

## ABSTRACT

Title of dissertation: OPTIMALITY OF EVENT-BASED POLICIES FOR  
DECENTRALIZED ESTIMATION OVER SHARED  
NETWORKS

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Cyber-physical systems often consist of multiple non-located components that sense, exchange information and act as a team through a network. Although this new paradigm provides convenience, flexibility and robustness to modern systems, design methods to achieve optimal performance are elusive as they must account for certain detrimental characteristics of the underlying network. These include constrained connectivity among agents, rate-limited communication links, physical noise at the antennas, packet drops and interference. We propose a new class of problems in optimal networked estimation where multiple sensors operating as a team communicate their measurements to a fusion center over an interference prone network modeled by a collision channel. Using a team decision theoretic approach, we characterize jointly optimal communication policies for one-shot problems under different performance criteria.

First we study the problem of estimating two independent continuous random variables observed by two different sensors communicating with a fusion center over

a collision channel. For a minimum mean squared estimation error criterion, we show that there exist team-optimal strategies where each sensor uses a threshold policy. This result is independent of the distribution of the observations and, can be extended to vector observations and to any number of sensors. Consequently, the existence of team-optimal threshold policies is a result of practical significance, because it can be applied to a wide class of systems without requiring collision avoidance protocols.

Next we study the problem of estimating independent discrete random variables over a collision channel. Using two different criteria involving the probability of estimation error, we show the existence of team-optimal strategies where the sensors either transmit all but the most likely observation; transmit only the second most likely observation; or remain always silent. These results are also independent of the distributions and are valid for any number of sensors. In our analysis, the proof of the structural result involves the minimization of a concave functional, which is an evidence of the inherent complexity of team decision problems with nonclassical information structure.

In the last part of the dissertation, the assumption on the cooperation among sensors is relaxed, and we show that similar structural results can also be obtained for systems with one or more selfish sensors. Finally the assumption of the independence is lifted by introducing the observation of a common random variable in addition to the private observations of each sensor. The structural result obtained provides valuable insights on the characterization of team-optimal policies for a general correlation structure between the observed random variables.

OPTIMALITY OF EVENT-BASED POLICIES FOR  
DECENTRALIZED ESTIMATION OVER SHARED NETWORKS

by

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## Dedication

*In memory of my grandfather **Hugo Guilherme Müller**:*

*the best engineer and teacher that I have ever met.*

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## Chapter 1: Introduction

State estimation is a fundamental component in stochastic control. Algorithms that estimate or track the state (or a function of the state) of a physical system provide essential information for monitoring, control and decision-making under uncertainty. Recent technological advancements have allowed the use of smart distributed sensing devices interconnected over a network to perform various complex tasks. Although sensor networks provide convenience, flexibility and robustness to modern systems, design methods to achieve optimal performance are elusive as they must account for certain detrimental characteristics of the underlying network. These include constrained connectivity among agents, rate-limited communication links, physical noise at the antennas, packet drops and interference. This dissertation studies a new class of problems of optimal networked estimation where *multiple sensors* operating as a *team* communicate their measurements to a fusion center over an interference prone network. Using a team decision theoretic approach, we characterize jointly optimal transmission policies under different cost criteria over a single time-step: also known as *one-shot* problems.

## 1.1 Networked estimation

Cyber-physical systems are often formed by multiple non-located components that sense, exchange information and act as a team through a network [1]. In such networked decision systems the agents are sensors, estimators, controllers and actuators, and are generically referred to as decision makers or DMs as in Fig. 1.1.

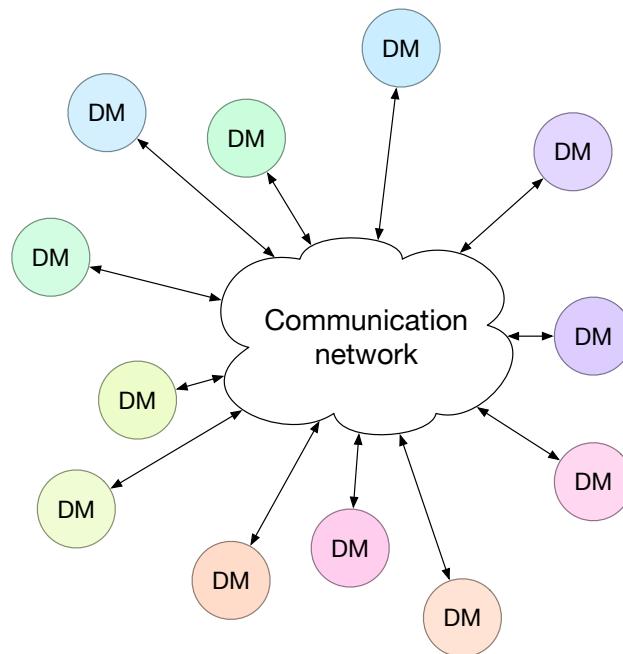


Figure 1.1: A networked decision system.

Regardless of their role in the system, a decision making node is typically capable of making local measurements and, sending and receiving information packets with other nodes in the network. This is illustrated in Fig. 1.2. The limitations introduced by the network may come in many forms: noisy or rate-limited communication links, delay, interference and packet drops, to name a few. Generally, the performance of the system is degraded by the network, imposing new challenges on the analysis and

design of modern systems.

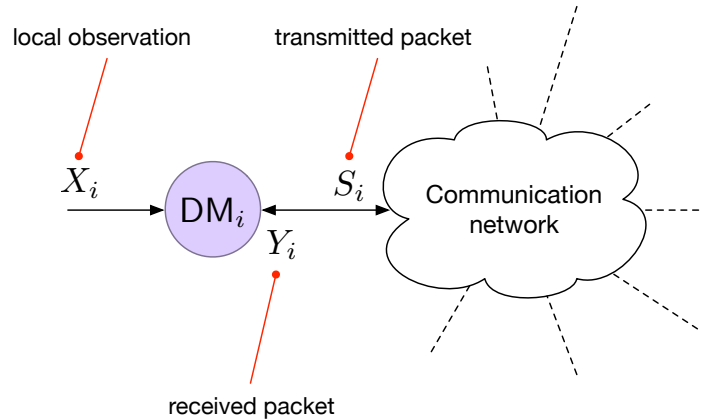


Figure 1.2: Basic functionalities of a decision making node.

Within this context, remote estimation systems are comprised by non-located sensing and estimation blocks that communicate over a constrained network. The goal of the system designer is to obtain communication and estimation policies that optimize a given performance criterion such as the mean squared or the probability of estimation error, subject to the constraints imposed by the network. Most of the existing results in this research area consist of characterizations of optimal communication policies for a single sensor and a single remote estimator in sequential decision making settings over a finite or infinite horizon [2]. However, remote sensing systems with multiple sensors operating under interference have not received much attention from researchers in the fields of control and estimation theory. What was not understood (or known) is whether there existed a tractable model for interference in a multi-sensor remote estimation scenario for which optimal solution could be characterized. This dissertation answers this and other questions by studying canonical one-shot problem formulations where the wireless network is modeled us-

ing a collision channel. As in the wireless communication networking literature, where this class of channels is widely used [3,4], we show that the collision channel is a useful model for decentralized networked control and estimation.

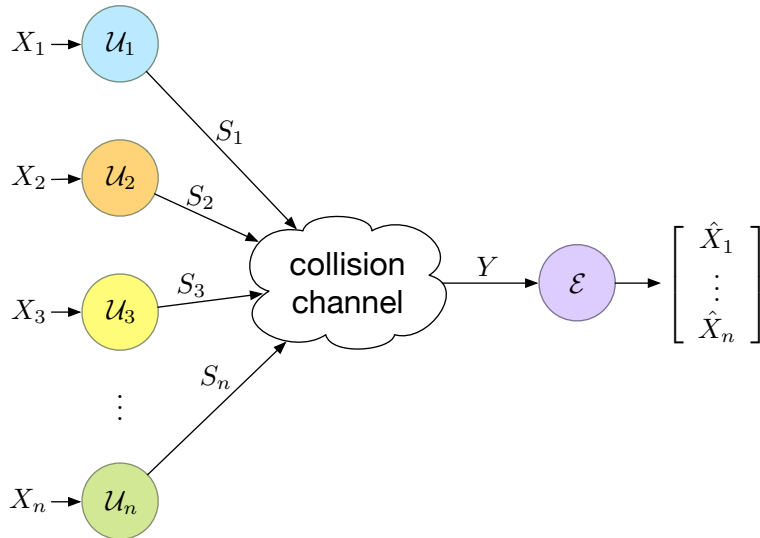


Figure 1.3: Basic framework.

The main framework used in the problems in this dissertation is depicted in Fig. 1.3 and is described as follows. Consider a decentralized remote sensing system where  $n \geq 2$  sensors make local measurements, denoted by  $X_1, \dots, X_n$ . Each sensor must decide whether or not a given measurement should be transmitted to the fusion center and communicate them in packets, denoted by  $S_1, \dots, S_n$ , over a wireless network. The group of sensors cooperate as a team to achieve the optimal performance, but are not allowed to communicate with each other. This decentralized operation mode is called *silent coordination* [5]. The network is modeled by a collision channel for which at most one transmitted packet can reliably reach the fusion center and multiple transmissions result in a collision. The fusion center  $\mathcal{E}$  observes the channel output  $Y$  and forms estimates of all the measurements,  $\hat{X}_1, \dots, \hat{X}_n$ . The goal



is to design the communication policies  $\mathcal{U}_1, \dots, \mathcal{U}_n$  at the sensors such that a given performance metric involving the estimation error is minimized.

## 1.2 Applications

The framework proposed here can be used to model a distributed sensor network in which measurements that are made by non-located sensors are wirelessly transmitted to a fusion center. One-shot problem formulations arise when the objective is to detect a one-time event of interest or estimating vital variables in real time, and with minimal delay. This is the case when multiple sensors monitor large structures or systems such as bridges, electric power grids and, oil and gas pipelines, which are subject to potentially catastrophic events. In such scenarios, the delay and additional infrastructure required for coordinating access to the network over large distances would be costly and impede swift detection and estimation. Here we adopt a one-shot formulation in which each sensor does not have time to coordinate or communicate with the other, and must decide whether to communicate based solely on its measurement. The observations are independent, but otherwise allowed to be arbitrarily distributed, possibly coming from different families of distributions.

The collision channel captures the effect of interference present in wireless networks where devices share the same frequency band for communication and do not follow any form of scheduling or random access protocols. Constraints like the ones addressed in this dissertation are common in settings where the devices are very

simple and can only perform elementary operations without sophisticated communication modules. Examples where this is the case include: nanoscale intra-body networks for health monitoring and drug delivery systems; networks for environmental monitoring of air pollution, water quality and biodiversity control [6, 7]. Remote estimation systems of this type can also be applied in scenarios where the devices are heterogeneous and there is a strict requirement for real-time wireless networking. For example, ad hoc networks such as the *Internet-of-Things* for the lack of necessity in agreement of a communication protocol among the devices [8]; systems such as data centers, which are subject to cascading power failures [9] or cyber-attacks [10], and must be detected in minimal time and as accurately as possible.

### 1.3 Related literature

Control and estimation over communication networks have been of great interest to control theorists and engineers during the last decade [11]. With the advent of cyber-physical systems as a new paradigm for system design, the development of new tools and models in networked control and estimation are as important now as ever. The components or blocks of a cyber-physical system are noncollocated and typically interconnected by a network. Moreover, these blocks have access to different information, which is often corrupted by noise, delayed or incomplete due to physical or operational constraints. Such problems can be cast in the framework of team decision theory and their analysis often combine tools from control, information and optimization theories [12].

Many channel and network models have been studied in the existing networked control and estimation literature. Apart from the traditional additive Gaussian noise (AWGN) and fading channels, the packet drop channel (also known as analog erasure channel) has attracted most of the attention of the research community in control. Most notably, the works of Sinopoli et al. [13] and Gupta et al. [14, 15] have become landmark references in the area. However, there have been only few studies that explicitly deal with the effects of interference in control and estimation over wireless networks [16–20]. Our work seeks to contribute in this growing field by modeling interference using a simple model for a communication medium shared by multiple devices known as the *collision channel* [21], which has been largely used, along with queueing theory, in the design and analysis of wireless networks [3].

We assume that the collision channel can only carry one packet and differs from the packet drop channel in the following fundamental aspect: the channel output alphabet has two distinct symbols to represent no-transmission (idle channel) and collision (simultaneous transmissions) events. Therefore, the receiver is always able to detect if the transmitters attempted to communicate or not, even though the colliding packets cannot be correctly decoded.

There exist various ways in which the collision channel model could be modified to incorporate features of more sophisticated systems. For instance, channels with asynchronous access [22] and multi-packet reception capabilities [23, 24]. There are also variants that assume sequential transmissions with and without feedback [25]. One of the possible variations is the collision channel with capture, where the sensors may also adjust the power used to transmit a packet, and in the event of a collision,

the packet transmitted with the largest power survives the collision and the others are lost [26].

Since the communication between the components in a cyber-physical system usually occurs over a network of limited capacity (or limited infrastructure), it is important to understand how these limitations may degrade the performance of the overall system. More importantly, the system designer must provide strategies that make the best use of the limited available communication resources. There are three main lines of research that consider the effects of communication links between sensors and estimators. The first class of problems corresponds to the characterization of fundamental limitations on performance caused by noisy communication channels [27]. Here we find the more traditional communication models such as the AWGN and fading channels [28–30]; as well as the packet drop channel of [13–15]. The second class of problems studies the effect of noiseless but rate limited channels in estimation and control, in which signals are quantized prior to transmission. Results known as *data-rate theorems* [31] establish the minimum quantization rate necessary to stabilize an unstable plant, while other works establish conditions for the existence of stable quantizer-estimator schemes [32].

Finally, a third class of problems studies the trade-off between communication rate and estimation performance over noisy or noiseless but costly communication channels. An interesting common feature of these problems is that threshold (event-based) policies *emerge* as solutions to optimization problems and are not an architecture *imposed* by the system designer. The first contributions in this field were done by Imer and Basar in [33], where a limit on the number of noiseless trans-

missions that can be made by the sensor over a finite horizon imposes an upper bound on the communication rate. The idea that event-based estimation/control systems can be used for *signalling* was first mentioned in [33], whose results were later complemented by [34]. A continuous time formulation of this problem was studied by Rabi et al. in [35]. Xu and Hespanha in [36] solved an infinite horizon problem whose objective functional combined the expected estimation error and a communication cost. Lipsa and Martins in [37] also considered a finite horizon problem with an objective functional that combines estimation error and communication costs and established the structure of jointly optimal communication and estimation policies. In particular, [37] shows that there exist optimal communication policies of the symmetric threshold type and the optimal estimator admits a simple recursive structure. In [38] the authors showed that this structure is preserved when the channel randomly drops packets. Nayyar et al. in [39] generalized [33] and [37] obtaining structural results when, in addition to communication costs, there is a stochastically varying energy budget, which reflects the sensor's ability to harvest energy from the environment in order to communicate. In the context of control of dynamical systems over communication networks, the work of Molin and Hirche in [40] shows that certainty equivalence controllers are optimal for point-to-point communication links are of the type in [33] and [37]. A model similar to the one presented here was used Gupta et al. in a game between a sensor and a jammer in a remote estimation problem [41].

## 1.4 Contributions

My doctoral research has introduced a new class of problems in remote estimation for which optimal solutions can be characterized analytically. One contribution in the area of networked control and estimation systems is the formal definition of a collision channel and its use to model interference among agents sharing a (wireless) communication network. The collision channel has been a widely used model in wireless networks. To the best of our knowledge, most of the current literature in networked control that uses this channel model assumes some form of collision avoidance mechanism. These protocols may cause delays and require additional infrastructure in networked systems. We chose to deal with packet collisions in a different way, by exploiting them for implicit communication (also known as signaling) among the agents.

Our system models are formulated using the framework of team decision problems, which are generally difficult to solve. However, the class of problems posed in this dissertation admits characterizations of team-optimal strategies. This constitutes a contribution to the field of team decision theory, where most of the problems that admit characterizations of team-optimal strategies belong to the class of linear quadratic Gaussian (LQG) or linear exponential Gaussian (LEG) teams.

For the problem of estimating independent continuous random variables with a mean squared error criterion, the existence of team-optimal strategies with a threshold structure is established. Our proof uses a technique based on Lagrange duality theory for generalized moment optimization problems with variable bounds.

This technique may also be useful in other decentralized control and estimation problems with quadratic cost functionals. As a side contribution, a new metric for quantization of continuous random variable with unequal distortion was introduced. To the best of our knowledge, this is the first time that such metric is reported in the literature. Due to the asymmetry of the distortion metric, the convergence of the modified Lloyd-Max algorithm devised to minimize this new cost does not follow from the classic sufficient condition by Fleischer [42]. We provide a proof for the convergence to a locally optimal quantizer in the Gaussian case. Finally, we show that the optimal policies for a system with identically distributed observations and symmetric probability density functions have asymmetric thresholds. This a major departure from the current literature on remote estimation.

For the problem of estimating independent discrete random variables over the collision channel, two performance criteria are used: the aggregate and total probability of error. In both cases, we show that the optimization over the policy space of one decision maker while keeping the other fixed is a concave minimization problem, which is known to be NP-hard. Typically, such problems are solved using approximation techniques which guarantee performances within a fixed bound from the optimal. However, we are able to solve these problems exactly using a “converse-achievability” approach, where we obtain a lower bound and provide a structured policy that achieves it. On one hand this illustrates the complexity of solving team problems with discrete observation and action spaces; and on the other it shows that there may be instances of such problems that are tractable using non-traditional techniques. One important contribution is the characterization of a team-optimal

strategies for the system with a total probability of estimation error, in which each sensor transmits all but the most likely of its observations.

In addition to this dissertation, my doctoral research has produced conference articles [43–46], journal papers [47, 48] and a book chapter [2].

## 1.5 Outline

The dissertation is structured in seven chapters, including this introduction. The rest of the dissertation is organized as follows.

Chapter 2 presents the basic team decision framework used in the analysis of the networked estimation problems posed in this dissertation. We define the concept of a static team decision problem with a non-classical information structure and, the notions of team-optimality and person-by-person optimality. This chapter introduces the jargon and the notation used throughout the dissertation.

Chapter 3 introduces the problem of estimating two independent continuous random variables over the collision channel. Using a person-by-person optimality approach, we show that there exists a team-optimal strategy where the sensors use threshold policies with respect to a mean squared estimation error criterion. The proof of this result relies on an analogy with a remote estimation subproblem with communication costs and the solution of a generalized moment optimization problem with variable bounds. This structural result is independent of the distribution of the observed random variables.

Chapter 4 shows that the computation of person-by-person optimal thresh-



old policies for the problem in Chapter 3 is equivalent to the design of a one-bit quantizer that minimizes a distortion metric that is unequal across quantization regions. We argue that asymmetric thresholds are optimal when it is advantageous to embed information in collision and no-transmission symbols. The numerical examples illustrate that the policy design is a non-convex problem in general, and optimal solutions can be symmetric or asymmetric, depending on the parameters that specify the problem. A numerical approach to compute locally optimal threshold policies is proposed based on a modified version of the Lloyd-Max algorithm. We present examples for the case when the variables are Gaussian that illustrate that the algorithm converges to a local minimum and that it can be used to compute person-by-person optimal solutions.

Chapter 5 considers the problem of estimating two independent discrete random variables over the collision channel. We obtain structural results of team optimal policies for two criteria: a convex combination of the individual probabilities of estimation errors for each observation; and a total probability of error. In the first case, we show that there exist team-optimal policies where the sensors either transmit all but the most likely observation; transmit only the second most likely observation; or always remain silent. In the second case, we show that the every sensor transmitting all but the most likely observation is a team-optimal strategy. In both cases, the proof consists of using the person-by-person approach involving the minimization of a concave functional. We solve these problems exactly by obtaining a lower bound and providing a policy that achieves it. The results can be extended to  $n$  sensors and are valid for any probability mass functions for the

observed random variables.

Chapter 6 extends the basic framework in two separate directions. First, we allow each sensor to minimize its own cost functional, which leads to a non-cooperative game formulation. We show that the search for Nash-equilibria can be constrained to the class of threshold policies. Second, we relax the independence assumption by considering that each sensor observes a common random variable in addition to their private observations. We show that there exist team-optimal strategy where each sensor uses a policy such that: for every realization of the common observation, is a threshold policy on the private observations.

Chapter 7 concludes the dissertation and outlines open research problems.

## Chapter 2: Fundamentals

The problems studied in this dissertation can be categorized as team decision problems, where multiple agents have access to limited local information and must choose their actions with the goal of jointly minimizing a common cost function. The origins of team decision theory can be traced back to the seminal work of Marschak and Radner in economics [49–51] and Witsenhausen in control theory [52]. An important aspect of this class of problems is the role played by information on their tractability. The different connections between economics, decentralized control and information theory in team decision problems can be found in the tutorial paper by Ho [53]. The complexity of solving problems in this class is in general NP-hard, a fact that was established by Tsitsiklis and Athans in [5].

In order to set the stage for the forthcoming chapters, here we introduce the main components of a stochastic team decision problem. There are several levels of sophistication at which this material could be presented. We have chosen to provide a definition that deliberately avoids technical details. A more in-depth definition is available in the book by Yuksel and Basar [12, ch. 2].

## 2.1 Static team decision problems

The team decision problems we consider in this dissertation are stochastic, which means that there is an underlying notion of uncertainty with a known probabilistic description characterized by a probability space  $(\Omega, \mathcal{A}, \mathbf{P})$ . Consider a random variable  $W$ , which denotes the *state of the world*, taking values on an alphabet  $\mathbb{W}$ . The random variable  $W$  is either continuous or discrete, and is distributed according to a probability density function  $f_W$  or a probability mass function  $p_W$ . A team consists of a set of  $n \geq 2$  *agents* or *decision makers*, where the  $i$ -th decision maker will be denoted by  $\text{DM}_i$ . It is assumed that all the agents agree on the underlying probability space in the problem. What distinguishes a team decision problem from other problems of decision making under uncertainty is that each agent only has *partial* knowledge about the state of the world  $W$ . In a *static* team decision problem, the observation of  $\text{DM}_i$  is denoted by a random variable  $X_i$  taking values on  $\mathbb{X}_i$  and is related to the state of the world by an *information function*  $\mathcal{H}_i : \mathbb{W} \rightarrow \mathbb{X}_i$  such that  $X_i = \mathcal{H}_i(W)$ . The set of information functions  $\mathcal{H} = (\mathcal{H}_1, \dots, \mathcal{H}_n)$  constitute an *information structure*. Under a *classical* (or centralized) information structure all the agents have the same information. Otherwise, a static team decision problem is said to have a *non-classical* (or decentralized) information structure.

Based solely on  $x_i$ , the  $\text{DM}_i$  chooses a control action  $u_i \in \mathbb{A}_i$  according to a *control policy*  $\mathcal{U}_i : \mathbb{X}_i \rightarrow \mathbb{A}_i$  such that  $u_i = \mathcal{U}_i(x_i)$ . The set of admissible control policies for  $\text{DM}_i$  is denoted by  $\mathbb{U}_i$ . Finally, for each set of control actions  $u = (u_1, \dots, u_n)$  and the state of the world  $w \in \mathbb{W}$ , a cost  $\mathcal{C}(u, w) \in \mathbb{R}$  is incurred. A

*team strategy* is denoted by  $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_n)$  and the strategy space is denoted by  $\mathbb{U} = \mathbb{U}_1 \times \dots \times \mathbb{U}_n$ . A stochastic *team decision problem* is stated formally below:

**Problem 2.1** (Stochastic team problem). *Given the probabilistic model of the uncertainty variable  $W$ , information structure  $\mathcal{H}$  and a cost function  $\mathcal{C}$ , find a team strategy  $\mathcal{U}^* \in \mathbb{U}$  that minimizes the cost functional  $\mathcal{J}(\mathcal{U})$  defined in Eq. (2.1):*

$$\mathcal{J}(\mathcal{U}) \stackrel{\text{def}}{=} \mathbf{E} \left[ \mathcal{C} \left( \mathcal{U}_1(\mathcal{H}_1(W)), \dots, \mathcal{U}_n(\mathcal{H}_n(W)), W \right) \right]. \quad (2.1)$$

## 2.2 Notions of optimality

Throughout the chapters of this dissertation we will be primarily interested in characterizations of the solutions of different instances of Problem 2.1. There are two solution concepts that we will often be referring to: the notion of *team-optimal* and *person-by-person optimal* solutions.

**Definition 2.1** (Team-optimal solutions). *For a given stochastic team decision problem, a team strategy  $\mathcal{U}^* = (\mathcal{U}_1^*, \dots, \mathcal{U}_n^*) \in \mathbb{U}$  is a team-optimal solution if*

$$\mathcal{J}(\mathcal{U}^*) = \inf_{\mathcal{U} \in \mathbb{U}} \mathcal{J}(\mathcal{U}). \quad (2.2)$$

*If such a strategy exists, the optimal cost is denoted by  $\mathcal{J}^*$ .*

Static team decision problems with a non-classical information structure are in general difficult solve, often leading to non-convex optimization problems. In particular, problems with discrete observation and control action sets are known to

be NP-hard [5]. However, in the continuous case, there are two important classes of teams with *linear* information functions for which the structure of the optimal solutions are known in closed form: teams with *quadratic* and *exponential quadratic* cost functions, and where the state of the world  $W$  is a *Gaussian* random vector (LQG and LEG teams, respectively). These classic results were obtained by Radner [50] and Krainak et al. [54, 55].

An alternative way to deal with the inherent complexity of solving a team decision problem is by the use of certain approximation techniques, which guarantee that a suboptimal strategy is within a fixed bound of the optimal [56]. However, it may be possible to establish structural properties of team-optimal policies by using a weaker notion of optimality known as *person-by-person* optimal solutions.

**Definition 2.2** (Person-by-person optimal solutions). *For a given stochastic team decision problem, a team strategy  $\mathcal{U}^* = (\mathcal{U}_1^*, \dots, \mathcal{U}_n^*) \in \mathbb{U}$  is a person-by-person optimal solution if*

$$\mathcal{J}(\mathcal{U}^*) \leq \mathcal{J}(\mathcal{U}_1^*, \dots, \mathcal{U}_{i-1}^*, \mathcal{U}_i, \mathcal{U}_{i+1}^*, \dots, \mathcal{U}_n^*), \quad \mathcal{U}_i \in \mathbb{U}_i, \quad i \in \{1, \dots, n\}. \quad (2.3)$$

Person-by-person optimality is a necessary but, in general, not a sufficient condition for team optimality. However, using the fact that if  $\mathcal{U}^*$  is team-optimal for Problem 2.1 then it is also person-by-person optimal, if a particular structure holds for every person-by-person optimal policy, it must also hold for team-optimal strategies. The idea of characterizing team-optimal solutions via person-by-person optimality is called the *person-by-person approach*. The advantage of using this

approach is to decompose a complicated problem into simpler subproblems for which a systematic analysis is often possible.

## 2.3 Notation

We use the terminology *sensor*, *agent* and *decision maker* (DM) interchangeably throughout this dissertation. We adopt the following notation: random variables and random vectors are represented using upper case letters, such as  $X$ . Realizations of random variables and random vectors are represented by the corresponding lower case letter, such as  $x$ . We denote the independence between two random variables  $X$  and  $Y$  by  $X \perp\!\!\!\perp Y$ . The probability density function of a continuous random variable  $X$ , provided that it is well defined, is denoted by  $f_X$ . Functions and functionals are denoted using calligraphic letters such as  $\mathcal{F}$  or  $\mathcal{F}$ . We use  $\mathcal{B}(\delta)$  and  $\mathcal{N}(m, \sigma^2)$  to represent the Bernoulli probability mass function of parameter  $\delta \in [0, 1]$  and the Gaussian probability distribution of mean  $m$  and variance  $\sigma^2$ , respectively. The real line is denoted by  $\mathbb{R}$ . Sets are represented in blackboard bold font, such as  $\mathbb{A}$ . The cardinality of a set  $\mathbb{A}$  is denoted by  $|\mathbb{A}|$ . The complement of a subset  $\mathbb{A} \subset \mathbb{B}$  is denoted by  $\mathbb{B} \setminus \mathbb{A}$ . The complement of a subset  $\mathbb{A} \subset \mathbb{R}$  is denoted by  $\mathbb{A}^c \stackrel{\text{def}}{=} \mathbb{R} \setminus \mathbb{A}$ . The empty set is denoted by  $\emptyset$ . The extended real line is defined as  $\bar{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty, +\infty\}$ . The probability of an event  $\mathfrak{E}$  is denoted by  $\mathbf{P}(\mathfrak{E})$ ; the expectation and variance of a random variable  $Z$  are denoted by  $\mathbf{E}[Z]$  and  $\mathbf{V}[Z]$ , respectively. The positive and negative parts of a real-valued function  $\mathcal{G}$  are defined as  $[\mathcal{G}(x)]^+ \stackrel{\text{def}}{=} \max\{0, \mathcal{G}(x)\}$  and  $[\mathcal{G}(x)]^- \stackrel{\text{def}}{=} \max\{0, -\mathcal{G}(x)\}$ . We denote by  $L_\mu^p(\mathbb{R})$  the

space of all  $\mu$ -measurable functions  $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}} |\mathcal{G}(x)|^p d\mu(x) < +\infty$ ,  $1 \leq p < \infty$ . The probability mass function of a discrete random variable  $X$  taking values on the set  $\mathbb{X}$  is denoted by  $p(x)$ , where  $x \in \mathbb{X}$ . We denote the reordering of the elements of  $\mathbb{X}$  according to decreasing probability by  $[\mathbb{X}] = \{x_{[1]}, x_{[2]}, \dots\}$ , where  $x_{[m]}$  precedes  $x_{[n]}$  if  $p(x_{[m]}) \geq p(x_{[n]})$ , with  $m, n \in \{1, \dots, |\mathbb{X}|\}$ . Denote  $p_{[n]} = p(x_{[n]})$ , for  $n \in \{1, \dots, |\mathbb{X}|\}$ . The indicator function of a set  $\mathbb{A}$  is denoted by  $\mathbf{1}_{\mathbb{A}}(x)$ .



## Chapter 3: Minimum mean squared error estimation over the collision channel

Consider a distributed sensing system that comprises two sensors, each observing a random variable, and a remote estimator. The goal of the remote estimator is to produce estimates of the random variables based on information transmitted to it by the sensors. The random variables are independent and information is transferred from the sensors to the estimator via a collision channel, which can only convey a single packet at a time. Each sensor has the authority to decide what and when to transmit, and simultaneous transmissions result in a collision event to be detected at the estimator. We assume that there is no communication between the sensors, which precludes the use of coordinated strategies. In this chapter, we use a person-by-person optimality approach to characterize the structure of team-optimal strategies at the sensors with respect to a mean squared error criterion. More specifically, we show that there exists a team-optimal strategy that uses deterministic threshold policies to decide when to transmit a measurement. This structural result is independent of the distributions of the observed random variables.

### 3.1 Motivation

Cyber-physical systems are often formed by multiple non-located components that sense, exchange information and act as a team through a network [1]. In the wireless case, the network may support only a finite number of simultaneous transmissions due to limitations such as interference. Here, we are interested in characterizing team-optimal policies when the maximal number of simultaneous transmissions is strictly less than the number of components sharing the same network. In order to obtain design principles that can be characterized analytically, we consider a simple configuration formed by a remote estimator that operates on information received from two sensors that measure a random variable each (see Fig. 3.1). Each sensor has the authority to decide whether to attempt a transmission or to remain silent based solely on the random variable it measures. In our formulation, we assume that information is conveyed through a *collision channel* for which at most one transmission can reliably reach the estimator and multiple transmissions result in a collision. We consider the design of policies that minimize a mean squared estimation error, subject to the communication constraint imposed by the collision channel. In particular, we will prove the optimality of policies with a threshold structure, which constitute an important class of event-based policies in problems of remote estimation and control.

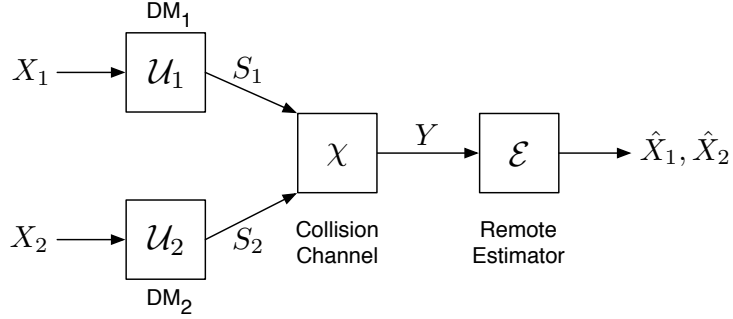


Figure 3.1: Block diagram representation for estimation over the collision channel.

### 3.2 System model

We adopt the basic framework depicted in Fig. 3.1, which comprises two sensors (or decision makers) labelled  $\mathcal{U}_1$  and  $\mathcal{U}_2$  and a remote estimator labelled  $\mathcal{E}$  connected by a collision channel  $\chi$ . Consider two independent continuous random variables  $X_1$  and  $X_2$  with distributions  $\mu_1$  and  $\mu_2$ , respectively. The random variable  $X_i$  is observed by  $\mathcal{U}_i$ , and we assume that  $\mathbf{E}[X_i] = 0$  and  $\mathbf{V}(X_i) < +\infty$ ,  $i \in \{1, 2\}$ . Each decision maker has the authority to decide whether to attempt a transmission of its measurement to the estimator. It is also important to notice from Fig. 3.1 that there is no communication between  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , which precludes policies that involve coordination. The collision channel defined below conveys information from the sensors to the estimator.

**Definition 3.1** (Collision Channel). *The channel input alphabet is  $\mathbb{S} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\emptyset\}$ , and the channel output alphabet is  $\mathbb{Y} \stackrel{\text{def}}{=}} (\{1, 2\} \times \mathbb{R}) \cup \{\emptyset, \mathfrak{C}\}$ , where  $\mathfrak{C}$  represents the occurrence of a collision. The symbol  $\emptyset$  indicates absence of transmission. The*

collision channel is a deterministic two-input map  $\chi : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{Y}$  defined as follows:

$$\chi(s_1, s_2) \stackrel{\text{def}}{=} \begin{cases} (1, s_1) & \text{if } s_1 \neq \emptyset, s_2 = \emptyset \\ (2, s_2) & \text{if } s_1 = \emptyset, s_2 \neq \emptyset \\ \mathfrak{C} & \text{if } s_1 \neq \emptyset, s_2 \neq \emptyset \\ \emptyset & \text{if } s_1 = \emptyset, s_2 = \emptyset. \end{cases} \quad (3.1)$$

The channel inputs are denoted by  $S_1$  and  $S_2$ , while  $Y$  designates the output that is defined as  $Y \stackrel{\text{def}}{=} \chi(S_1, S_2)$ .

