of the ancient Greeks in Euclidean geometry through to Hilbert's "axiomatic formalism" and beyond (see the discussion of the three worlds of mathematics in Tall, 2013). Having existed for these thousands of years, this storyline lives at the timescale[6] of a world system, rather than an educational system. When mathematics is viewed as an axiomatic system, there is a feeling that the mathematics itself (and/or the logic behind it) is the ultimate authority to which both the professor and the students are subject. There is something potentially equalizing or egalitarian about this view, in that both the professor and students have access to the axioms and to logic; neither group necessarily has "privileged" access to mathematics (Chazan, Callis, & Lehman, 2009). Within this storyline the students are positioned as having the right to challenge the professor if his logic is unclear or incorrect, and the professor is positioned as having the duty to uphold and communicate this logical progression. In fact, it might make more sense to refer to the professor and students simply as people or logical beings, rather than as professor and student, which imply positions of respective power. Thus, rewriting the previous statement, within this storyline any person has the right to challenge any other person if their logic is unclear or incorrect, and any person has the duty to uphold and communicate this logical progression.

Storyline C: The classroom is a shared space of mathematical work

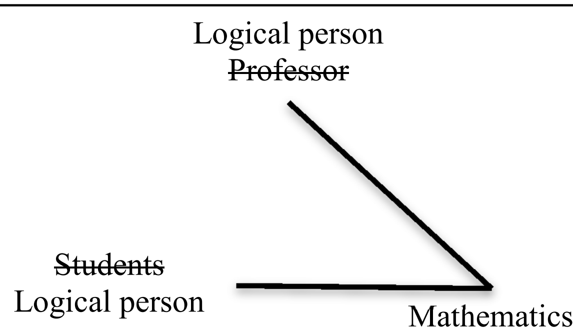
The other alternate storyline (Storyline C) is about the classroom as a shared space of mathematical work, and mathematics as the product of a shared classroom community. This storyline is more particular to a specific classroom, existing at the timescale of the semester[6] in this case; it can be seen as a goal or an ideal storyline within the "reform" mathematics education tradition (e.g., the second storyline in Tait-McCutcheon & Loveridge, 2016, p. 339). This storyline has an enculturation feel, in that the professor is both making visible his own thinking as a doer of mathematics and inviting students to participate and share their thinking. In such a space the authority is more shared as well, although not necessarily equally distributed; there is a sense of community and co-construction of understanding, such that the professor and students are accountable to one another. The professor has the duty to explain and justify his choices and his thinking, and students have the right to challenge him, to ask questions, and to contribute their own ideas.

Looking across the three storylines, a different aspect of mathematics classrooms is central to each storyline and gives each one its particular character. In storyline A, the professor is the central figure, mediating the relationship between the students and mathematics. In storyline B, the mathematics is central, and interactions between individuals are mediated by the authority of mathematics and logic. And in storyline C, the interactions between the professor and students (and potentially between students themselves) are central, as mathematics is constructed within and by a shared community. Figure 8 represents one summary of these three storylines in terms of the relationships they imply between the professor, students, and mathematics (using a simplified version of the instructional triangle).

Storyline A: Teaching mathematics is explaining



Storyline B: Mathematics is an axiomatic system



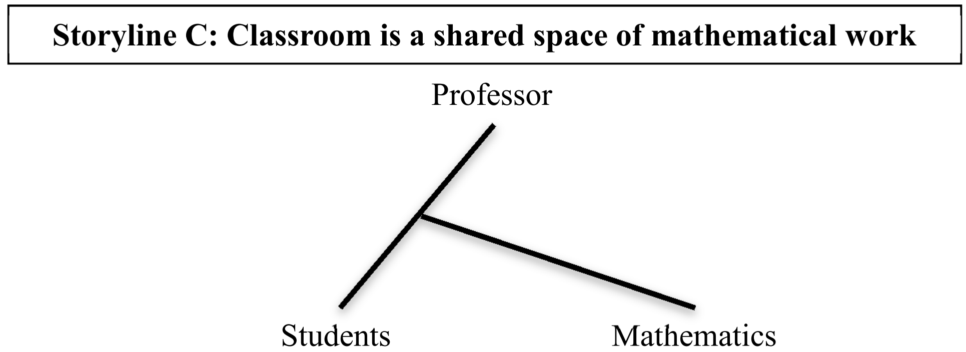


Figure 8. Storylines and the instructional triangle

Analyzing Positioning at Two Scales

As a conceptual lens, positioning theory can be and has been applied alongside a wide variety of analytic approaches (see Chapter 2). As explained earlier, I chose to examine positioning at two scales within this classroom. The first approach was to choose a significant moment in the semester and consider positioning within that interaction. This close discourse analytic approach of episodes of interaction is relatively commonly used in studies of positioning theory (e.g., Pinnow & Chval, 2015; Yamakawa et al., 2009), although this is often in order to explore positioning and identities of particular students over time. The second approach was to examine patterns of commonly used phrases across the semester and the positioning within these patterns. This approach was modeled closely on the work of Herbel-Eisenmann, Wagner, and Cortes (2010) and Herbel-Eisenmann and Wagner (2010), in which they examined lexical bundles in a corpus of secondary mathematics classroom transcripts. These two approaches each feel important and complementary – the former provides a rich contextual look at positioning in a moment, while the latter provides a more bird's-eye view of patterns in positioning across the semester.

This chapter and the following one play out these two approaches, respectively, attending to the second and third research questions:

2. How does one moment in the semester position students in relation to mathematics?
3. How do frequently used phrases across the semester position students in relation to mathematics?

One Moment in the Semester

An important type of interaction to analyze when considering positioning in the classroom is one in which a student expresses a different understanding from the professor. These interactions are likely to be illuminating because they represent a tension between the professor and student, and/or a potential challenge to the professor's authority. The fact that these interactions occurred at all in this classroom is already interesting to note, because it suggests a certain classroom environment in which it is acceptable or comfortable enough (for at least some students) to disagree with and challenge the professor.

Over the course of the semester, these disagreement interactions often took the form of student questions (18 out of 24 interactions), which were almost always negatively phrased, such as: Isn't it true that…? Shouldn't it be…? Don't we need…? In all of these interactions, the professor would disagree and try to briefly correct the student's understanding or clarify the statement at hand. The student's understanding was accepted as correct only three times; all three of these interactions involved some back-and-forth dialogue between student and professor, lasted more than a minute, and ended with the professor joking with the student and class and taking the correction in stride (Patrick and Identity Criterion on 10/23, Griffin and the extra PDF problem on 10/30, Patrick about monotone subsequence theorem on 12/7). These three interactions represented a very potent dynamic in terms of student positioning, and also featured rare extended back-and-forth between student and professor that would provide more material for analysis. I selected the first of these three to analyze because it happened earliest in the semester and thus had the most potential for novelty/tone setting, and because it occurred on an otherwise fairly "typical" day in the semester, unlike the other two (which were on a review day (10/30) and during the last week of class (12/7)).

This focal interaction took place on October 23rd, in the 8th week of the semester, in the time period between the first and second exam. Up to this point in the semester there had been four other lectures during which students had expressed a different understanding from the professor (accounting for 7 of the 24 total interactions of this type). Patrick, the student around whom this interaction centers, always sat in the front row on the left half of the room; he was a consistent participant in the class (at least once in 34 of 36 classes) and was especially/most active during this 8th week of the semester, which he suggested after the fact was because he liked this material the most ("when you get to differentiation I get a little bit more comfortable"). He was a White male but otherwise not necessarily a "typical" student in the class. He was a computer science major and a mathematics minor, so this course was not in fact required for his degree. He was 23, slightly older than an average student, and planning to graduate at the end of this fall semester and begin his PhD in computer science in the spring.

The content of this lecture in particular was focused on the Mean Value Theorem, considered one of the most important and fundamental theorems in calculus. The professor began the lecture by proving Rolle's Theorem, a particular case of the Mean Value Theorem, and then went on to prove the Mean Value Theorem. He then illustrated two applications of the Mean Value Theorem, one of which, the Identity Criterion, became the point of contention for this interaction. This exchange took place around 9:40, so there were about ten minutes left in class, including at least five that the professor wanted to use for a pop quiz. Having written up the statement of the Identity Criterion from the textbook on the board, the professor then summarized it to begin this exchange:

1 Professor: So when I have a function defined on an interval, then the function is constant if and only if the derivative is equal to zero everywhere
2 Patrick: Didn't we use this fact in Rolle's, in our proof of Rolle's Theorem though?
3 Professor: Absolutely not
4 Patrick: Okay
5 Professor: We never used it
6 Patrick: But didn't we state that if a was equal to b, then, and if the maximizer-
7 Professor: -But the function is not constant. It's telling you that (Writing on board) if you draw the graph of the function a and b, sorry a and b, then the value at a and the value at b are the same, but what happens in between is different
8 Patrick: Right, but then we assumed that the maximizer and the minimizer both happen at the end points, =and then-
9 Professor: =Oh! I- I sort of uh (Pause – 2 sec)
10 Damien: Yeah
11 Patrick: And you used this. Didn't you?
12 Professor: Okay I see what you're saying now. Yes I used it. =But, I can prove it
13 Patrick: =Okay, all right
(General laughter in the class)
14 Professor: Thank you for catching me (Pause)
15 Patrick: (While laughing) Just trying to hold you accountable, that's all

16 Professor: Absolutely (General laughter in the class) I like it. But then I would like to sort of tell you that we've done it actually, because this statement has two parts, right? There is the part about constant implies the derivative is zero, and derivative is zero imply constant

17 Patrick: Right

18 Professor: The part derivative imp- constant implies derivative equal zero, that part, I mean there is nothing there, we can prove it, right? And that's what I used, I did not use this part. I did not use that f prime of zero is equal to zero imply that the function is constant. I used the fact that the function is constant, therefore it has derivative everywhere and the derivative is zero. And that, even though we did not prove it formally, this is something we can go back to the definition and just prove, because the function is constant. Okay? So this part I don't want to prove, that's what I was going to say, that I didn't want to prove this, because that was sort of given. Okay? But you're right, uh, I never sort of wrote it formally, but this is something I assume we can all prove. Okay? So the real part of this lemma is what? …

The exchange focuses on a disagreement over the logic of the professor's presentation; although it could have been the case that Patrick had discovered a fatal logical error, in fact the heart of the disagreement is about the order in which the professor presented these theorems. The Identity Criterion the professor is presenting takes the logical format of a bi-conditional statement: if A then B, and if B then A. Patrick is raising the concern that one of those (if A then B) was used in an earlier proof, so to prove it now as a consequence of that earlier proof would be circular reasoning. The professor acknowledges that he did use it earlier, but explains that it can be proved independently of that theorem.

To analyze this interaction from a positioning theory lens, I attempted to map each speech turn onto possible positions and speech acts associated with the three storylines, using discourse analytic tools to provide evidence for these (to see the complete mapping, see Appendix G). This mapping represents a plausible interpretation of the positioning at play in this interaction; I consider it to be one of many plausible interpretations. In playing out interpretations from multiple storylines, I am trying to reveal possible interpretations by participants and to go beyond stating only a univocal interpretation (Herbel-Eisenmann et al., 2015). I am comfortable with the idea of having revealed a set of truths about this interaction that resonates with my experience in the classroom, rather than looking to identify one "truth" of how to interpret this interaction.

Turn 1

1 Professor: So when I have a function defined on an interval, then the function is constant if and only if the derivative is equal to zero everywhere

The interaction begins with a summary assertion by the professor about the Identity Criterion. Such an assertion about the content can be read within storyline A, where teaching mathematics is about explaining and summarizing a body of knowledge. The feeling of revealing true knowledge is present in the use of present tense "to be" verbs ("the function *is* constant," "the derivative *is equal to* zero") that make assertions of existence or relationships (Biber, 2006, p. 247) and that are seemingly beyond time and space (detemporalization). Once certain conditions are met (that the function is defined on an interval), then it no longer matters what the function is or when and where in time and space it is encountered; this relationship between the function and its derivative will always hold. Within storyline A, the professor is positioned as the primary mathematics doer ("when *I* have") who has the right to "have" these functions and explain their truths, and the student is the passive recipient who has the duty to follow along. The speech act or social meaning of this utterance is then that the professor is explaining or sharing an absolute truth with the students.

There are also flavors of the other two storylines in this one utterance. Storyline B about mathematics as axiomatic and logical can be heard in the formal conditional language ("if and only if") and in the lead-up to this exchange where the professor wrote the theorem on the board in formal mathematical language ("Let *I* be an open interval and suppose f: I $\rightarrow \mathbb{R}$ is differentiable. Then f is constant $\Leftrightarrow$ f'(x) = 0 $\forall$x in I").

More notably, storyline C about the classroom as a shared space of mathematical work can be heard in the use of the personal pronoun "I" at all, rather than saying "when *one* has a function" or even more impersonally "*Given* a function" or "*Let there be* a function," which would be more likely phrasings in a textbook (Herbel-Eisenmann & Wagner, 2007). In other words, his saying "when *I* have" could be interpreted as a way of making his thinking visible to the students and making mathematics feel like a human and personal activity, rather than reading it through storyline A where the "I" feels exclusionary and authoritative. In this reading, the speech act or social meaning of this utterance is that the professor is "translating" the theorem back into relatable terms for the whole class to engage with. This interpretation has gestural support as well, in that when he wrote the theorem on the board in formal mathematical language his back was to the students, and he has now turned back to the students and is speaking directly to them in order to make visible the "so what" or the gist of the theorem.

Both interpretations (through storyline A and storyline C) are viable; it is possible that different students heard it through one or the other, depending on their attitude, mood, and positioning that day. And it is probable that students heard it in other ways and through other storylines entirely.

Turn 2

2 Patrick: Didn't we use this fact in Rolle's, in our proof of Rolle's Theorem though?

Patrick's question can be read within storyline B about mathematics as an axiomatic system, in that he is challenging the logical progression of proofs. This storyline is indicated by Patrick's verb choice of "use" – he is attending to how the fact got *used* and thus asking a question about the logical progression, rather than about the fact or the content itself. Within this storyline you cannot use a fact before it is proved, and you especially can't say that a fact is a consequence of a theorem when you used that fact in its proof – this is circular reasoning and a severe breakdown in logic. As Patrick explained in an interview where we discussed this interaction, "oftentimes in math classes […] if ever you use something prior to proving it, to prove something else, you always run into the danger of using that thing to prove the thing and then you have a circular dependency." Within storyline B, Patrick's question has the potential to be interpreted as a fairly severe accusation; if Patrick were right, then substantial repair and/or clarification would be needed. Patrick's question positions mathematics and logic as the ultimate authority, Patrick as a whistleblower of sorts, and the professor as making a logical error.

And yet, Patrick's intention was for it to be a clarification question rather than a correction. As he explained after the fact: "I think that I brought it up because I didn't see why it wasn't in danger of being a sort of circular dependency and so I wanted to clarify." Indeed, Patrick's question is strikingly diplomatic rather than accusatory in its tone. In the first place it is phrased as a question rather than a statement ("Didn't we use…" vs. "We used…"), which makes the phrasing more tentative and allows the professor to still be an arbiter on what happened. Patrick also uses personal pronouns of "we" and "our," rather than "you" and "your," which help soften the question so that it does not sound like an accusation. The use of "we" and "our" calls to mind storyline C, in that even though Patrick did not actively participate in the earlier proof he still sees it as a shared object that exists now in a collective space. And the fact that he used personal pronouns rather than an entirely impersonal expression ("Didn't this fact get used in the proof of Rolle's Theorem though?") positions mathematics as a human activity, and both Patrick and the professor as mathematics doers. Within this collective classroom space, Patrick's question is not positioning the professor as wrong or mistaken (as he explained in the interview, he did not really think there was circular reasoning happening). Rather, he is calling for the professor to account for an inconsistency and to give a more complete representation of the logical relationship between these two proofs; Patrick wants this thinking to be made visible for him to follow. In this view the social meaning of Patrick's question could be read as: "you're not giving us the full story" or "there is something here I am not following."

Turns 3-5

3 Professor: Absolutely not

4 Patrick: Okay

5 Professor: We never used it

The professor's fairly abrupt and unusually curt response of "Absolutely not" suggests that he heard Patrick's statement more through storyline B and interpreted the question as a logical accusation of circular reasoning. He continues with Patrick's shared pronouns, saying "*we* never used it," though, which softens the defensiveness (compared to "I never used it"). But the absoluteness of his language ("absolutely," "never") really rings through and somewhat negates the possibility of storyline C and the shared classroom space (which I think would require more of a question back to Patrick, e.g. "I don't think so – how do you see us using it before?"). As Patrick commented when he listened to the interaction again, "He [the professor] seems so sure of himself." While continuing in storyline B, it seems that the professor has flipped the positioning so that while mathematics and logic are still the highest authority, the professor has *not* made a logical error, and Patrick is mistaken or confused. The social meaning is that the professor is correcting Patrick.

Turn 6

6 Patrick: But didn't we state that if a was equal to b, then, and if the maximizer-

However, Patrick persists in his question asking, a fairly remarkable and unusual occurrence (as noted in the context section above), suggesting both that he did not read the professor's comment as a harsh correction or personal commentary on Patrick's intelligence (and thus did not respond by being embarrassed/angry/upset) and that he was confident enough in his own understanding to be willing to dispute the professor's absolute statements. One possible way of reading Patrick's persistence is that within storyline B, mathematics and logic are still the highest authority, and so Patrick does not have to engage with the professor in a personal power struggle but rather in a more personally neutral disagreement about the logic of the proof (although of course the power differential between professor and student is still present and clearly shapes the dynamics of the interaction). This reading is somewhat supported by the specific logical details Patrick adds ("if a was equal to b, then" "if the maximizer"). It is also supported by Patrick's comment in an interview: "I think that in math and science more than in other fields there's a sort of playfulness associated with banter and argumentation […] it's sort of a more endearing quality to kind of challenge somebody, and it becomes not as big of a deal" (November interview).

Another possible reading is that the continuity of the first personal plural pronouns ("we") from lines two through six allows Patrick to remain in storyline C of the classroom as a shared space of mathematical work – although Patrick and the professor are not peers, they are still working on the same team of constructing shared mathematical understandings. This reading is somewhat supported by the fact that Patrick continues the tentative question phrasing and the first person plural pronouns with "didn't we state…", thereby continuing to position Patrick and the professor as co-constructors of mathematical understanding, and allowing the social force of the statement to be a clarification or reminder of previous work.

Turn 7

7 Professor: -But the function is not constant. It's telling you that (Writing on board) if you draw the graph of the function a and b, sorry a and b, then the value at a and the value at b are the same, but what happens in between is different

By interrupting Patrick and beginning with "But…" the professor seems to reassert authority, returning perhaps most strongly to storyline A where teaching mathematics is explaining truths. This shift makes sense within this storyline where the professor has expertise, both in mathematics and in explaining mathematics. He hears Patrick's comment within the framework of this expertise and assumes he understands it and the source of Patrick's confusion; perhaps he even hears it as a common student misunderstanding that he has seen before. The use of the second person pronoun "you" is also interesting because the "you" feels more like a generalized you that could refer to anyone, rather than a direct referent to Patrick (see Rowland, 1999). This also fits within a storyline A reading where the teacher is the explainer; rather than having a direct one-on-one negotiation with Patrick, he has shifted back to explaining the content of this theorem to the whole class.

It is also possible to read this interruption in the same vein as the professor's earlier contradiction of Patrick through storyline B; the professor is re-asserting his positioning that he has *not* made a logical error, and that Patrick is mistaken or confused.

Turn 8

8 Patrick: Right, but then we assumed that the maximizer and the minimizer both happen at the end points, =and then-

Patrick returns to the conversation seamlessly; both the lack of pause and the word "Right" indicate that he has followed the professor's clarification of the conditions of Rolle's Theorem (i.e., that it applies when f(a) = f(b) and not just when f is constant). Patrick persists with his point, using "but" to emphasize the sticking point for him, and returning to the first person plural "then we assumed…" to fill in the next detail from the proof of Rolle's Theorem. Once again the readings from storyline B and C are both reasonable; his remark can be read both as a correction of the logical breakdown and as a specific reminder of the shared classroom work. Patrick clearly is not taking up the possible positioning from the professor's commentary of being incorrect or confused about the mathematics, and in fact seems to respond by positioning himself more confidently, as evident in his use of a direct statement for the first time, rather than a question ("then we assumed" rather than "didn't we assume").

Turns 9-10

9 Professor: =Oh! I- I sort of uh (Pause – 2 sec)

10 Damien: Yeah

The professor interrupts Patrick again, presumably with the intent of correcting Patrick again or clarifying the steps from before, but then trails off as he looks at the work he just drew on the board. Into this silence Damien voices support for Patrick; he is another vocal student who is sitting in the back of the room on the opposite side. Although there is not much to interpret from a single “Yeah,” it indicates that at least one other student in the class is following this interaction closely; it is not a private communication understood only by the professor and Patrick (as it can feel when students ask certain ‘sidetrack’ questions). Indeed, the fact that Damien contributed at all to this discussion between the professor and Patrick seems to make sense within a storyline C reading of the classroom as a shared space of mathematical work where Damien positions himself, and presumably Patrick, as a co-constructor of mathematical understandings who can hold the professor accountable for making the proof process visible.

Turn 11

11 Patrick: And you used this. Didn’t you?

Patrick finishes making his point here, in the climactic moment of this exchange. There is a feeling of building momentum with his use of the word “and” (rather than “but”) and his switch to second person pronouns (“you” rather than “we”). Yet there is also an interesting vagueness in the use of the word “this,” which perhaps makes the moment more powerful in allowing Patrick to refer to “an understood but un-named mathematical referent” (Rowland, 1999, p. 20). Patrick is perhaps the closest to treating the professor as an equal in this moment (i.e., an individual “you”), in the vein of storyline C where they are both doing mathematics together. Patrick is pushing the professor to account for his presentation of this mathematics.

The follow-up of “Didn’t you?” is interesting because it could sound like an accusation, almost like a lawyer finishing up a cross-examination of a witness in front of a judge (perhaps mathematics/logic?). But Patrick's tone is far more tentative; his voice gets quieter, his pitch goes up at the end, and he says it quickly and almost slurred together. It sounds like a genuine question rather than an accusation, and feels like a restoration or potential return to traditional authority – as though Patrick realizes he is treading on fragile ground and wants to offer both the professor and himself a way out of this potential power struggle.

Turn 12

12 Professor: Okay I see what you're saying now. Yes I used it. =But, I can prove it

The professor's response, significantly, feels solidly within storyline C - Patrick is holding the professor accountable, and the professor responds in kind with a personal acknowledgment, both of Patrick's point ("I see what you're saying") and of its validity ("Yes I used it"). As the professor says these words, you can hear an exhalation from Patrick, and a general relaxation of tension, as though the entire class had been holding its breath.

The professor then tacks on, "but I can prove it," which feels readable both within storyline C, as a continued accounting of his work within the shared space, but also within storyline B, as justification that there is not a logical fallacy, and within storyline A, as a reassertion that as the teacher he can explain the actual proof of this statement.

Turn 13-14

13 Patrick: =Okay, all right

(General laughter in the class)

14 Professor: Thank you for catching me (Pause)

The general feeling of a release of tension continues with Patrick's "Okay, all right," and the general laughter in the class. The laughter and feeling of a shared joke suggest the extent to which this exchange is unusual and different from the standard mode of operating (i.e., Storyline A). For the professor to admit that Patrick's claim is correct, particularly after adamantly rejecting it at first ("Absolutely not"), and to even thank Patrick for "catching" him represents a dramatic departure from the typical professor-student dynamic of Storyline A. If this were more typical, it probably would not be met with such a shared response and with laughter. It also suggests a feeling of safety after a moment of tension – the professor could have chastised Patrick or responded with anger in some form, or he could have felt embarrassed and as though he "lost face" and been unable to accept Patrick's critique as valid. In the wake of these lingering possibilities, the professor's choosing to acknowledge Patrick's point personally and to thank him clearly signals that Patrick in particular and the class in general are "safe." Perhaps counterintuitively, the professor's thanking Patrick also preserves a certain sense of his authority because it positions Patrick as a responsible student who is fulfilling his duty of noticing the professor's mistakes (within storyline B where mathematics/logic is the ultimate authority).

Turn 15

15 Patrick: (While laughing) Just trying to hold you accountable, that's all

Patrick, perhaps pushing the moment, responds with a joke about wanting to hold the professor accountable. This joke sounds in fact like a reasonable statement when read through storyline C, where Patrick and the professor are positioned as co-constructors with the duty to be accountable to one another, or storyline B, where Patrick and the professor are both accountable to the higher authority of logic. So the fact that Patrick treats it as a joke (laughing as he says it) indicates how unusual or unlikely of a statement it is, and the pre-eminence of storyline A in the classroom. In particular, read within storyline A this statement makes no sense, because it implies a role reversal where the student is positioned as having the right to hold the professor accountable.

Turns 16-18

In the remainder of this exchange (lines 16 to 18), the professor provides his explanation of how the logical confusion can be resolved. There is a slightly defensive tone to his explanation, and the feeling of a need for him to justify himself. But the reason for the justification can be understood differently depending on the storyline – it can be to restore his authority (storyline A); to restore the sense of logic (storyline B); or to let the students into his own thinking about the proof so that they share a better understanding of the mathematics that has been proved and of what requires proving when (storyline C).

Conclusion

Having analyzed this interaction using positioning theory and discourse analysis tools, we can now return to the second research question: How does one moment in the semester position students in relation to mathematics? At the most overarching level, within the dominant and expected narrative of storyline A, the students are positioned as relatively passive followers while the professor reveals and explains a body of mathematical knowledge to them. This distribution of authority and agency is almost the implied storyline that goes with the words "teacher" and "students" (at least in the U.S.) – teachers are doers and students are listeners. It is not unique to mathematics or to undergraduate classrooms, although the vision of the teacher as imparting absolute truths of the universe is somewhat more specific to these contexts.

Yet this is not the only narrative available in this classroom; if it were, then this interaction might not have happened at all. Patrick seemed to invoke other storylines in order to express an understanding different from the professor's and thereby claim some agency. These other storylines do not flip Patrick's positioning in relation to mathematics on its head but instead represent subtler shifts. In storyline B, mathematics and logic are the highest authority to which both the professor and students are subject, which positions Patrick as having access to mathematics independent from the professor as a mediator. And in storyline C, the classroom is a shared space of mathematical work, in which the professor has the duty to make his reasoning visible to students and Patrick has the right to hold him accountable; both the professor and the students count as mathematics doers.

These storylines may be ones that Patrick has access to from other experiences and are particular to him as an individual, or they may be ones that are invoked within this classroom at various points and to varying degrees, or some combination of the two. This interaction played out with fairly minimal disruption to the overall class dynamic, and the professor seemed willing to relinquish some authority and politely accept the correction, both of which suggest that the storylines may have a larger foothold in this classroom than just for Patrick. But these storylines are not the only possible interpretations, as noted before, and do not encompass all of the dynamics of this interaction. When I asked Patrick why he persisted after the professor said "Absolutely not," which might have turned other students away, he explained:

> Well because I was pretty sure that I was right, and I was pretty sure that if I was right, he would recognize it, right? And if I was wrong he would recognize it (Laughs). Um, and I feel like I'm comfortable enough with him that I can do it, there are certainly professors who I, you know, wouldn't-wouldn't pull that with.

This explanation invokes a certain sense of personal confidence and entitlement, as well as a strong personal relationship with the professor. These facts seem to belong to more local storylines about Patrick's mathematics identity and his relationship with professors, rather than to the larger storylines described above. But both facts suggest a stronger position for Patrick in relation to mathematics, one where he has certain agency both in relation to assessing his own mathematical understanding and to expressing it to the professor, than would be suggested in the expected storyline A.

To summarize, most of the students may still be generally positioned in a fairly traditional role of passive consumers of mathematics in this classroom, at the very least because of the dominant lecture format of this class. But, this interaction suggests different routes for potentially shifting this positioning toward more student agency and ownership of mathematics—through an appeal to logical thinking and mathematics as a higher authority, through a sharing and accountability within the classroom space, and/or through personal relationship dynamics.

CHAPTER 6: PATTERNS OF POSITIONING OVER TIME

Analyzing a single interaction exchange is useful for an in-depth and in-context look at how positioning occurs in the classroom, but it has the limitation of being only a single moment in the semester. While this single moment has much to reveal about positioning in relation to mathematics, it raises questions about how positioning happens over the semester more broadly: what is specific to that interaction on that day with that student, what is relevant to larger patterns of interaction in the semester, and what is possibly missing from that interaction? Looking at patterns of interaction, though, presents its own challenges, especially around how to bound the analysis of 36 days' worth of field notes and audio recordings.

One approach used by Herbel-Eisenmann, Wagner, and Cortes (2010) is to analyze lexical bundles, groups of three or more words that frequently occur together, as windows into patterns of interaction. These bundles are a way of capturing a sense of "normal" discourse within a particular context (or "register"), such as university classroom teaching (Biber, 2006) or secondary mathematics classrooms (Herbel-Eisenmann et al., 2010; Herbel-Eisenmann & Wagner, 2010). "Lexical bundles allow one to focus on mundane combinations of words that often go unnoticed but that also have important structuring and signaling effects in the discourse" (Herbel-Eisenmann et al., 2010, p. 29). The subtler and less conscious undercurrents of repeated phrases are just as powerful and important to consider as jarring or significant events, in terms of analyzing how classroom interactions position students in relation to mathematics.

"Our discourse patterns [can] invisibly undermine the goals we have for our students" (Herbel-Eisenmann & Cirillo, 2009, p. 6).

Lexical Bundle Analysis

In order to identify lexical bundles within the semester, I transcribed nine class sessions (25% of the 36 audio recordings) from across the semester. I divided the semester into three time periods of roughly equal length (four to five weeks of audio) using exams as the boundary points (as significant shifts in terms of both content and potential participation), and then selected three days to transcribe from each time period. I excluded exam review days from selection because the dynamics and patterns of participation were so different on those days. Three of the transcripts I had already completed at the request of the professor; he was interested in looking at the days where the major theorems were introduced (Intermediate Value Theorem on 9/28, Mean Value Theorem on 10/23, Fundamental Theorem on 11/25). In general I tried to preserve representativeness while choosing a quasi-random sample of days.

Using this set of nine transcripts, I used the "word combinations" feature in MAXQDA, a qualitative data analysis program, to identify the most frequently occurring four-word bundles. (This program identified phrases with apostrophes as four-word phrases, such as "we're going to" and "I don't know," which I decided to exclude, in line with previous studies). My goal in this analysis, unlike that in Herbel-Eisenmann et al. (2010) or Biber (2006), was not to characterize a "register" but instead to capture frequently used phrases within this particular classroom, to gain insight into patterns of interaction over the semester. Therefore I decided that the bundles had to occur in at least five of the nine transcripts in order to be included in the analysis, to establish a certain consistency of use. (Certain phrases like "the Mean Value Theorem" appeared quite often but only in one transcript, for example). I also decided that the bundles had to appear at least 15 times (in the ~64,000 total words), in order to qualify as "frequently" used. For comparison, Herbel-Eisenmann et al. (2010) used a cut-point of at least 40 instances (in ~680,000 words).

Here is the complete list of 20 lexical bundles that met those constraints:

Table 2. Lexical bundles across nine transcripts

Bundle	**Number of instances**	**Number of transcripts**
is equal to zero	52	6
than or equal to	50	8
less than or equal	48	8
greater or equal to	47	5
to be equal to	44	8
is less than or	34	7
you guys with me	32	8
for all x in	29	7
or equal to zero	29	6
I want you to [a b]	27	8
is going to be [a b]	25	6
is that clear for everyone [1]	22	6
be equal to zero	19	6
I can find a(n)	18	7
will be equal to	18	6
it has to be	17	8
I would like to	17	7
is not equal to	16	5
it's going to be [a b]	15	7
I don't want to [a]	15	6
if you want to [a b]	15	5

[a] Appeared in Biber (2006) University Classroom Teaching corpus
[b] Appeared in Herbel-Eisenmann et al. (2010) Secondary Mathematics Classroom corpus
[1] Technically, this was two four-word bundles (*is that clear for* and *that clear for everyone*) each of which appeared 22 times. Those two bundles always occurred together, which is why I chose to condense them into one five-word bundle.