**Assumption 3.1.** *We assume that a transmission is successful if it conveys its real-valued measurement to the estimator. This is a realistic premise when the transmitted message has enough bits to represent a real number with negligible quantization error. According to Eq. (3.1), we also consider that each packet contains in its header the identification number of the sender. This allows the remote estimator to unambiguously determine the origin of a successful transmission.*

The following are precise definitions of the communication policies at the decision makers.

**Definition 3.2** (Communication policies). *A communication policy for the  $i$ -th sensor is determined by a map  $\mathcal{U}_i : \mathbb{R} \rightarrow [0, 1]$ . The actions  $U_1$  and  $U_2$  are binary random variables that satisfy the following probabilistic law:*

$$\mathbf{P}(U_i = 1 | X_i = x_i) \stackrel{\text{def}}{=} \mathcal{U}_i(x_i), \quad i \in \{1, 2\}, \quad (3.2)$$

where  $U_i = 1$  means that the  $i$ -th sensor will attempt transmission; and  $U_i = 0$  means that it will remain silent. We adopt independent randomization mechanisms to generate  $U_1$  and  $U_2$ , which guarantee that  $(X_1, U_1)$  is independent of  $(X_2, U_2)$ . Each  $U_i$  acts on the  $i$ -th channel input according to the following definition:

$$S_i \stackrel{\text{def}}{=} \begin{cases} X_i & \text{if } U_i = 1 \\ \emptyset & \text{if } U_i = 0 \end{cases}, \quad i \in \{1, 2\}. \quad (3.3)$$

The combined action of Eq. (3.3) and the channel in Eq. (3.1) leads to the following rule to determine the output of the channel:

$$Y = \begin{cases} (1, X_1) & \text{if } U_1 = 1, U_2 = 0 \\ (2, X_2) & \text{if } U_1 = 0, U_2 = 1 \\ \mathbf{c} & \text{if } U_1 = 1, U_2 = 1 \\ \emptyset & \text{if } U_1 = 0, U_2 = 0. \end{cases} \quad (3.4)$$

We can now precisely state the problem of optimal remote estimation over the collision channel.

**Problem 3.1.** *Let  $X_1$  and  $X_2$  be two independent continuous random variables with zero mean and finite variance. Consider the following cost*

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{E} \left[ (X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2 \right], \quad (3.5)$$

where each  $\hat{X}_i$  is defined below<sup>1</sup>:

$$\hat{X}_i \stackrel{\text{def}}{=} \mathcal{E}_i(Y), \quad \text{and} \quad \mathcal{E}_i(y) \stackrel{\text{def}}{=} \mathbf{E}[X_i|Y = y], \quad y \in \mathbb{Y}. \quad (3.6)$$

Solve the following optimization problem:

$$\underset{(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2}{\text{minimize}} \quad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2), \quad (3.7)$$

where  $\mathbb{U}_i$  represents the set of randomized policies defined as follows:

$$\mathbb{U}_i \stackrel{\text{def}}{=} \{\mathcal{U} \in L^2_{\mu_i}(\mathbb{R}) \mid \mathcal{U} : \mathbb{R} \rightarrow [0, 1]\}, \quad i \in \{1, 2\}. \quad (3.8)$$

**Remark 3.1.** *If  $X_1$  and  $X_2$  are non-constant then the two terms in Eq. (3.5) cannot be both zero because the collision channel can convey at most one sensor transmission to the estimator. This implies that there is a trade-off in Eq. (3.5) that causes the minimal cost to be always positive.*

### 3.3 Structural result: optimality of threshold policies

The goal of this subsection is to state our main result as Theorem 3.1, which guarantees the existence of team-optimal deterministic threshold policies as defined below. This is an important structural result because it shows that the infinite dimensional minimization stated in Problem 3.1 can be recast as a finite-dimensional search with respect to threshold limits. We will defer the proof of Theorem 3.1 until

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<sup>1</sup>Notice that the estimator in Eq. (3.6) is always optimal for the cost in Eq. (3.5).

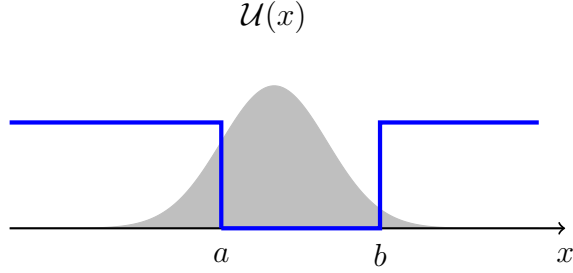


Figure 3.2: A deterministic threshold policy

the end of Section 3.4, in which we develop all the required auxiliary results.

**Definition 3.3** (Deterministic threshold policy). *A policy  $\mathcal{U}$  is of the deterministic threshold type if there are constants  $a$  and  $b$  in  $\bar{\mathbb{R}}$  for which the following holds:*

$$\mathcal{U}(x) = \begin{cases} 0 & \text{if } a \leq x \leq b \\ 1 & \text{otherwise.} \end{cases} \quad (3.9)$$

*If  $a = -b$  then the threshold policy is called symmetric, otherwise it is called asymmetric.*

**Theorem 3.1.** *There exists a pair of deterministic threshold policies  $(\check{\mathcal{U}}_1^*, \check{\mathcal{U}}_2^*)$  that is team-optimal for Problem 3.1.*

**Remark 3.2.** *Although we were unable to do so, we believe that the existence of an optimum could have been established using the results in [57]. Regardless of whether this interesting connection to [57] is possible or not, Theorem 3.1 is an indispensable result because it shows the existence of a team-optimal solution with a specific deterministic threshold structure.*

### 3.4 Estimation with communication costs

We will prove Theorem 3.1 using the person-by-person optimality approach, which consists of minimizing the cost over the policy space of each of the decision makers while keeping the other fixed. Using this approach, for any pair of policies we can construct a new pair with equal or better cost, where each policy is threshold. Here, we establish an analogy between the minimization of the cost in Eq. (3.5) with respect to either  $\mathcal{U}_1$  or  $\mathcal{U}_2$  while keeping the other fixed, and a problem of optimal remote estimation systems with communication costs. The intuition behind this correspondence is that if  $\mathcal{U}_{\{j:j \neq i\}}$  is fixed then the increase in the mean squared estimation error of  $X_j$  that results from the collisions caused by  $\mathcal{U}_i$  can be viewed, from the perspective of  $\mathcal{U}_i$ , as a communication cost. This analogy will be useful in the proof of our main result (Theorem 3.1) and, as we explain later, it also leads to a new class of problems of independent interest.

In order to make this analogy precise, without loss of generality, consider that  $\mathcal{U}_j^*$  is fixed to an arbitrary choice in  $\mathbb{U}_j$ , and let  $\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i)$  denote the resulting cost defined as follows:

$$\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i) \stackrel{\text{def}}{=} \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) \Big|_{\mathcal{U}_j = \mathcal{U}_j^*}, \quad i \neq j. \quad (3.10)$$

The following proposition unveils the underlying additive communication cost embedded in Eq. (3.10).



**Proposition 3.1.** *Given a preselected  $\mathcal{U}_j^*$ , and  $i \neq j$ , the following holds:*

$$\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i) = \mathbf{E} \left[ (X_i - \hat{X}_i)^2 \right] + \rho_{\mathcal{U}_j^*} \mathbf{P}(U_i = 1) + \theta_{\mathcal{U}_j^*}, \quad (3.11)$$

where  $\rho_{\mathcal{U}_j^*}$  and  $\theta_{\mathcal{U}_j^*}$  do not depend on  $\mathcal{U}_i$  and are given by:

$$\rho_{\mathcal{U}_j^*} = \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 1 \right] - \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 0 \right] \quad (3.12)$$

$$\theta_{\mathcal{U}_j^*} = \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 0 \right]. \quad (3.13)$$

*Proof.* Using total expectation, write:

$$\begin{aligned} \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \right] &= \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 1 \right] \mathbf{P}(U_i = 1) \\ &\quad + \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 0 \right] \mathbf{P}(U_i = 0). \end{aligned} \quad (3.14)$$

Since  $\mathbf{P}(U_i = 0) = 1 - \mathbf{P}(U_i = 1)$ , we have:

$$\begin{aligned} \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \right] &= \left( \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 1 \right] - \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 0 \right] \right) \mathbf{P}(U_i = 1) \\ &\quad + \mathbf{E} \left[ (X_j - \hat{X}_j)^2 \mid U_i = 0 \right]. \end{aligned} \quad (3.15)$$

□

The cost functional in Eq. (3.11) has three components: the first is the mean square estimation error of  $X_i$ , the second ascribes a cost  $\rho_{\mathcal{U}_j^*}$  to the probability of attempting a transmission and the third is constant with respect to  $\mathcal{U}_i$ . The following

proposition will be important later on.

**Proposition 3.2.** *For a given a preselected  $\mathcal{U}_j^*$ , it holds that  $\rho_{\mathcal{U}_j^*} \geq 0$ .*

*Proof.* Using iterated expectations and the fact that for  $i \neq j$ ,  $U_i$  and  $U_j$  are mutually independent, we write

$$\rho_{\mathcal{U}_j^*} = \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 1] - \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 0] \quad (3.16)$$

$$\begin{aligned} &= \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 1, U_j = 0] \mathbf{P}(U_j = 0) \\ &\quad - \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 0, U_j = 0] \mathbf{P}(U_j = 0) \\ &\quad + \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 1, U_j = 1] \mathbf{P}(U_j = 1) \\ &\quad - \mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 0, U_j = 1] \mathbf{P}(U_j = 1). \end{aligned} \quad (3.17)$$

When  $(U_i = 0, U_j = 1)$ , the estimator receives  $Y = (j, X_j)$ . Since  $\mathcal{E}_j(j, X_j) = X_j$ , we have that  $\mathbf{E}[(X_j - \hat{X}_j)^2 | U_i = 0, U_j = 1] = 0$ . In the cases where  $Y = (i, x_i)$  and  $Y = \emptyset$ , the optimal estimates are

$$\mathcal{E}_j(i, x_i) = \mathbf{E}[X_j | Y = (i, x_i)] \quad (3.18)$$

$$\stackrel{(a)}{=} \mathbf{E}[X_j | U_j = 0, U_i = 1, X_i = x_i] \quad (3.19)$$

$$\stackrel{(b)}{=} \mathbf{E}[X_j | U_j = 0] \quad (3.20)$$

and

$$\mathcal{E}_j(\emptyset) = \mathbf{E}[X_j|Y = \emptyset] \tag{3.21}$$

$$\stackrel{(c)}{=} \mathbf{E}[X_j|U_j = 0, U_i = 0] \tag{3.22}$$

$$\stackrel{(d)}{=} \mathbf{E}[X_j|U_j = 0], \tag{3.23}$$

where the equality (a) follows from the equivalence  $Y = (i, x_i) \Leftrightarrow (U_j = 0, U_i = 1, X_i = x_i)$ , equality (b) follows from the independence between  $X_j$  and  $(U_i, X_i)$ , equality (c) follows from the equivalence between  $Y = \emptyset \Leftrightarrow (U_i = 0, U_j = 0)$ , and equality (d) follows from the independence between  $X_j$  and  $U_i$ . Therefore, the following equality holds

$$\mathbf{E}[(X_j - \hat{X}_j)^2|U_i = 1, U_j = 0] = \mathbf{E}[(X_j - \hat{X}_j)^2|U_i = 0, U_j = 0]. \tag{3.24}$$

Finally, the communication cost is given by

$$\rho_{\mathcal{U}_j^*} = \mathbf{E} \left[ (X_j - \hat{X}_j)^2 | U_i = 1, U_j = 1 \right] \mathbf{P}(U_j = 1) \geq 0. \tag{3.25}$$

□

From Propositions 3.1 and 3.2, we conclude that the minimization of  $\mathcal{J}$  with respect to  $\mathcal{U}_i$ , while keeping  $\mathcal{U}_j^*$  fixed, can be cast as follows:

**Problem 3.2.** Consider that  $\beta$  in  $[0, 1]$  and a non-negative constant  $\varrho$  are given.

Let  $D$  and  $X$  be two independent random variables. The variable  $D$  is Bernoulli

with  $\mathbf{P}(D = 1) = \beta$  and  $X$  is a continuous random variable with distribution  $\mu$ , zero-mean and finite variance  $\sigma_X^2$ . Let  $\mathbb{U} \stackrel{\text{def}}{=} \{\mathcal{U} \in L_\mu^2(\mathbb{R}) \mid \mathcal{U} : \mathbb{R} \rightarrow [0, 1]\}$ . Find a solution to the following:

$$\underset{\mathcal{U} \in \mathbb{U}}{\text{minimize}} \mathcal{J}(\mathcal{U}) \quad (3.26)$$

where the cost is defined for any  $\mathcal{U}$  in  $\mathbb{U}$  as follows:

$$\mathcal{J}(\mathcal{U}) \stackrel{\text{def}}{=} \mathbf{E}[(X - \hat{X})^2] + \varrho \mathbf{P}(U = 1). \quad (3.27)$$

Here,  $U \in \{0, 1\}$ ,  $\mathbf{P}(U = 1 \mid X = x) \stackrel{\text{def}}{=} \mathcal{U}(x)$ , the pair  $(X, U)$  is independent of  $D$  and  $\hat{X} \stackrel{\text{def}}{=} \mathbf{E}[X \mid Z]$ , where  $Z$  is the output of the point-to-point collision channel defined as follows:

$$Z \stackrel{\text{def}}{=} \begin{cases} X & \text{if } U = 1, D = 0 \\ \mathfrak{C} & \text{if } U = 1, D = 1 \\ \emptyset & \text{if } U = 0. \end{cases} \quad (3.28)$$

**Remark 3.3.** From Propositions 3.1 and 3.2, we conclude that the minimization of Eq. (3.5) with respect to  $\mathcal{U}_i$ , while keeping a preselected  $\mathcal{U}_j^*$  fixed, is equivalent to Problem 3.2 provided that we recognize a correspondence between  $(X, U, D)$  and  $(X_i, U_i, U_j)$ . To complete this analogy, we can select  $\beta$  as  $\mathbf{P}(U_j = 1)$  and  $\varrho$  as  $\rho_{\mathcal{U}_j^*}$ .

## Optimality of deterministic threshold policies for Problem 3.2

Notice that the channel specified in Eq. (3.28), which is adopted in the formulation of Problem 3.2, is fundamentally different from the erasure model of [13].

Unlike the latter where erasures occur independently of the channel input, information loss in Eq. (3.28) results from collision events that depend on both  $U$  and the exogenous variable  $D$ . Our goal in what follows is to prove Theorem 3.2, which establishes that there is at least one deterministic threshold policy that is optimal. This is an important result for the solution of Problem 3.2 because it shows that the infinite dimensional optimization in Eq. (3.26) can be recast as a finite-dimensional minimization with respect to the thresholds. Equally important is Lemma 3.2, which is central for the proof of Theorem 3.1.

We start by stating the following proposition that can be derived from standard continuity arguments:

**Proposition 3.3.** *Consider a policy  $\mathcal{U}' \in \mathbb{U}$  for Problem 3.2. Given a positive real constant  $\epsilon$ , there is a policy  $\mathcal{U} \in \mathbb{U}$  satisfying the following two inequalities:*

$$|\mathcal{J}(\mathcal{U}) - \mathcal{J}(\mathcal{U}')| < \epsilon \tag{3.29a}$$

$$0 < \mathcal{U}(x) < 1, \quad \mu - \text{a.e.} \tag{3.29b}$$

The following lemma will be used on the proof of Lemma 3.2 and it states the solution of a moment minimization problem akin to what can be found in [58].

**Lemma 3.1.** *Consider that a random variable  $X$  with distribution  $\mu$ , constants*

$\gamma \in \mathbb{R}$  and  $\alpha \in (0, 1)$  are given, and consider the optimization problem

$$\begin{aligned}
& \underset{\mathcal{G} \in L^2_\mu(\mathbb{R})}{\text{minimize}} && \mathbf{E}[X^2 \mathcal{G}(X)] \\
& \text{subject to} && \mathbf{E}[X \mathcal{G}(X)] = \gamma \\
& && \mathbf{E}[\mathcal{G}(X)] = 1 \\
& && 0 \leq \mathcal{G}(x) \leq \frac{1}{1-\alpha}, \quad \mu - \text{a.e.}
\end{aligned} \tag{3.30}$$

If the problem in Eq. (3.30) has a feasible solution  $\mathcal{G}$  for which the last constraint is satisfied with strict inequalities then there is an optimal solution  $\check{\mathcal{G}}$  with the following threshold structure:

$$\check{\mathcal{G}}(x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } \check{a} \leq x \leq \check{b} \\ 0 & \text{otherwise,} \end{cases} \tag{3.31}$$

for some real constants  $\check{a}$  and  $\check{b}$ .

*Proof.* The proof uses a technique from [59, Section 5.7.3] adapted to infinite dimensional linear programming. We start by defining a new objective function  $\mathcal{C} : L^2_\mu(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  that incorporates the inequality constraints by making them implicit

$$\mathcal{C}(\mathcal{G}) = \begin{cases} \mathbf{E}[X^2 \mathcal{G}(X)] & \text{if } 0 \leq \mathcal{G}(x) \leq \frac{1}{1-\alpha}, \quad \mu - \text{a.e.} \\ +\infty & \text{otherwise.} \end{cases} \tag{3.32}$$

This leads to the following equivalent optimization problem:

$$\begin{aligned}
& \underset{\mathcal{G} \in L^2_{\mu}(\mathbb{R})}{\text{minimize}} && \mathcal{C}(\mathcal{G}) \\
& \text{subject to} && \mathbf{E}[X\mathcal{G}(X)] = \gamma \\
& && \mathbf{E}[\mathcal{G}(X)] = 1.
\end{aligned} \tag{3.33}$$

Letting  $\nu \in \mathbb{R}^2$  denote the vector of dual variables  $\nu = (\nu_0, \nu_1)$ , the Lagrange dual function for this problem is

$$\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\gamma + \inf_{0 \leq \mathcal{G}(x) \leq \frac{1}{1-\alpha}} \mathbf{E}[(X^2 + \nu_0 X + \nu_1)\mathcal{G}(X)], \tag{3.34}$$

where the bounds on  $\mathcal{G}$  hold  $\mu$ -a.e. The following function minimizes the last term in the right hand side of Eq. (3.34):

$$\mathcal{G}_{\nu}(x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } x^2 + \nu_0 x + \nu_1 \leq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.35}$$

which when substituted in Eq. (3.34) leads to the following expression for  $\mathcal{C}^*(\nu)$ :

$$\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\gamma - \frac{1}{1-\alpha} \mathbf{E}[[X^2 + \nu_0 X + \nu_1]^-]. \tag{3.36}$$

In Appendix A.2 it is shown in detail that, provided there exists a feasible solution  $\mathcal{G}$  for which the last constraint in Eq. (3.30) is satisfied with strict inequalities, then strong duality holds and there exists a vector  $\nu^* \in \mathbb{R}^2$  that maximizes  $\mathcal{C}^*(\nu)$ .

Hence, an optimal solution  $\mathcal{G}^*$  for the problem in Eq. (3.33) is obtained by substituting such a  $\nu^*$  in Eq. (3.35), or equivalently, by setting the equality  $\mathcal{G}^* = \mathcal{G}_{\nu^*}$ . From Appendix A.3, any such  $\nu^*$  leads to a polynomial  $x^2 + \nu_0^*x + \nu_1^*$  that always admits real roots, which we denote as  $a^*$  and  $b^*$ , with  $a^* \leq b^*$ . Since  $x^2 + \nu_0^*x + \nu_1^*$  is a convex parabola in  $x$ , we can further conclude that the test  $x^2 + \nu_0x + \nu_1 \leq 0$  can be replaced with  $a^* \leq x \leq b^*$ . Using these facts in conjunction with Eq. (3.35), we conclude that there is an optimal solution of the form in Eq. (3.31).  $\square$

**Lemma 3.2.** *Assume that  $\mathcal{U}' \in \mathbb{U}$  is a given policy for Problem 3.2. For every positive real constant  $\epsilon$ , there is a deterministic threshold policy  $\check{\mathcal{U}}$  for which  $\mathcal{J}(\check{\mathcal{U}}) < \mathcal{J}(\mathcal{U}') + \epsilon$ .*

*Proof.* Our overarching strategy is to view the problem in Eq. (3.30) as a version of Problem 3.2 with additional constraints so that we can use Lemma 3.1 to obtain the desired result.

From Proposition 3.3, we know that from the given  $\mathcal{U}'$ , we can construct  $\mathcal{U}''$  so that the following holds:

$$\mathcal{J}(\mathcal{U}'') < \mathcal{J}(\mathcal{U}') + \epsilon \tag{3.37a}$$

$$0 < \mathcal{U}''(x) < 1, \quad \mu - \text{a.e.} \tag{3.37b}$$



We now proceed by defining  $\gamma$  and  $\alpha$  as follows:

$$\gamma \stackrel{\text{def}}{=} \mathbf{E}[X|U'' = 0] \quad (3.38a)$$

$$\alpha \stackrel{\text{def}}{=} \mathbf{P}(U'' = 1) = \mathbf{E}[\mathcal{U}''(X)] \quad (3.38b)$$

where the action  $U''$  is generated from the policy  $\mathcal{U}''$  as described in Problem 3.2. The cases when  $\alpha = 0$  or  $\alpha = 1$  immediately correspond to optimal threshold policies  $\check{\mathcal{U}}(x) = 0$ ,  $\mu - \text{a.e.}$  and  $\check{\mathcal{U}}(x) = 1$ ,  $\mu - \text{a.e.}$ , respectively. So, without loss of generality, we consider that  $\alpha$  is in  $(0, 1)$ .

Fact 1: Given  $\gamma$  and  $\alpha$  defined in Eq. (3.38), we conclude that  $\mathcal{G}''$  defined as follows:

$$\mathcal{G}''(x) \stackrel{\text{def}}{=} \frac{1 - \mathcal{U}''(x)}{1 - \alpha}, \quad x \in \mathbb{R} \quad (3.39)$$

satisfies the constraints of Eq. (3.30), and from Eq. (3.37b) the variable bounds are satisfied with strict inequality. Hence, we conclude that the conditions for Lemma 3.1 are satisfied for the  $\gamma$  and  $\alpha$  defined in Eq. (3.38). Denote with  $\mathbb{U}_{\alpha, \gamma}$  the subset of policies  $\mathcal{U} \in \mathbb{U}$  for which the following holds:

$$\mathbf{E}[X|U = 0] = \gamma; \quad \mathbf{E}[\mathcal{U}(X)] = \alpha, \quad (3.40)$$

where  $U$  is the action generated from  $\mathcal{U}$  as described in Problem 3.2. For any  $\mathcal{U}$  in  $\mathbb{U}_{\alpha, \gamma}$ ,  $\mathcal{J}(\mathcal{U})$  can be written as:

$$\mathcal{J}(\mathcal{U}) = \mathbf{E}[\beta(X - \hat{x}_{\mathbf{c}})^2 + \varrho | U = 1] \alpha + \mathbf{E}[(X - \hat{x}_{\emptyset})^2 | U = 0] (1 - \alpha), \quad (3.41)$$

for which  $\hat{x}_{\mathfrak{C}}$  and  $\hat{x}_{\emptyset}$  are defined as:

$$\hat{x}_{\mathfrak{C}} \stackrel{\text{def}}{=} \mathbf{E}[X|Z = \mathfrak{C}]; \quad \hat{x}_{\emptyset} \stackrel{\text{def}}{=} \mathbf{E}[X|Z = \emptyset], \quad (3.42)$$

where  $Z$  is the channel output as described in Problem 3.2. Since  $\mathbf{E}[X|U = 1] = \hat{x}_{\mathfrak{C}}$ , we can rewrite the cost as:

$$\begin{aligned} \mathcal{J}(\mathcal{U}) &= (1 - \beta) \mathbf{E}[X^2 | U = 0] (1 - \alpha) \\ &\quad - \left[ \frac{(1 - \alpha)^2}{\alpha} \beta + (1 - \alpha) \right] \gamma^2 + \varrho\alpha + \beta\sigma_X^2, \end{aligned} \quad (3.43)$$

where  $\mathcal{U} \in \mathbb{U}_{\alpha, \gamma}$  and we used the facts that  $\hat{x}_{\mathfrak{C}}\mathbf{P}(U = 1) = -\hat{x}_{\emptyset}\mathbf{P}(U = 0)$  and  $\hat{x}_{\emptyset} = \mathbf{E}[X|U = 0] = \gamma$ .

Fact 2: Notice that for  $\mathcal{U}$  in  $\mathbb{U}_{\alpha, \gamma}$ ,  $\mathbf{E}[X^2|U = 0]$  and  $\mathbf{E}[X|U = 0]$  can be written as  $\mathbf{E}[X^2\mathcal{G}(X)]$  and  $\mathbf{E}[X\mathcal{G}(X)]$ , respectively, where  $\mathcal{G}$  is found from Bayes' law to be:

$$\mathcal{G}(x) = \frac{1 - \mathcal{U}(x)}{1 - \alpha}, \quad x \in \mathbb{R}. \quad (3.44)$$

Fact 3: From Fact 2, Eq. (3.44) and Section 3.4, we conclude that minimizing  $\mathcal{J}(\mathcal{U})$  with respect to  $\mathcal{U}$  constrained to Eq. (3.40) is equivalent to solving the problem in Eq. (3.30).

From Fact 1 we know that the conditions for the validity of Lemma 3.1 are satisfied. Hence, from Lemma 3.1, Fact 3 and Eq. (3.44) we conclude that there is a deterministic threshold policy  $\check{\mathcal{U}}$  that minimizes  $\mathcal{J}(\mathcal{U})$  subject to the constraints in Eq. (3.40). Such policy can be computed from the solution in Lemma 3.1 as

follows:

$$\check{\mathcal{U}}(x) = 1 + (\alpha - 1)\check{\mathcal{G}}(x), \quad x \in \mathbb{R}. \quad (3.45)$$

Since  $\check{\mathcal{U}}$  satisfies Eq. (3.40), by optimality we conclude that  $\mathcal{J}(\check{\mathcal{U}}) \leq \mathcal{J}(\mathcal{U}'')$  holds, which in conjunction with Eq. (3.37a) concludes the proof.  $\square$

**Theorem 3.2.** *There is a deterministic threshold policy  $\check{\mathcal{U}}^*$  that is optimal for Problem 3.2.*

*Proof.* Consider that the parameters that specifies an instance of Problem 3.2 are given and denote the minimum cost in Eq. (3.26) as  $\varsigma^*$ . Let  $\mathcal{U}_{(n)}$  be a sequence of policies such that  $\lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{U}_{(n)}) = \varsigma^*$ . From Lemma 3.2, we can define a sequence of threshold policies  $\check{\mathcal{U}}_{(n)}$ , such that

$$\mathcal{J}(\check{\mathcal{U}}_{(n)}) \leq \mathcal{J}(\mathcal{U}_{(n)}) + \frac{1}{n+1}. \quad (3.46)$$

Let  $\check{a}_{(n)}$  and  $\check{b}_{(n)}$  be the thresholds associated with  $\check{\mathcal{U}}_{(n)}(x)$ . We now proceed to study the convergence to an optimum based on the sequence  $\{(\check{a}_{(n)}, \check{b}_{(n)})\}_{n=0}^{\infty}$ . We start by remarking that the sequence  $\{(\check{a}_{(n)}, \check{b}_{(n)})\}_{n=0}^{\infty}$  has at least one subsequence  $\{(\check{a}_{(m_n)}, \check{b}_{(m_n)})\}_{n=0}^{\infty}$  for which  $\check{a}^* \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \check{a}_{(m_n)}$  and  $\check{b}^* \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \check{b}_{(m_n)}$  are well defined and take values in  $\bar{\mathbb{R}}$ , with  $\check{a}^* \leq \check{b}^*$ . The proof follows by using Eq. (3.46) and Proposition A.2 (Appendix A.1) to conclude that the thresholds  $\check{a}^*$  and  $\check{b}^*$  define an optimal policy for Problem 3.2, which we denote as  $\check{\mathcal{U}}^*$ .  $\square$

## Proof of Theorem 3.1

Our proof is organized in two main steps that hinge on the analogy developed in the first part of this section, which presents results of independent interest for an optimal point-to-point remote estimation paradigm that includes communication costs.

*Proof of Theorem 3.1.* For any parameter selection that specify an instance of Problem 3.1, let  $\varsigma^*$  be the minimum cost and select a sequence of policies  $\{(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)})\}_{n=0}^{\infty}$  for which the following holds:

$$\lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}) = \varsigma^*. \quad (3.47)$$

Step 1: From Remark 3.3 and Lemma 3.2, we conclude that there is a sequence of deterministic threshold policies  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$  for which the following holds:

$$\mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \mathcal{U}_{2,(n)}) \leq \mathcal{J}(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}) + \frac{1}{n+1}, \quad n \geq 0 \quad (3.48a)$$

$$\mathcal{J}(\mathcal{U}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) \leq \mathcal{J}(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}) + \frac{1}{n+1}, \quad n \geq 0. \quad (3.48b)$$

Step 2: We can repeat the method used in Step 1 to conclude that there is a sequence of deterministic threshold policies  $\{(\check{\check{\mathcal{U}}}_{1,(n)}, \check{\check{\mathcal{U}}}_{2,(n)})\}_{n=0}^{\infty}$  for which the

following holds:

$$\mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) \leq \mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \mathcal{U}_{2,(n)}) + \frac{1}{n+1}, \quad n \geq 0 \quad (3.49a)$$

$$\mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) \leq \mathcal{J}(\mathcal{U}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) + \frac{1}{n+1}, \quad n \geq 0. \quad (3.49b)$$

Our conclusion from Eqs. (3.47) to (3.49) is that the sequences  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$  and  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$  satisfy the following:

$$\lim_{n \rightarrow \infty} \mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) = \varsigma^* \quad (3.50a)$$

$$\lim_{n \rightarrow \infty} \mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) = \varsigma^*. \quad (3.50b)$$

Without loss of generality, we proceed to analyze the convergence of the sequence  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$  to an optimal solution. An equivalent argument could have been developed using  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$ . Let  $\check{a}_1^*$ ,  $\check{b}_1^*$ ,  $\check{a}_2^*$  and  $\check{b}_2^*$  be constants in  $\bar{\mathbb{R}}$  for which there is a subsequence  $\{(\check{\mathcal{U}}_{1,(m_n)}, \check{\mathcal{U}}_{2,(m_n)})\}_{n=0}^{\infty}$  whose associated thresholds satisfy  $\lim_{n \rightarrow \infty} \check{a}_{1,(m_n)} = \check{a}_1^*$ ,  $\lim_{n \rightarrow \infty} \check{b}_{1,(m_n)} = \check{b}_1^*$ ,  $\lim_{n \rightarrow \infty} \check{a}_{2,(m_n)} = \check{a}_2^*$  and  $\lim_{n \rightarrow \infty} \check{b}_{2,(m_n)} = \check{b}_2^*$ . The proof is concluded by invoking Proposition A.1 (Appendix A.1) to show that the thresholds  $\check{a}_1^*$ ,  $\check{b}_1^*$ ,  $\check{a}_2^*$  and  $\check{b}_2^*$  define an optimal policy, which we denote as  $(\check{\mathcal{U}}_1^*, \check{\mathcal{U}}_2^*)$ .  $\square$

**Remark 3.4.** *Note that the proofs of the structural results of Theorems 3.1 and 3.2 are completely independent of the distributions of  $X_1$  and  $X_2$ , as long as they are zero mean independent continuous random variables with finite variances. In fact,  $X_1$  and  $X_2$  may come from completely different families of distributions. The structural*

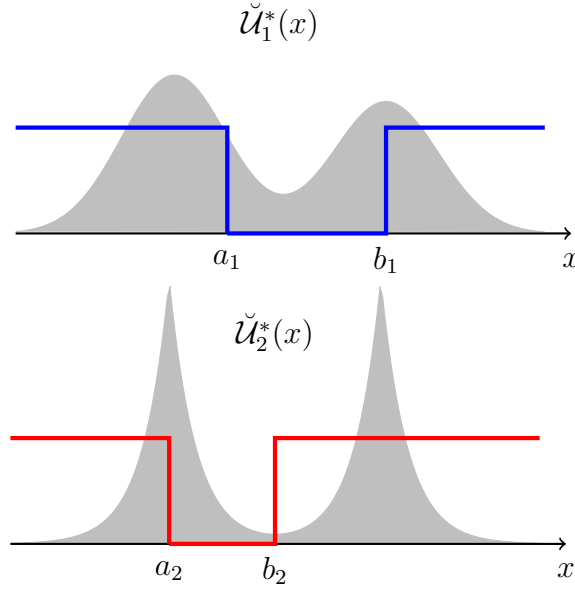


Figure 3.3: There exists an team-optimal pair  $(\check{\mathcal{U}}_1^*, \check{\mathcal{U}}_2^*)$  of threshold policies for Problem 3.1 even if the densities are asymmetric or multimodal.

*result of Theorem 3.1 is also true for a sensing system with any number of sensors measuring mutually independent random variables, under the additional assumption that the remote estimator can decode the indices of the sensors involved in a collision.*

### 3.5 Consequences for more general systems

In most sequential multi-agent decision making problems, it is useful to solve the problem for a single pair of agents in a single time step to characterize the structure of the team-optimal policies. Our results constitute a first step in solving a decentralized sequential estimation problem over a collision channel, providing valuable insights into the nature of its team-optimal solutions. Despite the apparent simplicity of our model, our results have implications for a wide class of systems. Firstly, the structural results herein also hold for continuous random vectors, re-

quiring only minor modifications in the proofs. We chose to present the results for scalar random variables to simplify the proofs and facilitate the visualization of the threshold policies in  $\mathbb{R}$ , since in  $\mathbb{R}^n$  they would become hyper-ellipsoidal surfaces. Another important feature of our formulation is that the results hold for an arbitrary number of sensor nodes measuring independent random variables. In order to see this, we need the additional assumption that the remote estimator can decode the index of the sensors that were involved in a collision. Then, when solving the person-by-person optimization problem, we can treat the sensors with fixed strategies as a “superuser” observing a random vector and occupying the channel with a given probability  $\beta$ . Following the same arguments of Section 3.4 we obtain the same structural result.

The one-shot problem we have solved is a fundamental building block for the sequential problem. The fact that the result is independent of the probability density functions of the measurements is particularly important in the sequential case with feedback because the state of a Gauss-Markov system conditioned on the channel outputs or acknowledgements has a distribution that is no longer Gaussian. Finally, we have shown that the optimal thresholds can be asymmetric for a single stage problem with even probability density functions. This implies that the restriction to symmetric threshold policies is suboptimal for more general sequential event-based estimation problems over the collision channel.

### 3.6 Summary

We studied the collision channel as a model for interference in a multi-sensor remote estimation problem. Our goal was to characterize team-optimal communication policies in a simple interference setting. Using a person-by-person optimality approach, we established the existence of a team-optimal strategy which consists of threshold policies. We showed that, from the perspective of a single decision maker, the aggregate quadratic cost can be decomposed in two terms: a mean squared estimation error and a communication cost. For this cost, we proved that there exist an optimal threshold communication policy. The proofs of our results hinge on Lagrange duality applied to a generalized moment optimization problem with variable bounds. The structural results obtained here are independent of the distributions of the random variables.



## Chapter 4: Numerical computation of optimal thresholds

In this chapter, we turn our focus to the design of optimal policies for Problem 3.2, which will ultimately lead to person-by-person optimal policies for Problem 3.1. We base our arguments on the observation that Problem 3.2 can be understood as an one-bit quantization problem with distinct quadratic distortion metrics across two quantization regions. The intuition behind this interpretation comes from the fact that the sensor’s decision of transmitting or not can be exploited for communication by embedding in the two possible actions an additional bit of information. When the transmission is successful, this additional bit is redundant because the received packet already contains all the relevant information for a perfect estimate. However, when the transmission fails due to the occurrence of a collision, the estimator forms  $\hat{X}$  based on this additional bit. The objective of the system’s designer is to “compress” in this bit (represented by  $\emptyset$  and  $\mathfrak{C}$ ) the maximum amount of information about  $X$  as possible. This situation does not occur if instead of collisions we had random erasures. The observation of an erasure does not reveal the sensor’s intent to communicate since they cannot be distinguished from “erasures” due to the absence of transmitted packets when the channel is idle.

## 4.1 Policy design via quantization theory

The structural result of Theorem Theorem 3.2 established the existence of an optimal deterministic threshold policy for Problem 3.2. Here we will make the analogy with one-bit quantization more precise. First, we will let  $\mathcal{U}$  act as a deterministic encoder, which partitions the real line  $\mathbb{R}$  into two measurable sets  $\mathbb{A}_0$  and  $\mathbb{A}_1 = \mathbb{A}_0^c$  such that  $\mathbb{A}_0 \stackrel{\text{def}}{=} \mathcal{U}^{-1}(0)$  and  $\mathbb{A}_1 \stackrel{\text{def}}{=} \mathcal{U}^{-1}(1)$ . We relax Problem 3.2 by letting the estimator  $\mathcal{E}$  lie in a class of admissible deterministic decoders, where

$$\mathcal{E}(z) = z, \quad \text{if } z \notin \{\emptyset, \mathfrak{C}\}. \quad (4.1)$$

Let  $\hat{x}_\emptyset, \hat{x}_\mathfrak{C} \in \mathbb{R}$ , define

$$\mathcal{E}(\emptyset) \stackrel{\text{def}}{=} \hat{x}_\emptyset \quad \text{and} \quad \mathcal{E}(\mathfrak{C}) \stackrel{\text{def}}{=} \hat{x}_\mathfrak{C}. \quad (4.2)$$

With the cost in Problem 3.2 now depending on  $\mathcal{U}$  and  $\mathcal{E}$ , and assuming that the random variable  $X$  has zero mean, finite variance  $\sigma_X^2$ , and admits a probability density function  $f_X(x)$ , we can rewrite the functional as

$$\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E}) = \int_{\mathbb{A}_0} (x - \hat{x}_\emptyset)^2 f_X(x) dx + \int_{\mathbb{A}_0^c} [\beta(x - \hat{x}_\mathfrak{C})^2 + \varrho] f_X(x) dx. \quad (4.3)$$

Our goal is to choose a partition of  $\mathbb{R}$  into a measurable set  $\mathbb{A}_0$  and its complement  $\mathbb{A}_0^c$  and their respective representation points  $\hat{x}_\emptyset$  and  $\hat{x}_\mathfrak{C}$  such as to minimize the average distortion quantified by  $\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E})$ .

Applying the structural result obtained in Chapter 3 the optimal partition is such that  $\mathbb{A}_0 = [a, b]$ , where  $a \leq b$  and  $a, b \in \bar{\mathbb{R}}$ . When they exist<sup>1</sup>, the optimal thresholds and representation symbols can be found by solving the optimization problem in Eq. (4.4) with variables in  $a, b, \hat{x}_\emptyset, \hat{x}_\mathfrak{e} \in \mathbb{R}$ :

$$\begin{aligned} \text{minimize} \quad & \int_{[a,b]} (x - \hat{x}_\emptyset)^2 f_X(x) dx + \int_{[a,b]^c} [\beta(x - \hat{x}_\mathfrak{e})^2 + \varrho] f_X(x) dx \\ \text{subject to} \quad & a \leq b \end{aligned} \tag{4.4}$$

In other words, the communication and estimation policies that jointly minimize the cost  $\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E})$  can be found by solving an optimal scalar quantization problem of a random variable  $X \sim f_X(x)$ , where representation symbols are penalized by distinct quadratic distortion functions.

**Remark 4.1.** *Relaxing the estimator to lie in a larger class of admissible estimators, rather than fixing it as the conditional expectation operator, will be important in order to obtain a numerical procedure for finding solutions for Problem 3.2.*

The nearest neighbor condition and an equivalent problem

Let  $\hat{x} \stackrel{\text{def}}{=} (\hat{x}_\emptyset, \hat{x}_\mathfrak{e}) \in \mathbb{R}^2$ . For any given  $\hat{x}$ , the set  $\mathbb{A}_0^*$  which yields the minimal cost must satisfy

$$x \in \mathbb{A}_0^* \Leftrightarrow (x - \hat{x}_\emptyset)^2 \leq \beta(x - \hat{x}_\mathfrak{e})^2 + \varrho. \tag{4.5}$$

---

<sup>1</sup>The existence of optimal thresholds for Gaussian observations was established in [45].

This is true regardless of the probability density function  $f_X(x)$ . Since  $\varrho \geq 0$ , for  $0 \leq \beta < 1$ , the second degree polynomial

$$\mathcal{P}_{\hat{x}}(x) \stackrel{\text{def}}{=} (x - \hat{x}_{\varnothing})^2 - \beta(x - \hat{x}_{\mathfrak{C}})^2 - \varrho \quad (4.6)$$

admits two distinct real roots<sup>2</sup>. We will denote the minimum of these roots by  $a(\hat{x})$  and the largest by  $b(\hat{x})$ . Therefore, without loss of optimality, we may assume that the no-transmission interval is, for a given  $\hat{x}$ ,

$$\mathbb{A}_0 = [a(\hat{x}), b(\hat{x})], \quad (4.7)$$

and the cost is reduced to a function  $\mathcal{J}_q : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as

$$\mathcal{J}_q(\hat{x}) \stackrel{\text{def}}{=} \int_{[a(\hat{x}), b(\hat{x})]} (x - \hat{x}_{\varnothing})^2 f_X(x) dx + \int_{[a(\hat{x}), b(\hat{x})]^c} [\beta(x - \hat{x}_{\mathfrak{C}})^2 + \varrho] f_X(x) dx, \quad (4.8)$$

where the two maps  $a : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $b : \mathbb{R}^2 \rightarrow \mathbb{R}$  are given by

$$a(\hat{x}) \stackrel{\text{def}}{=} \frac{1}{1 - \beta} \left[ (\hat{x}_{\varnothing} - \beta \hat{x}_{\mathfrak{C}}) - \sqrt{\beta(\hat{x}_{\varnothing} - \hat{x}_{\mathfrak{C}})^2 + (1 - \beta)\varrho} \right] \quad (4.9)$$

and

$$b(\hat{x}) \stackrel{\text{def}}{=} \frac{1}{1 - \beta} \left[ (\hat{x}_{\varnothing} - \beta \hat{x}_{\mathfrak{C}}) + \sqrt{\beta(\hat{x}_{\varnothing} - \hat{x}_{\mathfrak{C}})^2 + (1 - \beta)\varrho} \right]. \quad (4.10)$$

**Remark 4.2.** *The function  $\mathcal{J}_q(\hat{x})$  is twice continuously differentiable at every  $\hat{x} \in$*

---

<sup>2</sup>When  $\beta = 1$ , the polynomial  $\mathcal{P}_{\hat{x}}(x)$  admits a single root. This case can be arbitrarily well approximated by a sequence of problems with  $\beta_n \in [0, 1)$  such that  $\{\beta_n\} \uparrow 1$ .

$\mathbb{R}^2$ . Furthermore, there is no loss in optimality in minimizing  $\mathcal{J}_q(\hat{x})$  over  $\mathbb{R}^2$  instead of solving the problem in Eq. (4.4), which is defined over  $\mathbb{R}^4$  [42].

Therefore, we have an equivalent finite dimensional unconstrained optimization problem in terms of the pair of representation points  $\hat{x}$  that specify the estimator  $\mathcal{E}$ :

**Problem 4.1.** *Given the constants  $\varrho \geq 0$ ,  $\beta \in [0, 1)$  and  $f_X(x)$ , solve the unconstrained nonlinear optimization problem*

$$\underset{\hat{x} \in \mathbb{R}^2}{\text{minimize}} \quad \mathcal{J}_q(\hat{x}). \quad (4.11)$$

The centroid condition

We now obtain a set of necessary optimality conditions corresponding to  $\nabla \mathcal{J}_q(\hat{x}^*) = 0$ .

**Proposition 4.1.** *Any minimizing  $\hat{x}^* = (\hat{x}_\emptyset^*, \hat{x}_\mathfrak{E}^*)$  must satisfy*

$$\int_{[a(\hat{x}^*), b(\hat{x}^*)]} (x - \hat{x}_\emptyset^*) f_X(x) dx = 0 \quad (4.12)$$

$$\int_{[a(\hat{x}^*), b(\hat{x}^*)]^c} (x - \hat{x}_\mathfrak{E}^*) f_X(x) dx = 0. \quad (4.13)$$

Proposition 4.1 essentially states that the optimal representation points must be centroids of the interval  $\mathbb{A}_0^* = [a(\hat{x}^*), b(\hat{x}^*)]$ , defined by the two roots of  $\mathcal{P}_{\hat{x}^*}(x)$ , and its complement. If the density  $f_X$  has full support on  $\mathbb{R}$ , the conditions in

Proposition 4.1 can be written more compactly as

$$\hat{x}^* = \mathcal{F}(\hat{x}^*), \quad (4.14)$$

where  $\mathcal{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\mathcal{F}(\hat{x}) \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx} \\ \frac{-1}{1 - \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx} \end{bmatrix} \int_{a(\hat{x})}^{b(\hat{x})} x f_X(x) dx. \quad (4.15)$$

Hence, any critical point of  $\mathcal{J}_q(\hat{x})$  and, in particular, any optimal solution  $\hat{x}^*$  are fixed-points of the nonlinear map  $\mathcal{F}$ . The first and second components of the vector  $\mathcal{F}(\cdot)$  are denoted by  $\mathcal{F}_\emptyset(\cdot)$  and  $\mathcal{F}_\mathbf{e}(\cdot)$ , respectively.

**Proposition 4.2.** *If  $f_X$  has full support on  $\mathbb{R}$  and is even, the following statements about the map  $\mathcal{F}$  in Eq. (4.15) hold:*

1. *Any nonzero fixed point  $\hat{x}$  must satisfy*

$$\text{sgn}(\hat{x}_\emptyset) = -\text{sgn}(\hat{x}_\mathbf{e}) \quad (4.16)$$

2. *The vector  $\hat{x} = (0, 0)$  is always a fixed point*
3. *If  $\hat{x}$  is a fixed point then  $-\hat{x}$  is also a fixed point*
4. *The set*

$$\mathbb{L}_\beta \stackrel{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 \mid \hat{x}_\emptyset = \beta \hat{x}_\mathbf{e}\} \quad (4.17)$$

*is mapped into  $(0, 0)$*

5. Any fixed point  $\hat{x}$  satisfies

$$|\hat{x}_\emptyset| |\hat{x}_\mathbf{e}| \leq \sigma_X^2, \quad (4.18)$$

where  $\sigma_X^2 = \mathbf{V}(X)$ .

*Proof.* The first statement can be readily verified from the definition of  $\mathcal{F}$ . To prove the second statement we compute  $a(\cdot)$  and  $b(\cdot)$  at  $\hat{x} = (0, 0)$ , which yield

$$b(0, 0) = -a(0, 0) = \sqrt{\frac{\rho}{1 - \beta}}. \quad (4.19)$$

Since  $f_X(x)$  is an symmetric probability density function,  $\int_{-b}^b x f_X(x) dx = 0$ , which implies that  $\mathcal{F}(0, 0) = (0, 0)$ . The third statement can be shown by assuming that  $\mathcal{F}(\hat{x}) = \hat{x}$  and computing  $\mathcal{F}(-\hat{x})$ . It can be verified that  $a(-\hat{x}) = -b(\hat{x})$ . Therefore, with a change of variables we obtain

$$\int_{a(-\hat{x})}^{b(-\hat{x})} x f_X(x) dx = \int_{-b(\hat{x})}^{-a(\hat{x})} x f_X(x) dx = - \int_{a(\hat{x})}^{b(\hat{x})} x f_X(x) dx \quad (4.20)$$

and

$$\int_{a(-\hat{x})}^{b(-\hat{x})} f_X(x) dx = \int_{-b(\hat{x})}^{-a(\hat{x})} f_X(x) dx = \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx. \quad (4.21)$$

Substituting these in the expression of  $\mathcal{F}(-\hat{x})$ , we obtain

$$\mathcal{F}(-\hat{x}) = -\mathcal{F}(\hat{x}) = -\hat{x}. \quad (4.22)$$

The fourth statement can be verified by noticing that for  $\hat{x} \in \mathbb{L}_\beta$ , we have  $a(\hat{x}) = -b(\hat{x})$ . Since  $f_X(x)$  is even,

$$\int_{a(\hat{x})}^{b(\hat{x})} x f_X(x) dx = 0. \quad (4.23)$$

Finally, since  $\hat{x}_\emptyset = \mathbf{E}[X \mid a(\hat{x}) \leq X \leq b(\hat{x})]$ , the following inequality holds [60]:

$$|\hat{x}_\emptyset| \leq \sigma_X \left( \frac{1 - \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx}{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx} \right)^{\frac{1}{2}}. \quad (4.24)$$

From the definition of  $\mathcal{F}(\hat{x})$ ,

$$\hat{x}_\mathbf{e} = - \frac{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx}{1 - \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx} \hat{x}_\emptyset, \quad (4.25)$$

which implies that

$$|\hat{x}_\mathbf{e}| \leq \sigma_X \left( \frac{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx}{1 - \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx} \right)^{\frac{1}{2}}. \quad (4.26)$$

Eqs. (4.24) and (4.26) imply the inequality in Eq. (4.18).  $\square$

The following proposition reveals an important symmetry property of the cost function  $\mathcal{J}_q(\hat{x})$ .

**Proposition 4.3.** *If  $f_X(x)$  is an symmetric probability density function, then the cost  $\mathcal{J}_q(\hat{x})$  is an even function. In particular, if  $f_X(x) = \mathcal{N}(0, \sigma^2)$ , the cost  $\mathcal{J}_q(\hat{x})$  is an even function.*

*Proof.* The proof follows from the fact that  $a(-\hat{x}) = -b(\hat{x})$ , then performing a



change of variables in the integrals of the expression of  $\mathcal{J}_q(-\hat{x})$  with the assumed symmetry of  $f_X(x)$ .  $\square$

It is easy to show that if  $f_X$  is an even probability density function, then the cost  $\mathcal{J}_q(\hat{x})$  is an even function. In particular, if  $X \sim \mathcal{N}(0, \sigma_X^2)$  the cost function in Eq. (4.8) is even. One important consequence this fact together with Proposition 4.2 is that, for any even density  $f_X$ , the search for an optimal solution  $\hat{x}^*$  may be constrained to either

$$\mathbb{Q}_1 \stackrel{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 \mid \hat{x}_\emptyset \geq 0, \hat{x}_\mathbf{e} \leq 0\} \quad (4.27)$$

or

$$\mathbb{Q}_2 \stackrel{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 \mid \hat{x}_\emptyset \leq 0, \hat{x}_\mathbf{e} \geq 0\} \quad (4.28)$$

without loss of optimality. Figure 4.1 shows where the stationary points of  $\mathcal{F}$  may lie.

## 4.2 Examples

In this section we provide examples of optimal policies for Problem 3.2 obtained as solutions to Problem 4.1 when  $X \sim \mathcal{N}(0, \sigma_X^2)$ .

**Example 4.1** (Non-convexity of the cost function). *Consider the cost  $\mathcal{J}_q(\hat{x})$  for a typical choice of parameters: let  $X \sim \mathcal{N}(0, 1)$ ,  $\beta = 0.5$  and  $\varrho = 1$  in Problem 4.1. The plot of the cost function in log-scale is shown in Fig. 4.2 and its level curves are shown in Fig. 4.3. These two figures allow us to make two important observations. First, since the sublevel sets are not convex,  $\mathcal{J}_q(\hat{x})$  is neither convex nor quasi-*

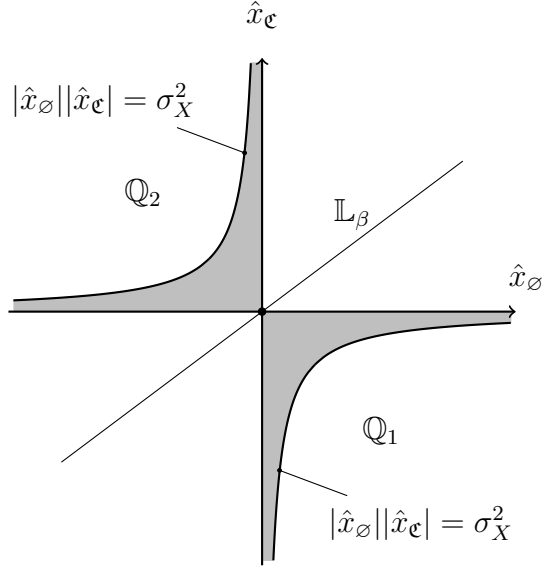


Figure 4.1: The shaded region of  $\mathbb{R}^2$  shown above contains all the critical points of  $\mathcal{J}_q(\hat{x})$  when  $f_X$  is an even density. The origin is always a critical point and the line  $\mathbb{L}_\beta$  is entirely mapped by  $\mathcal{F}$  into  $(0, 0)$ .

*convex. This is the case even if we constrain its domain to  $\mathbb{Q}_1$  or  $\mathbb{Q}_2$ . The second observation is the occurrence of a single minimum in each  $\mathbb{Q}_i$ ,  $i \in \{1, 2\}$ . However, due to the intricate structure of  $\mathcal{J}_q(\hat{x})$ , obtaining a proof of this fact remains an open problem.*

The optimal solutions to the various minimization problems considered in what follows were obtained using standard nonlinear programming solvers constraining  $\mathcal{J}_q(\hat{x})$  to  $\mathbb{Q}_1$ . More sophisticated algorithms for solving Problem 4.1 with optimality guarantees (such as the Branch-and-bound method) can be used along with the fact that  $\mathcal{J}_q(\hat{x})$  can be decomposed as a difference of convex functions since it is twice continuously differentiable [61].

**Example 4.2** (Optimality of asymmetric thresholds). *Let  $X \sim \mathcal{N}(0, 1)$ ,  $\beta = 0.5$  and  $\varrho = 1$  in Problem 4.1. A pair of representation points that minimize the cost*

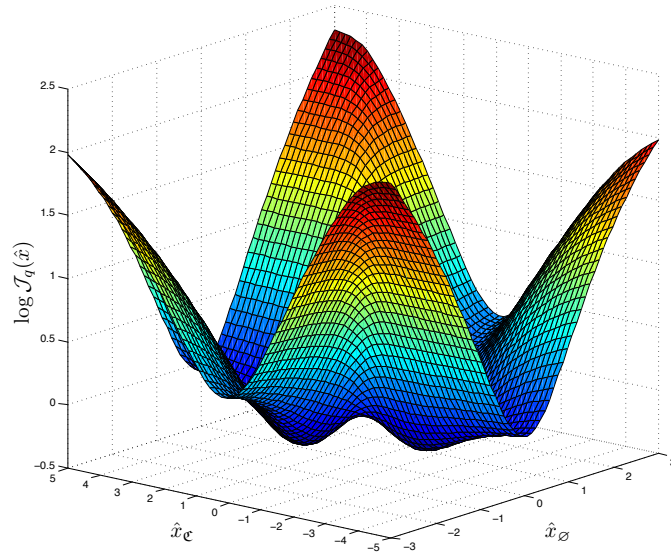


Figure 4.2: The cost function  $\mathcal{J}_q(\hat{x})$  in log-scale for  $\sigma_X^2 = 1$ ,  $\beta = 0.5$  and  $\varrho = 1$ . The log-scale helps us in visualizing the two minima.

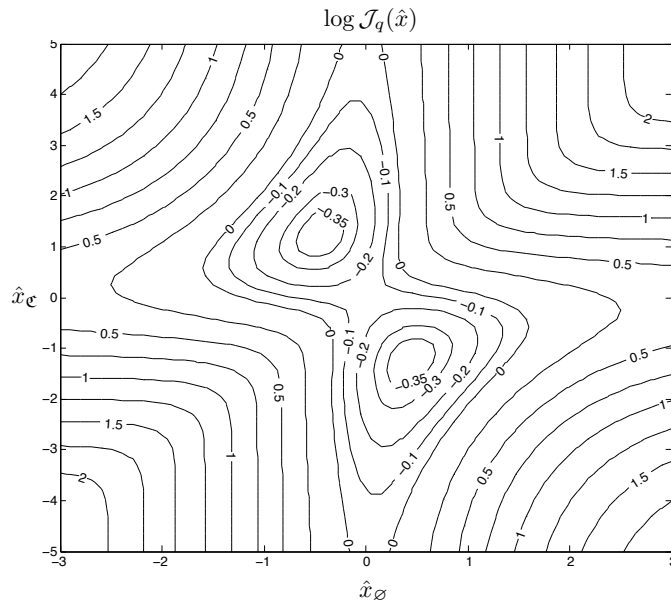


Figure 4.3: The level curves of cost function  $\mathcal{J}_q(\hat{x})$  in log-scale for  $\sigma_X^2 = 1$ ,  $\beta = 0.5$  and  $\varrho = 1$ . From the sublevel sets, we can conclude that  $\mathcal{J}_q(\hat{x})$  is neither convex nor quasi-convex. Despite this fact, we can observe that there is a single minimum in  $\mathbb{Q}_1$  and another in  $\mathbb{Q}_2$ .

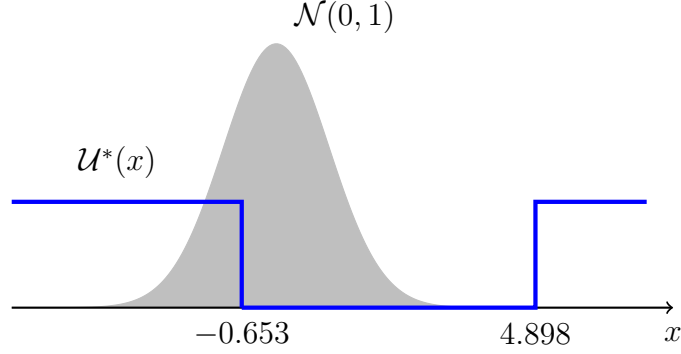


Figure 4.4: Optimal threshold policy for the problem in Example 4.2.

function in Eq. (4.8) is  $\hat{x}^* = (0.434, -1.255)$  corresponding to a cost  $\mathcal{J}_q^* = 0.681$ . By using the expressions in Eqs. (4.9) and (4.10), we obtain the values of the optimal thresholds of the optimal no-transmission interval  $\mathbb{A}_0^* = [-0.653, 4.898]$ . This policy is depicted in Fig. 4.4. Therefore, the optimal no-transmission interval is characterized by asymmetric thresholds. If, on the other hand, we consider only symmetric threshold policies, by the centroid conditions, their optimal representation points are  $(0, 0) \stackrel{\text{def}}{=} \hat{x}_{sym}$ . Hence, the optimal cost within the class of symmetric policies is

$$\mathcal{J}_{sym}^* \stackrel{\text{def}}{=} \mathcal{J}_q(0, 0) = 0.871 > \mathcal{J}_q^*. \quad (4.29)$$

We cast the observation about the asymmetry of the optimal thresholds drawn from Example 4.2 as the following remark.

**Remark 4.3.** For  $\beta > 0$  the optimal communication policies for Problem 3.2 have, in general, asymmetric thresholds.

This may lead us to erroneously assume that when  $\beta > 0$  the optimal policies must be asymmetric. The purpose of the next example is to show this is not always

the case.

**Example 4.3** (Optimality of symmetric thresholds). *Consider Problem 3.2 with  $X \sim \mathcal{N}(0, 1)$ ,  $\beta = 0.1$  and  $\varrho = 1$ . In this case, we can verify numerically that the pair of representation points that minimizes the cost function in Eq. (4.8) is  $\hat{x}^* = (0, 0)$ , yielding a cost  $\mathcal{J}_q^* = 0.595$ . Recovering the corresponding optimal no-transmission interval we have  $\mathbb{A}_0^* = [-1.054, 1.054]$ . This policy is depicted in Fig. 4.5.*

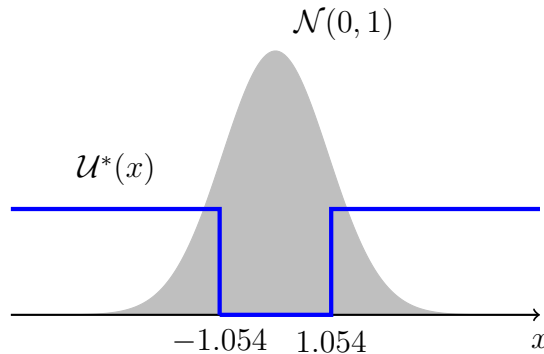


Figure 4.5: Optimal threshold policy for the problem in Example 4.3.

It is also interesting to observe how the optimal thresholds vary with the variance  $\sigma_X^2$ . Table 4.1 shows that when  $\sigma_X^2$  increases, the no-transmission interval has a positive drift and its length increases while its probability decreases. This means that transmissions will occur more often as the variance of the random variable  $X$  increases.

### 4.3 A modified Lloyd-Max algorithm

In this section of the chapter, we propose an iterative procedure inspired by the Lloyd-Max algorithm [62] to design optimal communication and estimation policies

$\sigma_X^2$	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathcal{J}_q^*$	$\mathbf{P}(U = 1)$
1	(0, 0)	[-1.054, 1.054]	0.595	0.292
2	(0.505, -0.565)	[-0.495, 1.743]	0.822	0.472
3	(1.124, -0.789)	[0.086, 2.586]	0.970	0.588
4	(1.658, -0.856)	[0.562, 3.313]	1.085	0.660
5	(2.130, -0.886)	[0.970, 3.960]	1.186	0.706

Table 4.1: Numerical results of the optimization Problem 3.2 when  $X \sim \mathcal{N}(0, \sigma_X^2)$ ,  $\beta = 0.1$  and  $\varrho = 1$  for different values of  $\sigma_X^2$ .

for Problem 3.2. We call this procedure the *Modified Lloyd-Max* (MLM) algorithm.

The MLM is an alternative to standard nonlinear solvers to find optimal solutions to Problem 4.1. The  $k$ -th iteration of the MLM algorithm consists of two steps:

- **Threshold update step:** For a fixed pair of representation points  $\hat{x}^{(k)} \in \mathbb{R}^2$ , update the thresholds that define the no-transmission interval according to

$$\mathbb{A}_0^{(k)} = [a(\hat{x}^{(k)}), b(\hat{x}^{(k)})] \quad (4.30)$$

- **Centroid computation step:** Obtain a new pair of representation points  $\hat{x}^{(k+1)}$  as the centroids of  $\mathbb{A}_0^{(k)}$  and its complement, i.e.,

$$\hat{x}_{\emptyset}^{(k+1)} = \mathbf{E} \left[ X \mid X \in \mathbb{A}_0^{(k)} \right], \quad (4.31)$$

and

$$\hat{x}_{\mathbf{c}}^{(k+1)} = \mathbf{E} \left[ X \mid X \notin \mathbb{A}_0^{(k)} \right]. \quad (4.32)$$

Henceforth, we will consider only the Gaussian case by assuming that  $X \sim \mathcal{N}(0, \sigma_X^2)$ .

This allows us to make important claims and observations about the MLM algo-

rithm, properties of fixed points and its convergence.

## An equivalent nonlinear autonomous dynamical system

The MLM algorithm outlined above can be understood as a nonlinear dynamical system described by successive applications of the map  $\mathcal{F}$  in Eq. (4.15). For a fixed  $\hat{x}^{(0)} \neq (0, 0)$ ,

$$\hat{x}^{(k+1)} = \mathcal{F}(\hat{x}^{(k)}), \quad k = 0, 1, \dots \quad (4.33)$$

It is important that the initial point  $\hat{x}^{(0)}$  is a nonzero vector, otherwise the algorithm outputs a sequence identically equal to zero. When  $X \sim \mathcal{N}(0, \sigma_X^2)$ , it can be shown that the sets  $\mathbb{Q}_1$  and  $\mathbb{Q}_2$  are invariant to the map  $\mathcal{F}$ , i.e.,

$$\mathcal{F}(\mathbb{Q}_i) \subset \mathbb{Q}_i, \quad i \in \{1, 2\}. \quad (4.34)$$

Therefore, a sequence of points generated by Eq. (4.33) will either belong to  $\mathbb{Q}_1$  or  $\mathbb{Q}_2$  depending on the initial condition  $\hat{x}^{(0)}$ . Furthermore, it is a well-known fact that the Lloyd-Max iterations generate a non-increasing sequence of values of the objective function [63], i.e.,

$$\mathcal{J}_q(\hat{x}^{(k+1)}) \leq \mathcal{J}_q(\hat{x}^{(k)}), \quad k = 0, 1, \dots \quad (4.35)$$

## On the convergence of the MLM algorithm

In general, unless it is known that  $\mathcal{F}$  is a Banach contraction, there are no guarantees that the dynamical system describing the MLM algorithm will converge to a unique fixed point. Moreover, empirical evidence shows that for a very large set of parameters, there are multiple fixed points. Therefore, it is unlikely that such contraction properties will hold for  $\mathcal{F}$ . However, the fact that the MLM is a descent algorithm together with the fact that the stationary points in  $\mathbb{Q}_i$ ,  $i \in \{1, 2\}$  are isolated indicate that convergence results to a local minimum may be proved. The paper by Du et al. [64] present several convergence results for Lloyd-Max type algorithms that can be used to establish convergence of the MLM. In particular, [64, Theorem 2.6] states:

*If the iterations in the Lloyd algorithm stay in a compact set where the Lloyd map  $\mathcal{F}$  is continuous, then the algorithm is globally convergent to a critical point of  $\mathcal{J}_q(\hat{x})$ .*

In Appendix B we show the existence of such a compact set when  $X \sim \mathcal{N}(0, \sigma_X^2)$ ,  $\beta \in [0, 1)$  and  $\varrho \geq 0$ . Under these conditions, the MLM is globally convergent to a local minimum of  $\mathcal{J}_q(\hat{x})$ .

**Remark 4.4.** *The classic sufficient condition due to Fleischer in [42] stating that if  $f_X$  is a log-concave density with full support on  $\mathbb{R}$ , the Lloyd-Max algorithm converges to a unique stationary point does not hold here due to the non-uniformity of the distortion metric in the quantization problem in Eq. (4.4).*



## 4.4 Numerical results

When  $X \sim \mathcal{N}(0, \sigma_X^2)$  the design of the transmission thresholds and representation points can be done by means of a globally convergent algorithm, which consists of iteratively applying a nonlinear map  $\mathcal{F}$  to a nonzero initial vector  $\hat{x}^{(0)} \in \mathbb{Q}_i$ ,  $i \in \{1, 2\}$ . Since  $\mathcal{J}_q(\hat{x})$  is non-convex we are not able to claim that a critical point found through the MLM algorithm is a global minimum, but from observing the general shape of  $\mathcal{J}_q(\hat{x})$  for several combination of parameters, we conjecture that  $\mathcal{F}$  will have at most two critical points in each  $\mathbb{Q}_i$ ,  $i = 1, 2$ . One of the stationary points is always  $(0, 0)$ , which can be a global minimum in some cases. For example, the trajectory of a sequence  $\{\hat{x}^{(k)}\} \rightarrow (0, 0)$  generated by the MLM applied to  $\hat{x}^{(0)} = (1, -1)$  when  $\beta = 0.1$  and  $\varrho = 1$  is shown in Fig. 4.6. The stopping criterion used was based on the magnitude of the gradient at  $\hat{x}^{(k)}$  as follows

$$\|\nabla \mathcal{J}_q(\hat{x}^{(k)})\| < 10^{-6}. \quad (4.36)$$

In most cases, however, the global minimum is a nonzero stationary point, which will correspond to asymmetric thresholds for the no-transmission interval  $\mathbb{A}_0^*$ . Fig. 4.7 illustrates the trajectory of points generated by  $\mathcal{F}$  when  $\beta = 0.3$ ,  $\varrho = 1$  and  $\sigma_X^2 = 1$ . The initial condition  $\hat{x}^{(0)}$  was chosen to lie on the curve  $\hat{x}_\varnothing \hat{x}_\mathbf{e} = -\sigma_X^2$ . In all the numerical examples of this section, the algorithm was initialized with  $\hat{x}^{(0)} = (\sigma_X, -\sigma_X)$ .