In analyzing these bundles, I worked to sort them around categories of how positioning potentially plays out, particularly around the traditional storyline and the two alternative storylines described in the previous chapter. A few of these bundles came up in previous analyses (as noted in Table 2), but most were unique to this set of transcripts. I began sorting the bundles using the categories described in Biber (2006), but given how many of the bundles found here were unique and given their use in context, it became clear that it would be important to refine those categories and to develop some new categories. The four resulting categories (as seen in Figure 9) are: Mathematical relations; Intention/prediction; Checking in; and Desire.

The mathematical relations bundles are unique to this data set, and encompass impersonal phrases expressing relationships between mathematical objects. The intention/prediction bundles are quite similar to the first in terms of the general focus on mathematical relationships, but slightly different in the implication of a human actor expressing an intention or making a prediction. The checking in bundles are also unique to this data set, and include questions the professor uses to check in with the class as a whole. Finally, the desire bundles are the most personal category, used by the professor to describe his (or the students') wants and plans.

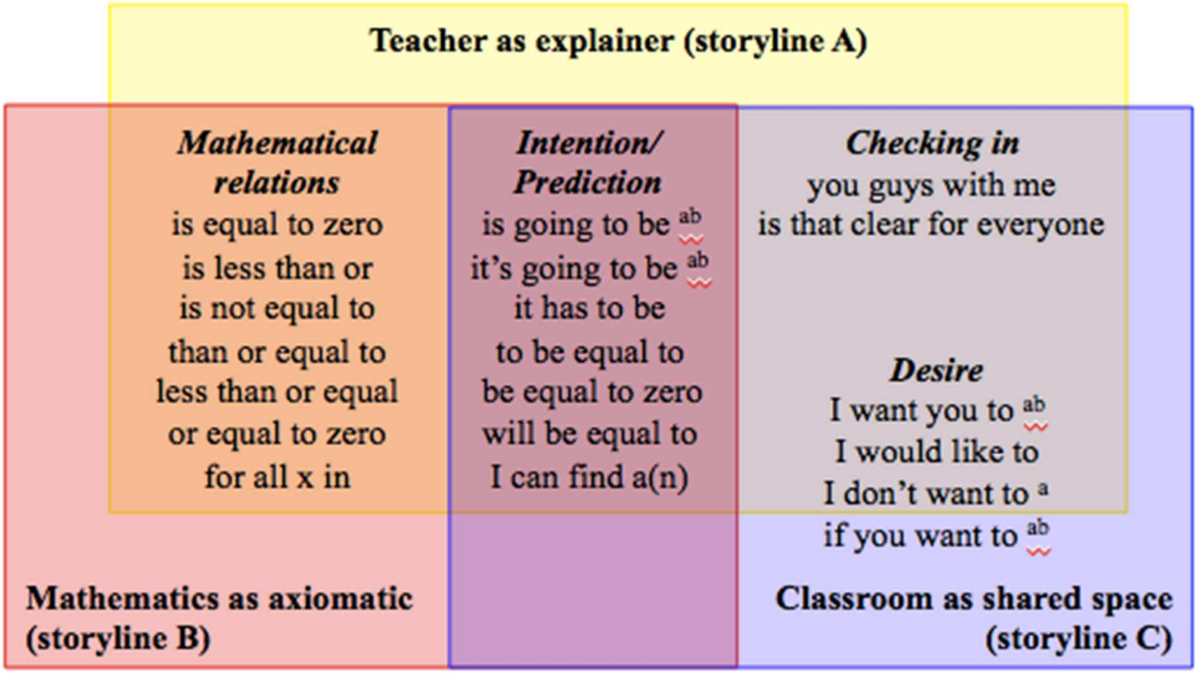


Figure 9. Categories of bundles organized by storyline

The following sections will provide more detail about each of the categories, as well as interpretations of how these phrases could position students in relation to mathematics in interactions, as read through the three storylines established in the previous chapter. In the following interpretations, it is important to keep in mind that the general pattern implied in these phrases is not deterministic of how they play out in context. As explained by Herbel-Eisenmann and Wagner (2010), "the researcher's role is to try to understand what the computer-identified pervasive patterns of speech *index* about the discourse in which they are used" (p. 44, emphasis mine). These phrases, then, can be thought of as markers or as windows into interactions; the significance of a phrase (the speech act) in any particular moment would depend on that interaction in context. The following discussion is focused on playing out potential positioning patterns, not on specifying any definite interpretations.

Mathematical relations bundles

The mathematical relation bundles read more or less as they would in a textbook; they seem to exist independent of space, time, or human activity. Six of these seven bundles are comparative expressions of two mathematical objects and whether they are equal or whether one is less than another (an equivalence relation and an ordering relation, respectively). Three of them include the word "is," a form of "to be," which establishes the sense of assertions of existence that are seemingly beyond time and space (Biber, 2006, p. 247). There is also one "quantifier" bundle, *for all x in*, which serves to specify when a particular relationship is true. While less universal than "is," this phrase still suggests a certain universality in that the subsequent truth will hold for *all* variables within this specified constraint.

As such, these phrases can generally be read as positioning the person who says them with significant mathematical authority. Within storyline A about teaching mathematics as explaining a body of knowledge, they invoke mathematics as a body of absolute impersonal knowledge, and position the person saying them as one who has the right and expertise to explain these facts. Since that person in this classroom was almost always the professor (with a few notable exceptions, as will be described below), these phrases position the professor as the expert and students as the audience; a possible speech act associated with these bundles in an interaction might be "explaining an absolute truth." Within storyline B about mathematics as an axiomatic system, the interpretation is fairly similar; these phrases invoke the sense of mathematics as a formal and impersonal game, and thus position the person saying them as a logical thinker, capable of piecing together precise relationships between mathematical objects. Again, because the professor mostly uses these phrases, this position is largely associated with him; the existence of the students as an audience is almost irrelevant to the use of these textbook-like phrases. The strong feeling of mathematical authority means these bundles are harder to read through storyline C about the classroom as a shared space of mathematical work.

Intention/prediction bundles

The second category of bundle is similar to but slightly different from the first, and includes phrases expressing intention or prediction, almost always about mathematical relations[7]. Two of these bundles (*it's going to be*; *is going to be*) are also found in the previous two corpuses; Biber (2006) categorized them as intention/prediction bundles (hence the category name) and described them as "impersonal, expressing predictions of future events that do not entail the volition of the speaker. These bundles are usually used when explaining a logical or mathematical process that involves several steps" (p. 142).

I have added several bundles to this category that did not appear in the other corpuses. The bundle *it has to be* reads very similarly to *it's going to be*, with perhaps more emphasis on the speaker's belief about this relationship rather than making a general prediction. The bundle *I can find a(n)* reads as a more personal statement of intention (what the professor will do next) or prediction (what it is possible for him to do). *I can find a(n)* was always followed by words like number, value, n, index, and

[7] There are a few exceptions related to events in the class (e.g., "the exam *is going to be* on Friday next week"). In the 59 uses of *it's going to be, it has to be,* and *is going to be*, only 6 of them are not related to mathematical statements

function, thus fitting the overall theme of intention/prediction about mathematical relations in particular.

Table 3. Verbs preceding "*to be equal to*"

Verb	**Count**	**Example in context**
set	15	"Then I'll set a_2 *to be equal to* what?" (10/28)
have	11	"The function has *to be equal to* 0 at some point" (11/11)
choose	6	"So if I choose m *to be equal to* this value, then…" (10/23)
be	4	"That's going *to be equal to* f(b) minus f(a)" (11/25)
define	2	"Define c *to be equal to* one over epsilon" (9/9)
take	2	"So if you take epsilon *to be equal to* one, you can find…" (9/16)
allow	1	"I can allow myself *to be equal to* that thing, okay?" (10/12)
draw	1	"Draw the function *to be equal to* 5" (10/23)
want	1	"If I want g(a) *to be equal to* g(b), what is that telling me?" (10/23)

The last three bundles that I included in this category, *to be equal to, be equal to 0*, and *will be equal to*, might seem more appropriate in the mathematical relations category because they are impersonal statements of equality relations. These bundles, though, are in the infinitive and the future tense, rather than the present tense "is" of the earlier bundles, which is important because it introduces a human doer. The infinitive "to be" requires a verb come before it and thus requires a person with an intention for things to be equal (see Table 3 for verbs coming before *to be equal to*). And the future tense "will be" implies a prediction and once again introduces the dimension of time and belief ("If I put this to be m, what happens here? You'll have f'(c) *will be equal to* g'(c) multiplied by m.") Therefore these three bundles belong more to the intention/prediction category, rather than the mathematical relations category, because they imply the existence of a person who has intentions, predictions, and/or beliefs about these mathematical relations.

Since these bundles express an attitude or belief the speaker holds, they read somewhat less authoritatively than the mathematical relations bundles. Rather than expressing relations between mathematical objects x and y, as the mathematical relations bundles do, these bundles are expressing relations between the mathematical objects and the individual mathematics doer. Within storyline A about teaching mathematics as explaining, these intention/prediction bundles position the person saying them as someone with expertise about this body of knowledge, in that the speaker is expressing their beliefs about these mathematical relationships. Within storyline B, they invoke mathematics as a logical proceeding of steps and express the speaker's right to make such a prediction of logical necessity (e.g., *it has to be*). As with the earlier bundles, given that the professor is almost always the one saying these bundles (although exceptions will be discussed later), he is being positioned as the mathematical expert and logical thinker in these two storylines, while the students are the audience.

But these bundles, unlike the mathematical relations ones, also invoke mathematics as a human activity, in the sense that a personal judgment is being made about these mathematical relations, and as a shared activity in this classroom (Storyline C). These bundles feel like a way for the professor to make his thinking and his stance towards this mathematics visible to the students. He is making predictions and intentions about what will come next, by way of these phrases, and thereby establishing a shared vision of the mathematics. From this storyline, these bundles can been seen to position students as a mathematical audience with the right to ask questions of the professor, and with the understanding that the mathematics being constructed is open to their understanding as well, rather than dictated and closed off by the professor.

Checking in bundles

The remaining bundles fall into two categories: Checking in and Desire. The two bundles around checking in both take the form of questions the professor would ask to the entire class (*you guys with me; is that clear for everyone*). Both questions were treated largely as rhetorical (although the professor would sometimes pause after the question to wait for any replies). In the 54 instances in which these two phrases were used, only once did the professor's question result in a student question, and in that instance the professor called on her by name (i.e., *Is that clear for everyone? Sara?*). They typically signaled a transition from one topic to the next, and acted as a way to lead the students through the lecture.

Both bundles have an implied expectation – you guys *should* be with me, it *should* be clear for everyone – and in the absence of a response, the assumed answer is yes. In order to respond, students would need to resist this expectation and express their confusion to the whole class. These questions have a similar feel to the tag questions (Ok? Right?) in calculus lectures described by Gerofsky (1999), which she argued served as a persuasive device, "used to obtain both agreement with a statement made by the lecturer ("It's positive ... OK?") and permission to move on to the next section of the lecture" (p. 42). Thus, within storyline A of the teacher as explainer, these bundles position students as compliant consumers of the professor's mathematical explanations, and read as a reminder of the duty of the student to be following the professor's explanations.

On the other hand, within storyline C about the classroom as a shared mathematical space, these bundles read fairly differently. The fact that the professor asks these checking in questions so frequently, even though students do not respond to them, suggests that he sees himself as accountable to the students. If they are not "with him" or it is not "clear for everyone," then there is no point in continuing, because the mathematics should be a shared product and not merely an explication to a separate audience. There is a certain feeling of caring that comes with these questions; whether or not they are effective in eliciting students' understanding, these bundles still communicate that the professor cares about whether or not the students are understanding. In this way the professor is positioned still as an authority, but the students are positioned as important partners or co-creators of mathematical understanding.

Storyline B about mathematics as an axiomatic system portrays people's interactions with mathematics as individualistic encounters, so these checking in bundles are fairly irrelevant in that narrative.

Desire bundles

Finally, there are the desire bundles. All four of these bundles feature personal pronouns (I, you) and verbs about desire (want, like), and therefore a sense of the professor and students as people with personal wishes and needs. These bundles are found in Biber (2006) and Herbel-Eisenmann et al. (2010), and are categorized by Biber (2006) as personal desire bundles (*if you want to; I don't want to*) and personal obligation bundles (*I want you to*). I chose to include all three in one category, and to add the new bundle *I would like to*, because the use of personal pronouns and expressions of desire make these four bundles seem quite similar to one another and quite different from the previous categories.

These bundles, though, are harder to summarize than the previous categories, in that all of them end in the word “to,” which requires a subsequent verb that can significantly shift the meaning of the expression. For example, in Table 4, I organized the 28 instances of the bundle *I want you to*, based on the verb that came next. In general, the professor is using this expression to communicate his expectations about wanting students to understand a proof (14/28 phrases). But, he does also use it more as a directive to tell them what to do (5/28 phrases), which is how the bundle was largely used in the Secondary Mathematics Classroom corpus (Herbel-Eisenmann & Wagner, 2010).

Similarly, when he says *I don’t want to*, 10 of the 15 times are related to his not wanting to do part of a proof (e.g., “*I don’t want to* do this [computation], you guys can do it”), often calling to mind time pressure and constraints, but the other 5 uses are quite different (e.g., “So how could I have gotten rid of this 1, if *I don't want to* use 1?”).

Table 4. Verbs following "*I want you to*"

Type of Verb	**Count**	**List of verbs**	**Example in context**
Understanding	14	think about, see, try to prove, try to see, play with, use your intuition	“Before I start [the proof], *I want you to* think about it in your own mind.” (12/9)
Attention	6	keep in mind, remember, ignore, go back to	“*I want you to* keep in mind these three diagrams” (10/12)
Action	5	do, make, find	“*I want you to* do the same thing when x-zero is greater than x” (11/25)
Communication	2	tell me, express	“And that’s what *I want you to* tell me” (11/11)
Certainty	1	be sure	“I don’t want you to think, *I want you to* be sure” (10/8)

Although hard to summarize, overall, these desire bundles can be read through storyline A as the professor giving directives and explaining his choices, on the basis of his authority as the teacher (except *I want you to*). They almost sound like the professor claiming his right to make decisions and give directions according to his desires and whims (i.e., I am not doing this part of the proof because I don't want to; you should prove this because I want you to). The students once again are positioned as relatively powerless in relation to mathematics; mathematics happens according to the desires of the professor.

On the other hand, through storyline C these bundles read much more openly, as the professor offering an account for the decisions and directions he is giving, because it is his duty to be accountable to the students. These bundles often seem to be used as a way for the professor to be clear about what students should be able to do and understand on their own (*I don't want to, I want you to, if you want to*), and what they should still need him to explain (*I would like to*). In this way the mathematics is happening in a shared space, where students have the right to understand the choices made by the professor and to know what they are accountable for understanding about the mathematics. Using expressions of personal desire also conveys a general (and perhaps surprising) sense that human desire is an important component of how mathematics happens.

Similarly to the checking in bundles, these desire bundles are also fairly irrelevant to Storyline B about mathematics as an axiomatic system, since desires and wishes do not figure prominently in that narrative where logic and reason are dominant.

Bundles spoken by students

Interestingly, of the 20 lexical bundles examined here, students only ever say six of them, all of which are either mathematical relations bundles or intention/prediction bundles. It is relatively unsurprising that students do not use the checking in or desire bundles, which are phrased using pronouns that only make sense for the professor in this classroom. It would be fairly strange and unexpected in this class for a student to say, "*I want you to* explain that again," and it would be almost nonsensical for a student to ask, "Is that clear for everyone?" (Although, one can imagine another mathematics classroom in which these are less surprising utterances for students.) Yet the mathematical relations and intention/prediction bundles feel so authoritative as assertions about mathematics that it might also seem unreasonable or unexpected for students, who generally seem to have little agency in relation to mathematics, to use them. Examining the instances of students speaking these bundles is therefore very interesting as a way to understand more about students' positioning in relation to mathematics in this classroom.