We obtained the optimal solutions of Problem 3.2 with the probability of

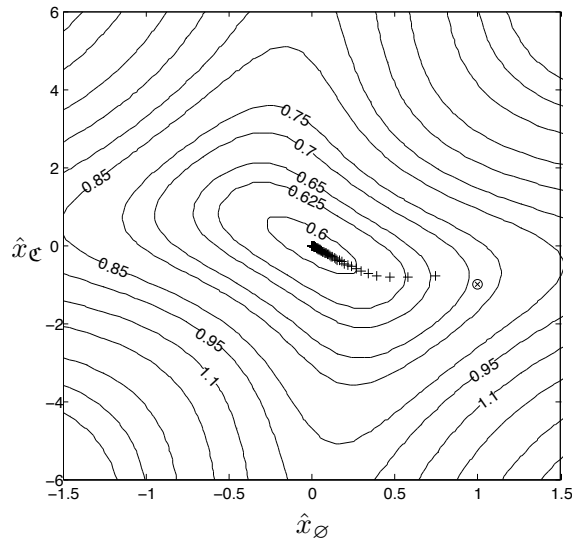


Figure 4.6: Trajectory of the sequence generated by the MLM algorithm when  $\beta = 0.1$ ,  $\varrho = 1$ ,  $\sigma_X^2 = 1$ . The level curves indicate that  $\mathcal{J}_q(\hat{x})$  has a single critical point at  $(0, 0)$ . The initial condition  $\hat{x}^{(0)}$  is shown in  $\otimes$  and the remaining points are displayed using  $+$ .

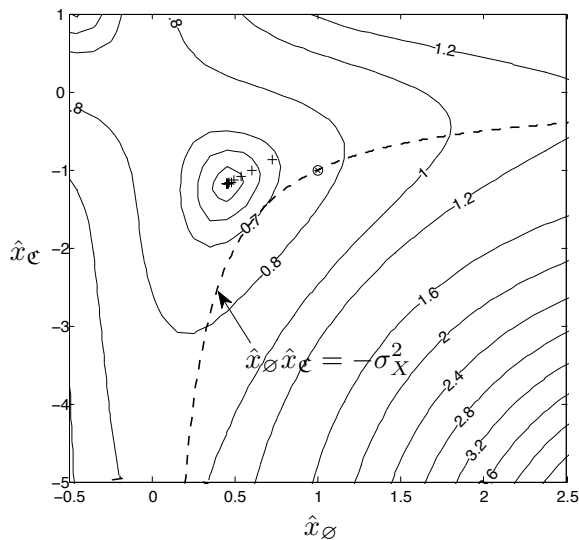


Figure 4.7: Sequence of points generated by  $\mathcal{F}$  converging to  $\hat{x}^* = (0.454, -1.183)$ . In this case, the parameters are  $\beta = 0.3$ ,  $\varrho = 1$  and  $\sigma_X^2 = 1$ . The initial condition, represented by  $\otimes$ , lies on the curve  $\hat{x}_\varnothing \hat{x}_c = -\sigma_X^2$ .

$\beta$	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathbf{P}(\mathbb{A}_0^*)$	$\mathcal{J}_q^*$	$N_{\text{it}}$
0	(0, 0)	[-1, 1]	0.683	0.516	39
0.1	(0, 0)	[-1.054, 1.054]	0.708	0.595	168
0.2	(0.408, -1.008)	[-0.608, 2.133]	0.712	0.649	27
0.3	(0.454, -1.183)	[-0.596, 2.908]	0.723	0.666	21
0.4	(0.447, -1.231)	[-0.624, 3.756]	0.734	0.675	22
0.5	(0.434, -1.255)	[-0.653, 4.898]	0.743	0.681	22
0.6	(0.421, -1.275)	[-0.680, 6.612]	0.752	0.687	22
0.7	(0.409, -1.295)	[-0.705, 9.477]	0.760	0.693	22
0.8	(0.399, -1.313)	[-0.729, 15.22]	0.767	0.699	22
0.9	(0.388, -1.330)	[-0.752, 32.47]	0.774	0.704	23
0.99	(0.380, -1.345)	[-0.772, 343.1]	0.780	0.709	23

Table 4.2: Numerical results of the optimization Problem 3.2 when  $\varrho = 1$ ,  $\sigma_X^2 = 1$  and different values of  $\beta$ .

collision  $\beta$  varying from zero to 0.99, which are displayed in Table 4.2. A few observations can be drawn from this table. First, we notice that when  $\beta = 0.1$  the number of iterations  $N_{\text{it}}$  to achieve the convergence criterion in Eq. (4.36) is much larger than for any other row. This is justified by the values of the cost function evaluated near the origin being very close to the minimum. Therefore, all the points around  $(0, 0)$  are nearly stationary, hence the slow convergence.

Another interesting observation is that when the probability of a concurrent transmission  $\beta$  approaches 1, the no-transmission interval  $\mathbb{A}_0^*$  tends to increase. However, we can see that its probability tends to a value bounded away from 1. Therefore, even when the collision event has probability 1, it may be worth paying the communication cost to transmit a packet because the optimal strategy always conveys one-bit of information through the collision and no-transmission symbols.

The dependency of the optimal solutions with the communication cost  $\varrho$  when  $\beta = 0.5$  and  $\sigma_X^2 = 5$  is shown in Table 4.3. We make the following observations. Even when communication is free ( $\varrho = 0$ ), the optimal no-transmission interval has

$\varrho$	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathbf{P}(\mathbb{A}_0^*)$	$\mathcal{J}_q^*$	$N_{\text{it}}$
0	(2.360, -1.279)	[0.853, 11.14]	0.351	1.244	29
0.1	(2.321, -1.309)	[0.800, 11.10]	0.361	1.309	29
0.2	(2.283, -1.339)	[0.744, 11.07]	0.370	1.372	28
0.5	(2.169, -1.432)	[0.580, 10.96]	0.398	1.557	28
1	(1.986, -1.591)	[0.311, 10.82]	0.445	1.846	25
2	(1.655, -1.917)	[-0.206, 10.66]	0.537	2.355	27
5	(0.970, -2.806)	[-1.460, 10.95]	0.743	3.406	23
10	(0.457, -3.866)	[-2.795, 12.35]	0.894	4.246	16
20	(0.135, -5.223)	[-4.377, 15.36]	0.975	4.792	10
50	(0.006, -7.686)	[-7.078, 22.47]	0.999	4.993	5

Table 4.3: Numerical results for Problem 3.2 when  $\beta = 0.5$  and  $\sigma_X^2 = 5$  for various values of  $\varrho$ .

a positive probability. This is because there is a probability that information will be lost due to a collision. In order to make the best use of the virtual signaling channel (see Fig. 4.9), the Shannon entropy  $H(U)$  must be nonzero, which forces  $\mathbf{P}(U = 0) = \mathbf{P}(\mathbb{A}_0^*) > 0$ .

When  $\varrho \rightarrow +\infty$  two notable things happen. One is that the number of iterations required to achieve the convergence criterion in Eq. (4.36) decreases sharply. Also, as opposed to the case when  $\beta \rightarrow 1$ , the probability of the no-transmission interval tends to one,  $\mathbf{P}(\mathbb{A}_0^*) \rightarrow 1$ , as  $\varrho$  increases. Therefore, not transmitting will turn out to be optimal in the regime of very large communication costs.

## 4.5 Person-by-person optimal policies for the Gaussian case

Table 4.4 illustrates that the results developed in the previous sections can be used to obtain person-by-person optimal policies for Problem 3.1 when  $X_i \sim \mathcal{N}(0, \sigma_i^2)$ ,  $i \in \{1, 2\}$ . Letting  $\sigma_2^2 = 1$  and varying  $\sigma_1^2$ , we applied the MLM algorithm, alternating between the optimization of the policies for DM<sub>1</sub> and DM<sub>2</sub> until

a fixed point was found. We do not have a proof that this procedure converges globally, but the policies obtained can be verified to be person-by-person optimal. The communication policy for  $DM_i$  is summarized by its no-transmission interval denoted by  $\mathbb{A}_{i,0}^*$ .

We observe that as the variance of the observations of  $DM_1$  increases, the person-by-person optimal policies are such that the channel will be more often accessed by  $DM_1$  and less often accessed by  $DM_2$ . This will cause a decrease in the probabilities of collisions and of an idle channel. It is interesting to note that all the policies listed in Table 4.4 outperform traditional sensor scheduling policies in which only the sensor measuring the random variable with the largest variance, which is the one that can reduce the cost the most, transmits. In that case, even with collisions, the person-by-person optimal policies when  $\sigma_1^2 = \sigma_2^2 = 1$  outperform the naive scheduling policy by approximately 46%. That is the case even when  $\sigma_1^2 = 5$ , which is considerably larger than  $\sigma_2^2 = 1$ , yielding a gain of approximately 3% over the scheduling policy. Also note that when  $\sigma_1^2 = \sigma_2^2 = 1$ , the framework of Problem 3.1 is completely symmetric, i.e.,  $X_1$  and  $X_2$  are identically distributed and  $f_X$  is an even probability density function. Despite these two facts, the person-by-person optimal policies listed in the first row of Table 4.4 have asymmetric thresholds. The pair of person-by-person optimal policies obtained for  $\sigma_1^2 = \sigma_2^2 = 1$  is depicted in Fig. 4.8. This is a major departure from the previous results in remote estimation from [37–41], all of which establish the optimality of symmetric threshold policies.

$\sigma_1^2$	$\mathbb{A}_{1,0}^*$	$\mathbb{A}_{2,0}^*$	$\mathbf{P}(\mathfrak{C})$	$\mathbf{P}(\emptyset)$	$\mathcal{J}^*$
1	[0.098, 5.359]	[0.098, 5.359]	0.290	0.213	0.540
2	[0.864, 4.534]	[-0.545, 10.50]	0.214	0.191	0.764
3	[1.877, 4.060]	[-1.260, 27.67]	0.090	0.116	0.889
4	[2.635, 4.158]	[-1.718, 56.71]	0.040	0.072	0.945
5	[3.236, 4.374]	[-2.051, 98.63]	0.019	0.048	0.971

Table 4.4: Person-by-person optimal policies for  $\text{DM}_1$  and  $\text{DM}_2$  in Problem 3.1 where the measurements are independently distributed as  $X_1 \sim \mathcal{N}(0, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(0, 1)$ .

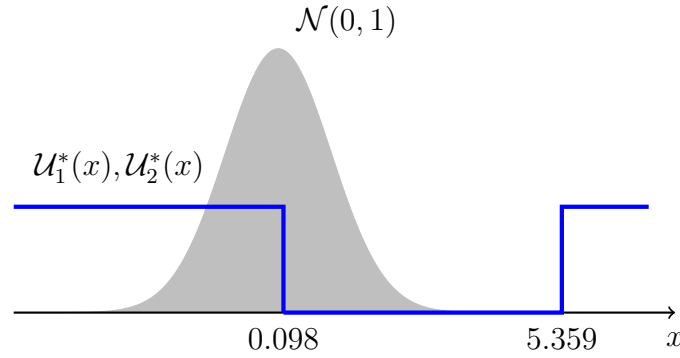


Figure 4.8: Person-by-person optimal policies obtained for Problem 3.1 when  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ .

## 4.6 On the optimality of asymmetric thresholds

An interesting feature of the person-by-person policies of Table 4.4 is the asymmetry of optimal thresholds. This can be intuitively justified by the presence of a collision symbol at the channel output, which can be used to convey information to the remote estimator. Note that when the probability of the channel being occupied in Problem 3.2 is zero, the optimal policies are always symmetric. The presence of two distinct symbols for collision and no-transmission creates an implicit noiseless channel between the sensor and estimator shown in Fig. 4.9. For a given communication cost, asymmetric communication policies can lower the variance of the estimation error in Problem 3.2 and, consequently, may be optimal for Problem 3.1. Notice that for the models in [13], in which collisions and communication costs are not considered, it is optimal to always transmit.

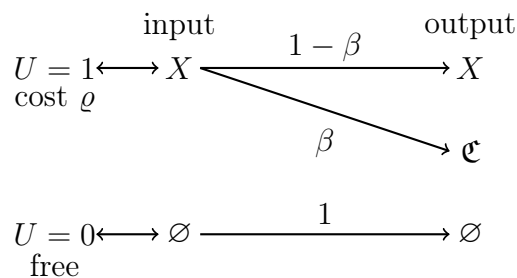


Figure 4.9: Implicit channel between the sensor and the remote estimator in Problem 3.2.

## 4.7 Summary

We considered the design of optimal thresholds of communication policies the remote estimation problem over the collision channel. First, we established the

connection between Problem 3.2 and a new class of one bit quantization problems. Then, we showed that the optimal thresholds can be symmetric or asymmetric, depending on the parameters that describe the optimization problem. We show that asymmetric threshold policies exploit the additional output symbol in the output of the channel to transmit valuable information to the remote estimator. Finally, a Lloyd-Max type algorithm is proposed and shown to be globally convergent to a locally optimal solution in the Gaussian case.



## Chapter 5: Maximum *a posteriori* probability estimation over the collision channel

So far, our results concern only the estimation of continuous random variables in the mean square sense over the collision channel. However, most modern systems have state spaces that either digital or discrete in nature. In this chapter, we consider a Bayesian estimation problem illustrated by the block diagram of Fig. 5.1: Two sensors observing independent discrete random variables, decide whether to communicate their measurements to a remote estimator over a collision channel according to stochastic communication policies. The communication constraint imposed by the collision channel is such that only one sensor can transmit its measurement perfectly; and if more than one sensor transmit simultaneously, a collision is declared. Upon observing the channel output, the estimator forms estimates of all the measured random variables. The goal is to design communication policies at the sensors and at the fusion center so as to form estimates of all of the observed random variables with two fidelity criteria: a convex combination of the individual probabilities of estimation errors, and a total probability of estimation error. We show that there exist person-by-person optimal policies for these problems with a certain deterministic structure involving only the two most likely outcomes of each

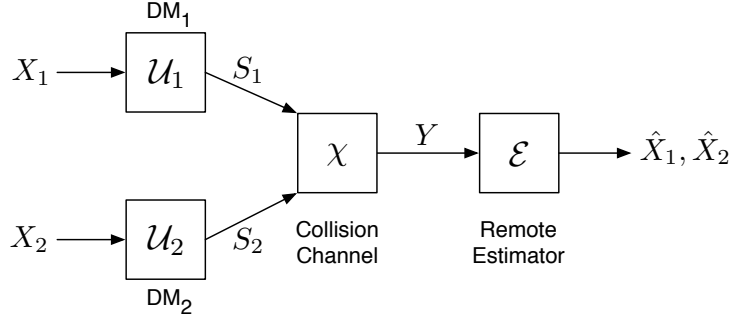


Figure 5.1: Block diagram representation for estimation over the collision channel.

random variable. This structure is then used to find team-optimal policies for any probability mass functions of the observed random variables. In particular, we find a team-optimal strategy for the case of minimum total probability of estimation error, which consists of each sensor transmitting all of the observations except the most likely one.

## 5.1 System model

Consider two independent discrete random variables  $X_1$  and  $X_2$  taking values on finite or countably infinite alphabets  $\mathbb{X}_1$  and  $\mathbb{X}_2$ , respectively. Let  $i \in \{1, 2\}$ , the random variable  $X_i$  is arbitrarily distributed according to a probability mass function  $p_i(x_i)$ , where  $x_i \in \mathbb{X}_i$ . Without loss of generality, we assume that every element of  $\mathbb{X}_i$  has a strictly positive probability, i.e.,  $p_i(x_i) > 0$  for all  $x_i \in \mathbb{X}_i$ . The decision maker  $DM_i$  observes the realization  $x_i$ , and must decide to attempt to transmit it or not to the remote estimator, based solely on its measurement, according to a communication policy  $\mathcal{U}_i$ . The decision to attempt a transmission or not is represented by a binary random variable  $U_i \in \{0, 1\}$ , where  $U_i = 1$  denotes an attempt to transmit and  $U_i = 0$  denotes the decision to remain silent.

**Definition 5.1** (Communication policies). *The communication policy for  $DM_i$  is a function  $\mathcal{U}_i : \mathbb{X}_i \rightarrow [0, 1]$  such that*

$$\mathbf{P}(U_i = 1 | X_i = x_i) \stackrel{\text{def}}{=} \mathcal{U}_i(x_i), \quad i \in \{1, 2\}. \quad (5.1)$$

*The set of all communication policies for  $DM_i$  is denoted by  $\mathbb{U}_i = [0, 1]^{\mathbb{X}_i}$ . The channel input  $S_i$  corresponds to a communication packet<sup>1</sup>, whose content is determined as follows:*

$$S_i = \begin{cases} X_i & \text{if } U_i = 1 \\ \emptyset & \text{if } U_i = 0 \end{cases}, \quad i \in \{1, 2\}, \quad (5.2)$$

*where the symbol  $\emptyset$  denotes no-transmission.*

**Remark 5.1.** *The transmitted packets always contain in their headers the identification number of its sender. This allows the estimator to unambiguously determine the origin of every successfully received packet.*

**Definition 5.2** (Collision Channel). *The channel input alphabet for  $DM_i$  is  $\mathbb{S}_i \stackrel{\text{def}}{=} \mathbb{X}_i \cup \{\emptyset\}$ , and the channel output alphabet is  $\mathbb{Y} \stackrel{\text{def}}{=} (\{1 \times \mathbb{X}_1\} \cup \{2 \times \mathbb{X}_2\}) \cup \{\emptyset, \mathfrak{C}\}$ , where  $\mathfrak{C}$  represents the occurrence of a collision. The symbol  $\emptyset$  indicates absence of transmission. The collision channel is a deterministic two-input map  $\chi : \mathbb{S}_1 \times \mathbb{S}_2 \rightarrow \mathbb{Y}$*

---

<sup>1</sup>The concept of *packet* used here is slightly different from the usual notion from the literature of communication networks. Here, a packet is not constrained on the number of bits it may carry.

defined as follows:

$$\chi(s_1, s_2) \stackrel{\text{def}}{=} \begin{cases} (1, s_1) & \text{if } s_1 \neq \emptyset, s_2 = \emptyset \\ (2, s_2) & \text{if } s_1 = \emptyset, s_2 \neq \emptyset \\ \emptyset & \text{if } s_1 = \emptyset, s_2 = \emptyset \\ \mathfrak{C} & \text{if } s_1 \neq \emptyset, s_2 \neq \emptyset. \end{cases} \quad (5.3)$$

**Remark 5.2.** *There is a fundamental difference between the collision channel described above and the erasure channel commonly found in the literature of remote control and estimation, e.g. [13]: there are two distinct symbols to represent no-transmission and collision events. This creates an opportunity to embed on these symbols information that can aid the fusion center in estimating the observed random variables even when the communication fails.*

### 5.1.1 Aggregate probability of estimation error

For any given pair of communication policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ , the estimator is interested in forming estimates  $\hat{X}_1$  and  $\hat{X}_2$  that minimize a fidelity criterion consisting of a convex combination of the individual probabilities of error of estimating  $X_1$  and  $X_2$ . Define  $\mathcal{J}_A : \mathbb{U}_1 \times \mathbb{U}_2 \rightarrow \mathbb{R}$  such that

$$\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \alpha_1 \mathbf{P}(X_1 \neq \hat{X}_1) + \alpha_2 \mathbf{P}(X_2 \neq \hat{X}_2), \quad (5.4)$$

where  $\alpha_1, \alpha_2 > 0$ , such that  $\alpha_1 + \alpha_2 = 1$ .

The motivation for using this criterion is that, in some situations, it is possible

that one of the random variables is more important to the estimator than the other. This cost allows the receiver to set the priority of each of the random variables it is interested in. It is straightforward to show that for any two communication policies, the receiver that minimizes the cost in Eq. (5.4) forms a *maximum a posteriori probability* (MAP) estimate of the random variable  $X_i$  given the observed channel output  $Y$  according to functions  $\mathcal{E}_i : \mathbb{Y} \rightarrow \mathbb{X}_i$  defined as

$$\mathcal{E}_i(y) \stackrel{\text{def}}{=} \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | Y = y), \quad i \in \{1, 2\}. \quad (5.5)$$

**Problem 5.1.** *Given a pair of probability mass functions  $p_1$  and  $p_2$ , find a pair of policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$  that jointly minimizes  $\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2)$  in Eq. (5.4) subject to the communication constraint imposed by the collision channel of Eq. (5.3) and that the estimator employs the MAP rule of Eq. (5.5).*

### 5.1.2 Total probability of estimation error

For any given pair of communication policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ , the estimator is interested in forming estimates  $\hat{X}_1$  and  $\hat{X}_2$  that minimize the probability of the union of the individual estimation error events  $X_1 \neq \hat{X}_1$  and  $X_2 \neq \hat{X}_2$ . Define  $\mathcal{J}_B : \mathbb{U}_1 \times \mathbb{U}_2 \rightarrow \mathbb{R}$  such that

$$\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{P}(\{X_1 \neq \hat{X}_1\} \cup \{X_2 \neq \hat{X}_2\}) \quad (5.6)$$

The interpretation behind this choice for the objective function is that there

are cases in which the observed random variables are components of a vector source and the goal of the fusion center is to estimate the entire source with minimum probability error. In this case, for any two communication policies, the receiver that minimizes the cost in Eq. (5.6) forms a MAP estimate of the random variables  $(X_1, X_2)$  given the observed channel output  $Y$  according to a function  $\mathcal{E} : \mathbb{Y} \rightarrow \mathbb{X}_1 \times \mathbb{X}_2$  defined as

$$\mathcal{E}(y) \stackrel{\text{def}}{=} \arg \max_{(x_1, x_2) \in \mathbb{X}_1 \times \mathbb{X}_2} \mathbf{P}(X_1 = x_1, X_2 = x_2 | Y = y). \quad (5.7)$$

**Problem 5.2.** *Given a pair of probability mass functions  $p_1$  and  $p_2$ , find a pair of policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$  that jointly minimizes  $\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2)$  in Eq. (5.6) subject to the communication constraint imposed by the collision channel of Eq. (5.3) and that the estimator employs the MAP rule of Eq. (5.7).*

**Remark 5.3.** *One important feature of the information structure in the problems considered here is that it is non-classical: the action of  $DM_1$  cannot be perfectly predicted by  $DM_2$ , and vice versa.*

### 5.1.3 A motivating example

The collision channel in Eq. (5.3) can only transmit perfectly a single communication packet at a time. One way to guarantee that collisions never occur is by means of a sensor scheduling policy (also known as a collision avoidance protocol). Sensor scheduling however is not a truly decentralized strategy in the sense that the system designer enforces all but one agent to remain silent while a single sensor

transmits. Consider the simple scenario where  $X_1$  and  $X_2$  are independent Bernoulli random variables with probability mass functions  $p_1$  and  $p_2$ , respectively. Using a sensor scheduling policy where only one sensor is allowed to access the channel in Problem 5.1, the best possible performance is given by

$$\mathcal{J}_A^{\text{sch}} \stackrel{\text{def}}{=} 1 - \max_{i \in \{1,2\}} \max_{x \in \{0,1\}} \alpha_i p_i(x) > 0. \quad (5.8)$$

However, it is possible to achieve zero aggregate probability of error using the following pair of deterministic policies  $(\mathcal{U}_1^*, \mathcal{U}_2^*)$ :

$$\mathcal{U}_i^*(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{if } x_i = 1 \end{cases}, \quad i \in \{1, 2\}. \quad (5.9)$$

This holds for any pair of Bernoulli random variables with probability mass functions  $p_1$  and  $p_2$ . The reason behind this is the fact that  $Y = \emptyset \Leftrightarrow (X_1 = 0, X_2 = 0)$  and, similarly,  $Y = \mathfrak{C} \Leftrightarrow (X_1 = 1, X_2 = 1)$ . This team-optimal pair of policies makes use of the distinction between no-transmissions and collisions to convey information about the observations to the remote estimator. We are interested in answering the following question: *Is there a similar strategy that is team-optimal for any two arbitrarily distributed random variables?*

## 5.2 Structural results

In this chapter we characterize the structure of team-optimal communication policies for Problems 5.1 and 5.2. One important feature of the results below is that

they are independent of the distributions of the observations, and are valid even when the alphabets are countably infinite.

**Theorem 5.1** (Team-optimal solutions for Problem 5.1). *There exists a pair of team-optimal policies for Problem 5.1 where each sensor either transmits all but the most likely of its observations; transmits only the second most likely of its observations; or remains always silent.*

**Theorem 5.2** (Team-optimal solutions for Problem 5.2). *There exists a pair of team-optimal policies for Problem 5.2 where each sensor transmits all but the most likely of its observations.*

From here on, we will prove Theorems 5.1 and 5.2 using the person-by-person optimality approach. We will show that the optimization subproblem faced by a single DM while keeping the policies of the other DM fixed is a concave minimization problem. Such problems are NP-hard. However, we are able to solve these concave minimization problems exactly using a two-step approach: first, we obtain a lower bound that holds for any feasible policy (the converse part) and then we provide a structured deterministic policy that achieves this lower bound (the achievability part).

### 5.3 Solution to Problem 5.1

In order to characterize the structure of a class of communication policies that minimize the cost in Eq. (5.6), we will re-express it in a more convenient form using Bayes' rule from the point of view of a single DM. Let  $i, j \in \{1, 2\}$  such that  $j \neq i$ ,



from the perspective of  $\text{DM}_i$  and assuming that the policy used by  $\text{DM}_j$  is arbitrarily fixed as  $\mathcal{U}_j^* \in \mathbb{U}_j$ , we have:

$$\mathcal{J}_A(\mathcal{U}_i, \mathcal{U}_j^*) = \alpha_i \mathbf{P}(X_i \neq \hat{X}_i) + \alpha_j (\rho_{\mathcal{U}_j^*} \mathbf{P}(U_i = 1) + \theta_{\mathcal{U}_j^*}), \quad (5.10)$$

where

$$\rho_{\mathcal{U}_j^*} \stackrel{\text{def}}{=} \mathbf{P}(X_j \neq \hat{X}_j | U_i = 1) - \mathbf{P}(X_j \neq \hat{X}_j | U_i = 0) \quad (5.11)$$

and

$$\theta_{\mathcal{U}_j^*} \stackrel{\text{def}}{=} \mathbf{P}(X_j \neq \hat{X}_j | U_i = 0). \quad (5.12)$$

The terms  $\rho_{\mathcal{U}_j^*}$  and  $\theta_{\mathcal{U}_j^*}$  are constant in  $\mathcal{U}_i$ . In particular,  $\rho_{\mathcal{U}_j^*}$  can be interpreted as a communication cost incurred by  $\text{DM}_i$  when it attempts to transmit its measurement. A similar interpretation has been used in [47] and relates this problem to the multi-stage estimation case with limited actions solved in [34].

### 5.3.1 Communication cost

We proceed to characterize the communication cost and the offset terms in further detail: first by showing that they are constant in  $\mathcal{U}_i$  and then establishing that they are non-negative and upper bounded by 1. These facts will be subsequently used in the proof of Theorem 1.

**Proposition 5.1.** *Provided that  $X_1$  and  $X_2$  are mutually independent, the term that corresponds to the communication cost  $\rho_{\mathcal{U}_j^*}$  and the off-set term  $\theta_{\mathcal{U}_j^*}$ , where*

$j \in \{1, 2\}$ , in Eqs. (5.11) and (5.12), are given by

$$\rho_{\mathcal{U}_j^*} = \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x) - \max_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x). \quad (5.13)$$

and

$$\theta_{\mathcal{U}_j^*} = \sum_{x \in \mathbb{X}_j} (1 - \mathcal{U}_j^*(x)) p_j(x) - \max_{x \in \mathbb{X}_j} (1 - \mathcal{U}_j^*(x)) p_j(x). \quad (5.14)$$

Consequently,

$$0 \leq \rho_{\mathcal{U}_j^*}, \theta_{\mathcal{U}_j^*} \leq 1. \quad (5.15)$$

*Proof.* First, we need to show that, for  $i, j \in \{1, 2\}$  and  $j \neq i$ , the following holds

$$\mathcal{E}_i((j, x_j)) = \mathcal{E}_i(\emptyset), \quad x_j \in \mathbb{X}_j. \quad (5.16)$$

In other words, for the purpose of estimating  $X_i$ , the observation of  $Y = (j, x_j)$  at the fusion center is equivalent to receiving  $Y = \emptyset$ .

From the definition of the MAP estimator  $\mathcal{E}_i$  in Eq. (5.5), we write

$$\begin{aligned} \mathcal{E}_i(\emptyset) &= \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | Y = \emptyset) \\ &= \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | U_i = 0, U_j = 0) \\ &\stackrel{(a)}{=} \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | U_i = 0). \end{aligned} \quad (5.17)$$

Similarly,

$$\begin{aligned}
\mathcal{E}_i((j, x_j)) &= \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | Y = (j, x_j)) \\
&= \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | U_i = 0, U_j = 1, X_j = x_j) \\
&\stackrel{(b)}{=} \arg \max_{x_i \in \mathbb{X}_i} \mathbf{P}(X_i = x_i | U_i = 0) \\
&= \mathcal{E}_i(\emptyset). \tag{5.18}
\end{aligned}$$

The equalities (a) and (b) follow from the fact that, since  $X_1$  and  $X_2$  are mutually independent, the following Markov chain relationship holds

$$X_1 \leftrightarrow U_1 \leftrightarrow U_2 \leftrightarrow X_2. \tag{5.19}$$

Consequently, the probability of estimation error obtained by using the MAP estimator conditioned on the events  $(U_i = 1, U_j = 0)$  and  $(U_i = 0, U_j = 0)$  are the same, i.e.,

$$\mathbf{P}(X_j \neq \hat{X}_j | U_i = 1, U_j = 0) = \mathbf{P}(X_j \neq \hat{X}_j | U_i = 0, U_j = 0). \tag{5.20}$$

Finally, given that  $(U_i = 0, U_j = 1) \Leftrightarrow (Y = (j, X_j))$ , we have

$$\mathbf{P}(X_j \neq \hat{X}_j | U_i = 0, U_j = 1) \equiv 0. \tag{5.21}$$

Expressing  $\rho_{\mathcal{U}_j^*}$  using the law of total probability and, Eqs. (5.20) and (5.21), we

obtain

$$\rho_{\mathcal{U}_j^*} = \mathbf{P}(X_j \neq \hat{X}_j | U_i = 1, U_j = 1) \mathbf{P}(U_j = 1) \quad (5.22)$$

$$= \mathbf{P}(X_j \neq \mathcal{E}_j(\mathfrak{C}), U_j = 1). \quad (5.23)$$

Using the definition of the MAP estimator and expressing the result in terms of the policy  $\mathcal{U}_j^*$ , we have

$$\rho_{\mathcal{U}_j^*} = \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x) - \max_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x). \quad (5.24)$$

Following similar steps, we can show that the off-set term  $\theta_{\mathcal{U}_j^*}$  is given by:

$$\theta_{\mathcal{U}_j^*} = \mathbf{P}(X_j \neq \mathcal{E}_j(\emptyset), U_j = 0), \quad (5.25)$$

which expressed in terms of  $\mathcal{U}_j^*$  is

$$\theta_{\mathcal{U}_j^*} = \sum_{x \in \mathbb{X}_j} (1 - \mathcal{U}_j^*(x)) p_j(x) - \max_{x \in \mathbb{X}_j} (1 - \mathcal{U}_j^*(x)) p_j(x). \quad (5.26)$$

□

### 5.3.2 Single decision maker subproblem

Consider a different estimation problem depicted in Fig. 5.2 where a single sensor observes a random variable  $X$  and must decide whether to transmit a measurement  $x$  over a *stochastic* collision channel to a remote estimator, which forms

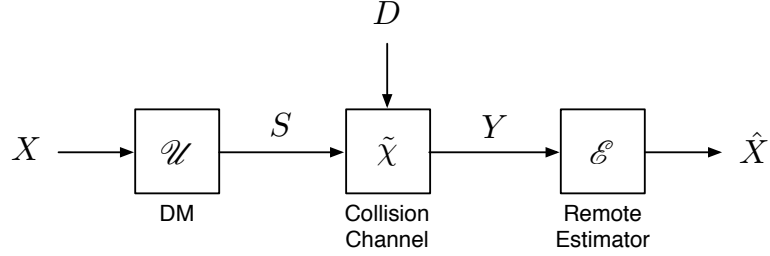


Figure 5.2: An equivalent single DM estimation problem over a collision channel.

an estimate  $\hat{X}$  on the basis of the channel output  $Y$ . We would like to find policies at the DM and the estimator that minimize a cost that combines the probability of estimation error and a communication cost. The solution to this problem will have implications to the solution of Problem 5.1.

We will now make the statement of the subproblem precise. Let the input  $S$  to the channel be determined according to

$$S = \begin{cases} X & \text{if } U = 1 \\ \emptyset & \text{if } U = 0, \end{cases} \quad (5.27)$$

where the probability distribution of the binary random variable  $U$  is given by

$$\mathbf{P}(U = 1|X = x) = \mathcal{U}(x), \quad \mathcal{U} \in \mathbb{U}, \quad (5.28)$$

where  $\mathcal{U}$  is the communication policy used by the DM and  $\mathbb{U} \stackrel{\text{def}}{=} \{\mathcal{U} \mid \mathcal{U} : \mathbb{X} \rightarrow [0, 1]\}$  is the set of admissible policies. The collision channel we consider now is stochastic and can be in one out of two states controlled by a Bernoulli random variable  $D \sim \mathcal{B}(\beta)$ . When  $D = 0$  the channel is not occupied, and if the DM decides to transmit, its packet will reach the destination; when  $D = 1$ , the channel is occupied

and any transmission attempted by the DM will result in a collision.

**Definition 5.3** (Stochastic point-to-point collision channel). *Let  $D \sim \mathcal{B}(\beta)$ . The output of the point-to-point collision channel  $Y = \tilde{\chi}(S, D)$  is given by the following map*

$$\tilde{\chi}(S, D) \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } S = \emptyset \\ S & \text{if } S \neq \emptyset, D = 0 \\ \mathfrak{C} & \text{if } S \neq \emptyset, D = 1. \end{cases} \quad (5.29)$$

Finally, we must define the cost to be minimized by the DM. Let  $\mathcal{J}(\mathcal{U}) : \mathbb{U} \rightarrow \mathbb{R}$  such that

$$\mathcal{J}(\mathcal{U}) \stackrel{\text{def}}{=} \mathbf{P}(X \neq \hat{X}) + \varrho \mathbf{P}(U = 1). \quad (5.30)$$

**Problem 5.3.** *For given  $\beta \in [0, 1]$ ,  $\varrho \geq 0$  and  $X \sim p(x)$ ,  $x \in \mathbb{X}$ , find a policy  $\mathcal{U} \in \mathbb{U}$  that minimizes the cost  $\mathcal{J}(\mathcal{U})$  in Eq. (5.30) subject to the constraint imposed by the channel in Eq. (5.29) and that the estimator forms  $\hat{X}$  according to the following MAP rule:*

$$\mathcal{E}(y) \stackrel{\text{def}}{=} \arg \max_{x \in \mathbb{X}} \mathbf{P}(X = x | Y = y). \quad (5.31)$$

We will provide a solution to Problem 5.3 using the following two lemmas.

Lemma 5.1 establishes an important property of the cost.

**Lemma 5.1.** *The cost  $\mathcal{J}(\mathcal{U})$  is concave on  $\mathbb{U}$ .*

*Proof.* Using the law of total probability, we rewrite the cost  $\mathcal{J}(\mathcal{U})$  as:

$$\begin{aligned}\mathcal{J}(\mathcal{U}) &= \left( \beta \mathbf{P}(X \neq \hat{X} | U = 1, D = 1) + \varrho \right) \mathbf{P}(U = 1) \\ &\quad + \mathbf{P}(X \neq \hat{X} | U = 0) \mathbf{P}(U = 0).\end{aligned}\tag{5.32}$$

Simplifying this expression using the relationships developed in the previous sections, we get:

$$\begin{aligned}\mathcal{J}(\mathcal{U}) &= 1 + (\varrho + \beta - 1) \mathbf{P}(U = 1) - \mathbf{P}(X = \mathcal{E}(\emptyset) | U = 0) \mathbf{P}(U = 0) \\ &\quad - \beta \mathbf{P}(X = \mathcal{E}(\mathfrak{C}) | U = 1) \mathbf{P}(U = 1).\end{aligned}\tag{5.33}$$

Using the definition of the MAP estimator, we can write the following probabilities in terms of  $\mathcal{U}$ :

$$\mathbf{P}(X = \mathcal{E}(\emptyset) | U = 0) = \max_{x \in \mathbb{X}} \frac{(1 - \mathcal{U}(x))p(x)}{\mathbf{P}(U = 0)}\tag{5.34}$$

and

$$\mathbf{P}(X = \mathcal{E}(\mathfrak{C}) | U = 1) = \max_{x \in \mathbb{X}} \frac{\mathcal{U}(x)p(x)}{\mathbf{P}(U = 1)}.\tag{5.35}$$

Finally, after some algebraic manipulation, the cost can be rewritten as

$$\begin{aligned}\mathcal{J}(\mathcal{U}) &= 1 + (\varrho + \beta - 1) \sum_{x \in \mathbb{X}} \mathcal{U}(x)p(x) - \max_{x \in \mathbb{X}} (1 - \mathcal{U}(x))p(x) \\ &\quad - \beta \max_{x \in \mathbb{X}} \mathcal{U}(x)p(x).\end{aligned}\tag{5.36}$$

The rest of proof follows by standard arguments found in [59, ch. 3]. □

**Lemma 5.2.** *For  $\beta \in [0, 1]$  and  $\varrho \geq 0$ , the following policy minimizes  $\mathcal{J}(\mathcal{U})$ :*

$$\mathcal{U}_{\beta, \varrho}^*(x) = \begin{cases} \mathbf{1}_{\mathbb{X} \setminus \{x_{[1]}\}}(x) & \text{if } 0 \leq \varrho \leq 1 - \beta \\ \mathbf{1}_{\{x_{[2]}\}}(x) & \text{if } 1 - \beta < \varrho \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (5.37)$$

*Proof.* Since  $\mathcal{J}(\mathcal{U})$  is continuous and  $\mathbb{U}$  is compact with respect to the weak\* topology<sup>2</sup>, the minimizer exists [65]. Due to the concavity of  $\mathcal{J}(\mathcal{U})$  established in Lemma 5.1, the minimizer must lie on the boundary of the feasible set. Moreover, the search can be further constrained to the corners of the  $|\mathbb{X}|$ -dimensional hypercube that describes the feasible set. This implies that Problem 5.3 admits an optimal deterministic policy. Without loss of optimality, we constrain the search for an optimal policy by considering partitions of the alphabet  $\mathbb{X} = \mathbb{X}_0^{\mathcal{U}} \cup \mathbb{X}_1^{\mathcal{U}}$ , where

$$\mathbb{X}_k^{\mathcal{U}} \stackrel{\text{def}}{=} \{x \in \mathbb{X} \mid \mathcal{U}(x) = k\}, \quad k \in \{0, 1\}. \quad (5.38)$$

Searching over the set of possible partitions has exponential complexity. However, we may still explore the structure of the cost to obtain an optimal solution to this problem. Let

$$\mathcal{J}(\mathcal{U}) = 1 + (\varrho + \beta - 1) \sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x) - \beta \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x). \quad (5.39)$$

---

<sup>2</sup>This technical detail can be ignored when  $|\mathbb{X}| < \infty$ .



We will obtain a lower bound that holds for every deterministic policy  $\mathcal{U} \in \mathbb{U}$ , and show that  $\mathcal{U}_{\beta,\varrho}^*$  always achieves it.

First, consider the case when  $\varrho > 1 - \beta$ . Since

$$\sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \geq \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \quad (5.40)$$

the cost satisfies the following lower bound:

$$\mathcal{J}(\mathcal{U}) \geq 1 - (1 - \varrho) \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x). \quad (5.41)$$

The right hand side of the inequality above can be minimized by assigning  $x_{[1]}$  to the set  $\mathbb{X}_0^{\mathcal{U}}$ . If  $1 - \varrho \geq 0$ , we assign  $x_{[2]}$  to the set  $\mathbb{X}_1^{\mathcal{U}}$ , otherwise we set  $\mathbb{X}_1^{\mathcal{U}} = \emptyset$ .

Therefore, we obtain the following lower bound for the cost:

$$\mathcal{J}(\mathcal{U}) \geq 1 - [1 - \varrho]^+ p_{[2]} - p_{[1]}. \quad (5.42)$$

This lower bound is met with equality by the policy  $\mathcal{U}_{\beta,\varrho}^*$ :

$$\mathcal{J}(\mathcal{U}_{\beta,\varrho}^*) = \begin{cases} 1 - (1 - \varrho)p_{[2]} - p_{[1]} & \text{if } 1 - \beta \leq \varrho \leq 1 \\ 1 - p_{[1]} & \text{if } \varrho > 1. \end{cases} \quad (5.43)$$

Similarly, when  $0 \leq \varrho \leq 1 - \beta$  we have

$$\sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \leq 1 - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x). \quad (5.44)$$

Therefore, for every  $\mathcal{U} \in \mathbb{U}$ , we establish the following lower bound on the cost:

$$\mathcal{J}(\mathcal{U}) \geq (\varrho + \beta)(1 - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x)) - \beta \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \quad (5.45)$$

The right hand side of the inequality above can be minimized by assigning  $x_{[1]}$  to the set  $\mathbb{X}_0^{\mathcal{U}}$ . If  $1 - \varrho \geq 0$ , we assign  $x_{[2]}$  to the set  $\mathbb{X}_1^{\mathcal{U}}$ . Therefore, we obtain the following lower bound for the cost:

$$\mathcal{J}(\mathcal{U}) \geq (\varrho + \beta)(1 - p_{[1]}) - \beta p_{[2]}. \quad (5.46)$$

The policy  $\mathcal{U}_{\beta, \varrho}^*$  achieves this lower bound:

$$\mathcal{J}(\mathcal{U}_{\beta, \varrho}^*) = 1 + (\varrho + \beta - 1) \sum_{x \in \mathbb{X} \setminus \{x_{[1]}\}} p(x) - \max_{x \in \{x_{[1]}\}} p(x) - \beta \max_{x \in \mathbb{X} \setminus \{x_{[1]}\}} p(x) \quad (5.47)$$

$$= (\varrho + \beta)(1 - p_{[1]}) - \beta p_{[2]}. \quad (5.48)$$

□

**Remark 5.4.** *Lemma 5.2 provides a solution to Problem 5.3 described only in terms of the two most likely outcomes of the observed random variable  $X$ . We observe that the optimal policy depends on  $\beta$  and  $\varrho$ . As a particular case: when  $\beta = 0$  and any  $\varrho \in [0, 1]$ , the optimal policy is*

$$\mathcal{U}^*(x) = \begin{cases} 0 & \text{if } x = x_{[1]} \\ 1 & \text{otherwise.} \end{cases} \quad (5.49)$$

This result is related to a similar problem solved by Imer and Basar in [34].

### 5.3.3 Team-optimal policies for Problem 5.1

We will now use the results in Section 5.3.2 to reduce the search space of possible optimal strategies for each DM in Problem 5.1. The strategy is to use a person-by-person optimality approach together with Lemma 5.2 to prove that, without loss in optimality, the search can be constrained to three policies for each DM.

*Proof of Theorem 5.1.* Consider the cost  $\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2)$  in Problem 5.1. Arbitrarily fixing the policy of DM<sub>j</sub> we have

$$\mathcal{J}_A(\mathcal{U}_i, \mathcal{U}_j^*) \propto \mathbf{P}(X_i \neq \hat{X}_i) + \frac{\alpha_j}{\alpha_i} (\rho_{\mathcal{U}_j^*} \mathbf{P}(U_i = 1) + \theta_{\mathcal{U}_j^*}). \quad (5.50)$$

The problem of minimizing  $\mathcal{J}_A(\mathcal{U}_i, \mathcal{U}_j^*)$  over  $\mathcal{U}_i \in \mathbb{U}_i$  is equivalent to solving an instance of Problem 5.3 with parameters  $\varrho$  and  $\beta$  defined as

$$\varrho \stackrel{\text{def}}{=} \frac{\alpha_j}{\alpha_i} \rho_{\mathcal{U}_j^*} \quad \text{and} \quad \beta \stackrel{\text{def}}{=} \mathbf{P}(U_j = 1) = \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x). \quad (5.51)$$

From Lemmas 5.1 and 5.2, for every policy  $\mathcal{U}_j^* \in \mathbb{U}_j$  there exists an optimal policy  $\mathcal{U}_i^*$  for DM<sub>i</sub> where either the sensor attempts to transmit every measurement with the exception of the most likely observation; attempts to transmit just the second most likely observation; or it remains always silent. Since this is true for every  $\mathcal{U}_j^*$  it must also hold when  $\mathcal{U}_j^*$  is such that  $(\mathcal{U}_i^*, \mathcal{U}_j^*)$  is a person-by-person optimal solution.

Since every team-optimal solution is also person-by-person optimal, there exists a team-optimal pair of policies where each policy has one of the structures outlined above.  $\square$

**Remark 5.5.** *There may be other optimal solutions that do not display the same structure of the policies in Lemma 5.2. One implication of our result is that the performance of the optimal remote estimation system is determined by the probabilities of the two most likely outcomes of  $X_1$  and  $X_2$ . Also, the optimal performance of a system with binary observations is always zero, i.e., independent binary observations can be estimated perfectly from the output of the collision channel with two users. The pair of team-optimal policies described in the motivating example of Section 5.1.3 also fits in the structure of the team-optimal policies of Theorem 5.1.*

Since there is no loss in optimality in constraining the search over policies with the structure given by Theorem 5.1, we define the following candidate policies for  $\text{DM}_i$ :

$$\mathcal{U}_i^1(x) \stackrel{\text{def}}{=} \mathbf{1}_{\mathbb{X}_i \setminus \{x_{i,[1]}\}}(x) \quad (5.52)$$

$$\mathcal{U}_i^2(x) \stackrel{\text{def}}{=} \mathbf{1}_{\{x_{i,[2]}\}}(x) \quad (5.53)$$

$$\mathcal{U}_i^3(x) \stackrel{\text{def}}{=} 0 \quad (5.54)$$

for  $i \in \{1, 2\}$ . Therefore, the search space is reduced to a set of 9 pairs of policies.

We proceed by evaluating the performance of each of the pairs  $(\mathcal{U}_1, \mathcal{U}_2)$  using the

expressions in Eqs. (5.11) and (5.12). Let  $i \in \{1, 2\}$  and define the quantity

$$t_i \stackrel{\text{def}}{=} 1 - p_{i,[1]} - p_{i,[2]}, \quad (5.55)$$

we have the following:

- If  $\text{DM}_i$  chooses to use  $\mathcal{U}_i^1(x)$ , then

$$\mathbf{P}(U_i = 1) = 1 - p_{i,[1]} \quad (5.56)$$

$$\rho_{\mathcal{U}_i} = t_i \quad \text{and} \quad \theta_{\mathcal{U}_i} = 0; \quad (5.57)$$

- If  $\text{DM}_i$  chooses to use  $\mathcal{U}_i^2(x)$ , then

$$\mathbf{P}(U_i = 1) = p_{i,[2]} \quad (5.58)$$

$$\rho_{\mathcal{U}_i} = 0 \quad \text{and} \quad \theta_{\mathcal{U}_i} = t_i; \quad (5.59)$$

- If  $\text{DM}_i$  chooses to use  $\mathcal{U}_i^3(x)$ , then

$$\mathbf{P}(U_i = 1) = 0 \quad (5.60)$$

$$\rho_{\mathcal{U}_i} = 0 \quad \text{and} \quad \theta_{\mathcal{U}_i} = 1 - p_{i,[1]}. \quad (5.61)$$

**Corollary 5.1.** *The optimal cost obtained from solving Problem 5.1 is given by*

$$\mathcal{J}_A^* = \min \left\{ \begin{array}{l} \alpha_1 t_1 (1 - p_{2,[1]}) + \alpha_2 t_2 (1 - p_{1,[1]}) \\ \alpha_1 t_1 p_{2,[2]} + \alpha_2 t_2 \\ \alpha_1 t_1 + \alpha_2 t_2 p_{1,[2]} \\ \alpha_1 (1 - p_{1,[1]}) \\ \alpha_2 (1 - p_{2,[1]}) \end{array} \right\}. \quad (5.62)$$

*Proof.* Using the notation developed in the previous sections, we have:

$$\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2) = \alpha_1 (\rho_{\mathcal{U}_1} \mathbf{P}(U_2 = 1) + \theta_{\mathcal{U}_1}) + \alpha_2 (\rho_{\mathcal{U}_2} \mathbf{P}(U_1 = 1) + \theta_{\mathcal{U}_2}), \quad (5.63)$$

where

$$\rho_{\mathcal{U}_i} = \sum_{x \in \mathbb{X}_{i,1}^{\mathcal{U}_i}} p_i(x) - \max_{x \in \mathbb{X}_{i,1}^{\mathcal{U}_i}} p_i(x), \quad (5.64)$$

and

$$\theta_{\mathcal{U}_i} = \sum_{x \in \mathbb{X}_{i,0}^{\mathcal{U}_i}} p_i(x) - \max_{x \in \mathbb{X}_{i,0}^{\mathcal{U}_i}} p_i(x). \quad (5.65)$$

Construct Table 5.1 containing the values of the objective function for each choice of policies. It can be verified by inspection that the pairs  $m = 6, 7, 8$  and  $9$  are always outperformed by the other pairs of policies and can be discarded from our search, which can be done by searching over a list of 5 possible pairs.

□

$m$	$(\mathcal{U}_1, \mathcal{U}_2)$	$\mathcal{J}_A^{(m)}(\mathcal{U}_1, \mathcal{U}_2)$
1	$(\mathcal{U}_1^1, \mathcal{U}_2^1)$	$\alpha_1 t_1(1 - p_{2,[1]}) + \alpha_2 t_2(1 - p_{1,[1]})$
2	$(\mathcal{U}_1^1, \mathcal{U}_2^2)$	$\alpha_1 t_1 p_{2,[2]} + \alpha_2 t_2$
3	$(\mathcal{U}_1^2, \mathcal{U}_2^1)$	$\alpha_1 t_1 + \alpha_2 t_2 p_{1,[2]}$
4	$(\mathcal{U}_1^1, \mathcal{U}_2^3)$	$\alpha_2(1 - p_{2,[1]})$
5	$(\mathcal{U}_1^3, \mathcal{U}_2^1)$	$\alpha_1(1 - p_{1,[1]})$
6	$(\mathcal{U}_1^2, \mathcal{U}_2^2)$	$\alpha_1 t_1 + \alpha_2 t_2$
7	$(\mathcal{U}_1^2, \mathcal{U}_2^3)$	$\alpha_1 t_1 + \alpha_2(1 - p_{2,[1]})$
8	$(\mathcal{U}_1^3, \mathcal{U}_2^2)$	$\alpha_1(1 - p_{1,[1]}) + \alpha_2 t_2$
9	$(\mathcal{U}_1^3, \mathcal{U}_2^3)$	$\alpha_1(1 - p_{1,[1]}) + \alpha_2(1 - p_{2,[1]})$

Table 5.1: Value of the cost function  $\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2)$  evaluated at each of the 9 pairs of candidate solutions.

### 5.3.4 Examples

We explore the role of the probability distributions in determining which of the 5 pairs of policies ( $m = 1$  through 5 in Table 5.1) is team-optimal. In Examples 1, 2 and 3 below, we assume that  $\alpha_1 = \alpha_2^3$ , which further reduces our search to policy pairs  $m = 1, 2$  and 3. We will use the following quantities:

$$\mathcal{J}_A^{(2)} - \mathcal{J}_A^{(3)} = -t_1(1 - p_{2,[2]}) + t_2(1 - p_{1,[2]}) \quad (5.66)$$

$$\mathcal{J}_A^{(2)} - \mathcal{J}_A^{(1)} = t_2(p_{1,[1]} - t_1) \quad (5.67)$$

$$\mathcal{J}_A^{(3)} - \mathcal{J}_A^{(1)} = t_1(p_{2,[1]} - t_2). \quad (5.68)$$

### Uniform random variables

For uniformly distributed observations, we have

$$p_i(x) = \frac{1}{N_i}, \quad x = 1, 2, \dots, N_i. \quad (5.69)$$

---

<sup>3</sup>In this case, the weights  $\alpha_1$  and  $\alpha_2$  are irrelevant and we may assume that they are both equal to 1.

Hence, the probabilities of the two most likely outcomes are

$$p_{i,[1]} = p_{i,[2]} = \frac{1}{N_i} \quad (5.70)$$

and the probability of the remaining symbols is given by

$$t_i = 1 - \frac{2}{N_i}, \quad i \in \{1, 2\}. \quad (5.71)$$

Without loss of generality we assume that  $N_1, N_2 \geq 3$  and  $N_1 \leq N_2$ . Since

$$\mathcal{J}_A^{(2)} - \mathcal{J}_A^{(3)} = \frac{1}{N_1} - \frac{1}{N_2} \geq 0 \quad (5.72)$$

$$\mathcal{J}_A^{(3)} - \mathcal{J}_A^{(1)} = \left(1 - \frac{2}{N_1}\right) \cdot \left(\frac{3}{N_2} - 1\right) \leq 0, \quad (5.73)$$

our assumptions imply that  $\mathcal{J}_A^* = \mathcal{J}_A^{(3)}$  and the pair of policies corresponding to  $m = 3$  is team-optimal. In other words, for uniformly distributed random observations, a team-optimal strategy is: Each DM arbitrarily chooses a priori one of the possible outcomes in their respective alphabets; the DM observing the random variable with the largest support transmits every measurement that does not match its chosen symbol; and the DM observing the random variable with smaller support only transmits only the measurements that match its chosen symbol.



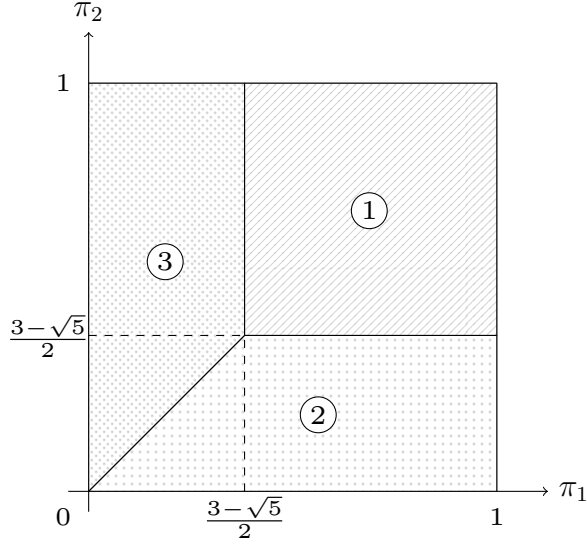


Figure 5.3: Partition of the parameter space indicating where each of the policy pairs is team-optimal when the observations are geometrically distributed. The circled number corresponds to  $m$  in Table 5.1.

### Geometric random variables

For a geometrically distributed random variable with parameter  $\pi_i$ , we have

$$p_i(x) = (1 - \pi_i)^x \pi_i, \quad x = 0, 1, \dots \quad (5.74)$$

The probabilities of the two most likely outcomes are

$$p_{i,[1]} = \pi_i \quad (5.75)$$

$$p_{i,[2]} = (1 - \pi_i)\pi_i \quad (5.76)$$

and the probability of the remaining symbols is given by

$$t_i = (1 - \pi_i)^2, \quad i \in \{1, 2\}. \quad (5.77)$$

Note that  $\mathcal{J}_A^{(1)} \leq \mathcal{J}_A^{(2)}, \mathcal{J}_A^{(3)}$  if and only if  $p_{i,[1]} - t_i \geq 0, i \in \{1, 2\}$ , i.e.,

$$-\pi_i^2 + 3\pi_i - 1 \geq 0, \quad i \in \{1, 2\}. \quad (5.78)$$

Also,  $\mathcal{J}_A^{(2)} \leq \mathcal{J}_A^{(3)}$  if and only if

$$(1 - \pi_2)^2 \pi_1 \leq (1 - \pi_1)^2 \pi_2, \quad (5.79)$$

which is satisfied if  $\pi_1 \leq \pi_2$ . This yields the partitioning of the parameter space  $(\pi_1, \pi_2) \in [0, 1]^2$  into the three regions depicted in Fig. 5.3 indicating which pair of policies is team-optimal in the corresponding region.

## Poisson random variables

For a Poisson distributed observation with parameter  $\lambda_i \geq 1$ , we have

$$p_i(x) = \frac{\lambda_i^x}{x!} e^{-\lambda_i}, \quad x = 0, 1, \dots \quad (5.80)$$

The probabilities of the two most likely outcomes are

$$p_{i,[1]} = p_{i,[2]} = \frac{\lambda_i^{\lfloor \lambda_i \rfloor}}{\lfloor \lambda_i \rfloor!} e^{-\lambda_i} \quad (5.81)$$

and the probability of the tail is given by

$$t_i = 1 - 2 \frac{\lambda_i^{\lfloor \lambda_i \rfloor}}{\lfloor \lambda_i \rfloor!} e^{-\lambda_i}, \quad i \in \{1, 2\}. \quad (5.82)$$

Using the same argument as in the previous example, note that  $\mathcal{J}_A^{(1)} \leq \mathcal{J}_A^{(2)}, \mathcal{J}_A^{(3)}$  if and only if  $p_{i,[1]} - t_i \geq 0$ , i.e.,

$$\frac{\lambda_i^{[\lambda_i]}}{[\lambda_i]!} e^{-\lambda_i} \geq \frac{1}{3}, \quad i \in \{1, 2\}. \quad (5.83)$$

Defining the following function

$$\mathcal{F}(\lambda) \stackrel{\text{def}}{=} \frac{\lambda^{[\lambda]}}{[\lambda]!} e^{-\lambda} - \frac{1}{3}, \quad (5.84)$$

the value of  $\bar{\lambda}$  for which  $\mathcal{F}(\bar{\lambda}) = 0$  can be found numerically and is approximately equal to  $\bar{\lambda} = 1.5121$ . Also,  $\mathcal{J}_A^{(2)} \leq \mathcal{J}_A^{(3)}$  if and only if

$$\frac{\lambda_2^{[\lambda_2]}}{[\lambda_2]!} e^{-\lambda_2} \geq \frac{\lambda_1^{[\lambda_1]}}{[\lambda_1]!} e^{-\lambda_1}, \quad (5.85)$$

which is satisfied when  $\lambda_1 \geq \lambda_2$ .

Identically distributed observations

When the observations are identically distributed, i.e.,  $\mathbb{X}_1 = \mathbb{X}_2 \stackrel{\text{def}}{=} \mathbb{X}$ , and  $p_1(x) = p_2(x) \stackrel{\text{def}}{=} p(x)$ ,  $x \in \mathbb{X}$ , we have:

$$\mathcal{J}_A^{(2)} = \mathcal{J}_A^{(3)} = (1 - p_{[1]} - p_{[2]}) \cdot (1 + p_{[2]}) \quad (5.86)$$

and

$$\mathcal{J}_A^{(1)} = (1 - p_{[1]} - p_{[2]}) \cdot (2 - 2p_{[1]}) \quad (5.87)$$

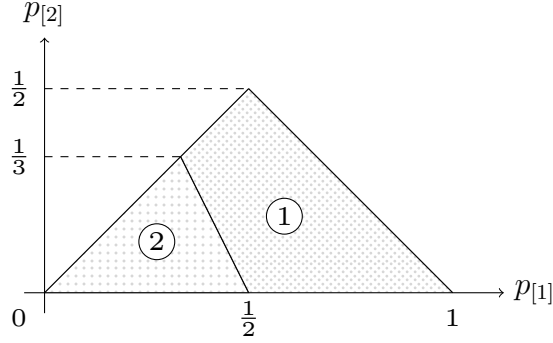


Figure 5.4: Partition of the parameter space indicating where each of the policies is team-optimal in the case of identically distributed observations. The circled number corresponds to the optimal  $m$  in Table 5.1.

Therefore,  $\mathcal{J}_A^{(2)} \leq \mathcal{J}_A^{(1)}$  if and only if

$$2p_{[1]} + p_{[2]} \leq 1. \tag{5.88}$$

Recalling that  $p_{[1]} \geq p_{[2]}$ , we have the partitioning of the parameter space  $[0, 1]^2$  according to Fig. 5.4.

## 5.4 Solution to Problem 5.2

In this section we consider the optimization of a different cost, the total probability of an error event. The interpretation is that the random variables  $X_1$  and  $X_2$  are the components of a vector source  $W = (X_1, X_2)$  where  $W \sim p_W(x_1, x_2) = p_1(x_1)p_2(x_2)$ , where  $(x_1, x_2) \in \mathbb{X}_1 \times \mathbb{X}_2$ , with each component being observed by a different sensor. The goal is to find policies at the sensors such as to minimize the probability of error in estimating the entire vector  $W$ :

$$\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{P}(W \neq \hat{W}). \tag{5.89}$$

In order to minimize the probability of error criterion, the estimator also implements the MAP rule with respect to the joint conditional probability, i.e.,  $\hat{W} = \mathcal{E}(Y)$  where

$$\mathcal{E}(y) = \arg \max_{w \in \mathbb{W}} \mathbf{P}(W = w | Y = y), \quad (5.90)$$

where  $\mathbb{W} = \mathbb{X}_1 \times \mathbb{X}_2$ .

The overall proof strategy is to obtain a characterization of team-optimal strategies using a person-by-person optimality approach. Unfortunately, the cost  $\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2)$  does not admit a nice decomposition similar to the one used in Problem 5.1. On the other hand, through our analysis we obtain a team-optimal solution for Problem 5.2.

We start by expanding the cost  $\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2)$  using the law of total probability to obtain an expression that will serve as the basis for identifying the dependencies of the cost in terms of the communication policies  $\mathcal{U}_1$  and  $\mathcal{U}_2$ .

**Proposition 5.2.** *When the MAP estimator is used in Problem 5.2, the following holds:*

$$\mathbf{P}(W = \hat{W} | Y = \mathfrak{C}) = \max_{x \in \mathbb{X}_1} \mathbf{P}(X_1 = x | U_1 = 1) \max_{x \in \mathbb{X}_2} \mathbf{P}(X_2 = x | U_2 = 1); \quad (5.91)$$

and

$$\mathbf{P}(W = \hat{W} | Y = \emptyset) = \max_{x \in \mathbb{X}_1} \mathbf{P}(X_1 = x | U_1 = 0) \max_{x \in \mathbb{X}_2} \mathbf{P}(X_2 = x | U_2 = 0). \quad (5.92)$$

*Proof.* The conditional probability of a correct estimate conditioned on the event of

a collision can be computed as:

$$\begin{aligned}
\mathbf{P}(W = \hat{W}|Y = \mathfrak{C}) &\stackrel{(a)}{=} \max_{w \in \mathbb{W}} \mathbf{P}(W = w|Y = \mathfrak{C}) \\
&\stackrel{(b)}{=} \max_{w \in \mathbb{W}} \mathbf{P}(W = w|U_1 = 1, U_2 = 1) \\
&\stackrel{(c)}{=} \max_{x \in \mathbb{X}_1} \mathbf{P}(X_1 = x|U_1 = 1) \max_{x \in \mathbb{X}_2} \mathbf{P}(X_2 = x|U_2 = 1),
\end{aligned}$$

where (a) follows from the definition of a MAP estimate; the equality (b) follows from the equivalence of the events  $Y = \mathfrak{C} \Leftrightarrow (U_1 = 1, U_2 = 1)$ ; finally, (c) follows from the independence of  $X_1$  and  $X_2$  and the fact that  $U_i$  is independent of  $X_j$ , for  $i, j \in \{1, 2\}$  such that  $i \neq j$ .

The proof of the second equality can be derived from the equivalence of the event  $Y = \emptyset \Leftrightarrow (U_1 = 0, U_2 = 0)$  followed by the same sequence of steps.  $\square$

**Proposition 5.3.** *When the MAP estimator is used in Problem 5.2, for  $i, j \in \{1, 2\}$  and  $i \neq j$ , the following holds:*

$$\mathbf{P}(W = \hat{W}|Y = (i, X_i)) \stackrel{\text{w.p.1}}{=} \max_{x \in \mathbb{X}_j} \mathbf{P}(X_j = x|U_j = 0). \quad (5.93)$$

*Proof.* It suffices to show that for every  $\tilde{x} \in \mathbb{X}_i$ , the following equalities hold:

$$\begin{aligned}
\mathbf{P}(W = \hat{W}|Y = (i, \tilde{x})) &\stackrel{(a)}{=} \max_{w \in \mathbb{W}} \mathbf{P}(W = w|Y = (i, \tilde{x})) \\
&\stackrel{(b)}{=} \max_{w \in \mathbb{W}} \mathbf{P}(W = w|U_i = 1, X_i = \tilde{x}, U_j = 0) \\
&\stackrel{(c)}{=} \max_{x \in \mathbb{X}_i} \mathbf{P}(X_i = x|U_i = 1, X_i = \tilde{x}) \max_{x \in \mathbb{X}_j} \mathbf{P}(X_j = x|U_j = 0) \\
&= \max_{x \in \mathbb{X}_j} \mathbf{P}(X_j = x|U_j = 0), \quad (5.94)
\end{aligned}$$

where (a) follows from the definition of the MAP estimator; the equality in (b) follows from the equivalence of the events  $Y = (i, \tilde{x}) \Leftrightarrow (U_i = 1, X_i = \tilde{x}, U_j = 0)$ ; and (c) follows from the independence of  $X_1$  and  $X_2$ .  $\square$

### 5.4.1 Single decision maker subproblem

Unlike Problem 5.1, where the additive nature of the cost allowed us to make an analogy with a single DM remote estimation problem with a communication cost, Problem 5.2 does not admit an additive decomposition. However, we can still use the same techniques with an equivalent cost somewhat less insightful than the one in the previous section. We proceed to define the auxiliary abstract problem solved by each of the decision makers using a person-by-person problem approach.