When examined in context, the mathematical relations bundles came up three times, when students were giving formal, textbook-like definitions or steps of a proof. For example, on September 16th, Nico used a mathematical relations bundle when giving a definition in response to the professor's question:

> Professor: What does it mean for S to be bounded above? So this is what? (Writing on board) What does it mean for S to be bounded above?
> Nico: There exists a number-
> Professor: -So there exists a number m-one
> Nico: Uh that is greater *than or equal to* every element in S
> Professor: So that x is *less than or equal to* m-one for all x in S.

And on October 28th, Jie used a mathematical relations bundle while giving the steps of a proof as the professor writes them on the board:

> Professor: So let me sort of see your argument and then let's come back to this and then we'll prove it this way again. Okay go ahead, tell me what you do
> Jie: Uh it was like, if there is a c
> Professor: (Writing on board) If there exists a c
> Jie: Then f'(c) *is not equal to* 0
> Professor: Oh we have already f prime of uh-
> Jie: f prime of c
> Professor: Oh so (Writing on board) f prime of any other c, okay
> Jie: Of c is not equal to 0. So if c is positive, f prime of c is gonna be greater than zero, because f prime is [unclear]
> Professor: (Writing on board) So if c is bigger than zero, than f prime of c will be bigger than zero

The intention/prediction bundles came up four times, when students were giving ideas for proofs, rather than more formal steps, or generally expressing more tentative understandings. These statements seemed to belong more to the student, as an expression of their own understanding, rather than a more direct replication of a memorized fact or proof from the textbook. For example, on September 9th, Damien used one of these intention bundles in responding to a question from the professor about how to get started on a proof:

> Professor: So what am I supposed to do here? I'm supposed to give- to be given any arbitrarily small number epsilon, right, positive number, and I need to figure out an n for which this statement holds, right? Mhm?
> Damien: Um we can assume that a is true and we can choose epsilon *to be equal to* one over c. No?
> Professor: Okay uh I think you're pretty close…

Similarly, on October 12th when Nate was giving an extended idea for a proof, different from that in the textbook, he used a prediction bundle:

> Professor: Okay? You all with him? (Pause) So he's saying that if the absolute value of u is positive, if this is in absolute value

greater than that guy, then this is going to be strictly positive. But if not?

Nate: Then v is greater than u, and v *is going to be* greater than that sum, so then v times v *is going to be* greater than u times that sum

Professor: So don't you want to use this part in that case?

Nate: Oh yeah

Professor: You solved it right, I mean, the way you're doing it is fine, but I'm just- there's probably another way of doing it…

Looking across the instances of students using these bundles, it seems that for students to take up these phrases positions them as mathematical doers in an important way. There may be a subtle difference between students using the mathematical relations phrases, which seem to position them more as mathematical replicators, and the intention/prediction phrases, which position them more as mathematical thinkers. But there is still a general sense that these phrases have a certain mathematical power, and that students speaking these phrases is one way for them to claim or take up some of this mathematical power. These phrases were used by six students, all of whom were significant participants in class in terms of interactions around proofs, supporting the sense from these bundles that these students tended to position themselves as relatively confident mathematics doers.

Conclusion

Having examined these categories of lexical bundles using positioning theory and discourse analysis tools, we can now return to the third research question: How do frequently used phrases across the semester position students in relation to mathematics? In looking back at the intersection of bundles and storylines (Figure 9), it is notable that all of the bundles (again, except *if you want to*) can be read within storyline A, in which the students are positioned as relatively passive consumers of the professor's mathematical explanations. This is not to say that these phrases are inherently problematic and should not be used; indeed, it would be hard to imagine a mathematics class that did not use phrases like *is not equal to*. Rather, it is the repetition and strength of these phrases that I want to draw attention to. It is important to recognize that these subtle, unconscious patterns of language use can be seen as an undercurrent that reinforces the professor's mathematical authority while closing options for students' agency in relation to mathematics.

There are also two other interpretations of these lexical bundles that are important to consider, taken from the other two storylines. The first is that the impersonal language of some of these bundles, while potentially seeming remote or even dehumanizing, can be a way for (some) students to try out their voice as mathematicians (when they are invited to enter the mathematical conversation). They are unlikely to use bundles related to personal understanding or desire, which would be a significant deviation from this classroom's norms. But these other bundles, which make no personal claims but instead are expressions of logical relationships between mathematical objects, feel accessible to (some) students. As seen through storyline B, these bundles suggest a direct interaction between the student and mathematics, rather than being mediated through the professor. When a student says, "v *is going to be* greater than u," they are expressing their personal conviction and judgment about the mathematics. Additionally, the fact that both the professor and the students use the same expressions nurtures the germinating bud of an idea that the students can talk the talk of mathematicians.

The other important interpretation to consider is that these expressions reveal a sense of the professor's obligations to the students, particularly in the checking in and desire bundles. When the professor communicates directly to the students (as indicated by the use of personal pronouns in these two categories of bundles), he is not often giving directions or telling students what to do, but rather he is explaining his own decisions and expressing his expectations for students. The frequent use of these kinds of phrases can be seen as way that the professor is reaching out to the students – trying to invite them into his thinking, invite them into a shared space of mathematical creation. This is not to say that the professor is ceding his authority (which would in fact be false to do in this space given the larger constraints of grading and assessment), but rather that he is acknowledging his accountability to the students. As seen through storyline C, these bundles position the professor and students in a shared space of mathematical work; the students are positioned as co-constructors of understanding, with the right to hold the professor accountable for making his thinking and expectations visible.

Stepping back to look across these two chapters, it seems clear that there will not a singular answer to the broader overarching question of how classroom interactions can be seen as positioning students in relation to mathematics in this particular classroom. The different lenses of the three storylines offer overlapping and intersecting interpretations of interactions, both in one moment and in patterns of often-used phrases. From the traditional and expected storyline A, teaching mathematics is an act of explaining, which places mathematics within the control of the professor, to be copied and reproduced by the students. But in storylines B and C, there are ways for students to be positioned in more direct contact with mathematics, by appeal to egalitarian notions of logical and rational thought (storyline B), or by establishing a sense of the classroom as a shared space for doing mathematics (storyline C). To me these different storylines offer a tentative hope, to be discussed in more detail in the upcoming discussion, that small shifts in classroom discourse and interactions can also shift how students are positioned in relation to mathematics, in ways that give students more agency and voice as doers of mathematics.

CHAPTER 7: DISCUSSION

It is worthwhile to take a moment to look back at where we started from, as a frame within which to consider the conclusions of this study. Mathematics is a powerful discipline, as an intellectual achievement of rationality, a gatekeeper within institutions, and a culturally valued expression of intelligence. Given this power, the question of what it feels like to learn mathematics in classrooms can be a particularly fraught one. At the undergraduate level, students who previously had positive relationships with mathematics often experience a sense of rupture and alienation—myself included. Rather than focusing on characteristics of the student as explanations for these negative experiences, this study took classroom interactions as the central focus. I was motivated by the assumption (and hope) that better understanding classroom interactions in an upper-level undergraduate mathematics course would help us to better understand students' relationships with mathematics, and thus be better positioned to potentially shift and improve these interactions, students' relationships with mathematics, and the social-cultural forces that shape them.

Conclusions

In studying the interactions in a particular upper-level undergraduate mathematics course for an entire semester, several kinds of conclusions came to the surface that are nested within one another. The first is that advanced mathematics classes using a traditional lecture format offer few opportunities for students to develop positive relationships with mathematics (similar to Paoletti et al., 2017). The second is that within this traditional lecture format, there is possibility for variation (as found in Mills, 2011; Fukawa-Connelly, 2012), and the professor in this class was able to engage students in different kinds of classroom interactions and establish a relatively comfortable classroom environment. The third is that within these different kinds of interactions, different resources and storylines were available that could potentially position students as doers of mathematics. The fourth and final conclusion, within which the other three are nested, is that constraints on the course made it hard to shift away from a typical lecture format that efficiently covered the most content. Let us walk through each of these conclusions.

In traditional university mathematics instruction, the professor is in control of the discourse, the interactions, and the mathematics. Although prior research has uncovered variation within such lectures in terms of the interaction and questioning patterns, the genuine invitations for students to participate are still quite limited (Paoletti et al., 2017). Thus the more stereotype-like storyline A stills holds, where the professor's role is to explain the mathematics content at hand to his audience of students, who are in turn responsible for processing this content and producing evidence of their understanding in the form of homework and exams. Although students may not be actively participating very often, there is still an array of possible student responses—students can already understand the material and be barely listening or on their phones; students can already understand the material and follow along to confirm their own understanding; students can be confused about the material and listen to the lecture in the hopes of clearing up some confusion; students can be overwhelmed by the material and copy down the work at the board with plan of making sense of it after class; students can shut down entirely and stop listening or taking notes. But the notable dynamic I want to draw attention to here is that the opportunities for students to improve their relationship with mathematics are significantly limited—students with already positive relationships might maintain them, but students who come in unsure about "proofs" or who quickly get lost have really no opportunities for interactions within the classroom that might shift them. The students' access to mathematics in this classroom space is mediated almost entirely through the professor.

Although in broad brushstrokes the classroom I studied could be described as above, an important and striking feature of the analysis was that this professor was doing something different. From learning the students' names to inviting them to participate in proofs to accepting disagreement from students, the professor was able to establish an environment in the classroom that felt safe and accessible for (at least some) students. Consider these three quotes from students who were doing very well, doing okay, and struggling, respectively:

> "[This professor is] one of the best people like for making us feel comfortable. And I think that you're not gonna see it better than what it's been in this class" (Nate, December interview)
>
> "I think that he's probably the first good professor that I've honestly had all of-all of math in college" (Anne, November interview)
>
> "I think that's something I love about [the professor], that he's like respectful, like he'll give ideas like a chance" (Rohit, November interview)

The sense of both appreciation and of unusualness in all three quotes is striking. It suggests that something in the moment-to-moment interactions was allowing students to develop a different relationship with the professor, and, as I will argue below, possibly with mathematics as well.

The close analysis of the patterns of interaction and of the positioning possibilities suggest that there were a few resources available for positioning students as doers of mathematics. The first was students' relationships with the professor; he made it clear that he valued them as people and as students, and invited them to take risks and try out mathematical thinking in this public space. The second was students' prior relationship with mathematics; students who came into the class with a certain sense of confidence and/or a sense of being deeply invested in the mathematics tended to take up opportunities to participate (this claim would need to be further investigated and supported). The other two resources are storylines B and C; these larger narratives about mathematics could allow students to position themselves as logical thinkers (storyline B) and fellow doers of mathematics (storyline C). The first two resources might feel hard to access or draw on because they seemed to live in the professor's personality and in the students' mathematics identities, but these two storylines feel like they have the potential to be more broadly accessible and influential. As a larger question, I wonder: Are there ways to foster these storylines in other upper-level undergraduate classrooms? And do they ring true to professors and to students as potentially helpful resources?

But, a discussion of a classroom in an institutional setting would not be complete without acknowledgment of the constraints, both practical and cultural, on possible interactions. As expressed by one student who struggled, "I feel like a lot of problems I had with the course were not really like [the professor's] fault, but, really the structure of the course. And I feel like the math department needs to do something about it" (Sara, December interview). The first and most obvious constraint on interactions was related to time and pacing; there is a set amount of content the course is expected to cover, in order for students from all sections of the course to be similarly prepared for the subsequent course (*Introduction to Analysis II*). This constraint is by no means a new or unique one; Davis and Hersh (1981) described it several decades ago:

> Ideally, mathematical instruction says, 'Come, let us reason together.' But what comes from the mouth of the lecturer is often, 'Look, I tell you this is the way it is.' This is proof by coercion. There are several reasons for this to happen. First of all, there is the shortage of time. We must accomplish (or think we must) a certain amount in a semester so that the student is prepared for the next course in mathematics or for Physics 15. Therefore we cannot afford to linger lovingly over any of the difficulties but must rush breathlessly through our set piece. (p. 282)

A second and perhaps easier to overlook constraint is the practical and cultural expectations around assessments; the high-stakes exams are required in courses like this one, and the expectation that most students will not do well and that the exams and final grades will be curved is also almost built into these courses. A third constraint is the requirement for this particular course that all mathematics majors must pass the course, regardless of concentration; the content of the course might be applicable to all of the different concentrations, but the manner in which it is taught and assessed is really most relevant only for the pure mathematics students who intend to go to graduate school. These constraints all restrict the available interactions and foster storylines of exclusion and competition rather than of shared mathematics and accountability.

So, how can classroom interactions in an undergraduate mathematics class be viewed as positioning students in relation to mathematics? Within the status quo of limited classroom interactions there is not much space for students to do more than consume or receive mathematics. There are many constraints that make the status quo hard to shift, but there are hints of possibilities for opening up classroom interactions and establishing resources and narratives that can position students as doers of mathematics: students' relationship with the professor, students' prior relationships with mathematics, storyline B about mathematics as an axiomatic system, and storyline C about the classroom as a shared space of mathematical work.

Speaking Back to Positioning Theory

I also want to discuss conclusions of a different kind from this study, which are about speaking back to the conceptual framework of positioning theory and its affordances and limitations. How does applying positioning theory help shed light on what is happening in classroom interactions? Are there aspects of applying it that are particularly fruitful or possibly problematic?

The Affordances

One clear affordance of positioning theory is that it helped me consider issues of power and access in relation to mathematics in classroom interactions. Patterns in classroom interactions, particularly for those involved in them day-to-day, can be taken for granted and normalized to an extent that it is hard to question them, to notice who is privileged by them and who is left out, and to imagine alternatives. Using positioning theory allowed me to place the power dynamics of these patterns of interaction front and center and stake claims to them; it allowed me to attend closely to language and to interactions in this classroom, staying true to the focus that brought me into this work, while providing a framework that allowed me to speak beyond just this one classroom.

Storylines in particular seemed to offer a powerful contribution in this regard. At times I became so caught up in noticing patterns in classroom interactions that it began to feel as though the study was about noticing patterns for the sake of noticing patterns. For example, noticing the shifts in personal pronouns in the interaction analyzed in Chapter 5 was so interesting and exciting to me that at first I was satisfied just pointing them out. Having a conceptual framework, and positioning theory in particular, prompted me to move beyond the patterns of this one interaction and to connect them to larger narratives. Rather than focusing only on the dynamics of Patrick and the professor in that moment, connecting their positioning to storylines was a way for me to consider the larger narratives the shape moments like this one; no interaction occurs in isolation from a conversational and cultural history. As suggested by Herbel-Eisenmann and colleagues (2015), "more attention to storylines can help the field better understand systems beyond interpersonal interactions and how the systems are brought to bear in interactions in mathematics learning contexts" (p. 201).

One other related and significant affordance of positioning theory is the way it can help to connect in-the-moment interactions to longer-term relationships and identities. Anderson (2009) draws particular attention to this idea of positioning over time in considering what it means for a student to be positioned as not competent in a mathematics classroom:

> …somewhere between one action being deemed 'failure' and a person being called 'a failure' lies a discursive process that brings named acts of failing close enough to rub up against the sense of a person as a failure—close enough that it sticks. How many times must a student fail to succeed at math exercises to be considered a failure at math? (p. 291)

Wortham (2004), similarly, considers a student's "trajectory of participation" in order to see how moment-to-moment positioning gradually "thickens" or converges around a particular identity for that student; "The thickening of identity happens across a trajectory of events as certain categories of identity come to identify an individual" (p. 185). Although my study did not focus on the trajectory or identity of a particular student, these ideas about moments "sticking" and "thickening" over time were influential to this study and to future research ideas (see below).

In particular, I began this study with a strong interest in and motivation around students' relationships with mathematics, but focused in the study on students' positioning in relation to mathematics in classroom interactions, a slightly different phenomenon. To me, the two are connected in that positioning in the moment is a resource and a starting point that helps to understand longer-term and more stable relationships with mathematics. Students enter the course with a mathematics identity and with a particular history of experiences and emotions toward mathematics; the interactions they experience in the class are patterns and moments that have the potential to confirm or disrupt or shift their relationships with mathematics; then students leave the course with a different mathematics identity (whether slightly different or dramatically so) and a new set of experiences, whether good or bad. In order to better understand and make claims about shifts in students' relationships with mathematics, it seems critical to have an understanding of moment-to-moment interactions and the positions and storylines that come up in them.