Let  $\mathbb{X}$  be a discrete, finite or countably infinite alphabet, and  $p(x)$  a probability mass function defined on  $\mathbb{X}$ . Let  $\mathbb{U}$  be the space of all functions  $\mathcal{U} : \mathbb{X} \rightarrow [0, 1]$ , and define  $\mathcal{J}_B : \mathbb{U} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \mathcal{J}_B(\mathcal{U}) \stackrel{\text{def}}{=} & 1 - \tau \max_{x \in \mathbb{X}} \mathcal{U}(x)p(x) - \varrho \sum_{x \in \mathbb{X}} \mathcal{U}(x)p(x) \\ & - (\varrho + \beta) \max_{x \in \mathbb{X}} (1 - \mathcal{U}(x))p(x). \end{aligned} \tag{5.95}$$

**Lemma 5.3.** *For nonnegative constants  $\varrho, \tau$  and  $\beta$ , the cost  $\mathcal{J}_B$  is concave on  $\mathbb{U}$ .*

*Proof.* The proof follows from standard arguments that can be found in [59, ch. 3].  $\square$

**Lemma 5.4.** *Given nonnegative constants  $\varrho, \tau$  and  $\beta \in \mathbb{R}$  such that  $\tau \leq \beta$ , and a*

probability mass function  $p(x)$  with  $x \in \mathbb{X}$ , the cost  $\mathcal{J}_B$  in Eq. (5.95) is minimized by the following policy:

$$\mathcal{U}^*(x) = \begin{cases} 0 & \text{if } x = x_{[1]} \\ 1 & \text{otherwise.} \end{cases} \quad (5.96)$$

*Proof.* From Lemma 5.3, the cost is concave in  $\mathcal{U}$ . Therefore, without loss in optimality, we can constrain the optimization to the class of deterministic strategies. For any deterministic policy  $\mathcal{U} \in \mathbb{U}$ , define

$$\mathbb{X}_k^{\mathcal{U}} \stackrel{\text{def}}{=} \{x \in \mathbb{X} \mid \mathcal{U}(x) = k\}, \quad k \in \{0, 1\}. \quad (5.97)$$

Constraining the policies to be deterministic and using the notation defined above, the cost becomes

$$\mathcal{J}_B(\mathcal{U}) = 1 - \tau \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \varrho \sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - (\varrho + \beta) \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x). \quad (5.98)$$

Since

$$\sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \leq 1 - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x), \quad (5.99)$$

we obtain the following inequality, which holds for every deterministic policy  $\mathcal{U} \in \mathbb{U}$ :

$$\mathcal{J}_B(\mathcal{U}) \geq 1 - \varrho - \tau \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \beta \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x). \quad (5.100)$$

The lower bound on the right hand side of the inequality above can be mini-



mized. If  $\tau \leq \beta$ , we assign the symbol  $x_{[1]}$  to  $\mathbb{X}_0^{\mathcal{U}}$  and  $x_{[2]}$  to  $\mathbb{X}_1^{\mathcal{U}}$ , yielding:

$$\mathcal{J}_B(\mathcal{U}) \geq 1 - \varrho - \tau p_{[2]} - \beta p_{[1]}. \quad (5.101)$$

Evaluating the cost of the policy  $\mathcal{U}^*$ , the lower bound is achieved with equality and therefore it is optimal.  $\square$

#### 5.4.2 Team-optimal policies for Problem 5.2

Theorem 5.2 states that a pair of communication policies at the sensors that jointly minimizes a probability of estimation error criterion consists of sending every measurement with exception of the most likely one for both sensors. This structure is independent of the probability mass function of the measurements. In other words, the following pair of policies  $(\mathcal{U}_1^*, \mathcal{U}_2^*)$  is team-optimal for Problem 5.2:

$$\mathcal{U}_i^*(x) = \begin{cases} 0 & \text{if } x = x_{i,[1]} \\ 1 & \text{otherwise} \end{cases}, \quad i \in \{1, 2\}. \quad (5.102)$$

*Proof of Theorem 5.2.* Using the person-by-person optimality approach, we will write the cost from the perspective of a single decision maker. Using the law of total probability and the results in Propositions 5.2 and 5.3, we can re-express the cost

as follows:

$$\begin{aligned}
\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) &= 1 - \max_{x \in \mathbb{X}_1} \mathcal{U}_1(x) p_1(x) \max_{x \in \mathbb{X}_2} \mathcal{U}_2(x) p_2(x) \\
&\quad - \max_{x \in \mathbb{X}_1} (1 - \mathcal{U}_1(x)) p_1(x) \max_{x \in \mathbb{X}_2} (1 - \mathcal{U}_2(x)) p_2(x) \\
&\quad - \sum_{x \in \mathbb{X}_1} \mathcal{U}_1(x) p_1(x) \max_{x \in \mathbb{X}_2} (1 - \mathcal{U}_2(x)) p_2(x) \\
&\quad - \sum_{x \in \mathbb{X}_2} \mathcal{U}_2(x) p_2(x) \max_{x \in \mathbb{X}_1} (1 - \mathcal{U}_1(x)) p_1(x). \tag{5.103}
\end{aligned}$$

Let  $i, j \in \{1, 2\}$  such that  $i \neq j$ . Fixing the communication policy of sensor  $j$ ,  $\mathcal{U}_j^* \in \mathbb{U}_j$ , from the perspective of  $\text{DM}_i$  we have the following cost

$$\begin{aligned}
\mathcal{J}_B(\mathcal{U}_i, \mathcal{U}_j^*) &= 1 - \tau_{\mathcal{U}_j^*} \max_{x \in \mathbb{X}_i} \mathcal{U}_i(x) p_i(x) - \varrho_{\mathcal{U}_j^*} \sum_{x \in \mathbb{X}_i} \mathcal{U}_i(x) p_i(x) \\
&\quad - (\varrho_{\mathcal{U}_j^*} + \beta_{\mathcal{U}_j^*}) \max_{x \in \mathbb{X}_i} (1 - \mathcal{U}_i(x)) p_i(x), \tag{5.104}
\end{aligned}$$

where

$$\beta_{\mathcal{U}_j^*} \stackrel{\text{def}}{=} \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x), \tag{5.105}$$

$$\varrho_{\mathcal{U}_j^*} \stackrel{\text{def}}{=} \max_{x \in \mathbb{X}_j} (1 - \mathcal{U}_j^*(x)) p_j(x) \tag{5.106}$$

and

$$\tau_{\mathcal{U}_j^*} \stackrel{\text{def}}{=} \max_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x). \tag{5.107}$$

Note that for any given  $\mathcal{U}_j^* \in \mathbb{U}_j$ , we have  $\beta_{\mathcal{U}_j^*} \geq \tau_{\mathcal{U}_j^*}$ . From Lemma 5.4, the policy

$\mathcal{U}_i^*$  that minimizes  $\mathcal{J}(\mathcal{U}_i, \mathcal{U}_j^*)$  is:

$$\mathcal{U}_i^*(x) = \begin{cases} 0 & \text{if } x = x_{i,[1]} \\ 1 & \text{otherwise.} \end{cases} \quad (5.108)$$

Since this is true for every  $\mathcal{U}_j^*$  it must also hold when  $\mathcal{U}_j^*$  is such that  $(\mathcal{U}_i^*, \mathcal{U}_j^*)$  is a person-by-person optimal solution. Since every team-optimal solution is also person-by-person optimal, there exists a team-optimal pair of policies with the structure outlined above.  $\square$

## 5.5 Extension to teams of $n$ sensors

The problem formulation involving only two sensors may seem too restrictive, but it is a fundamental step in going from a centralized problem with a single decision maker to a decentralized setup. In this section, we extend the results of this paper to a team of  $n$  sensors observing independent observations and communicating over a collision channel which can only support one transmitted packet. Let  $U = (U_1, \dots, U_n)$  denote the vector of binary decision variables, and  $U_{-i}$  denote vector of decision variables other than  $U_i$ . Similarly, let  $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_n)$  denote a vector of policies and  $\mathcal{U}_{-i}$  denote the vector of policies other than  $\mathcal{U}_i$ . We will need the following assumption:

**Assumption 5.1.** *If two or more sensors transmit simultaneously, the remote estimator receives a collision symbol but is able to decode the index of the transmitting sensors. This enables the receiver to determine if  $DM_i$  was silent or not when a*

collision occurs,  $i \in \{1, \dots, n\}$ .

The ability of identifying nodes involved in collisions in systems with packet capture has been demonstrated empirically in [66].

### Extension of Theorem 5.1

The result of Theorem 5.1 can be extended to a team of  $n$  sensors using the same person-by-person approach of the previous sections. In this section we provide a sketch of the proof. The problem statement for a team with  $n$  sensors is identical to the one in Section 5.3. Here we just specify the new cost:

$$\mathcal{J}_A(\mathcal{U}) \stackrel{\text{def}}{=} \sum_{k=1}^n \alpha_k \mathbf{P}(X_k \neq \hat{X}_k) \quad (5.109)$$

and make explicit that a collision occurs when more than one sensor attempts to transmit, i.e.,

$$Y = \mathfrak{C} \Leftrightarrow \sum_{k=1}^n U_k \geq 2. \quad (5.110)$$

**Theorem 5.3.** *There exists a team-optimal policy  $\mathcal{U}^*$  such that the policy of each sensor  $\mathcal{U}_i^*$ ,  $i \in \{1, \dots, n\}$ , has one of the following structures: the sensor transmits all except its most likely observation; transmits only its second most likely observation; or remains always silent.*

*Proof.* Fixing the policies of all the sensors except  $\mathcal{DM}_i$ , we can write:

$$\mathcal{J}_A(\mathcal{U}_i, \mathcal{U}_{-i}^*) = \alpha_i \mathbf{P}(X_i \neq \hat{X}_i) + \rho_{\mathcal{U}_{-i}^*} \mathbf{P}(U_i = 1) + \theta_{\mathcal{U}_{-i}^*}, \quad (5.111)$$

where the communication cost and offset terms are given by

$$\rho_{\mathcal{U}_{-i}^*} \stackrel{\text{def}}{=} \sum_{j \neq i} \alpha_j (\mathbf{P}(X_j \neq \hat{X}_j | U_i = 1) - \mathbf{P}(X_j \neq \hat{X}_j | U_i = 0)) \quad (5.112)$$

and

$$\theta_{\mathcal{U}_{-i}^*} \stackrel{\text{def}}{=} \sum_{j \neq i} \alpha_j \mathbf{P}(X_j \neq \hat{X}_j | U_i = 0). \quad (5.113)$$

Then, the problem from the perspective of  $\text{DM}_i$  is equivalent to Problem 5.3 with  $\varrho$  given by

$$\varrho \stackrel{\text{def}}{=} \frac{\rho_{\mathcal{U}_{-i}^*}}{\alpha_i}, \quad (5.114)$$

and the probability of collision of a packet from  $\text{DM}_i$  with packets coming from other sensors in the team is:

$$\beta \stackrel{\text{def}}{=} 1 - \prod_{j \neq i} \mathbf{P}(U_j = 0). \quad (5.115)$$

From Lemma 5.2, the policy  $\mathcal{U}_i^*$  that minimizes the cost  $\mathcal{J}_A(\mathcal{U}_i, \mathcal{U}_{-i}^*)$  for any choice of  $\mathcal{U}_{-i}^*$  is either to transmit all but the most likely observation; transmit only the second most likely observation; or to remain always silent. Since every person-by-person optimal solution admits an equivalent solution of this form, and every team-optimal solution is person-by-person optimal, there must exist a team-optimal solution with this structure.  $\square$

## Extension of Theorem 5.2

Let  $W = (X_1, \dots, X_n)$  be distributed according to

$$p_W(x_1, \dots, x_n) = \prod_{k=1}^n p_k(x_k). \quad (5.116)$$

The remote receiver is interested in forming an estimate vector  $\hat{W} = (\hat{X}_1, \dots, \hat{X}_n)$  such that the following cost is minimized

$$\mathcal{J}_B(\mathcal{U}) = \mathbf{P}(W \neq \hat{W}). \quad (5.117)$$

**Theorem 5.4.** *For a problem with  $n$  sensors observing independent random variables and communicating over the collision channel there exists a team-optimal solution  $\mathcal{U}^*$  such that each sensor transmits all observations except its most likely one.*

*Proof.* Using the definition of the MAP estimator, we can express the following probabilities:

$$\mathbf{P}(W = \hat{W} | Y = \emptyset) = \prod_{k=1}^n \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x | U_k = 0) \quad (5.118)$$

and

$$\mathbf{P}(W = \hat{W} | Y = (j, \tilde{x})) = \prod_{k \neq j} \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x | U_k = 0). \quad (5.119)$$

When a collision occurs, our assumption implies that the receiver is able to identify

the realization of the random vector  $U = (u_1, \dots, u_n)$ . Therefore,

$$\mathbf{P}(W = \hat{W} | Y = \mathfrak{C}) = \prod_{k=1}^n \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x | U_k = u_k). \quad (5.120)$$

Using total probability we can express the cost as:

$$\begin{aligned} \mathcal{J}_B(\mathcal{U}) &= 1 - \prod_{k=1}^n \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x, U_k = 0) \\ &\quad - \sum_{j=1}^n \left( \prod_{k \neq j} \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x, U_k = 0) \right) \mathbf{P}(U_j = 1) \\ &\quad - \sum_{u: \|u\| \geq 2} \left( \prod_{k=1}^n \max_{x \in \mathbb{X}_k} \mathbf{P}(X_k = x, U_k = u_k) \right). \end{aligned} \quad (5.121)$$

In terms of the policies  $\mathcal{U}_k$ ,  $k \in \{1, \dots, n\}$ , the expression above is equal to:

$$\begin{aligned} \mathcal{J}_B(\mathcal{U}) &= 1 - \prod_{k=1}^n \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k(x)) p_k(x) \\ &\quad - \sum_{j=1}^n \left( \prod_{k \neq j} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k(x)) p_k(x) \right) \sum_{x \in \mathbb{X}_j} \mathcal{U}_j(x) p_j(x) \\ &\quad - \sum_{u: \|u\| \geq 2} \left( \prod_{k: u_k=0} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k(x)) p_k(x) \right) \left( \prod_{k: u_k=1} \max_{x \in \mathbb{X}_k} \mathcal{U}_k(x) p_k(x) \right). \end{aligned} \quad (5.122)$$

Fixing the policies of every DM except DM<sub>*i*</sub>, we have:

$$\begin{aligned} \mathcal{J}_B(\mathcal{U}_i, \mathcal{U}_{-i}^*) &= 1 - \tau_{\mathcal{U}_{-i}^*} \max_{x \in \mathbb{X}_i} \mathcal{U}_i(x) p_i(x) - \varrho_{\mathcal{U}_{-i}^*} \sum_{x \in \mathbb{X}_i} \mathcal{U}_i(x) p_i(x) \\ &\quad - (\varrho_{\mathcal{U}_{-i}^*} + \beta_{\mathcal{U}_{-i}^*}) \max_{x \in \mathbb{X}_i} (1 - \mathcal{U}_i(x)) p_i(x), \end{aligned} \quad (5.123)$$

where the coefficients  $\varrho_{\mathcal{U}_{-i}^*}$ ,  $\tau_{\mathcal{U}_{-i}^*}$  and  $\beta_{\mathcal{U}_{-i}^*}$  are given by:

$$\varrho_{\mathcal{U}_{-i}^*} \stackrel{\text{def}}{=} \prod_{k \neq i} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x); \quad (5.124)$$

$$\tau_{\mathcal{U}_{-i}^*} \stackrel{\text{def}}{=} \sum_{u_{-i}: \|u_{-i}\| \geq 1} \left( \prod_{k: u_k=0} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \left( \prod_{k: u_k=1} \max_{x \in \mathbb{X}_k} \mathcal{U}_k^*(x) p_k(x) \right); \quad (5.125)$$

and

$$\begin{aligned} \beta_{\mathcal{U}_{-i}^*} \stackrel{\text{def}}{=} & \sum_{u_{-i}: \|u_{-i}\| \geq 2} \left( \prod_{\substack{k \neq i: \\ u_k=0}} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \left( \prod_{\substack{k \neq i: \\ u_k=1}} \max_{x \in \mathbb{X}_k} \mathcal{U}_k^*(x) p_k(x) \right) \\ & + \sum_{j \neq i} \left( \prod_{k \neq i, j} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x). \end{aligned} \quad (5.126)$$

It remains to show that  $\beta_{\mathcal{U}_{-i}^*} \geq \tau_{\mathcal{U}_{-i}^*}$ . Consider the following quantity:

$$\begin{aligned} \tau_{\mathcal{U}_{-i}^*} - \beta_{\mathcal{U}_{-i}^*} &= \sum_{u_{-i}: \|u_{-i}\|=1} \left( \prod_{\substack{k \neq i: \\ u_k=0}} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \left( \prod_{\substack{k \neq i: \\ u_k=1}} \max_{x \in \mathbb{X}_k} \mathcal{U}_k^*(x) p_k(x) \right) \\ &\quad - \sum_{j \neq i} \left( \prod_{k \neq i, j} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x) \end{aligned} \quad (5.127)$$

$$\begin{aligned} &= \sum_{j \neq i} \left( \prod_{\substack{k \neq i, j \\ x \in \mathbb{X}_k}} \max_{x \in \mathbb{X}_k} (1 - \mathcal{U}_k^*(x)) p_k(x) \right) \\ &\quad \cdot \left( \max_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x) - \sum_{x \in \mathbb{X}_j} \mathcal{U}_j^*(x) p_j(x) \right) \end{aligned} \quad (5.128)$$

$$\leq 0. \quad (5.129)$$

From Lemma 5.4, the policy  $\mathcal{U}_i^*$  that minimizes  $\mathcal{J}_B(\mathcal{U}_i, \mathcal{U}_{-i}^*)$  for any fixed  $\mathcal{U}_{-i}^*$



is:

$$\mathcal{U}_i^*(x) = \begin{cases} 0 & \text{if } x = x_{i,[1]} \\ 1 & \text{otherwise.} \end{cases} \quad (5.130)$$

Therefore, there exists a team-optimal solution  $\mathcal{U}^*$  such that each decision maker transmits all the observations except the most likely one.  $\square$

### Illustrative example

Consider the case of a system where  $n$  sensors observe independent identically distributed random variables with pmf  $p(x)$ ,  $x \in \mathbb{X}$ . If the sensors optimize their strategies with respect to cost  $\mathcal{J}_B(\mathcal{U})$ , Theorem 5.4 states that transmitting all but the most likely observation is a team-optimal solution. The performance of the team can be computed as a function of the probability of the two most likely symbols  $p_{[1]}$  and  $p_{[2]}$ :

$$\mathcal{J}_B(\mathcal{U}^*) = 1 - np_{[1]}^{n-1}(1 - p_{[1]} - p_{[2]}) - (p_{[1]} + p_{[2]})^n. \quad (5.131)$$

When the observations are binary random variables,  $p_{[1]} + p_{[2]} = 1$ , therefore the optimal cost is equal to zero. This is true for any number of sensors, i.e.,

$$\mathcal{J}_B^{\text{bin}}(\mathcal{U}^*) = 0. \quad (5.132)$$

However, when  $|\mathbb{X}| \geq 3$  we have  $p_{[1]} + p_{[2]} < 1$  and the performance of the system degrades when the number of sensors  $n$  increases, i.e.,

$$\lim_{n \rightarrow \infty} \mathcal{J}_B(\mathcal{U}^*) = 1. \quad (5.133)$$

In order to illustrate how the performance degrades with  $n$ , assume that each  $X_k$  is geometrically distributed with parameter  $\pi \in [0, 1]$ ,  $k = 1, \dots, n$ . Figure 5.5 shows the minimum total probability of error for a team with  $n = 2, 4, 8, 16$  and 64 sensors sharing a common medium modeled by a collision channel. Note that the probability of error is always close to 1 when  $\pi$  is sufficiently small, but falls sharply to 0 as  $\pi$  increases.

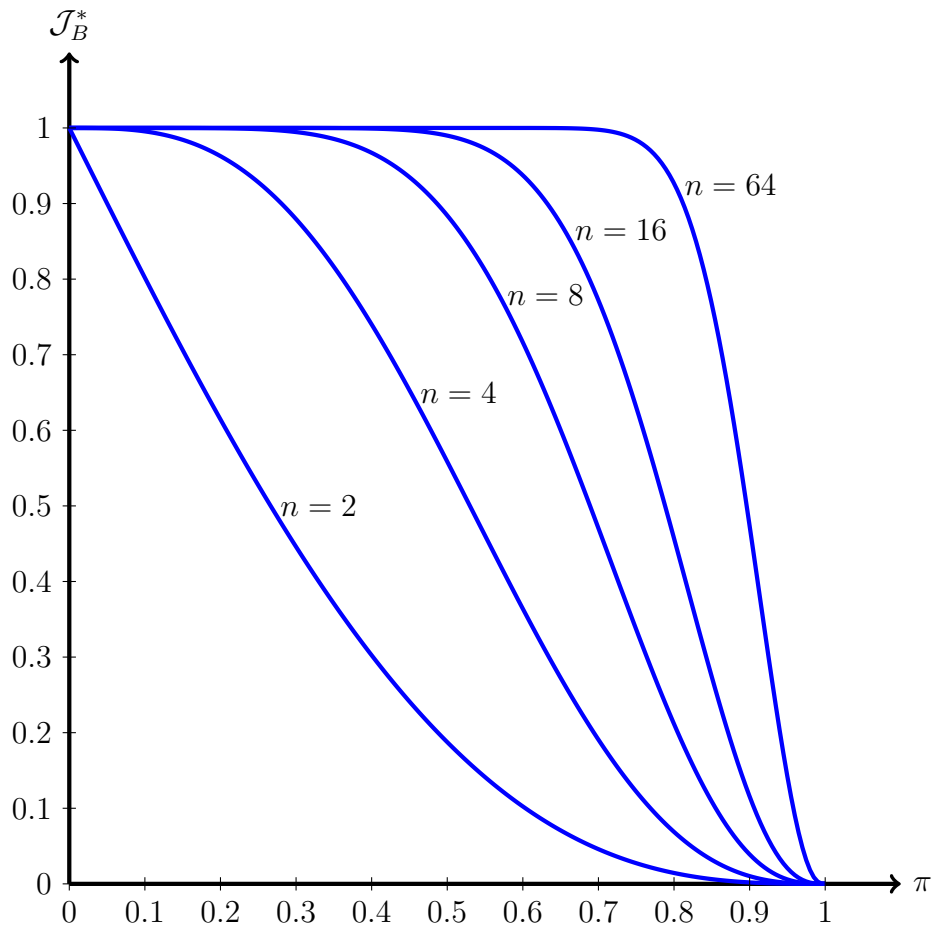


Figure 5.5: Optimal performance of a team with  $n$  sensors observing i.i.d. geometric random variables with parameter  $\pi$  and minimizing the total probability of error criterion.

## 5.6 Summary

In this chapter we studied a class of team-decision problems motivated by remote estimation of independent discrete random variables over a wireless network modeled by a collision channel. As a performance metric, we used two variations of the probability of estimation error criterion. For an aggregate probability of error, we obtained the structure of person-by-person optimal policies and used it to reduce the search space of candidate team-optimal policies. For the total probability of estimation error, we obtained a team-optimal solution. Our results are valid to arbitrarily distributed random variables on finite or countably infinite alphabets, and can be extended to any number of sensors under the assumption that the receiver can identify which sensors are involved in a collision. We showed that the performance of the overall system only depends on the probabilities of the two most likely symbols of each source and provided several examples.

## Chapter 6: Extensions

In the systems considered in this dissertation so far, we assumed that the agents make independent observations and have a common objective, working as members of a team. The purpose of this chapter is to generalize these two assumptions on the basic model and show that this framework can be used in other less restrictive scenarios. In the first part of this chapter, we will allow each decision maker to have its own objective functional, leading to a non-cooperative remote estimation game formulation. In the second part, we will consider a problem with dependent measurements, where each sensor observes a vector consisting of a common and a private component.

### 6.1 Remote estimation games

Consider a system where two sensors and two remote estimators share the same network, which can only support the perfect communication of a single packet between one sensor-estimator pair at a time. Each sensor observes a random variable and must decide when to send a measurement to its corresponding remote estimator. Each remote estimator forms an estimate of the random variable observed by its corresponding sensor according to the minimum mean squared error criterion.

The problem setup is inspired by the conventional Gaussian interference channel model [67]. In particular, the work of Berry and Tse in [68], which uses a game theoretic approach to characterize the trade-offs in the capacity region of the Gaussian interference channel.

### 6.1.1 System model

The basic framework for the problem considered in this section is illustrated in Fig. 6.1. The  $DM_i$  observes a realization of a random variable  $X_i$  where  $X_i \sim \mathcal{N}(0, \sigma_i^2)$ ,  $i \in \{1, 2\}$  such that  $X_1 \perp\!\!\!\perp X_2$ . The  $DM_i$  then decides if it will transmit or not a packet containing its measurement over the channel. The decision variable  $U_i = 1$  denotes that  $DM_i$  will attempt transmission and  $U_i = 0$  denotes the decision to remain silent. We refer to  $\mathcal{U}_i$  as the *communication policy* of  $DM_i$ .

**Definition 6.1** (Communication policies). *The communication policy for  $DM_i$  is a measurable function  $\mathcal{U}_i : \mathbb{R} \rightarrow [0, 1]$  such that*

$$\mathbf{P}(U_i = 1 | X_i = x_i) \stackrel{\text{def}}{=} \mathcal{U}_i(x_i), \quad i \in \{1, 2\}. \quad (6.1)$$

*The set of all communication policies for  $DM_i$  is denoted by  $\mathbb{U}_i$ . The channel input  $S_i$  corresponds to a communication packet, whose content is determined as follows:*

$$S_i = \begin{cases} X_i & \text{if } U_i = 1 \\ \emptyset & \text{if } U_i = 0 \end{cases}, \quad i \in \{1, 2\}, \quad (6.2)$$

*where the symbol  $\emptyset$  denotes no-transmission.*

The DMs are connected to the estimators by a wireless network modeled as a collision channel.

**Definition 6.2** (Collision channel). *The collision channel is defined by the deterministic map  $\chi : \mathbb{S}^2 \rightarrow \mathbb{Y}^2$ , where  $\mathbb{S} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\emptyset\}$  and  $\mathbb{Y} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\emptyset, \mathfrak{C}\}$  such that*

$$\chi(s_1, s_2) \stackrel{\text{def}}{=} \begin{cases} (\emptyset, \emptyset) & \text{if } s_1 = \emptyset, s_2 = \emptyset \\ (x_1, \emptyset) & \text{if } s_1 = x_1, s_2 = \emptyset \\ (\emptyset, x_2) & \text{if } s_1 = \emptyset, s_2 = x_2 \\ (\mathfrak{C}, \mathfrak{C}) & \text{if } s_1 = x_1, s_2 = x_2, \end{cases} \quad (6.3)$$

where the symbol  $\mathfrak{C}$  represents the occurrence of a collision.

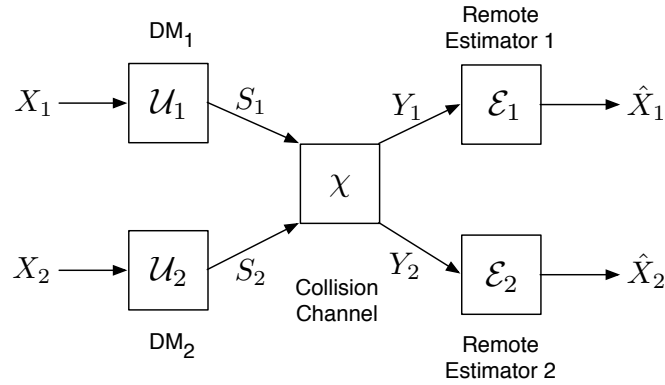


Figure 6.1: General setup for a non-cooperative remote estimation game over the collision channel.

The channel output is given by  $(Y_1, Y_2) = \chi(S_1, S_2)$ . The  $i$ -th estimator forms an estimate  $\hat{X}_i$  based on its corresponding channel output  $Y_i$  according to a measurable map  $\mathcal{E}_i : \mathbb{Y} \rightarrow \mathbb{R}$ . The function  $\mathcal{E}_i$  is called the *estimation policy* of estimator  $i$ . The goal of each sensor-estimator pair is to minimize a corresponding mean squared estimation error. Therefore, we assume without loss of optimality that for a given

pair of communication policies, the estimation policies are the conditional mean, i.e.,

$$\mathcal{E}_i(y) = \mathbf{E}[X_i|Y_i = y], \quad y \in \mathbb{Y}. \quad (6.4)$$

The costs functionals are maps  $\mathcal{J}_i : \mathbb{U}_1 \times \mathbb{U}_2 \rightarrow \mathbb{R}$  such that

$$\mathcal{J}_1(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{E}[(X_1 - \hat{X}_1)^2] \quad (6.5)$$

and

$$\mathcal{J}_2(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{E}[(X_2 - \hat{X}_2)^2], \quad (6.6)$$

for DM<sub>1</sub> and DM<sub>2</sub>, respectively. The expectation in each  $\mathcal{J}_i$  is taken with respect to both  $X_1$  and  $X_2$  and the goal of each sensor is to minimize its own cost.

### 6.1.2 Solution concepts

There are several ways to define the notion of solution of a non-cooperative game. The most typical solution concepts are: security policies (also known as minimax) and Nash-equilibrium solutions.

**Definition 6.3** (Security policies). *Let  $i, j \in \{1, 2\}$  and  $j \neq i$ , the security policy for DM <sub>$i$</sub>  is obtained by solving the following optimization problem:*

$$\min_{\mathcal{U}_i \in \mathbb{U}_i} \max_{\mathcal{U}_j \in \mathbb{U}_j} \mathcal{J}_i(\mathcal{U}_i, \mathcal{U}_j). \quad (6.7)$$

The security policy for DM <sub>$i$</sub>  is obtained by assuming that its opponent is

making the worst decision possible (from  $DM_i$ 's perspective). It is a robust solution concept that establishes an upper bound for the performance known as the *security level* for  $DM_i$ .

**Definition 6.4** (Nash-equilibrium). *A Nash-equilibrium solution consists of a pair of policies  $(\mathcal{U}_1^*, \mathcal{U}_2^*)$  satisfying the following pair of conditions*

$$\mathcal{J}_1(\mathcal{U}_1^*, \mathcal{U}_2^*) \leq \mathcal{J}_1(\mathcal{U}_1, \mathcal{U}_2^*), \quad \mathcal{U}_1 \in \mathbb{U}_1 \quad (6.8)$$

$$\mathcal{J}_2(\mathcal{U}_1^*, \mathcal{U}_2^*) \leq \mathcal{J}_2(\mathcal{U}_1^*, \mathcal{U}_2), \quad \mathcal{U}_2 \in \mathbb{U}_2. \quad (6.9)$$

In other words, at a Nash-equilibrium solution, there is no incentive for either DMs to unilaterally change their communication policies.

### 6.1.3 Structural results

#### Security policies

**Theorem 6.1.** *Consider the remote estimation game over the collision channel with independent Gaussian observations  $X_i \sim \mathcal{N}(0, \sigma_i^2)$ ,  $i \in \{1, 2\}$ . The policy*

$$\mathcal{U}_i^{sec}(x_i) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases} \quad (6.10)$$



is a security solution for  $DM_i$ ,  $i \in \{1, 2\}$ . The corresponding estimation policy is given by

$$\mathcal{E}_i^{\text{sec}}(y) = \begin{cases} \mathbf{E}[X_i|X_i \geq 0] = +\sqrt{\frac{2}{\pi}}\sigma_i & \text{if } y = \mathfrak{C} \\ \mathbf{E}[X_i|X_i < 0] = -\sqrt{\frac{2}{\pi}}\sigma_i & \text{if } y = \emptyset. \end{cases} \quad (6.11)$$

The security level for  $DM_i$  given by

$$\bar{\mathcal{J}}_i \stackrel{\text{def}}{=} \left(1 - \frac{2}{\pi}\right) \sigma_i^2. \quad (6.12)$$

*Proof.* From the perspective of  $DM_i$ , the worst case scenario is when its opponent,  $DM_j$ , attempts to access the channel regardless of its measurement, i.e.,  $DM_j$  uses a *selfish* policy  $\mathcal{U}_j^{\text{self}}(x) \equiv 1$ . In this case, the channel is occupied with probability  $\mathbf{P}(U_j = 1) = 1$  and there is no chance of getting a packet through the collision channel  $\chi$  to its corresponding remote estimator. In this case, the best course of action is to signal 1 bit of information about  $X_i$  using solely the no-transmission and collision symbols  $\emptyset$  and  $\mathfrak{C}$ . When a single bit is available to describe a zero mean Gaussian variable, one strategy that minimizes the mean square error is to transmit a packet when  $X_i \geq 0$  and not to transmit when  $X_i < 0$ . This choice is clearly not unique.  $\square$

**Example 6.1.** Consider the game over the collision channel with  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 2)$ . The security policy for  $DM_1$  is the single threshold policy described above and its corresponding cost is

$$\mathcal{J}_1(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{self}}) = 0.3634. \quad (6.13)$$

If  $DM_2$  indeed chooses to adopt the selfish policy of always transmitting while its opponent adopts the afore mentioned security policy, the transmissions made by  $DM_2$  will succeed and fail with probabilities 0.5, respectively. When the transmission fails, the best the remote estimator 2 can do is to output  $\hat{x}_2 = \mathbf{E}[X_2] = 0$  then incurring in a cost of

$$\mathcal{J}_2(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{self}}) = 1. \quad (6.14)$$

If  $DM_2$  decides to abandon its selfish policy in favor of using its own security policy, it is not difficult to show that

$$\mathcal{J}_1^{\text{sec}} \stackrel{\text{def}}{=} \mathcal{J}_1(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{sec}}) = \frac{3}{4} \left(1 - \frac{2}{\pi}\right) \sigma_1^2 = 0.2725 \quad (6.15)$$

and

$$\mathcal{J}_2^{\text{sec}} \stackrel{\text{def}}{=} \mathcal{J}_2(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{sec}}) = \frac{3}{4} \left(1 - \frac{2}{\pi}\right) \sigma_2^2 = 0.5450. \quad (6.16)$$

Figure 6.2 illustrates the security and selfish policies in Example 6.1.

## Nash-equilibrium policies

Example 6.1 shows that there is an incentive for  $DM_2$  not to use a selfish policy and share the channel with  $DM_1$ . Therefore, the selfish policy  $\mathcal{U}_2^{\text{self}}$  is not in *Nash-equilibrium* with the security policy  $\mathcal{U}_1^{\text{sec}}$ . In this section, we make use of the results from Chapter 3 to show that for the remote estimation game considered here, one may constrain the search for Nash-equilibria over the class of deterministic threshold policies.

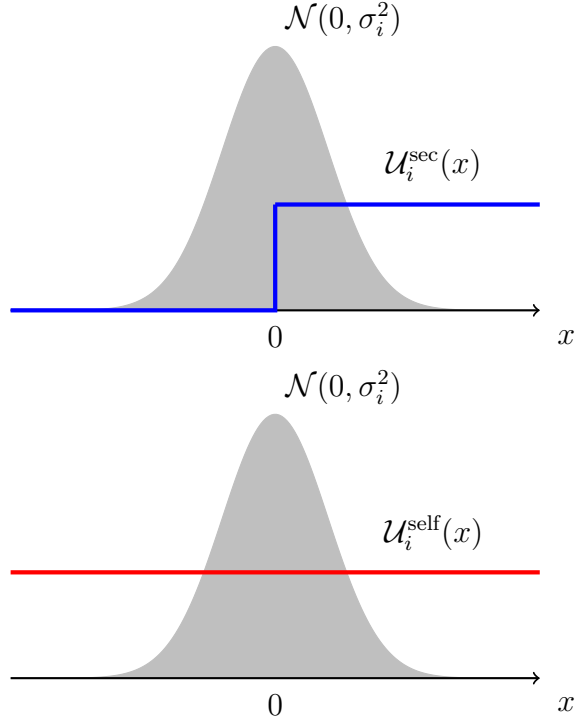


Figure 6.2: Security and selfish policies for  $DM_i$  when  $X_i \sim \mathcal{N}(0, \sigma_i^2)$ ,  $i \in \{1, 2\}$ .

**Theorem 6.2.** *Let  $(\mathcal{U}_1^*, \mathcal{U}_2^*)$  be a Nash-equilibrium solution for the remote estimation game with independent observations  $X_i$ , where  $X_i \sim \mathcal{N}(0, \sigma_i^2)$  for  $i \in \{1, 2\}$ . There exists a pair of Nash-equilibrium deterministic threshold policies  $(\check{\mathcal{U}}_1^*, \check{\mathcal{U}}_2^*)$  that attains the same costs.*

*Proof.* In order to establish this structural result, it is enough to show that there exists a deterministic threshold policy that is optimal for the problem

$$\min_{\mathcal{U}_i \in \mathbb{U}_i} \mathcal{J}_i(\mathcal{U}_i, \mathcal{U}_j^*), \quad (6.17)$$

with  $\mathcal{U}_j^* \in \mathbb{U}_j$  arbitrarily fixed, where  $i \neq j$ . If we find a class of policies with a particular structure that contains the optimal solution for every possible choice of  $\mathcal{U}_j^*$ , the structure will also hold when  $(\mathcal{U}_i^*, \mathcal{U}_j^*)$  is a Nash-equilibrium solution.