The Challenges

One challenge I experienced in using positioning theory as a conceptual framework was related to identifying speech acts, positions, and storylines in interactions. Herbel-Eisenmann and colleagues (2015) critiqued previous studies of positioning as follows: "the authors do not say how they knew a position or a storyline when they saw it in data. Instead, the authors offered many excerpts of data with interpretations and described general processes of analysis" (p. 191-2). With this critique in mind, I pushed myself to justify and provide evidence for my identification of speech acts, positions, and storylines.

It felt to me that speech acts and positions were somewhat easier to justify using tools of discourse analysis. It was an important challenge to keep pushing myself to be explicit and make conscious decisions about speech acts and positions in each turn of the interaction in Chapter 6, for example, rather than just skating over positioning in general terms. Taking this theory seriously means that it should be possible to map each moment onto this triad; it is not a general framework that sometimes applies and other times does not. But storylines felt much harder to justify. I kept asking myself: How did I know a storyline when I saw one?

This question feels important to answer, as a documentation of my process for other researchers, but also as an acknowledgment of its difficulty. I went into the analysis without particular storylines in mind, but at the same time somewhat "primed" by the preparation I had done - reading other articles using positioning theory, collecting storylines about mathematics while writing my proposal, and in general being a reader of mathematics education literature and an observer of this particular classroom for a semester. In that sense the storylines emerged from a combination of these perspectives, shaped by who I am as a researcher, in terms of what I have read and the experiences I have had. However, of the storylines that could possibly emerge, the ones that I chose and that were given status as applying in this situation needed to be more than just personally referenced. That is why it was important to check their face validity and content validity, which I did by identifying similar storylines in the literature on math teaching and learning and by talking to others.

In doing the analysis I was very actively looking to identify storylines - it was a primary focus in both the analysis of a moment (Chapter 5) and in the lexical bundle analysis (Chapter 6). In that sense the storylines emerged in a similar fashion to any pattern or thematic identification in qualitative analysis; I would consider possible categorizations of ideas, which would be discarded or strengthened depending on how other evidence played out. In the lexical bundle analysis I drew on previous literature in particular to help think through different possible storylines and then adjusted these ideas to fit more closely with my data. In the analysis of a moment I started with the lexical bundle ideas in mind and then really pushed myself to identify at least one storyline-position-speech act that went with each turn, refining the ideas of possible storylines as I went through the event repeated times.

Conversation with others was also an important dynamic of identifying storylines for me. I shared my work on lexical bundles with the Physics Education Research Group; they pushed me to take the storylines up a level and differentiate them more explicitly from positions, and they suggested many alternate interpretations and new storylines to consider. More informally, as I worked I was frequently discussing the ideas with those around me—from family members to fellow graduate students to my advisor. Testing storylines with a variety of people felt important, given that storylines are supposed to exist in the world in a larger sense and should be widely accessible and understandable (although perhaps varying in degree depending on level of exposure to the particular culture of undergraduate mathematics classrooms that is the focus here).

Looking back over this process of identifying storylines, I recognize that it is not an operationalized, repeatable analytic process (which is perhaps true of any type of qualitative analysis). It seems like it may be less important to operationalize the key terms (storylines, positions, speech acts) and how to identify them, and more important to attend to the recommendations from Herbel-Eisenmann and colleagues (2015) for how to speak across and beyond positioning theory work. For example, they recommend being sure to attend to and provide evidence for all three aspects of the theory, describing positions in terms of "rights and duties," being explicit about the timescales of various storylines, and providing multiple interpretations of possible positions and storylines rather than just one.

Another challenge to operationalizing and identifying the three key terms is that positioning theory can be used with a broad array of analytic approaches. Determining an analytic framework(s) appropriate for working within/alongside this conceptual framework did feel challenging and potentially overwhelming (especially as a novice researcher). I find it less important for positioning theory researchers to be prescriptive about how to identify a position or a storyline, and more useful to hear reflections and suggestions about different analytic methods that can be fruitful - including those I used of lexical bundle and discourse analysis. The analysis of interactions can feel unbounded, particularly with a large data set, and one way to place boundaries on it is by choosing an appropriate analytic framework.

Practical Implications

This study has practical implications that speak to several audiences: instructors of upper-level undergraduate mathematics courses (that is to say, mathematicians); mathematics departments; and perhaps teachers and learners of mathematics more broadly.

For Mathematicians

My hope for mathematicians is that they would take away a better sense of how to meet students halfway. Mathematicians often seem to perceive their students as unwilling or unable to do the mathematically rigorous work of proofs. The results of this study suggest that mathematicians should be pushed to question this narrative about students, and to recognize aspects of their own pedagogical and discourse choices as mattering, and shaping students' relationships with and access to mathematics. As suggestions from this study, for example, learning students' names and using them is a small but vital tool for building relationships with students; inviting students to contribute ideas to proofs is a way to open up the classroom to student thinking but takes time and can be risky for students. After reading the portrait of the semester, the professor of this class reflected: “There are things I think I should do (more engagement and giving more students a say in course issues) and others I should be careful about, like how casual and informal I am about grades” (personal communication, 3/3/2017). Reflections such as these feel like important directions and small steps towards mathematicians considering students’ course experiences as important and considering their role in these courses as important (beyond delivering content and assigning grades).

For Mathematics Departments

For mathematics departments, this study raises many questions about the constraints on courses like this one (i.e., required upper-level proofs-based courses). The struggles students experience in such courses are often addressed by focusing on the prerequisite course (Barr & Tagg, 1995); for example, at this university the suggested prerequisite course was recently made into a required prerequisite, and the content of this prerequisite course now covers the first few chapters of this course so that students see the material twice. This focus suggests that students are coming in unprepared, or that students are coming into the course who “should not” be there. Rather than focusing on the students as the source of the problem in this way, I argue that departments and instructors need to reflect on the way structures within that department constrain students’ opportunities to learn and to develop productive relationships with mathematics. This study suggests a few important structures to question, including whether the content coverage is appropriate and necessary, and how to make grading practices more transparent and consistent.

Another important consideration is how to meet the needs of students from all concentrations/tracks who are required to take the course. A preliminary study I conducted using the student interview data suggested that students who were not in the pure mathematics concentration experienced negative shifts in their overall mathematics confidence and identities from the beginning to the end of the course (Fleming, 2016). Secondary education majors, in particular, typically view courses like this one as unhelpful; some researchers are exploring changes that could be made to adapt a real analysis course to meet their needs (Wasserman, Weber, & McGuffey, 2017). Of course, creating separate courses for each of the separate concentrations raises its own concerns – about the feasibility of staffing and enrolling appropriate numbers in each course, about the equivalent "rigor" of different versions of the course, and about boxing students into a particular concentration. It seems, therefore, to be an open and important question for departments to consider about whether and how to re-structure upper-level courses like this one and/or the requirements for the major.

One final consideration for mathematics departments is about the professional development that is available or required for instructors. Although professional development resources are available (e.g., programs from professional associations like the Mathematical Association of America), the main pedagogical training that mathematicians are likely to have received was during their graduate programs, perhaps as an orientation or a summer workshop (as suggested in Speer & Hald, 2008; research on this topic is scarce). This reality is an extreme disservice to both the instructors and, more importantly, their students. Given the wealth of knowledge about empirically validated teaching practices and their potential to improve student learning outcomes (e.g., active learning in Freeman et al., 2014), it is critical that more of these practices are given attention and support, and that instructors are supported in implementing them (PCAST, 2012). The results from this study, in particular, show that particular discourse choices can matter significantly for positioning students differently in relation to mathematics. I could imagine implementing a professional development experience for instructors in all sections of this course where instructors video-record a "typical" lesson or two and then we analyze these videos together to identify common discourse practices (e.g., question types, responses to students, pronouns, common phrases) and discuss the possible impact of these practices and ways to potentially improve them (for a similar example in K-12 professional development see Herbel-Eisenmann & Cirillo, 2009). As a related or separate idea, I could imagine using the portrait of the semester as a starting point for facilitating a conversation with instructors (building on work with cases in K-12 professional development as described in Smith & Friel, 2008; Stein, Smith, Henningsen, & Silver, 2009). This narrative (or portions of it) could be used as an exemplar case to start discussions with professors (and possibly with students as well) about what they think is possible in such classrooms, what improvements they would like to see in their own classrooms, and perhaps about what it means to teach and learn mathematics in this classroom setting (drawing on the three storylines).

For Mathematics Teachers and Learners

At the broadest level, the three storylines described in this study exemplify certain views about the teaching and learning of mathematics that I think would be worthwhile for all teachers and learners of mathematics to consider. There are many taken-for-granted assumptions in classrooms about what it means to teach mathematics, to learn mathematics, to do mathematics, to do school mathematics, about what mathematics itself is, and so forth. I argue that noticing, naming, and discussing these assumptions is a powerful step in addressing conflicts that arise in classrooms and in imagining different possibilities of what classrooms can look like. The storylines that are taken up in classrooms to position teachers and learners matter for who has access to mathematics, who gets to do mathematics, and who is allowed to "belong." By making reference to common narratives about mathematics, we can begin to question how teachers and students are positioned in relation to mathematics (e.g., Tait-McCutcheon & Loveridge, 2016), how mathematics education is discussed in public conversation (e.g., in the media, see Rodney, Rouleau, & Sinclair, 2016), and what the future of mathematics education could look like.

> How can mathematics be taught humanely, that is, as a human endeavor, calling upon human powers and corresponding to individual desires and hopes? (Mason, 2001, p. 83)
>
> We [must] tell a story that puts a human face to participation in mathematics. (Thomas, 2001, p. 35)

Future Research

Several lines of future research emerge as fruitful possibilities to pursue. This study focused on the question of positioning in relation to mathematics and identified three storylines that were particularly relevant in this classroom. Examining positioning in other mathematics classrooms, particularly at the undergraduate level, with these storylines in mind would be an interesting exploration of their validity and usefulness. In describing these storylines as potential resources earlier, I wondered about ways to foster these storylines in other upper-level undergraduate classrooms, and whether they seem helpful to professors and to students. Conducting a study in another classroom(s), along with interviews with professors and students, would be a worthwhile way to further flesh out and test these storylines as resources.

There are also many other dynamics of positioning and storylines that could be explored. For example, I think storylines about race and gender (i.e., Whites and Asians are good at math and other races are bad at math; men are good at math and women are bad at math; women are quiet and submissive) would be interesting to analyze in the context of this classroom, because some of these storylines felt disrupted while others felt reinforced. For example, Nico and Damien, both students of color, were the most active participants and were seen as competent, but not as competent as a Korean student, Joon, the only student who did not participate for the entire semester, or as Griffin and Oliver, students who did not take notes and whom I would identify as White or mixed race Asian students. Jie, a Chinese student, was a strong female participant, but her contributions were often dismissed by her peers in interviews, perhaps because of her perceived status as a non-native English speaker.

Arielle and Sara, both women of color, were paid a lot of attention by the professor but in ways that often seemed to backfire and solidify their status as struggling students. Particularly for the students with whom I did interviews, I think that exploring these questions around classroom interactions, positioning, identity, and issues of race/gender would be powerful. Tracing the positioning of one student over the semester would be particularly interesting, in order to see how the moment-to-moment dynamics relate to that student's more stable identity and relationship with mathematics (as described earlier in Anderson, 2009; Wortham, 2004).

Another direction for future research using these or new data would be to consider authority as a particular theme. One particular kind of interaction that would be interesting for illuminating issues of authority is the professor's jokes in the classroom. The element of humor has been identified as an important component of how lectures are perceived (Fritze & Nordkvelle, 2003 cited in Bergsten, 2007). Jokes can be a way to bridge the gap between those with power and those without, but they can also reinforce or emphasize the breadth of that gap. This professor's particular jokes often showed his own tension or discomfort around being the ultimate authority in terms of grading and assessing students—a storyline about authority that goes beyond the classroom and into the structures of the department/university/school more broadly. There also seemed to be a potential cultural clash between the professor and the students around these jokes; the awkwardness had the feeling of something perhaps being lost in translation. Given the repeated nature and particular format of these jokes, as well as the different kind of authority and cultural storylines they reference, it would be an interesting focus for further development of ideas about authority in the classroom.

One final idea for future research would be to expand data collection on classroom interactions to a larger set of upper-level classrooms, across different professors and perhaps different content areas, with a more limited immersion over the semester (i.e., observe and record the class for a randomly selected 3-5 lessons over the semester). This expanded data collection could be useful in several ways. For example, it could be used to develop a corpus of classroom transcripts at the undergraduate level (in a similar manner to and for comparison with the Secondary Mathematics Classroom corpus in Herbel-Eisenmann et al., 2010). It could also be used to identify broader patterns in discourse in interactions. For example, Paoletti and colleagues (2017) categorized 11 lecturers' questions in upper-level proofs-based mathematics courses; I could try out their coding framework using these data and see whether similar patterns emerge, and provide feedback on their framework. Or, I could code for discourse moves developed in K-12 classroom contexts (e.g., Chapin, O'Connor, & Anderson, 2009) and consider the patterns in these moves that are evident in an undergraduate classroom as a way to advance our understanding of discourse moves and of discourse practices in lecture-based upper-level mathematics courses. Examining interactions at a larger scale, rather than at the level of a single classroom, would certainly have a lot to offer in terms of expanding our understanding of what is happening in upper-level undergraduate classrooms and how it can be improved.

APPENDICES

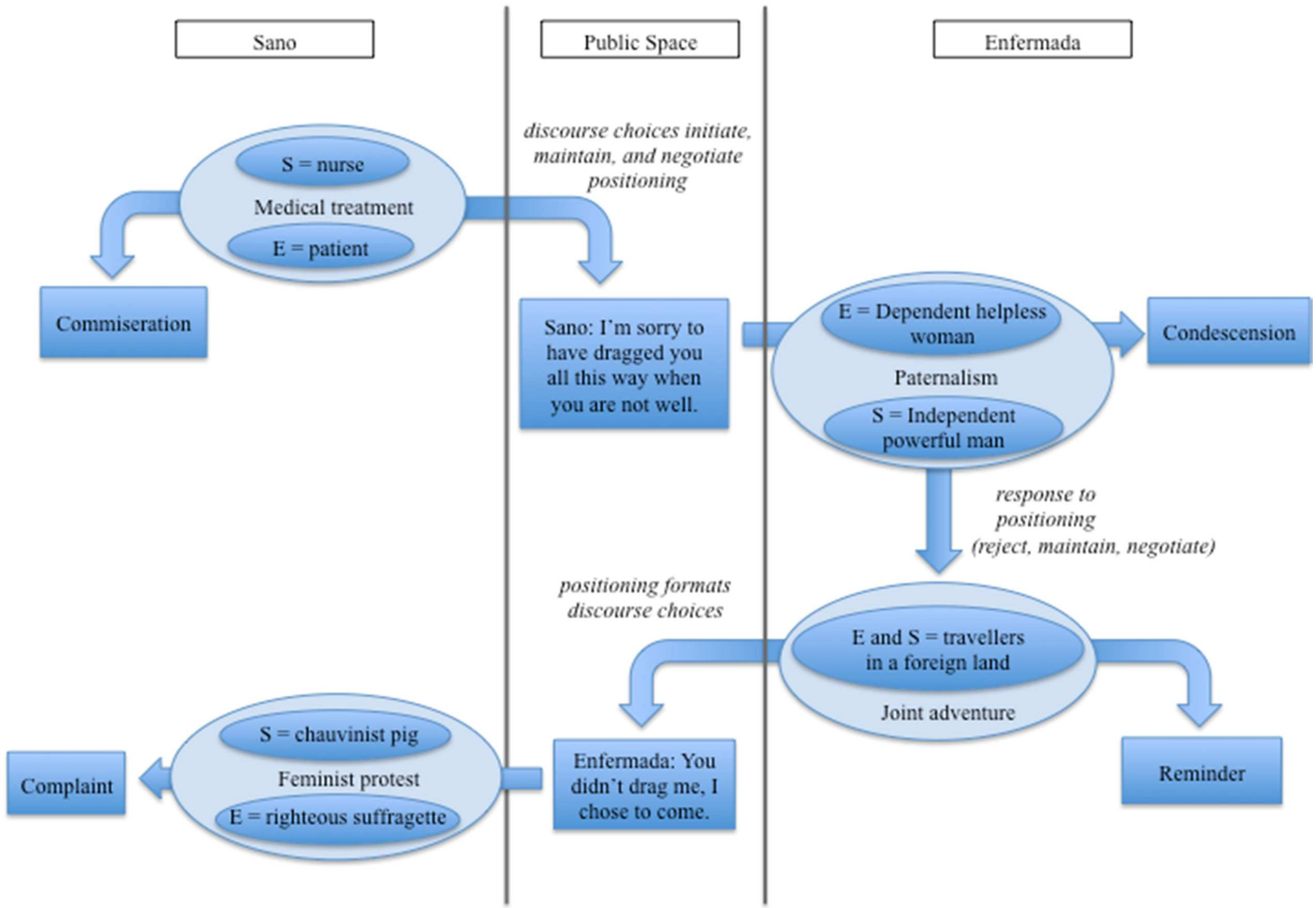
Appendix A – Worked Example of Positioning Theory

This appendix contains a worked example of positioning theory, along with my own diagram.