As we have done previously in Chapters 3 and 5, we analyze the problem by constructing an equivalent single DM subproblem. Let  $\mathcal{U}_j^* \in \mathbb{U}_j$  be arbitrarily fixed, the collision channel will be occupied by  $\text{DM}_j$  with probability  $\mathbf{P}(U_j = 1) = \beta$ . The cost to be minimized by the DM is

$$\mathcal{J}_i(\mathcal{U}_i, \mathcal{U}_j^*) = \beta \mathbf{E}[(X_i - \hat{X}_i)^2 | U_i = 1] \mathbf{P}(U_i = 1) + \mathbf{E}[(X_i - \hat{X}_i)^2 | U_i = 0] \mathbf{P}(U_i = 0), \quad (6.18)$$

which can be related to problem as an instance of Problem 3.2 with  $\beta = \mathbf{P}(U_j = 1)$  and  $\varrho = 0$ . Theorem 3.2 guarantees the existence of a deterministic threshold policy  $\check{\mathcal{U}}_i^*$  that is optimal for this problem. The same argument can be repeated for  $\text{DM}_j$ ,  $j \neq i$ , leading to a deterministic policy  $\check{\mathcal{U}}_j^*$ .  $\square$

**Example 6.2** (Nash-equilibrium policies). *Consider the game over the collision channel without capture where  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 2)$ . Using the Modified Lloyd-Max algorithm from Chapter 4, we may find Nash-equilibrium policies for the game over the collision channel by iteratively using it to find policies which constitute a fixed point of the following procedure: First, we fix  $\beta \in (0, 1)$  and apply the MLM algorithm for  $\text{DM}_1$  until we find a critical point, which corresponds to a policy  $\mathcal{U}_1$ . From  $\mathcal{U}_1$ , we obtain the corresponding  $\alpha = \mathbf{P}(U_1 = 1)$  and repeat the same steps for  $\text{DM}_2$ , where  $\alpha$  now plays the role of probability of the collision channel being occupied. Whenever a fixed point to this procedure is found, we stop and output the pair  $(\mathcal{U}_1^*, \mathcal{U}_2^*)$ . It is not known if this procedure will always produce a Nash-equilibrium. Using the procedure outlined above, we obtained the following pair of communication policies summarized in Table 6.1.*

$i$	$\mathbb{A}_{i0}^*$	$\hat{x}_{i\emptyset}^*$	$\hat{x}_{i\mathfrak{C}}^*$	$\mathcal{J}_i^{\text{nash}}$
1	[0.274,6.683]	0.980	-0.632	0.278
2	[0.387,9.451]	1.386	-0.894	0.557

Table 6.1: An example of a Nash-equilibrium

**Remark 6.1.** *It can be numerically verified that  $(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{sec}})$  is not a Nash-equilibrium.*

*Note that despite the fact that the policies in Example 6.2 are in Nash-equilibrium, their performance is strictly worse than if both DMs act according to their security policies, i.e.,*

$$\mathcal{J}_i^{\text{sec}} < \mathcal{J}_i^{\text{nash}}, \quad i \in \{1, 2\}. \quad (6.19)$$

*However, we cannot guarantee that there is no other Nash-equilibrium with a better performance than  $(\mathcal{U}_1^{\text{sec}}, \mathcal{U}_2^{\text{sec}})$ . This is an open question for future investigation.*

**Remark 6.2.** *The probability of transmission for DM<sub>1</sub> and DM<sub>2</sub> in the Nash-equilibrium policies of Example 6.2 are both equal to*

$$\mathbf{P}(U_i = 1) = 0.608, \quad i \in \{1, 2\}. \quad (6.20)$$

*Also note that the costs at this Nash-equilibrium scale linearly with the ratio between the variances of the measurements, i.e.,*

$$\mathcal{J}_2^{\text{nash}} = \left( \frac{\sigma_2^2}{\sigma_1^2} \right) \mathcal{J}_1^{\text{nash}}. \quad (6.21)$$

*We conjecture that these curious observations will always hold, but their proofs remain open for future work.*

#### 6.1.4 Summary

We have presented preliminary results on a problem where multiple sensors *compete* for the access to a collision channel in a non-cooperative game. Using the results from Chapter 3, we have established the structure of security policies and showed that, whenever a pair of Nash-equilibrium policies exists, there exists another equilibrium consisting of threshold policies attaining the same costs. In order to illustrate our results, we provided an example where the equilibrium policies were explicitly computed with the aid of the Modified Lloyd-Max algorithm. The main message here is the following altruistic result: even when sensors *do not cooperate* as members of a team, there is an incentive to share the communication resources among the agents.

## 6.2 Sensors with private and common observations

In the second part of this chapter we consider the Bayesian estimation problem illustrated by the block diagram of Fig. 6.3. Two sensors, each observing a private and a common random variable, decide whether to transmit their measurements to a remote estimator over a collision channel according to possibly stochastic communication policies. The communication constraint imposed by the collision channel is such that only one transmission may reach the estimator and, if more than one sensor transmits, a collision is declared. Upon observing the channel output, the estimator forms estimates of all the measured random variables. Our goal is to characterize communication policies that minimize a mean squared error criterion.

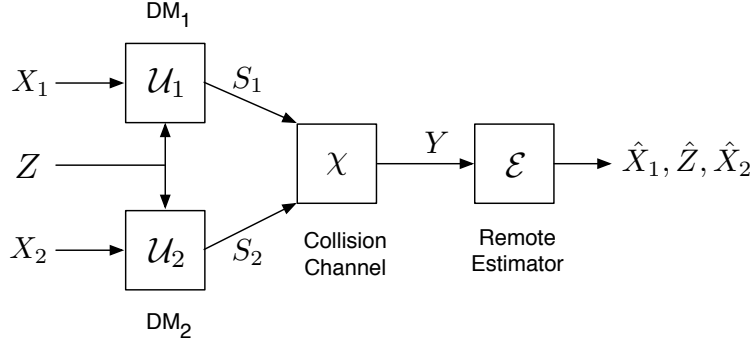


Figure 6.3: Schematic representation of decentralized estimation over a collision channel with private and common observations.

### 6.2.1 System model

Consider a random vector  $W$  with three components  $X_1, X_2$  and  $Z$  taking values on alphabets denoted by  $\mathbb{X}_1, \mathbb{X}_2$  and  $\mathbb{Z}$ , respectively. The vector  $W$  represents the state of the stochastic system that we wish to sense remotely over a wireless network. We assume that the probability density function of  $W$  has the following structure:

$$f_W(x_1, z, x_2) = f_Z(z) \cdot f_{X_1|Z}(x_1|z) \cdot f_{X_2|Z}(x_2|z), \quad (6.22)$$

for all  $(x_1, z, x_2) \in \mathbb{X}_1 \times \mathbb{Z} \times \mathbb{X}_2$ , i.e., the random variables  $X_1$  and  $X_2$  are conditionally independent given  $Z$ . The state  $W$  is jointly monitored by two sensors, which have access to partial observations: DM<sub>*i*</sub> has access to

$$W_i \stackrel{\text{def}}{=} (X_i, Z), \quad (6.23)$$

where  $X_i$  denotes its *private information* and  $Z$  denotes the *common information*,  $i \in \{1, 2\}$ . Note that, unless  $Z$  is deterministic,  $W_1$  and  $W_2$  are dependent, or

equivalently, they satisfy  $W_1 \not\perp W_2$ . From the perspective of  $DM_i$ , the fact that  $W_1$  and  $W_2$  may be dependent leads to the following structural differences relative to Chapter 3, where observations are assumed independent:

- The event that there is a concurring transmission may not be independent of  $W_i$ .
- A successful transmission made by  $DM_j$ ,  $j \neq i$ , may contain valuable side information for the estimation of  $X_i$ .

The decision maker  $DM_i$  observes the realization  $(x_i, z)$ , and must decide whether to communicate it to the remote estimator based solely on its measurement according to a communication policy  $\mathcal{U}_i$ ,  $i \in \{1, 2\}$ . The decision to communicate or not is represented by a binary random variable  $U_i \in \{0, 1\}$ , where  $U_i = 1$  denotes an attempt to communicate and  $U_i = 0$  denotes the decision to remain silent.

**Definition 6.5** (Communication Policies). *The communication policy for  $DM_i$  is a measurable function  $\mathcal{U}_i : \mathbb{X}_i \times \mathbb{Z} \rightarrow [0, 1]$  such that*

$$\mathbf{P}(U_i = 1 | X_i = x_i, Z = z) \stackrel{\text{def}}{=} \mathcal{U}_i(x_i, z), \quad i \in \{1, 2\}. \quad (6.24)$$

*The set of all communication policies for  $DM_i$  is denoted by  $\mathbb{U}_i$ . When a sensor decides to transmit, it sends its identification number, its private and common observations to the remote estimator. Otherwise, it remains silent. The channel input*



$S_i$  is determined as follows:

$$S_i = \begin{cases} (X_i, Z) & \text{if } U_i = 1 \\ \emptyset & \text{if } U_i = 0 \end{cases}, \quad i \in \{1, 2\}, \quad (6.25)$$

where the symbol  $\emptyset$  denotes a no-transmission.

**Definition 6.6** (Collision Channel). *Let the channel input alphabet be denoted by  $\mathbb{S}_1 \times \mathbb{S}_2$ , where  $\mathbb{S}_i = \{\mathbb{X}_i \times \mathbb{Z}\} \cup \{\emptyset\}$  and the channel output alphabet be denoted by  $\mathbb{Y} = \{1 \times \mathbb{S}_1\} \cup \{2 \times \mathbb{S}_2\} \cup \{\emptyset, \mathfrak{C}\}$ . Given the input random variables  $S_1$  and  $S_2$ , the collision channel output  $Y = \chi(S_1, S_2)$ , where  $\chi$  is given by the following deterministic map:*

$$\chi(s_1, s_2) \stackrel{\text{def}}{=} \begin{cases} (1, s_1) & \text{if } s_1 \neq \emptyset, s_2 = \emptyset \\ (2, s_2) & \text{if } s_1 = \emptyset, s_2 \neq \emptyset \\ \emptyset & \text{if } s_1 = \emptyset, s_2 = \emptyset \\ \mathfrak{C} & \text{if } s_1 \neq \emptyset, s_2 \neq \emptyset. \end{cases} \quad (6.26)$$

The symbol  $\mathfrak{C}$  denotes the occurrence of a collision between two simultaneous transmissions.

**Remark 6.3.** *The identification number allows the estimator to unambiguously determine the origin of every successful transmission.*

**Remark 6.4.** *There is a fundamental difference between the collision channel described above and the erasure channel commonly found in the literature of remote control and estimation, e.g. [13]: here there are two distinct symbols to represent*

*no-transmission and collision events. This provides the sensors the opportunity to use the no-transmission and collision symbols to transmit information using signaling [69].*

For any given pair of communication policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ , the estimator is interested in forming an estimate  $\hat{W}$  that minimizes a mean squared error criterion. Define  $\mathcal{J} : \mathbb{U}_1 \times \mathbb{U}_2 \rightarrow \mathbb{R}$  such that

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{E} \left[ (W - \hat{W})^T (W - \hat{W}) \right] \quad (6.27)$$

It is straightforward to show that for any two communication policies, the receiver that minimizes the cost in Eq. (6.27) forms a *minimum mean squared error* (MMSE) estimate of the random variable  $W$  given the observed channel output  $Y$ , i.e.,

$$\hat{W} \stackrel{\text{def}}{=} \mathcal{E}(Y), \quad \text{and} \quad \mathcal{E}(y) \stackrel{\text{def}}{=} \mathbf{E}[W|Y = y], \quad y \in \mathbb{Y}. \quad (6.28)$$

We are now ready to state the optimization problem for which the desired structural properties of optimal solutions will be established.

**Problem 6.1.** *Find a pair of policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$  that jointly minimize  $\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$  subject to the communication constraints imposed by the collision channel of Eq. (6.26) and the MMSE estimation rule of Eq. (6.28).*

## 6.2.2 The common information approach

We characterize solutions to Problem 6.1 using the common information approach proposed by Nayyar et al. in [70], which consists of expanding the cost by writing it as

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E} \left[ \mathbf{E} \left[ (W - \hat{W})^T (W - \hat{W}) | Z \right] \right] \quad (6.29)$$

and for each realization of the common information  $z$ , minimizing the conditional cost defined as

$$\mathcal{J}^z(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{E} \left[ (W - \hat{W})^T (W - \hat{W}) | Z = z \right] \quad (6.30)$$

over  $\mathbb{U}_1 \times \mathbb{U}_2$ . The idea is to look at the problem from the perspective of a fictitious agent called the *coordinator* that observes the common information and chooses the policies that each DM will use on their private information random variables [70].

This concept is illustrated in Fig. 6.4.

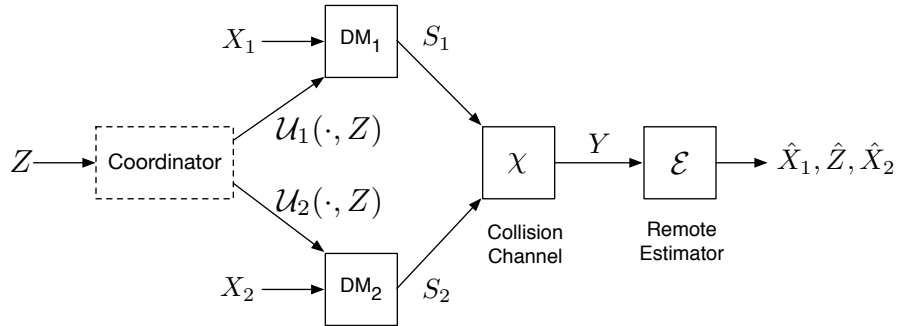


Figure 6.4: The common information approach applied to the collision channel with common and private observations.

### 6.2.3 Structural result

The main contribution in this section is to show that there are optimal policies that, for each realization of the common random variable, can be cast as a threshold policy on the private random variable. This generalizes the results of Chapter 3. We proceed to formally define the class of threshold policies on private information.

**Definition 6.7** (Threshold policy on private information). *A policy  $\mathcal{U}$  is a deterministic threshold policy on private information when, for every  $z \in \mathbb{Z}$ , there are constants  $a(z)$  and  $b(z) \in \bar{\mathbb{R}}$  for which the following holds:*

$$\mathcal{U}(x, z) = \begin{cases} 0 & \text{if } a(z) \leq x \leq b(z) \\ 1 & \text{otherwise} \end{cases}, \quad x \in \mathbb{X}. \quad (6.31)$$

*If  $a(z) = -b(z)$ ,  $z \in \mathbb{Z}$ , the threshold policy is called symmetric, otherwise it is called asymmetric.*

In other words, when a threshold policy on private information is used, the observations are transmitted over the channel if the private information is above or below specific thresholds, otherwise the sensor remains silent.

**Theorem 6.3.** *If the minimizer of  $\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$  exists, there is a pair of threshold policies on common information that attains the optimal cost.*

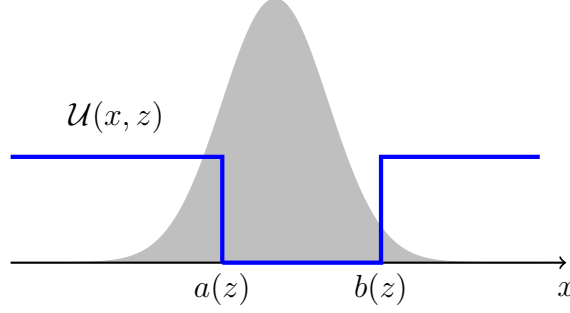


Figure 6.5: A threshold policy on private information for a fixed  $z \in \mathbb{Z}$ .

### Structure of the optimal estimator

Before proving our main result, we must develop a few auxiliary results. The first step is to characterize the structure of the MMSE estimator  $\mathcal{E}$  in Problem 6.1.

**Definition 6.8** (Class of admissible estimators). *An estimator  $\mathcal{E} : \mathbb{Y} \rightarrow \mathbb{R}^3$  is admissible if it has the following structure*

$$\mathcal{E}(y) = \begin{cases} [\hat{x}_{1\emptyset} \ \hat{z}_{\emptyset} \ \hat{x}_{2\emptyset}] & \text{if } y = \emptyset \\ [\hat{x}_{1\mathfrak{C}} \ \hat{z}_{\mathfrak{C}} \ \hat{x}_{2\mathfrak{C}}] & \text{if } y = \mathfrak{C} \\ [x_1 \ z \ \hat{f}_{2\emptyset}(z)] & \text{if } y = (1, x_1, z) \\ [\hat{f}_{1\emptyset}(z) \ z \ x_2] & \text{if } y = (2, x_2, z) \end{cases} \quad (6.32)$$

with  $\hat{x}_{i\emptyset}, \hat{x}_{i\mathfrak{C}} \in \mathbb{R}$ ,  $\hat{f}_{i\emptyset} : \mathbb{Z} \rightarrow \mathbb{R}$ ,  $i \in \{1, 2\}$  and  $\hat{z}_{\emptyset}, \hat{z}_{\mathfrak{C}} \in \mathbb{R}$ . The set of all admissible estimator is denoted by  $\mathbb{E}$ .

**Lemma 6.1.** *For any pair of policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ , the MMSE estimator belongs to the class of admissible estimators  $\mathbb{E}$ .*

*Proof.* For a fixed pair of policies  $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ , the MMSE estimator of  $W$  given the channel output  $Y = y$  is  $\mathcal{E}(y) = \mathbf{E}[W|Y = y]$ . Note that computing the conditional expectation for each possible value of  $y \in \mathbb{Y}$  is equivalent to computing the following quantities:

$$\mathcal{E}(\emptyset) = \mathbf{E}[W|U_1 = 0, U_2 = 0] \quad (6.33)$$

$$\mathcal{E}(\mathfrak{C}) = \mathbf{E}[W|U_1 = 1, U_2 = 1] \quad (6.34)$$

$$\mathcal{E}((1, x_1, z)) = \mathbf{E}[W|U_1 = 1, U_2 = 0, X_1 = x_1, Z = z] \quad (6.35)$$

$$\mathcal{E}((2, x_2, z)) = \mathbf{E}[W|U_1 = 0, U_2 = 1, X_2 = x_2, Z = z]. \quad (6.36)$$

Let the following conditional expectations be denoted by

$$\hat{x}_{i\emptyset} \stackrel{\text{def}}{=} \mathbf{E}[X_i|U_1 = 0, U_2 = 0] \in \mathbb{R}, \quad i \in \{1, 2\} \quad (6.37)$$

and

$$\hat{z}_{\emptyset} \stackrel{\text{def}}{=} \mathbf{E}[Z|U_1 = 0, U_2 = 0] \in \mathbb{R}. \quad (6.38)$$

Similarly,

$$\hat{x}_{i\mathfrak{C}} \stackrel{\text{def}}{=} \mathbf{E}[X_i|U_1 = 1, U_2 = 1] \in \mathbb{R}, \quad i \in \{1, 2\} \quad (6.39)$$

and

$$\hat{z}_{\mathfrak{C}} \stackrel{\text{def}}{=} \mathbf{E}[Z|U_1 = 1, U_2 = 1] \in \mathbb{R}. \quad (6.40)$$

When the estimator receives  $Y = (i, x_i, z)$ , the conditional expectations of  $X_i$  and  $Z$  are

equal to  $x_i$  and  $z$ , respectively. Finally, for  $i \neq j$  we have:

$$\begin{aligned}
\hat{f}_{i\emptyset}(z) &\stackrel{\text{def}}{=} \mathbf{E}[X_i|Y = (j, x_j, z)] \\
&= \mathbf{E}[X_i|U_i = 0, U_j = 1, X_j = x_j, Z = z] \\
&= \mathbf{E}[X_i|U_i = 0, Z = z],
\end{aligned} \tag{6.41}$$

where the last equality follows from the conditional independence of  $X_1$  and  $X_2$  given  $Z$ . □

In the proof of Theorem 6.3 we allow the estimator to be arbitrarily fixed within the class of maps which have the same structure as the MMSE estimator. The idea is to show that there exists a pair of deterministic threshold policies on private information for any fixed estimator  $\mathcal{E}$  with this structure.

### Single decision maker subproblem

From now on, we assume that the estimator is arbitrarily fixed in the class of admissible estimators  $\mathbb{E}$  and does not depend on the communication policies. In order to apply the person-by-person optimality approach, we first need to express the cost from the perspective of a single decision maker, assuming that the communication policy of the other sensor is arbitrarily fixed. For  $i, j \in \{1, 2\}$  such that  $i \neq j$ , we can write the conditional cost for  $\text{DM}_i$  for any fixed choice of  $\mathcal{U}_j \in \mathbb{U}_j$  as follows:

$$\mathcal{J}^z(\mathcal{U}_i, \mathcal{U}_j) = \mathbf{E} \left[ (X_i - \hat{X}_i)^2 + (Z - \hat{Z})^2 | Z = z \right] + \rho_j^z \mathbf{P}(U_i = 1 | Z = z) + \theta_j^z, \tag{6.42}$$

where  $\rho_j^z$  and  $\theta_j^z$  are given by:

$$\rho_j^z \stackrel{\text{def}}{=} \mathbf{E} \left[ (X_j - \hat{X}_j)^2 | U_i = 1, Z = z \right] - \mathbf{E} \left[ (X_j - \hat{X}_j)^2 | U_i = 0, Z = z \right] \quad (6.43)$$

and

$$\theta_j^z \stackrel{\text{def}}{=} \mathbf{E} \left[ (X_j - \hat{X}_j)^2 | U_i = 0, Z = z \right]. \quad (6.44)$$

**Proposition 6.1.** *Let  $i, j \in \{1, 2\}$  such that  $i \neq j$ . For any fixed  $\mathcal{U}_j \in \mathbb{U}_j$  and  $\mathcal{E} \in \mathbb{E}$ , the values of  $\rho_j^z$  and  $\theta_j^z$  are constant in  $\mathcal{U}_i \in \mathbb{U}_i$ .*

*Proof.* The key idea to prove this result is that the following Markov chain relationship holds

$$X_1, U_1 \leftrightarrow Z \leftrightarrow X_2, U_2, \quad (6.45)$$

i.e.,  $X_1, U_1$  and  $X_2, U_2$  are conditionally independent given  $Z$ . Using the law of total expectation and Eq. (6.45), we have

$$\begin{aligned} \rho_j^z &= \mathbf{E}[(X_j - \hat{x}_{j\mathcal{E}})^2 | U_j = 1, Z = z] \mathbf{P}(U_j = 1 | Z = z) \\ &\quad - \mathbf{E}[(X_j - \hat{x}_{j\emptyset})^2 | U_j = 0, Z = z] \mathbf{P}(U_j = 0 | Z = z) \\ &\quad + \mathbf{E}[(X_j - \hat{f}_{j\emptyset}(z))^2 | U_j = 0, Z = z] \mathbf{P}(U_j = 0 | Z = z). \end{aligned} \quad (6.46)$$

Similarly,

$$\theta_j^z = \mathbf{E}[(X_j - \hat{x}_{j\emptyset})^2 | U_j = 0, Z = z] \mathbf{P}(U_j = 0 | Z = z). \quad (6.47)$$

Therefore,  $\rho_j^z, \theta_j^z$  do not depend on the choice of  $\mathcal{U}_i$ . □



This means that we can define a new equivalent cost  $\mathcal{J}_j^z : \mathbb{U}_i \rightarrow \mathbb{R}$  such that

$$\mathcal{J}_j^z(\mathcal{U}_i) \stackrel{\text{def}}{=} \mathbf{E} \left[ (W_i - \hat{W}_i)^T (W_i - \hat{W}_i) | Z = z \right] + \rho_j^z \mathbf{P}(U_i = 1 | Z = z), \quad (6.48)$$

which has the interpretation that, from the perspective of  $\text{DM}_i$ , the cost has two components: a mean square estimation error of the observed random vector  $W_i = (X_i, Z)$  and a communication cost that accounts for loss in estimation of the private information of  $\text{DM}_j$ . The single decision maker subproblem from the perspective of  $\text{DM}_i$  is to minimize  $\mathcal{J}_j^z$  in Eq. (6.48) over  $\mathbb{U}_i$  assuming that the policy  $\mathcal{U}_j$  used by  $\text{DM}_j$  is fixed.

*Proof of Theorem 6.3 .* Assume that the estimator  $\mathcal{E} \in \mathbb{E}$ , i.e., has the structure in Eq. (6.32). Let  $i, j \in \{1, 2\}$  such that  $i \neq j$ . For every realization  $z \in \mathbb{Z}$  and any fixed  $\mathcal{U}_j \in \mathbb{U}_j$ , we will show that there exists a threshold policy on private information that minimizes  $\mathcal{J}_j^z(\mathcal{U}_i)$ . The conditional cost from the perspective of  $\text{DM}_i$  in Eq. (6.48) can be further expanded and expressed as

$$\begin{aligned} \mathcal{J}_j^z(\mathcal{U}_i) &= \mathbf{E} \left[ \beta_j^z \{ (X_i - \hat{x}_{i\mathcal{E}})^2 + (z - \hat{z}_{\mathcal{E}})^2 \} + \rho_j^z | U_i = 1, Z = z \right] \times \mathbf{P}(U_i = 1 | Z = z) \\ &\quad + \left( \mathbf{E} \left[ (1 - \beta_j^z) \{ (X_i - \hat{x}_{i\emptyset})^2 + (z - \hat{z}_{\emptyset})^2 \} | U_i = 0, Z = z \right] \right. \\ &\quad \left. + \mathbf{E} \left[ \beta_j^z (X_i - \hat{f}_{i\emptyset}(z))^2 | U_i = 0, Z = z \right] \right) \times \mathbf{P}(U_i = 0 | Z = z), \end{aligned} \quad (6.49)$$

where

$$\beta_j^z \stackrel{\text{def}}{=} \mathbf{P}(U_j = 1 | Z = z). \quad (6.50)$$

The conditional distribution of the private information  $X_i$  given  $Z = z$  and  $U_i = 1$  is

$$f_{X_i|U_i,Z}(x|1, z) = \frac{\mathcal{U}_i(x, z)f_{X_i|Z}(x|z)}{\mathbf{P}(U_i = 1|Z = z)}. \quad (6.51)$$

Similarly, given  $Z = z$  and  $U_i = 0$ , we have

$$f_{X_i|U_i,Z}(x|0, z) = \frac{(1 - \mathcal{U}_i(x, z))f_{X_i|Z}(x|z)}{\mathbf{P}(U_i = 0|Z = z)}. \quad (6.52)$$

Defining the following polynomials:

$$\mathcal{P}_0(x) \stackrel{\text{def}}{=} (1 - \beta_j^z) [(x - \hat{x}_{i\emptyset})^2 + (z - \hat{z}_\emptyset)^2] + \beta_j^z (x - \hat{f}_\emptyset(z))^2 \quad (6.53)$$

and

$$\mathcal{P}_1(x) \stackrel{\text{def}}{=} \beta_j^z (x - \hat{x}_{i\mathbf{e}})^2 + (z - \hat{z}_{\mathbf{e}})^2 + \rho_j^z, \quad (6.54)$$

Eq. (6.49) becomes:

$$\mathcal{J}_j^z(\mathcal{U}_i) = \int_{\mathbb{X}_i} (\mathcal{P}_1(x) - \mathcal{P}_0(x))\mathcal{U}_i(x, z)f_{X_i|Z}(x|z)dx + \int_{\mathbb{X}_i} \mathcal{P}_0(x)f_{X_i|Z}(x|z)dx. \quad (6.55)$$

For a fixed  $z \in \mathbb{Z}$  and  $\mathcal{U}_j \in \mathbb{U}_j$ , our optimization problem is:

$$\begin{aligned} & \underset{\mathcal{U}_i \in \mathbb{U}_i}{\text{minimize}} && \mathcal{J}_j^z(\mathcal{U}_i) \\ & \text{subject to} && 0 \leq \mathcal{U}_i(x, z) \leq 1, \quad x \in \mathbb{X}_i. \end{aligned} \quad (6.56)$$

The cost is linear in  $\mathcal{U}_i$ , and is minimized by the following policy:

$$\mathcal{U}_i^*(x, z) = \begin{cases} 0 & \text{if } \mathcal{P}_1(x) - \mathcal{P}_0(x) \geq 0 \\ 1 & \text{otherwise.} \end{cases} \quad (6.57)$$

Since

$$\nabla_x^2(\mathcal{P}_1(x) - \mathcal{P}_0(x)) = \beta_j^z - 1 \leq 0, \quad (6.58)$$

the set of points  $x \in \mathbb{R}$  such that  $\mathcal{P}_1(x) - \mathcal{P}_0(x) \geq 0$  is convex and can be represented by a closed interval, i.e.,

$$\mathcal{U}_i^*(x, z) = \begin{cases} 0 & \text{if } a(z) \leq x \leq b(z) \\ 1 & \text{otherwise,} \end{cases} \quad (6.59)$$

where  $a(z), b(z) \in \bar{\mathbb{R}}$ . This structure must hold for any estimator  $\mathcal{E} \in \mathbb{E}$ . From Lemma 1, the MMSE estimator belongs to  $\mathbb{E}$  and, therefore, the threshold structure with private information is also optimal when  $\mathcal{E}$  is given by Eq. (6.28). Finally, for any given  $z \in \mathbb{Z}$ , from any given pair of person-by-person optimal solutions for  $\mathcal{J}^z(\mathcal{U}_1, \mathcal{U}_2)$  we can find has a pair of threshold policies that attains the optimal cost  $\mathcal{J}^{z*}$ . Since every pair of team-optimal solutions is also person-by-person optimal, if a team-optimal solution exists, we can construct a pair of threshold policies on private information that attains the optimal cost  $\mathcal{J}^*$ .  $\square$

**Remark 6.5.** *When  $\mathbb{X}_1, \mathbb{Z}$  and  $\mathbb{X}_2$  are finite alphabets, a team-optimal solution is guaranteed to exist. When either one of the alphabets is of infinite cardinality, the question of existence of team-optimal solutions becomes rather technical and is*

beyond the scope of this dissertation. We refer the readers to [71] for a comprehensive treatment on this topic.

#### 6.2.4 Computation of person-by-person optimal policies

The structural result of Theorem 6.3 is useful because we can constrain the search for optimal solutions over a smaller strategy space. In particular, when  $\mathbb{Z}$  is a finite alphabet, the optimization can be performed over a finite dimensional space, rather than an infinite dimensional one. In this section, we use the structure to derive expressions for the computation of person-by-person optimal policies for  $\text{DM}_1$  and  $\text{DM}_2$ . For a fixed pair of threshold policies on private information  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , we obtain expressions for the computation of the optimal estimates. For a given threshold policy on common information  $\mathcal{U}_i$ , we denote the *no-transmission sets* by

$$\mathbb{A}_i(z) \stackrel{\text{def}}{=} \{x \in \mathbb{X}_i \mid a_i(z) \leq x \leq b_i(z)\} \quad i \in \{1, 2\}. \quad (6.60)$$

The MMSE estimate of  $X_i$  when the estimator observes a no-transmission symbol is

$$\begin{aligned} \hat{x}_{i\emptyset} &= \mathbf{E}[X_i | Y = \emptyset] \\ &= \mathbf{E}[\mathbf{E}[X_i | U_i = 0, U_j = 0, Z]] \\ &= \mathbf{E}[\mathbf{E}[X_i | U_i = 0, Z]] \\ &= \mathbf{E}[\hat{f}_{i\emptyset}(Z)]. \end{aligned} \quad (6.61)$$

Recall from Eq. (6.41) that  $\hat{f}_{i\emptyset}(z)$  is the optimal estimate of the private information of  $\text{DM}_i$  when the estimator receives  $Y = (j, x_j, z)$ ,  $j \neq i$ . The interpretation is that  $\hat{f}_{i\emptyset}(z)$  is a refinement of the estimate of  $X_i$  given that  $\text{DM}_i$  sent a no-transmission symbol in the presence of common information provided by  $Z = z$ . Using the structure of the optimal policies, we can write

$$\hat{f}_{i\emptyset}(z) = \frac{\int_{a_i(z)}^{b_i(z)} x \cdot f_{X_i|Z}(x|z) dx}{\int_{a_i(z)}^{b_i(z)} f_{X_i|Z}(x|z) dx}, \quad z \in \mathbb{Z}. \quad (6.62)$$

Repeating these steps for the case when the estimator observes a collision symbol, we have:

$$\begin{aligned} \hat{x}_{i\mathfrak{C}} &= \mathbf{E}[X_i|Y = \mathfrak{C}] \\ &= \mathbf{E}[\mathbf{E}[X_i|U_i = 1, U_j = 1|Z]] \\ &= \mathbf{E}[\mathbf{E}[X_i|U_i = 1, Z]]. \end{aligned} \quad (6.63)$$

Note that the estimator never observes that  $\text{DM}_i$  attempted to transmit along with the common information  $Z$ . However, we define the auxiliary estimate functions  $\hat{f}_{i\mathfrak{C}} : \mathbb{Z} \rightarrow \mathbb{R}$  such that

$$\hat{f}_{i\mathfrak{C}}(z) \stackrel{\text{def}}{=} \mathbf{E}[X_i|U_i = 1, Z = z], \quad (6.64)$$

which because of the thresholds structure of the optimal policies, can be written as

$$\hat{f}_{i\mathfrak{C}}(z) = \frac{\int_{\mathbb{R} \setminus [a_i(z), b_i(z)]} x \cdot f_{X_i|Z}(x|z) dx}{\int_{\mathbb{R} \setminus [a_i(z), b_i(z)]} f_{X_i|Z}(x|z) dx}, \quad z \in \mathbb{Z}. \quad (6.65)$$

Finally, the estimator averages these auxiliary functions to obtain:

$$\hat{x}_{i\mathbf{e}} = \mathbf{E} \left[ \hat{f}_{i\mathbf{e}}(Z) \right]. \quad (6.66)$$

The estimates above are computed for the cases when communication either fails as a result of a collision between two simultaneous transmissions or when the channel is idle. When exactly one sensor successfully transmits its measurements, i.e.,  $Y = (j, x_j, z)$ , the common information aids the estimation of  $X_i$ , acting as side information.