Example (Davies & Harré, 1999, pp. 45-47):

The best way to recommend our proposal is to demonstrate its analytical power in a worked example. The story is about two characters, Sano and Enfermada, who, at the point the story begins, are at a conference. It is a winter's day in a strange city and they are looking for a chemist's shop to try to buy some medicine for Enfermada. A subzero wind blows down the long street. Enfermada suggests they ask for directions rather than conducting a random search. Sano, as befits the one in good health, and accompanied by Enfermada, darts into shops to make enquiries. After some time it becomes clear that there is no such shop in the neighborhood and they agree to call a halt to their search. Sano then says 'I'm sorry to have dragged you all this way when you're not well'. His choice of words surprises Enfermada, who replies 'You didn't drag me, I chose to come', occasioning some surprise in turn to Sano.
[...]
What speech-acts have occurred? To answer this question we have first to identify the storylines of which the utterances of S and E are moments. Only relative to those storylines can the speech-actions crystallize as relatively determinate speech-acts.

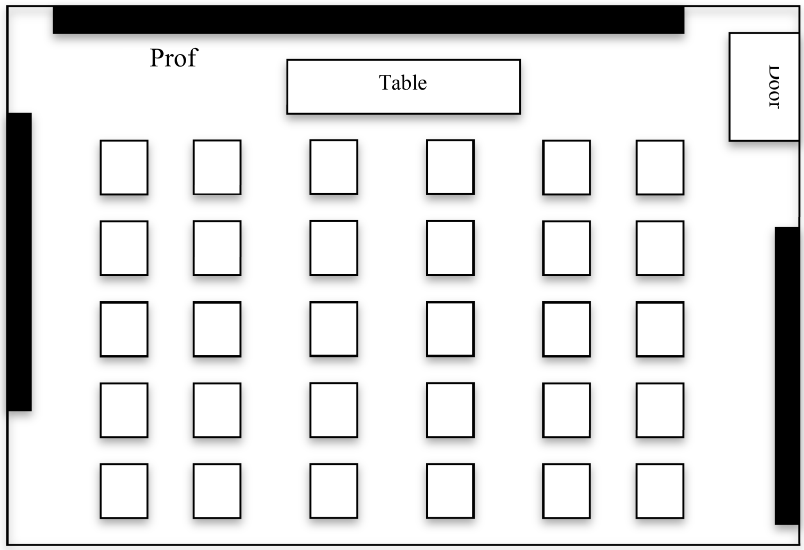
> *S's line as perceived by S*: medical treatment with associated positions on S = nurse and E = patient. In this story the speech act of "I'm sorry..." = commiseration
>
> *S's line as perceived by E*: Paternalism with associated positions of S = independent powerful man and E = dependent helpless woman. In this story the speech act of "I'm sorry..." = condescension.
>
> *E's line as perceived by E*: joint adventure with associated positions of S and E as travellers in a foreign land. In this story the speech act of "You didn't drag me..." = reminder in relation to this storyline
>
> *E's line as perceived by S*: feminist protest with associated positions of S = chauvinist pig and E = righteous suffragette. In this story the speech act of "You didn't drag me..." = complaint


Sano
Public Space
Enfermada
discourse choices initiate, maintain, and negotiate positioning
S = nurse
Medical treatment
E = patient
Commiseration
Sano: I'm sorry to have dragged you all this way when you are not well.
E = Dependent helpless woman
Paternalism
S = Independent powerful man
Condescension
response to positioning (reject, maintain, negotiate)
positioning formats discourse choices
E and S = travellers in a foreign land
Joint adventure
Reminder
S = chauvinist pig
Feminist protest
E = righteous suffragette
Complaint
Enfermada: You didn't drag me, I chose to come.

Appendix B – Observation Protocol

Date: Class Size and Make-up:

Seating Chart:



After-Class Summary

Timeline:

Start Time	**End Time**	**Description of Activity/Interactions**

Overall, was this class a "representative lesson" – did it follow the typical arc? Why or why not?

Time	Observation Notes	My comments	Of Note

Of Note column abbreviations:
IQ – Instructor Question
SQ – Student Question
SC – Student Comment
★ – Significant Event

Appendix C – Interview Participants

List of Interview Participants:

Students	Interview A	Interview B	Interview C
Rohit	X	X	X
Michael	X	X	X
Damien	X	X	X
Nico	X	X	X
Anne	X	X	X
Jamie	X	X	X
Arielle	X	X	X
Sara	X		X
Isaiah	X		X
Patrick		X	X
Nate		X	X
Joon			X

Participation Categories, other notes for interview selection:

Yellow – Asked in first round and agreed.
Red – Asked in first round, no response.
Green – Added in second round (significant shift in participation)
Blue – Added in third round (only student who was never called on)

Participation Category (after 2 weeks)	Students who gave consent	Gender	Race	Other Notes
Often	Nico	M	Latino	
	Damien	M	Black	
	Ryan	M	White	
Sometimes	Anne	F	White	
	Sara	F	Indian - mixed	
	Arielle	F	Black - mixed	Education concentration
	Paul	M	White	
	Patrick	M	White	
Once	Rohit	M	Indian	
	Michael	M	White	Friends with Jamie
	Nate	M	White	
	Jared	M	White	
Not at all	Jamie	F	White	Friends with Michael
	Joon	M	Korean	
	Jie	F	Chinese	
	Isaiah	M	Black	Dropped/Withdrew
	Becky	F	White	Dropped/Withdrew
	Esther	F	Chinese	Dropped/Withdrew
	Melissa	F	White	Dropped/Withdrew

Appendix D – Interview Guiding Questions

Interview A (beginning weeks of semester) – Getting to know you

1. I'd like to start by getting some background information about you. So please tell me a little about yourself – such as your year in school, what other math courses you've taken, and how you decided to be a math major.
    a. (Follow-up) How did you end up in this section of the course?
    b. (Follow-up) What are your plans for after this course – in terms of other courses you want to take, or in terms of bigger career ideas?

2. So now I'd like to hear more about your experience in the course – how's it going? What's it like for you so far? What do you like, what do you dislike about your experience so far?
    a. (Follow-up) How do you like his calling on people during lecture?
    b. (Follow-up) What were you expecting coming into the course – and is it going how you were expecting?

3. Thinking more specifically about your experience in the classes so far – is there anything you've noticed while in class, such as a particular moment that stands out in your mind? Maybe a story you've told to your friends or your family about classes so far? Or just something interesting, or a moment you were proud of?
    a. (Prompt if needed) What did you think about lecture today/yesterday?
    b. (Sometimes follow-up) How have the homeworks and quizzes been so far?
    c. (Sometimes follow-up) Have you been to office hours so far?
    d. (Sometimes follow-up) How have you been studying for class? On your own?

4. We're just about done, but I want to ask a few more broad questions about you before we finish. In general, how would you describe yourself in relation to math? Or as a math student?
    a. (Follow-up) Has that been consistent, or has it changed over time?
    b. (Follow-up) How would you describe your comfort or confidence with math?

5. Is there anything else you'd like to tell me or think I should know in terms of your experience in the course so far?

6. In terms of demographic information, would you tell me your age, and the race or races with which you identify?

Interview B (middle of semester) – What happens in class

1. This interview I'd like to dive right in to hearing about your experience in the course – how's it going? What's it been like since we last talked?
    a. (Prompt if needed) What do you like, what do you dislike about your experience so far?
    b. (Follow-up) Is the course still going (not) how you were expecting?

2. Is there anything you've noticed while in class these last few weeks, or from the semester so far – are there any particular moments that stand out in your mind, that maybe you've talked about with friends or other students in the class?
    a. How do you approach the homework – do you go online at all?
    b. How do you feel feel about the grading and feedback on the homework? Are you learning from it?

3. I'm going to share with you now a few moments that I've noticed, and ask for your take on them. (Play audio clip – see table below)
    a. First, can you just summarize for me what happened in that clip? What would you say was going on here?
    b. Do you remember this moment happening? If so, do you remember how you felt about it at the time? If not, how did you feel listening to it just now?
    c. Would you say this is typical – in terms of this class? In terms of other math classes?

4. Is there anything else you'd like to tell me or think I should know in terms of your experience in the course so far?

Students	**First Clip**	**Second Clip**	**Third Clip**	**Fourth Clip**
Rohit	10/14 After 1st exam	10/30 Anne and Nate	11/4 Rohit	
Michael	10/14 After 1st exam	10/28 Arielle and Michael	11/4 Patrick	
Damien	10/14 After 1st exam	10/28 Arielle and Michael	10/30 Sara and Nate	
Nico	10/14 After 1st exam	10/26 Nico	10/28 Nico	10/30 Anne and Nate
Anne	10/30 Anne and Nate	10/28 Nico	10/14 After 1st exam	
Jamie	10/14 After 1st exam	10/21 Jamie	10/23 Patrick	10/30 Anne and Nate
Patrick	10/23 Patrick	11/4 Patrick	10/14 After 1st exam	
Nate	10/14 After 1st	10/23 Patrick	10/28 Arielle	11/4 Rohit

	exam		and Michael	

Interview C (end of semester) – Your course experience overall

1. I'd like to begin this interview by hearing about your experience in the course overall – how was it? What was it like for you, overall? How does it feel to be (almost) done?
    a. (Prompt if needed) What did you like, what did you dislike about your experience?
    b. (Follow-up) Did the course end up going how you were expecting?
    c. (Follow-up) How was the third exam? How did you feel about the review for that exam?
    d. (Follow-up) How have you felt about the pace these last few weeks – leading up to and then after Thanksgiving? Has it felt different to come to class, or not really?

2. Is there anything you've noticed while in class these last few weeks, or from the whole semester – are there any particular moments that stand out in your mind, that maybe you've talked about with friends or other students in the class?
    a. On 11/23, Monday before Thanksgiving [12:02], professor was asking about how to establish that a function is integrable. James started explaining it but stumbled, the professor said he should be able to say it well, and then James wrote it on the board. Do you remember that moment? What did you think about it?
    b. (If applicable) Do you remember this particular moment? (See list below)

3. I also want to ask about participation in class overall – I know we talked about it some, but I wanted to get an overall sense, How you felt about participation in class? For yourself or for other people?
    a. (Prompt if needed) Did this class feel different for you from other math classes?
    b. (If needed) Did you feel comfortable participating in class?
    c. What are your thoughts on taking notes in class as a way of participating?
    d. If you had to categorize students in the class or organize them in some way, how would you do it? Is there a natural grouping that comes to mind?
    e. (Sometimes follow-up) Are there students in the class that stand out to you?
    f. (Sometimes follow-up) I felt like there was more diversity in this class, like there were more women than I was expecting, and more people from like different majors, and I wondered if it felt that way to you at all, or if you thought about it at all?

4. Thinking about the course overall,
    a. What advice would you give to students going into this course?
    b. What suggestions or improvements would you make to this course?

   c. What are your plans now that this course is done – in terms of other courses you want to take, or in terms of bigger career ideas?

5. In the first interview I asked about how you would describe yourself in relation to math or as a math student, and I was wondering if your experience in the course has changed that in any way?
   a. (Follow-up) How would you describe your comfort or confidence with math now?

6. Is there anything else you'd like to tell me or think I should know in terms of your experience in the course so far?

(Shut off recorder)
De-briefing: Thank you very much for doing this. This is what I'm thinking and what I've noticed so far – it's been so helpful to hear your story.

Moments Used in Question 2b of Interview C for Specific Students

Nate: 12/7
- Professor commented at 9:45 [51:45] about being able to read his mind?
- And his question at [56:30] about pointwise convergence of particular function. What prompted him to ask question in that moment

Patrick: 12/7 – his question at 9:20 M2 "Wasn't that because it's monotone? Could always extract monotone subsequence?" Professor asks what he means – writes up $n_1 < n_2 < \ldots$ - M2 adds equals signs – Professor then writes up $a_{n1} < a_{n2} < \ldots$ - and M2 says no that is what he meant, that you can always get a monotone subsequence, "Wasn't that one of our first theorems?"

Nico: 12/7 – his contribution at 9:15. Asks how to do Lemma 9.3, M16 gives an idea → that after N, it's bounded by epsilon. And only finitely many before that. Professor likes the idea, writes up Cauchy definition on board. M16 "Wouldn't you choose the max from 1 to index N?"

Appendix E – Coding Table

Code	Description	Count[8]	Example (of the lowest level sub-code)
Answered a question	The professor asked a factual question and a student responded	255	
Answered non-math related question	The professor asked a question not related to math content and a student responded	35	Professor: So last time I said I want you to do number twenty- is it twenty-two or twenty-three? Do you remember? Michael: It's twenty-three Professor: Twenty-three. (10/28)
Answered key idea question	The professor asked a question about the main idea needed in a proof (i.e., "key word") and a student responded	38	Professor: So what's the key word here? Why is this true? In just a single- uh let's see (Pause) in three words (Some laughs) Luke: Mean Value Theorem? Professor: Mean Value, no! That's what we're going to prove! (10/23)
Answered another kind of question	The professor asked a different kind of factual question and a student responded	182	Professor: So what would be the difference? The maximum of a set and the supremum? Patrick: The maximum necessarily lies inside of the set Professor: So the maximum will be inside the set and the supremum doesn't have to be inside the set. (9/9)
Contributed to a proof	A student contributed an idea for a step or approach to a proof	210	
Correct	A student contributed an idea to a proof and it was evaluated (implicitly or explicitly) as correct by the professor	80	Professor: So how do you prove it? (Pause) Michael: Suppose there is one Professor: Okay. So, proof. Okay um, so suppose there exists k in Z so that k is in n, n plus one. (9/9)

[8] The count for a high-level code is the sum of the counts for its sub-codes.

Partially correct	A student contributed an idea to a proof and it was evaluated (implicitly or explicitly) as having some parts correct but some parts incorrect by the professor	64	Anne: Um so if b is the least upper bound, then you know that b minus one half is gonna be in that set. So if b minus one half is in it, then b minus one half plus one will also be in it, cause it's natural numbers Professor: Okay. You start very well (Laughter) but there was a small thing. (9/9)
Incorrect	A student contributed an idea to a proof and it was evaluated (implicitly or explicitly) as incorrect by the professor	61	Professor: So I'll define a new function for which the assumption on that side will sort of have to be true. So what's that function g? Nate: It would be equal to f minus b… Professor: So- so- let- why- why don't we try this way? (10/23)
Not clear	A student contributed an idea to a proof and it was not clearly evaluated by the professor	5	Professor: So what can you say from here? (Pause) Damien: c(z) equals s prime- Professor: So the first thing I can say is that this, s(2) in absolute value is two absolute value of c(z). (11/11)
Checked understanding	A student asked a question or made a comment to the professor	143	
High-level		83	
Checking own understanding	A student asked a question or made a comment to confirm their understanding of a step in the proof or of a connection to something they already know	29	Professor: Either the maximum will be at an interior point or the minimum will be at an interior point. Both cannot be at- at the end point. Yes? Patrick: So since f(a) equals f(b) the only case where the maximizer and minimizer will both occur at the end points is when it's a horizontal line Professor: Exactly, and that's the only case that we just have to throw out. (10/23)
Different approach to proof	A student made a comment or asked a question about taking a different approach to a proof	12	Damien asks if you could do the problem by choosing two sequences y_n and x_n Professor: What do you think? Damien: That would work as well, right? Professor: That would work as well, you answered your own question (10/17)

Different understanding from professor	A student made a comment or asked a question that expressed a different understanding from the professor	24	Nate: Don't you need it to be twice differentiable on [a,b]? Professor: No, just differentiable Nate: Oh, okay (10/30)
Boundary testing	A student asked a questions about the domain and range of a definition or a proof	18	Griffin: Wait so, is anything ever differentiable on the end point? […] Professor: Oh yes! I could have sort of chosen like (Writing on board) uh sorry- 0 to 1, and to R, and just assigned x to x^2, right? The function x^2 is differentiable everywhere, so it could- just sort of depend on the function. So the only thing I don't care about here is that, the end points are the two things that I don't care about Griffin: But when we were defining differentiability- Professor: Okay so, so you have a good point… (10/23)
Low-level question	A student asked a question of confusion about a proof or definition (i.e., How do you get this - why is it this - is it this or this)	17	Professor: We good? Michael? Michael: It was just um, when you made $f(x_0)$ zero up there, I was just- Professor: Oh because here I said (Writing on board) f-k-zero less than k less than or equal to n minus one, so f zero, the zero-th derivative of the function is just the function itself. So saying that this at x-zero- is x-zero is zero, that's what this statement means Michael: Oh okay (10/28)
Clarified board work	A student asked a question or made a comment to clarify work on the board	18	Jie: Is that supposed to be n minus 1? 2 to the n minus 1? Professor: No I think this is- so I have to repeat this one n times, so it should be to the power n, because b_1 minus a_1 is b minus a divided by 2. Jie: b_1 minus a_1 is b minus a. Because you said a_1 equals to a and b_1 equals to b Professor: Thank you. (Correcting work on board) (9/28)

Homework/exams	A student asked a question about expectations for homework or exams, or asked to review a particular problem for homework or an exam	25	Professor: So you have five minutes for the quiz. Should be easy to do. So remember to put your name on it. Griffin: Wait so, we just have to compute the derivative? Professor: Yes that's all Griffin: Okay (10/23)
Called on by the professor	The professor called on a specific student by name or by gesturing. The student does not need to respond.	82	
To contribute to a proof	The professor called on a specific student to ask them to contribute or help with a proof	45	Professor: I need some help. How do I get here? (Pause) Nate? (9/28)
To see if student is following	The professor called on a specific student to see if he/she is following or has any questions	37	Professor: So S stops exactly before m, at m. Arielle? Is that good? Yes? (9/9)
Responded to another student	A student responded directly to another student, without the professor speaking in the interim	5	Professor: So what do you call a monotone subsequence? Patrick: Wasn't that one of our theorems, we can always find- Oliver: We only have plus or minus one though. Or is it one and zero- we only have the numbers one and zero, though. (11/4)

Appendix F – Memo about Theme One

Interactions in the class are often organized around the goal of students learning to do proofs. I see this goal as a process/practice/habit of mind goal, fitting in with the notion of this course as a "mathematical maturity" course or one in which students are held accountable for learning to think "like mathematicians." This goal works well to organize many of the kinds of interactions that occurred in the class.

In establishing this goal as a useful framework for organizing interactions, I found it helpful to first consider the interactions that were led by the instructor related to this goal, and then consider how students took up this goal in different ways in their interactions with the professor. While it is a shared and negotiated goal, I would argue that the professor is principally responsible for establishing the goal in interactions, while the students have a more responsive role in taking up this goal (or not) in interactions.

I also found it helpful to consider how explicitly the goal was addressed in the interactions (e.g., was the professor directly talking about learning how to do proofs). I used the idea of a continuum ranging from most structured to least, in the sense of there being an explicit structure related to the goal of doing proofs, along which I could roughly locate particular interactions and interaction patterns.

Instructor-led Interactions

On the more structured end of the continuum, I placed any interactions where the professor made comments about how he was approaching a proof, about his expectations for what they should be able to prove, or other commentary about how to do proofs. Perhaps the most explicit example of the professor's expectation came in a brief speech he gave during the fifth week:

> So let's try to establish this together. So we've been at this thing for maybe three or four weeks now, so at this point my hope is that you start sort of guessing how we start, at least how we get started on the proof of a statement you have no idea about. I mean this is- this is different from the homework, I want to take you on the spot, and try to see if you can help me- guide me through what are the different steps in trying to establish this theorem. I want you to start sort of developing your intuition about what is it that we want to do. (Pause – 2 sec) So I don't know- I don't know what we need to use yet- well okay I do, but if I were you in your position, what things are there- what is it that I'm going to have to use? (9/30 – 9:18 – 21:00)

In this speech the professor made it clear that these frequent interactions around completing proofs together were intended as an exercise for students to become comfortable doing proofs, particularly in the area of getting started on a proof, which students consistently said in interviews was the most challenging.

Similarly, two classes later, after writing up a theorem about a continuous function from Chapter 3 in the textbook, the professor tried to be explicit about the steps to take in order to start a proof:

> So I need somebody to help me get started. (Pause – 3 sec) (Erasing) So in any proof, the first thing I'll always do is to write down what I want to try to prove, and then we're going to try to see how to use each of the assumptions in the statement to prove that. So what is it that we need to prove? (10/5 – 9:20 – 30:00)

This kind of explicit commentary from the professor about how to approach proofs occurred quite frequently prior to the first exam (on 10/9), though it continued to a lesser degree throughout the semester.

In the middle of the continuum are interactions that may not directly refer to learning to do proofs, but are structured around key components of how to do proofs. For example, there were particular kinds of questions that the professor asked during proofs, which by their repetition (and therefore identification as a "kind" of question) conveyed the message that students should be using these strategies to learn to approach proofs. These questions fell into kinds such as: what's the key word, what's your intuition/feeling, what is given, and what are we trying to prove.

Focusing on key idea questions in particular, during the third week the professor began using this type of question. He used it three times on September 14th, each of which illustrated a slightly different perspective on this kind of interaction. He used it first after writing up the statement of the theorem that the sequence {1/n} converges to zero:

> Professor: So in one sentence, can anyone tell me why this is the case?
> Patrick: Because of the Archimedean Property
> Professor: Archimedean Property, thank you very much. (9/14 – 9:17 – 26:20)

He used it again closer to the end of class, after writing up the statement of the theorem about the limit of a sum of sequences:

> Professor: Can everybody prove this statement? (Pause – 4 sec) So what's the key thing to use here? Just two words
> Nico: Triangle inequality?
> Professor: Triangle inequality, right? That's all we need, right? […] I don't want to prove it, I think you guys can prove it (9/14 – 9:43 – 54:00)

Finally, with just a few minutes left in class, the professor used this question type again after writing up the statement of a theorem about two sequences converging to zero:

> Makes sense to everyone? (Pause – 4 sec) Clear, right? I shouldn't sort of worry about this, right? So can anyone give me the proof in just one line? (Writing on the board) Proof. (Pause – 4 sec) Let epsilon be bigger than zero… (9/14 – 9:47 – 57:30)

These key idea questions were all asked directly after writing up the statement of a theorem and prior to starting the formal proof, so in that sense the interactions served the function of introducing students to the big idea of the proof and helping them develop a strategy for how to get started. On some occasions, as in the second example above, the professor used this question in order to elicit the main idea and give a sketch of the proof without going through all the formal steps. While the interactions can thus function as components of the larger goal about learning to do proofs, it is notable that the professor often asked for a particular number of words, as in the second example above, which made the questions feel less genuine and more like just guessing what's on his mind or what strategy he's about to use. The authority still resided with the professor, and the questions created an opportunity for students to match their understanding up against his, rather than an opportunity for exploring different approaches to the proof. The professor used this kind of question throughout the semester and actually asked them most often later in the semester, after the second exam (16 of the 36 total key idea questions). But also after the second exam he was much more likely to answer his own question (8 of the 16 after the second exam) or to just complete the proof himself, as in the third example, which relates to the third theme of constraints on the course and content coverage.

Moving further along the continuum toward less structured interactions related to this goal, the professor frequently called on students by name to ask if they would contribute to a proof. These interactions were about learning to do proofs in the sense that being able to do proofs requires being able to come up with steps on your own (at least in this class-department-institutional context), and the professor was providing the opportunity for students to experience this tacit component of what it means to do proofs. Calling on students to contribute to proofs conveyed the message that students should be learning how to do proofs rather than just having proofs belong to the professor, and that he expected students to be able to contribute something to the proof. This practice of calling on particular students began on the fourth day of class and after that point occurred on most days before the first and second exam (16 out of 21 days) but dropped off significantly after the second exam (4 out of 12 days). Roughly two-thirds of the students (17 out of 26) were called on to contribute to a proof at least once in the semester; three students were called on a maximum of five times (Anne, Nate, and Sara). Students' responses to being called on varied significantly, from no response at all to a hesitant or brief response to an extended back-and-forth interaction with the professor.

The following is a more-or-less typical example of how the professor would call on students – the phrasing of "want to help me" was particularly common. In this example from the ninth week of class, he had written up a proposition with two components and talked through what each of the two components means.

> Professor: I don't think we need to prove the first part, right? Because the first part follows exactly from the lemma. How do we get that, Sara, want to help me? How do we get this part? Or sorry, how do we get this first part out of what we just did here?
> Sara: (Pause – 10 sec) You could use that?
> Professor: For which function, for what? So we can just change this guy to what? How can we change this guy? So instead of saying this is equal to that, we can move this to the other side, right? (Writing on the board)
> Sara: Mhm (10/26 – 9:23 – 24:30)

The most implicit interactions related to this goal came when the professor asked general questions about how to start a proof or about what step came next and students contributed to the proof. In these interactions, almost nothing was directly said or structured in such a way that made learning to do proofs an explicit goal;

however, the experience of participating in providing the steps of a proof can be argued to be the most direct way of students' learning how to do proofs in an interaction. Additionally, the professor typically provided some form of evaluation of students' contributions to proofs, which provided an informal assessment of how well they were learning to do proofs. It was rare for him to outright dismiss a suggestion for a proof as wrong; it was much more common for him to nudge a suggestion in a particular direction or pull out a word or two that were on the right track, which felt significant as a way to keep students engaged in this overall goal of doing proofs.

This type of interaction was extremely prevalent; after the first day of class, there was at least one student who contributed to a proof on every day. There was significant variation in the students who participated in proofs, how often they participated, and how often their contributions were evaluated as right or partially right. As an example of this variation, Rohit only contributed to proofs voluntarily three times, all of which were evaluated by the professor as incorrect or not helpful to the proof, while Nico contributed to proofs 25 times, 20 of which were evaluated as correct or partially correct.

Given this variation, it makes little sense to identify a typical contribution to a proof; instead I have identified an episode from the 8th week in which several students contributed to a proof in order to illustrate what the student contributions can look like and different ways in which the professor implicitly or explicitly evaluated their contributions. The professor wrote up the proposition, if f is differentiable at x_0 then f is continuous at x_0, made a few other remarks, and then launched into this sequence:

> Professor: All right, so let's prove this statement. (Erasing) How do we prove that? That f is continuous at x_0, what do we need to do?

Damien: We can start with the definition of differentiability
Professor: And then?
Damien: And then from there we need to show that if x um, if x converge to x_0 then f(x) converge to $f(x_0)$
Professor: So we need to look, so the proof (Writing on board) is to show that the limit as x approach x_0 of f(x) is equal to $f(x_0)$ but this is the same as proving that the limit as x approach x_0 of f(x) minus $f(x_0)$ is equal zero, right? So how do we go from the differentiability to this fact? (Pause – 6 sec)

This contribution from Damien helped to provide the structure of the proof; he identified the given of differentiability and then what it means to prove that a function is continuous. In providing these steps, Damien was framing the beginning and end of the proof, just as the professor often did in his proof presentations. And in providing these framing steps in response to the professor's open-ended question about proving the statement, Damien's response further suggested that Damien has learned something about how to do proofs (or at least how to do them in the context of these in-class interactions). The professor implicitly evaluated Damien's contribution as correct by writing up his idea and repeating his language, though without any overt acknowledgement that he sometimes used. Continuing in the episode, after the 6-second pause the professor called on a student (who I'm pretty positive had raised his hand):

Professor: Ryan?
Ryan: Um well the first thing I think would be to switch it to- choose a- choose a sequence that converges to x. Uh and-
Professor: To x_0 you mean?
Ryan: To x_0, yeah
Professor: Uh-huh
Ryan: Um and re-write it in terms of that sequence, going to infinity, uh and then (Pause – 5 sec) Oh. We want to prove that. Hold on.

In this interaction with Ryan, the professor did some of the "nudging" I mentioned earlier, making a small correction to Ryan's statement that could potentially help his contribution move in the right direction. The professor also provided a fair amount of leeway, allowing Ryan to pause for several seconds, which resulted in Ryan sort of concluding for himself that his contribution was not immediately helpful, rather than requiring an overt evaluation from the professor. Thus, although I would consider Ryan's contribution to be overall incorrect or not helpful in the shared task of completing the proof, the professor allowed time and space for Ryan to participate in the proof and potentially learn from his own confusion. In the last section of this episode, the professor did not respond to Ryan but immediately called on another student with his hand raised:

Professor: Nate?
Nate: Uh the limit as x approaches x_0 of the absolute value of f(x) minus $f(x_0)$ over x minus x_0 equals L-
Professor: -Why- why- why absolute value? Do you need absolute value?
Nate: Uhh, no. No, but by the definition of derivative, that divided by x minus x_0 is equal to some L
Professor: What's some L? It's not some L, it's something you *know*
Nate: Some defined number
Professor: Yes, which is?
Nate: Which is finite,
Professor: =But what's that number?
Nate: =Bounded. What?
Professor: What's that number?
Nate: Doesn't it change based on f(x)?
Professor: Yes but what's that number?
Nate: The derivative?
Professor: Yes! So it's a number, a fixed number. At x_0 it's $f'(x_0)$. It's not just a number L, it's $f'(x_0)$. Right?
Nate: Yeah
Professor: So how do I use that information then?
Nate: So then if you multiply both sides by x minus x_0
Professor: Exactly so this is what I'm trying to prove, thank you. So I can look at the limit (Writing on board) … (10/19 – 9:12 – 17:00)

In this final part of this episode, Nate's contribution was essentially correct, and the professor's corrections and set of leading questions were attending to the precision of details in the proof. The first question from the professor about absolute value, while phrased as a question, was fairly clearly an editorial comment; students often incorrectly described brackets around a limit expression as absolute value, a point which the professor emphasized again as he formally wrote up the proof. The next set of leading questions from the professor were also related to a particular detail that the professor wanted Nate to recognize, which does not substantially change the steps of the proof. The general correctness of Nate's approach was clear from the professor's positive evaluation at the end, with "exactly" and "thank you."

Throughout this episode of students contributing to proofs, we can see the different ways that students were learning to participate in both the framing and steps of the proof, as well as the different ways the professor took up and evaluated their contributions.

Student-Initiated Interactions

Examining student-initiated interactions adds an additional layer to this goal. As mentioned earlier, it was acknowledged as a shared goal, but students had different ways of taking up this goal and their different interactions suggested different stories about how they interpret it.

The most explicit response from students related to this goal came when they asked questions about what would count as a proof or what would need to be included in a proof on a homework or an exam (particularly Damien). For example, during the third class they were working on the proof that the square root of two is irrational and students wanted to claim that if $p^2 = 2q^2$ then p must be even, which led to the following exchange:

> Professor: You can prove the following statement that if you have an even integer then its square should be even. And if the square of an

integer is even then the integer itself must be even. That's a different statement

Damien: Do we need to prove that step before we go-

Professor: Uh right now I'm just going to assume it

Damien: -Or we can just assume it? Oh okay

Professor: But I'm going to ask you guys to sort of establish this. So here it's a small problem for you guys to do… (9/4 - 9:15 – 22:00)

Damien's question explicitly raises the issue of the steps of the proof and what needs to be proved when, which is an essential issue that students wrestle with when learning to do proofs (CITE) – what can be assumed when? Similarly, on the review day before the first exam, Damien raises this issue of what needs to be in a proof again, asking:

Damien: From there, could we just have assumed that- do we need to go over all this? Can we just assume that a_n converge to a from there?

Professor: From here?

Damien: Yeah. No? If we say that we don't get full credit?

Professor: No, not full credit. I think I want to see like uh you see sort of use a formal argument. Yes it's clear at this point that they converge, but I want to see it written formally. Okay? Um, you conclude from here I'll probably, um

Damien: It's already obvious

Professor: Yes it's obvious because- but we didn't state it in this way, so. We state it in this form. […] So we can use it if we have proved it in the following format […] But we did not prove this statement yet. (10/7 – 9:20 – 36:00)

Questions like this one from Damien made explicit the fact that students were learning to do proofs in a particular way in order to satisfy the external demands of grading and achieving certain academic outcomes, at least in part. That is, the goal of learning to do proofs as demonstrated in classroom interactions included not just a "mastery" orientation toward proofs but also a "performance" component.

[Note to self: Want to add more examples/types of student-initiated interactions here that move towards the implicit/less structured end of the continuum.]

Significance of the Goal for the Professor and Students

The professor addressed this goal in our interview at the end of the semester in a few different but related ways. The most direct statement that I see as tying in to the goal of learning how to do proofs came as part of his response to my question about how he thought about planning his lectures:

> One thing that was constant that I wanted to sort of bring out is: whatever I ask you to prove, at least sort of know where to start, sort of know where to sort of, what is it that I know, where am I going, should I sort of prove it by contradiction, is it a direct proof…um I don't think it's um, I don't think it's easy. (Professor interview)

This comment drew out many of the themes already seen in the interactions above around the kinds of questions he asked the students in order to engage them in the process of doing a proof. I also appreciated his acknowledgement of the difficulty that students encounter in learning how to do proofs because it provided a weight and importance to this goal as something that legitimately challenges students.

On a different but related note, the professor also addressed the value he finds in students' participation in the process of completing proofs:

> If I get feedback, sometimes like uh, it sort of shows that either they're thinking about it the wrong way, or maybe sometimes I have like a way to sort of think about the problem and I come like, a few times I came to sort of prove something in a way, and then somebody suggested a way that was way better than the way I was thinking, and so those type of thing I sort of like, to sort of get them to do. (Professor interview)

For him the participation in doing proofs served an assessment purpose of revealing possible student confusion. But also it provided the opportunity for students to make their own contributions to proofs.

Finally, in thinking about teaching this course in the future and improvements he wants to make, the professor addressed this goal of learning to do proofs and think logically as the key importance of the course that he wanted students to value:

> Rather than people just having to dread taking this class, um, hopefully they come in with a mindset of this is an important class that you learn, I mean in which you can put logical steps together. (Professor interview)

Students were also clearly aware of this goal. In their interviews several of them mentioned moments when no students would respond to the professor's requests for contributions to a proof as notable – sometimes it made them feel awkward because they were disappointing him, sometimes it seemed like an unreasonable request for them to have synthesized information extremely quickly. Several of them evaluated the class/their performance in the class as related to this goal (e.g., Patrick's comment about not feeling like he's learning problem solving, other students like Nate and Anne and Jamie feeling more confident about proofs). Others critiqued the professor's approach to teaching them how to do proofs – most notably Damien, who gave several examples from the end of the semester where the professor moved through proofs without providing motivation, examples, or evidence of how someone would come up with that proof.

Appendix G – Analysis of Interaction

In the diagrams below:

Speech actions (words spoken) are represented by square boxes
Speech acts (social meaning/force of the words) are represented by gray ovals
Positions are represented by rounded boxes
Storylines are represented by the three columns

Words and positions associated with the **professor** are in blue
Words and positions associated with the **students** are in green

1 Professor: So when I have a function defined on an interval, then the function is constant if and only if the derivative is equal to zero everywhere

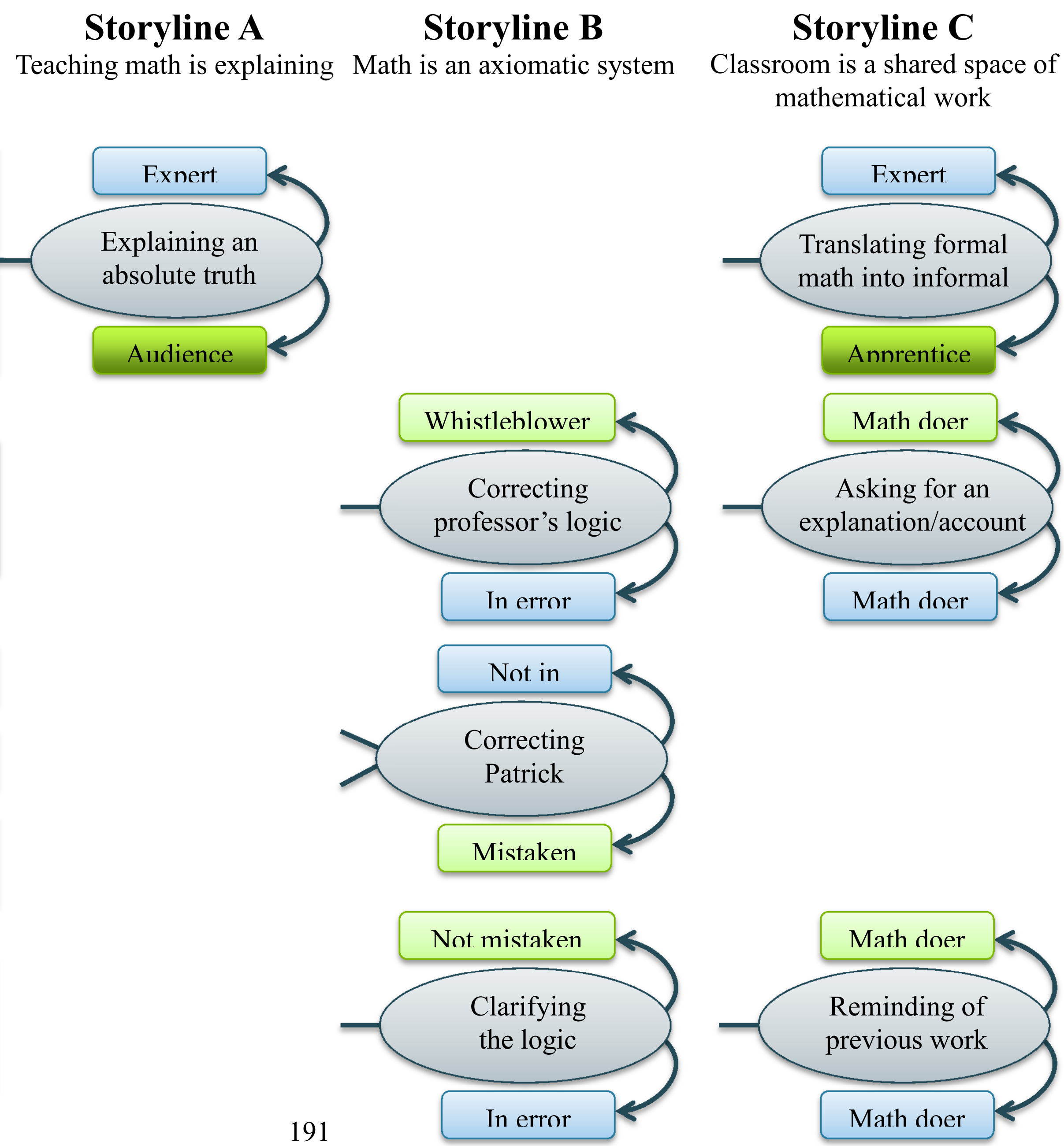
2 Patrick: Didn't we use this fact in Rolle's, in our proof of Rolle's Theorem though?

3 Professor: Absolutely not

4 Patrick: Okay

5 Professor: We never used it

6 Patrick: But didn't we state that if a was equal to b, then, and if the maximizer-

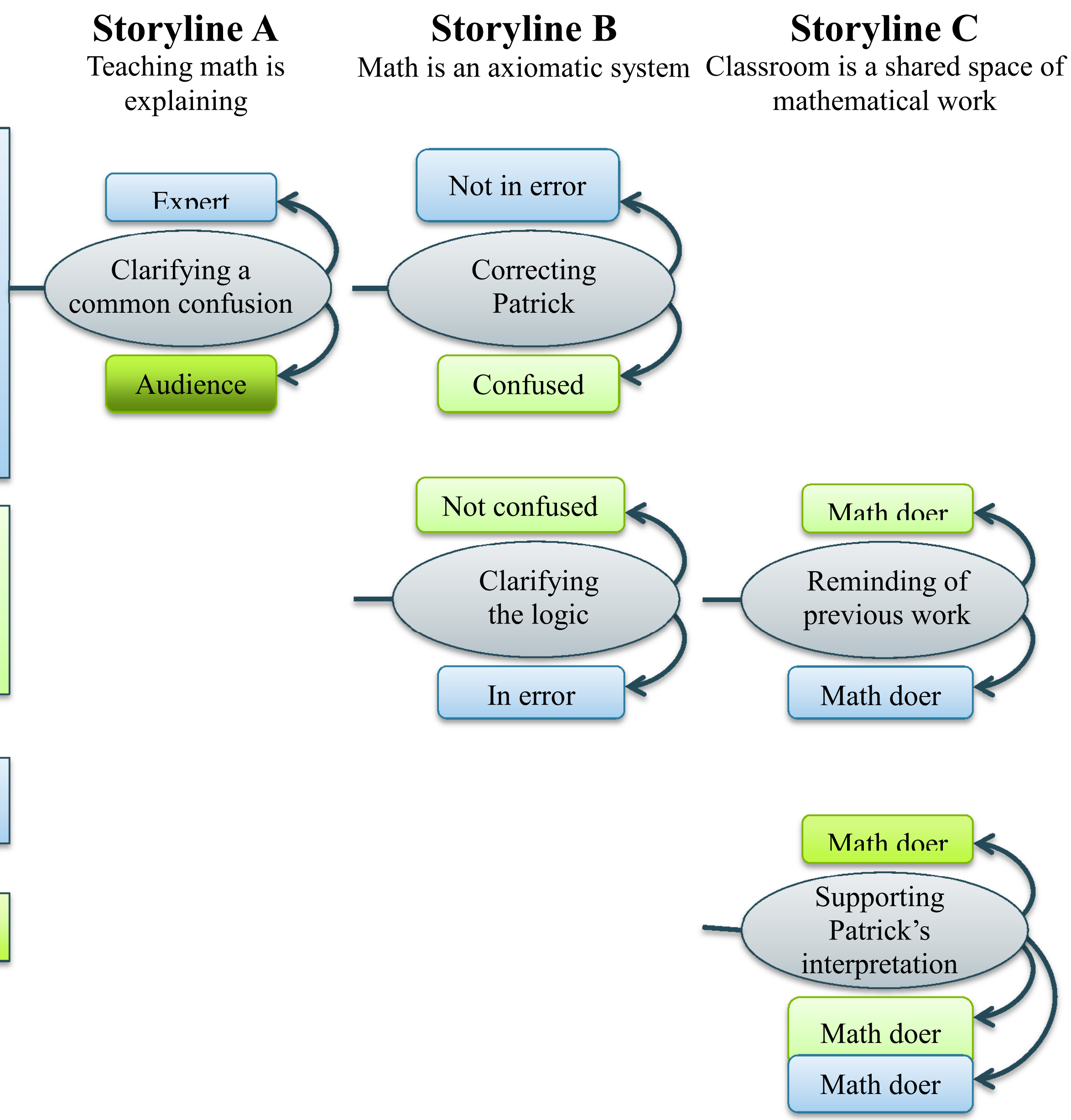


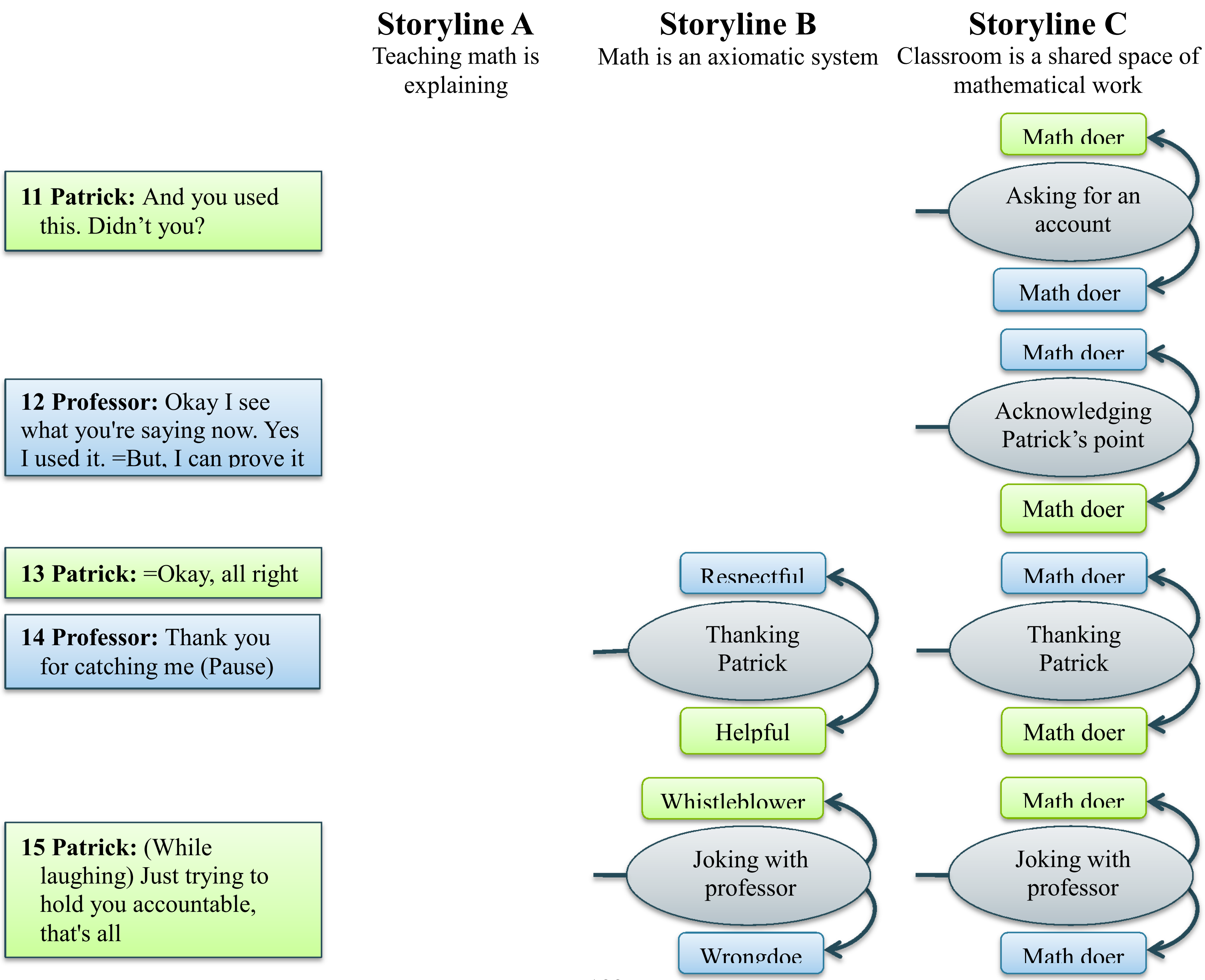
7 Professor: -But the function is not constant. It's telling you that (Writing on board) if you draw the graph of the function a and b, sorry a and b, then the value at a and the value at b are the same, but what happens in between is different

8 Patrick: Right, but then we assumed that the maximizer and the minimizer both happen at the end points, =and then-

9 Professor: =Oh! I- I sort of uh (Pause – 2 sec)

10 Damien: Yeah




Storyline A
Teaching math is explaining
Storyline B
Math is an axiomatic system
Storyline C
Classroom is a shared space of mathematical work
11 Patrick: And you used this. Didn't you?
Math doer
Asking for an account
Math doer
12 Professor: Okay I see what you're saying now. Yes I used it. =But, I can prove it
Math doer
Acknowledging Patrick's point
Math doer
13 Patrick: =Okay, all right
14 Professor: Thank you for catching me (Pause)
Respectful
Thanking Patrick
Helpful
Math doer
Thanking Patrick
Math doer
15 Patrick: (While laughing) Just trying to hold you accountable, that's all
Whistleblower
Joking with professor
Wrongdoe
Math doer
Joking with professor
Math doer

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