The expression for the estimate of the common information in the case of a collision is derived from

$$\hat{z}_{\mathbf{e}} = \mathbf{E}[Z|U_1 = 1, U_2 = 1], \quad (6.67)$$

which is computed as follows:

$$\hat{z}_{\mathbf{e}} = \frac{\int_{\mathbb{Z}} z \cdot \beta_1^z \beta_2^z f_Z(z) dz}{\int_{\mathbb{Z}} \beta_1^z \beta_2^z f_Z(z) dz}. \quad (6.68)$$

Finally, we compute the optimal estimate of the common observation in the case of a collision from

$$\hat{z}_{\emptyset} = \mathbf{E}[Z|U_1 = 0, U_2 = 0], \quad (6.69)$$

which corresponds to:

$$\hat{z}_{\emptyset} = \frac{\int_{\mathbb{Z}} z \cdot (1 - \beta_1^z)(1 - \beta_2^z) f_Z(z) dz}{\int_{\mathbb{Z}} (1 - \beta_1^z)(1 - \beta_2^z) f_Z(z) dz}. \quad (6.70)$$

The natural iterative procedure to search for a person-by-person optimal solution is to alternately optimize  $\mathcal{U}_1$  and  $\mathcal{U}_2$  keeping one of them constant until a fixed point is found. There are no guarantees that this procedure will converge. For  $i, j \in \{1, 2\}$  such that  $i \neq j$ , an iteration of the numerical procedure consists of the following steps:

- **Step 0**: Choose a threshold policy on private information  $\mathcal{U}_j \in \mathbb{U}_j$ . Fixing the communication policy of DM<sub>*j*</sub> corresponds to fixing the sets  $\mathbb{A}_j(z)$ ,  $z \in \mathbb{Z}$ .
- **Step 1**: Compute  $\hat{f}_{j\emptyset}(z)$  and  $\hat{f}_{j\mathfrak{C}}(z)$ ,  $z \in \mathbb{Z}$ , according to Eqs. (6.62) and (6.65), then compute  $\hat{x}_{j\emptyset}$  and  $\hat{x}_{j\mathfrak{C}}$  using Eqs. (6.61) and (6.66).
- **Step 2**: Compute the conditional probabilities of transmission, the communication costs and off-set terms according to:

$$\beta_j^z = 1 - \int_{a_j(z)}^{b_j(z)} f_{X_j|Z}(x|z) dx, \quad (6.71)$$

$$\begin{aligned} \rho_j^z &= \int_{a_j(z)}^{b_j(z)} (x - \hat{f}_{j\emptyset}(z))^2 f_{X_j|Z}(x|z) dx \\ &\quad - \int_{a_j(z)}^{b_j(z)} (x - \hat{x}_{j\emptyset})^2 f_{X_j|Z}(x|z) dx \\ &\quad + \int_{\bar{\mathbb{R}} \setminus [a_j(z), b_j(z)]} (x - \hat{x}_{j\mathfrak{C}})^2 f_{X_j|Z}(x|z) dx \end{aligned} \quad (6.72)$$

and

$$\theta_j^z = \int_{a_j(z)}^{b_j(z)} (x - \hat{x}_{j\emptyset})^2 f_{X_j|Z}(x|z) dx, \quad z \in \mathbb{Z}. \quad (6.73)$$

- **Step 3:** Provided that for all  $z \in \mathbb{Z}$ ,  $\beta_j^z, \rho_j^z$  and  $\theta_j^z$  are fixed; that  $\hat{x}_{i\emptyset}, \hat{x}_{i\mathfrak{C}}, \hat{z}_\emptyset$  and  $\hat{z}_\mathfrak{C}$  are given by Eqs. (6.61), (6.66), (6.68) and (6.70), solve the problem in Eq. (6.74):

$$\begin{aligned} & \underset{\mathcal{U}_i \in \mathbb{U}_i}{\text{minimize}} \quad \mathbf{E} \left[ \mathcal{J}^Z(\mathcal{U}_i, \mathcal{U}_j) \right] \\ & \text{subject to} \quad a_i(z) \leq b_i(z), \quad z \in \mathbb{Z}, \end{aligned} \quad (6.74)$$

with variables  $a_i(z), b_i(z), \hat{f}_{i\emptyset}(z)$  and  $\hat{f}_{i\mathfrak{C}}(z)$ ,  $z \in \mathbb{Z}$ ; where where  $\mathcal{J}^z(\mathcal{U}_i, \mathcal{U}_j)$  is given by the expression in Eq. (6.75):

$$\begin{aligned} \mathcal{J}^z(\mathcal{U}_i, \mathcal{U}_j) & \stackrel{\text{def}}{=} \int_{a_i(z)}^{b_i(z)} \left[ \beta_j^z [(x - \hat{x}_{i\mathfrak{C}})^2 + (z - \hat{z}_\mathfrak{C})^2] + \rho_j^z \right] f_{X_i|Z}(x|z) dx \\ & + \int_{\mathbb{R} \setminus [a_i(z), b_i(z)]} (1 - \beta_j^z) [(x - \hat{x}_{i\emptyset})^2 + (z - \hat{z}_\emptyset)^2] f_{X_i|Z}(x|z) dx \\ & + \int_{\mathbb{R} \setminus [a_i(z), b_i(z)]} \beta_j^z (x - \hat{f}_{i\emptyset}(z))^2 f_{X_i|Z}(x|z) dx + \theta_j^z. \end{aligned} \quad (6.75)$$

- **Step 4:** Fix the policy of  $\text{DM}_i$  according to the solution of the problem in Eq. (6.74) and follow the steps above for the optimization of  $\mathcal{U}_j$ . Repeat steps 1 through 4 until the cost cannot be further reduced and a fixed point is found.

**Remark 6.6.** *The optimization problem in Eq. (6.74) is defined over a finite dimensional space if  $\mathbb{Z}$  is a finite alphabet and may be solved using nonlinear programming solvers. The solution of this problem when  $Z$  is a continuous random variable is a*



*topic for future investigation.*

### 6.2.5 Example

Assume that  $X_1$ ,  $X_2$  and  $Z$  are mutually independent. The random variables  $X_1, X_2 \sim \mathcal{N}(0, 1)$  and  $Z \in \{-1, +1\}$  is distributed according to:

$$Z = \begin{cases} -1 & \text{with probability } 1 - p \\ +1 & \text{with probability } p. \end{cases} \quad (6.76)$$

Implementing the numerical procedure outlined in the previous section for different values of the probability  $p$ , we obtain the pairs of person-by-person optimal solutions shown in Table 6.2. As the parameter  $p$  approaches 0.5, the variance of the the common information  $Z$  increases to 1. The dependence between  $W_1$  and  $W_2$  can be measured using the RV coefficient [72], which in the case of this example is given by

$$\text{RV}(W_1, W_2) = \frac{(4p(1-p))^2}{1 + (4p(1-p))^2}. \quad (6.77)$$

Note that when  $p = 0.5$ , the RV coefficient between  $W_1$  and  $W_2$  achieves its maximum value. However, as the observations become more dependent, the variance of  $Z$  also increases, causing the overall mean squared estimation error to be larger. That explains why the minimum cost in Table 6.2 increases with  $p$ , even though the observations become more dependent.

The structural result from Theorem 6.3 states that the optimal communica-

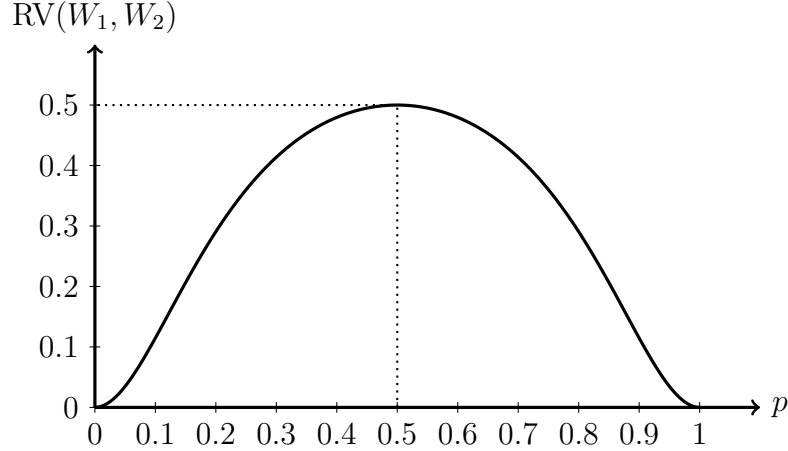


Figure 6.6: The RV coefficient between  $W_1$  and  $W_2$  in our problem.  $RV(W_1, W_2)$  is a measure of statistical dependence that generalizes the correlation coefficient for scalar random variables.

tion policies are event-based. In general, the person-by-person optimal policies in Table 6.2 show us that the thresholds used to define these event-based policies are asymmetric. This is attributed to the ability of the DMs to encode information in the collision and no-transmission symbols in order to further reduce the cost. We also note that the cost of using a time-sharing policy, in which the sensors take turns transmitting and remaining silent and thus, avoiding collisions, is equal to 1. In the worst case scenario, when  $p = 0.5$ , using the person-by-person optimal threshold policy in the last row of Table 6.2 shows an improvement of 21.6% over the scheduling policy. Therefore, this approach leads to a considerable reduction in cost over pure collision avoidance protocols. We also observe that  $DM_1$  employs a combination of scheduling and event-based policies. This shows that the common observation acts as a switch that schedules the transmission of  $DM_1$ : when  $Z = +1$ , it always transmits regardless of its private information; when  $Z = -1$ , it uses an asymmetric threshold policy.

$p$	$\mathcal{U}_1$	$\mathcal{U}_2$	$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$
0	[+0.0977, +5.3593] –	[+0.0976, +5.3593] –	0.5397
0.1	[+0.1006, +5.3386] $\emptyset$	[+0.0939, +5.3653] [–1.3931, + $\infty$ ]	0.5866
0.2	[+0.1123, +5.2634] $\emptyset$	[+0.0803, +5.3975] [–1.3625, + $\infty$ ]	0.6349
0.3	[+0.1388, +5.1130] $\emptyset$	[+0.0517, +5.4920] [–1.3207, + $\infty$ ]	0.6844
0.4	[+0.1872, +4.8772] $\emptyset$	[0.0032, +5.7099] [–1.2664, + $\infty$ ]	0.7344
0.5	[+0.2621, +4.5763] $\emptyset$	[–0.0668, +6.1340] [–1.2038, + $\infty$ ]	0.7838

Table 6.2: Person-by-person optimal policies for  $DM_1$  and  $DM_2$  in Problem 6.1 where the measurements are independently distributed as  $X_1, X_2 \sim \mathcal{N}(0, 1)$  and  $Z \sim \mathcal{B}(p)$ . Each policy is represented by a pair of no-transmission intervals  $\mathbb{A}_i(-1)$  and  $\mathbb{A}_i(+1)$ .

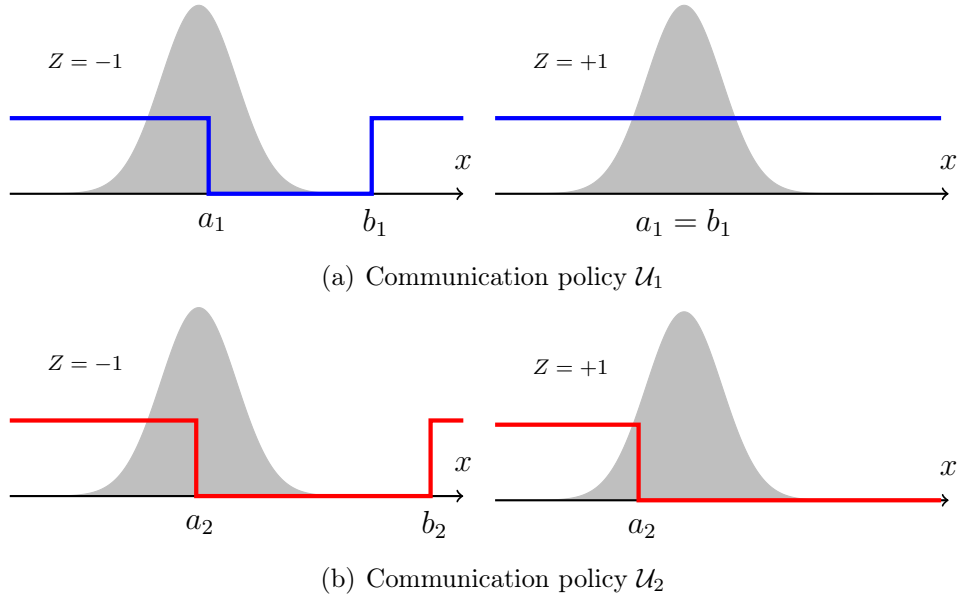


Figure 6.7: Structure of person-by-person optimal communication policies from in Table 6.2.

### 6.2.6 Summary

We took a first step to generalize the estimation problem over the collision channel to the case of dependent observations by considering the case where the

sensors observe common and private random variables. Using a combination of the common information and person-by-person optimality approaches, we showed that the search for team-optimal policies can be performed within the class of deterministic threshold on private information communication policies. Using this result, we obtained expressions for the MMSE estimates, outline an iterative procedure to compute person-by-person optimal policies and provide a numerical example. The results contained in this section may be useful for solving the dynamic case with feedback, when common information in the form of acknowledgements is available to the sensors prior to transmission.

## Chapter 7: Conclusion and future research

### 7.1 Conclusion

Estimation and control over wireless networks has been a very active research area with potential applications of great economic impact, e.g. cyber-physical systems. Within the context of remote estimation, systems with a single sensor are very well understood for a wide range of channel models and communication constraints. However, little is known when the sensing task is distributed among multiple sensors sharing a common communication medium. Part of the difficulties in obtaining results for the multiple sensor scenarios lies on finding a model that captures the essence of the interference phenomenon while retaining the overall tractability of the research problem. Another aspect is to use the right formalism to solve the problem and obtaining structural results leading to design principles for real systems. This dissertation proposes a new class of canonical decentralized remote estimation problems which are based on an abstraction for the wireless medium known as the collision channel. We derive structural results for the optimal solutions to these problems using the formalism of team decision theory.

The first part of the dissertation focuses on the problem of estimating two independent continuous random variables observed by two different sensors commu-

nicating with a fusion center over a collision channel. For a minimum mean squared estimation error criterion, we show that there exist team-optimal strategies where each sensor employs a threshold policy. Moreover, this is independent of the distribution of the observations irrespective of assumptions on modality and symmetry of the probability density functions. This result can be extended to vector observations and, under an additional assumption on the channel, to any number of sensors. Consequently, the existence of optimal policies with an event-based structure is a result of practical significance, because it can be applied to a wide class of systems where the network is modeled by a collision channel without any assumptions on collision avoidance protocols such as sensor scheduling.

The second part of the dissertation focuses on the problem of estimating two independent discrete random variables observed by two different sensors communicating with a fusion center over a collision channel. Using two criteria involving the probability of estimation error, we also show the existence of team-optimal strategies with a particular form of event-based structure characteristic to problems with discrete observations. These results are also independent of the distributions and are valid for any number of sensors, under an additional assumption on the channel. In our analysis, the proof of the structural result involves the minimization of a concave function, which is an evidence of the inherent complexity of such team decision problems. However, we are still able to solve the problem exactly without using any approximation techniques.

In the third part of the dissertation, the assumptions on the cooperation among sensors is relaxed, showing that similar structural results can also be obtained for

systems with one or more selfish sensors. Finally, the assumption on the independence is lifted by introducing the observation of a common random variable in addition to the private observations of each sensor. The structural result obtained may lead to characterizations of team-optimal policies for a general correlation structure between the observed random variables.

## 7.2 Future research

There are many opportunities for future research stemming from the problems posed in this dissertation. The most important question at this point is how to extend the problem formulation and the structural results from a one-shot to the sequential case. This is a challenging problem that would have many important consequences in estimation and control theory, generalizing the notion of Kalman filtering with intermittent observations to deal with collisions, as well as the structural result of Lipsa and Martins [37] to the multi-sensor case. The answer to this question may have connections with sequential one-bit quantization schemes such as sigma-delta modulators.

Another important question is to obtain ways to verify if a person-by-person optimal solution obtained for Problem 3.1 is in fact team-optimal or not. We suggest two possible ways to do this: the first would be to show if a set of technical conditions known as *stationarity conditions* [12] hold in our problem. If such conditions are satisfied, person-by-person optimality implies team-optimality. Another way that this can be done is to obtain a bound on estimation error akin to Cramer-Rao lower

bounds, and verify if a particular pair of person-by-person optimal policies achieves it. In this case, the achieving pair of person-by-person optimal policies would be team-optimal.

A natural extension of the model presented here is to allow the channel to support  $n$  users and a collision event when  $m < n$  simultaneous transmissions are made. Another open question is to determine the structure of optimal policies when observations are correlated instead of independent. Another interesting problem occurs in the  $n$  sensor case, when the assumption on the ability of the remote estimator to identify colliding nodes is removed. In that case, there is ambiguity on who transmitted a packet when a collision is observed. The problem formulation of Chapter 5 can also be extended to the case of sequential estimation of discrete Markov sources over the collision channel with feedback in the form of acknowledgements. On the numerical aspects of the problem with common and private observations in Chapter 6, more work is needed on how to efficiently perform the optimization procedure when  $Z$  is a continuous random variable. The results contained in that section may be useful for solving the dynamic case with feedback, when common information in the form of acknowledgements is available to the sensors at each time.

Finally, there has been an increasing interest in designing control systems robust to so-called *cyber-attacks*. Jamming can be seen as a denial-of-service attack on a wireless channel, in which the attacker blocks the communication between the legitimate parties by congesting the network with random data or injecting extra noise in the channel [41]. Problems of estimation in the presence of an intelligent jammer have been studied in the context of the Gaussian channel, e.g. [73, 74].



We argue that the collision channel could also be used to study estimation and control in the presence of malicious jammers. A new interesting scenario aligned to the system models in this dissertation is: a legitimate DM communicates with a remote estimator through a network modeled by a collision channel shared with a jammer. The jammer has access to side information about the DM's observation and chooses its actions strategically. The goal of the sensor-estimator pair is to choose a communication policy to minimize the mean squared estimation error and the jammer's purpose is to perform an attack on the channel with the intent to maximize it.

## Appendix A: Continuity and strong duality results

### A.1 Auxiliary results on continuity

This Appendix includes two propositions that state important continuity properties of the costs for Problems 3.1 and 3.2. In particular, they state that when evaluated for deterministic threshold policies, the cost varies continuously with respect to the thresholds. This observation is key to show the existence of an optimum in Theorems 3.1 and 3.2.

**Proposition A.1.** *Let  $(\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2)$  be a given pair of deterministic threshold policies characterized by thresholds  $\bar{a}_1, \bar{b}_1, \bar{a}_2$  and  $\bar{b}_2$  in  $\bar{\mathbb{R}}$ . Let  $\{(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)})\}_{n=0}^\infty$  be a given sequence of policies with associated thresholds  $\{\check{a}_{1,(n)}\}_{n=0}^\infty, \{\check{b}_{1,(n)}\}_{n=0}^\infty, \{\check{a}_{2,(n)}\}_{n=0}^\infty$  and  $\{\check{b}_{2,(n)}\}_{n=0}^\infty$ . If  $\lim_{n \rightarrow \infty} \check{a}_{1,(n)} = \bar{a}_1, \lim_{n \rightarrow \infty} \check{b}_{1,(n)} = \bar{b}_1, \lim_{n \rightarrow \infty} \check{a}_{2,(n)} = \bar{a}_2$  and  $\lim_{n \rightarrow \infty} \check{b}_{2,(n)} = \bar{b}_2$  holds then the following also holds:*

$$\lim_{n \rightarrow \infty} \mathcal{J}(\check{\mathcal{U}}_{1,(n)}, \check{\mathcal{U}}_{2,(n)}) = \mathcal{J}(\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2). \quad (\text{A.1})$$

**Proposition A.2.** *Let  $\bar{\mathcal{W}}$  be a deterministic threshold policy characterized by thresholds  $\bar{a}$  and  $\bar{b}$  in  $\bar{\mathbb{R}}$ , with  $\bar{a} \leq \bar{b}$ . Let  $\mathcal{W}_{(n)}$  be a sequence of policies for problem 3.2 with associated thresholds  $a_{(n)}$  and  $b_{(n)}$  that satisfy  $\lim_{n \rightarrow \infty} a_{(n)} = \bar{a}$  and  $\lim_{n \rightarrow \infty} b_{(n)} = \bar{b}$ .*

The following holds:

$$\lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{U}_{(n)}) = \mathcal{J}(\bar{\mathcal{U}}). \quad (\text{A.2})$$

## A.2 Strong duality

The purpose of this Appendix is to provide a proof that, under the conditions of Lemma 3.1, strong duality holds for the problem in Eq. (3.30). This is important since, as opposed to their finite dimensional counterparts, strong duality for infinite dimensional linear programs does not necessarily hold. Our proof will hinge on a result due to Borwein and Lewis [75] adapted by Limber and Goodrich in [76]. The constraint qualification under which strong duality holds involves the concepts of quasi interior (**qi**) and quasi-relative interior (**qri**) of a set. The relative interior of a set is denoted by **ri**.

**Theorem A.1** (Limber and Goodrich [76] - Theorem 4.1). *Let  $\mathbb{G}$  be a Banach space,  $\mathcal{J} : \mathbb{G} \rightarrow (-\infty, +\infty]$  a convex functional,  $\mathcal{A} : \mathbb{G} \rightarrow \mathbb{R}^n$  a linear continuous map, and  $\mathbb{G}_c \subset \mathbb{G}$  a closed convex set. Let  $b \in \mathbb{R}^n$  be a fixed vector such that  $b \in \mathbf{ri} \mathcal{A}(\mathbb{G}_c)$  and suppose that*

$$p^* = \inf \{ \mathcal{J}(\mathcal{G}) \mid \mathcal{A}(\mathcal{G}) = b, \mathcal{G} \in \mathbb{G}_c \} \quad (\text{A.3})$$

is finite. If

$$d^* = \sup_{\nu \in \mathbb{R}^n} \left\{ b^\top \nu + \inf_{\mathcal{G} \in \mathbb{G}_c} \{ \mathcal{J}(\mathcal{G}) - \nu^\top \mathcal{A}(\mathcal{G}) \} \right\}, \quad (\text{A.4})$$

then  $p^* = d^*$ , i.e., strong duality holds and the maximum is attained at some  $\nu^* \in \mathbb{R}^n$ .

Since  $\mathbf{ri} \mathcal{A}(\mathbb{G}_c) = \mathcal{A}(\mathbf{qri} \mathbb{G}_c)$ , when  $\mathbf{qri} \mathbb{G}_c \neq \emptyset$ , this can be restated as follows: if

$$\exists \mathcal{G} \in \mathbf{qri} \mathbb{G}_c \text{ such that } A(\mathcal{G}) = b, \quad (\text{A.5})$$

then  $p^* = d^*$ .

*Proof.* The reader is referred to [76] and [75]. □

We will verify that the conditions of Theorem A.1 are indeed satisfied for the optimization problem in Eq. (3.30).

- (i). The space  $L^2_\mu(\mathbb{R})$  is a Banach space.
- (ii). The objective functional is linear in  $\mathcal{G}$  and therefore convex.
- (iii). The map  $\mathcal{A} : L^2_\mu(\mathbb{R}) \rightarrow \mathbb{R}^2$ , where  $\mathcal{A}(\mathcal{G}) \stackrel{\text{def}}{=} [\mathbf{E}[X\mathcal{G}(X)] \ \mathbf{E}[\mathcal{G}(X)]]^\top$  is linear in  $\mathcal{G}$ , and it is also bounded and therefore continuous. Boundedness can be verified as follows

$$\|\mathcal{A}(\mathcal{G})\|_2^2 = |\mathbf{E}[X\mathcal{G}(X)]|^2 + |\mathbf{E}[\mathcal{G}(X)]|^2 \quad (\text{A.6})$$

$$\leq (\mathbf{E}[X^2] + 1) \mathbf{E}[|\mathcal{G}(X)|^2] \quad (\text{A.7})$$

$$< +\infty. \quad (\text{A.8})$$

The first inequality follows from the Cauchy-Schwarz inequality applied to the first term and Jensen's inequality applied to the second term. The strict inequality in the last step follows from the fact that  $X$  has finite second moment and  $\mathcal{G} \in L^2_\mu(\mathbb{R})$ .

(iv). The set

$$\mathbb{G}_c = \{\mathcal{G} \in L^2_\mu(\mathbb{R}) \mid 0 \leq \mathcal{G}(x) \leq \frac{1}{1-\alpha}, \mu - \text{a.e.}\} \quad (\text{A.9})$$

is closed and convex.

(v). Assuming the existence of a feasible point we have,

$$p^* \leq \mathbf{E}[X^2 \mathcal{G}(X)] \leq \frac{1}{1-\alpha} \mathbf{E}[X^2] < +\infty, \quad (\text{A.10})$$

where the strict inequality follows from  $X$  having finite second moment.

(vi). Finally, we must check if the following constraint qualification is satisfied

$$\exists \mathcal{G} \in \mathbf{qri} \mathbb{G}_c \text{ such that } \mathcal{A}(\mathcal{G}) = b, \quad (\text{A.11})$$

which corresponds to Borwein-Lewis' constraint qualification in [76]. Therefore, in order to have strong duality, there must be feasible point  $\mathcal{G} \in \mathbf{qri} \mathbb{G}_c$  such that  $\mathcal{A}(\mathcal{G}) = [\gamma \ 1]^\top$ . From [76, Theorem 2.1, Example 2.2], we have

$$\mathbf{qri} \mathbb{G}_c = \left\{ \mathcal{G} \in L^2_\mu(\mathbb{R}) \mid 0 < \mathcal{G}(x) < \frac{1}{1-\alpha}, \mu - \text{a.e.} \right\}, \quad (\text{A.12})$$

which is the condition that must be satisfied for Lemma 3.1 to hold.

### A.3 Suboptimality of sensor scheduling

In this Appendix we state and prove the following result used in the proof of Lemma Lemma 3.1.

**Proposition A.3.** *If  $\nu^*$  is a maximizer of the the Lagrange dual function in Eq. (3.34), the polynomial  $x^2 + \nu_0^*x + \nu_1^*$  always admits distinct real roots.*

*Proof.* We will show that  $(\nu_0^*)^2 > 4\nu_1^*$ . Suppose that  $\nu$  satisfies  $\nu_0^2 \leq 4\nu_1$ , implying that  $[x^2 + \nu_0x + \nu_1]^- \equiv 0$ . The Lagrange dual function becomes  $\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\hat{x}_\emptyset$ , its supremum subject to  $4\nu_1 \geq \nu_0^2$  is equal to  $\hat{x}_\emptyset^2$  and it is achieved by  $\nu_0^* = -2\hat{x}_\emptyset$  and  $\nu_1^* = \hat{x}_\emptyset^2$ .

When  $\nu_0^2 > 4\nu_1$ , the polynomial  $x^2 + \nu_0x + \nu_1$  admits two distinct real roots denoted by  $a(\nu)$  and  $b(\nu)$ :

$$a(\nu), b(\nu) = \frac{-\nu_0 \pm \sqrt{\nu_0^2 - 4\nu_1}}{2}. \quad (\text{A.13})$$

Let  $\nu_0 = -2\hat{x}_\emptyset$  and  $\nu_1 = \hat{x}_\emptyset^2 - \delta$ , for some  $\delta > 0$ . Clearly,  $\nu_0^2 - 4\nu_1 = 4\delta > 0$ . We will show that  $\exists \delta > 0$  such that  $\mathcal{C}^*(\nu) > \hat{x}_\emptyset^2$  and therefore there is no loss in optimality in restricting the dual problem to  $\{\nu \in \mathbb{R}^2 \mid \nu_0^2 > 4\nu_1\}$ . We start with

$$\mathcal{C}^*(\nu) \Big|_{\substack{\nu_0 = -2\hat{x}_\emptyset \\ \nu_1 = \hat{x}_\emptyset^2 - \delta}} = \hat{x}_\emptyset^2 + \delta + \frac{1}{1 - \alpha} \int_{\hat{x}_\emptyset - \sqrt{\delta}}^{\hat{x}_\emptyset + \sqrt{\delta}} [(x - \hat{x}_\emptyset)^2 - \delta] f_X(x) dx, \quad (\text{A.14})$$

and define

$$\int_{\hat{x}_\emptyset - \sqrt{\delta}}^{\hat{x}_\emptyset + \sqrt{\delta}} (x - \hat{x}_\emptyset)^2 f_X(x) dx \stackrel{\text{def}}{=} \mathcal{W}(\delta). \quad (\text{A.15})$$

When  $x$  varies from  $\hat{x}_\emptyset - \sqrt{\delta}$  to  $\hat{x}_\emptyset + \sqrt{\delta}$ , the quantity  $(x - \hat{x}_\emptyset)^2$  varies from 0 to  $\delta$ .

Therefore,

$$0 \leq \mathcal{W}(\delta) \leq \int_{\hat{x}_\emptyset - \sqrt{\delta}}^{\hat{x}_\emptyset + \sqrt{\delta}} \delta f_X(x) dx \stackrel{\text{def}}{=} \mathcal{V}(\delta). \quad (\text{A.16})$$

Since

$$\frac{\mathcal{V}(\delta)}{\delta} = \int_{\hat{x}_\emptyset - \sqrt{\delta}}^{\hat{x}_\emptyset + \sqrt{\delta}} f_X(x) dx, \quad (\text{A.17})$$

the limit  $\delta \downarrow 0$  yields  $\frac{\mathcal{V}(\delta)}{\delta} \rightarrow 0$ . Therefore,  $\mathcal{V}(\delta) = o(\delta)$  and consequently,  $\mathcal{W}(\delta) = o(\delta)$ . Implying that

$$C^*(\nu) \Big|_{\substack{\nu_0 = -2\hat{x}_\emptyset \\ \nu_1 = \hat{x}_\emptyset^2 - \delta}} = \hat{x}_\emptyset^2 + \delta + o(\delta) > \hat{x}_\emptyset^2. \quad (\text{A.18})$$

□

This proposition implies that *always-transmit* and *never-transmit* degenerate strategies used in sensor scheduling are strictly suboptimal for Problem 3.1.

## Appendix B: Convergence of the Modified Lloyd-Max algorithm

**Theorem B.1.** *Assume that  $X \sim \mathcal{N}(0, \sigma^2)$ . There exists a compact set  $\mathbb{C}$ , which contains all the critical points of  $\mathcal{J}_q(\hat{x})$ , such that  $\mathcal{F}(\mathbb{C}) \subset \mathbb{C}$ . Consequently, the modified Lloyd-Max algorithm is globally convergent to a critical point of  $\mathcal{J}_q(\hat{x})$ .*

*Proof.* Without loss of generality, we can constrain our analysis to  $\mathbb{Q}_1$  (or  $\mathbb{Q}_2$ ).

Proposition 4.2 implies that  $\mathcal{F}$  maps every  $\hat{x} \in \mathbb{Q}_1$  into

$$\mathbb{H}_1 \stackrel{\text{def}}{=} \mathbb{Q}_1 \cap \{\hat{x} \in \mathbb{R}^2 \mid \hat{x}_\varnothing \hat{x}_\mathfrak{c} \geq -\sigma^2\}. \quad (\text{B.1})$$

To construct a compact set  $\mathbb{C}_1$  invariant with respect to  $\mathcal{F}$  we will intersect  $\mathbb{H}_1$  with  $\{\hat{x} \in \mathbb{R}^2 \mid \|\hat{x}\|_\infty \leq \ell\}$ , where  $\ell$  is an arbitrarily large, finite and positive real number. Hence,

$$\mathbb{C}_1 \stackrel{\text{def}}{=} \mathbb{H}_1 \cap \{\hat{x} \in \mathbb{R}^2 \mid \|\hat{x}\|_\infty \leq \ell\}. \quad (\text{B.2})$$

Points  $\hat{x} \in \mathbb{C}_1$  such that  $\ell \gg \hat{x}_\varnothing$  and  $-\ell \ll \hat{x}_\mathfrak{c}$  will never map outside of  $\mathbb{C}_1$ .

This can be shown using Eqs. (4.24) and (4.26) and is omitted for brevity. We will now show that the points in the two regions:

1.  $\hat{x} \in \mathbb{H}_1$  with  $\hat{x}_\varnothing \approx \ell$  such that  $\hat{x}_\varnothing \leq \ell$  and  $\hat{x}_\mathfrak{c} \approx 0$
2.  $\hat{x} \in \mathbb{H}_1$  with  $\hat{x}_\mathfrak{c} \approx -\ell$  such that  $\hat{x}_\mathfrak{c} \geq -\ell$  and  $\hat{x}_\varnothing \approx 0$ .



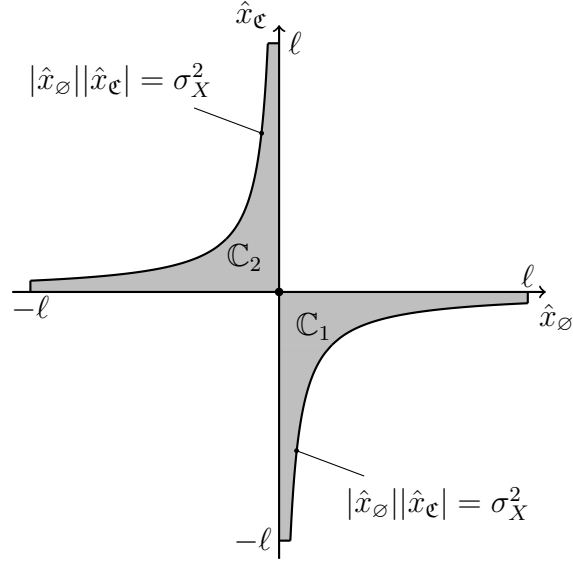


Figure B.1: The shaded region is the set  $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$  containing all the critical points of  $\mathcal{F}(\hat{x})$  on  $\mathbb{R}^2$ .

will map to points inside of  $\mathbb{C}_1$ .

Here we provide the detailed proof for region 1. The proof for region 2 is analogous and omitted for brevity. First we note that in the asymptotic regime of region 1,

$$a(\hat{x}) \approx \frac{\hat{x}_\emptyset}{1 + \sqrt{\beta}} \quad \text{and} \quad b(\hat{x}) \approx \frac{\hat{x}_\emptyset}{1 - \sqrt{\beta}}. \quad (\text{B.3})$$

Therefore, when  $\hat{x}_\emptyset \approx \ell$  and  $\hat{x}_\epsilon \approx 0$ , both  $a(\hat{x})$  and  $b(\hat{x})$  are large. Defining

$$\mathcal{L}(\hat{x}) \stackrel{\text{def}}{=} \int_{a(\hat{x})}^{b(\hat{x})} x f_X(x) dx - \ell \int_{a(\hat{x})}^{b(\hat{x})} f_X(x) dx \quad (\text{B.4})$$

$$= \sqrt{\frac{\sigma^2}{2\pi}} \left[ e^{-\frac{a^2(\hat{x})}{2\sigma^2}} - e^{-\frac{b^2(\hat{x})}{2\sigma^2}} \right] - \frac{\ell}{2} \left[ \text{erfc} \left( \frac{a(\hat{x})}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{b(\hat{x})}{\sqrt{2\sigma^2}} \right) \right]. \quad (\text{B.5})$$

Using the fact that  $\text{erfc}(t) \approx \frac{1}{\sqrt{\pi}} \frac{e^{-t^2}}{t}$ , for large values of  $t$ , we have

$$\mathcal{L}(\hat{x}) \approx \sqrt{\frac{\sigma^2}{2\pi}} \left[ \left( 1 - \frac{\ell}{a(\hat{x})} \right) e^{-\frac{a^2(\hat{x})}{2\sigma^2}} - \left( 1 + \frac{\ell}{b(\hat{x})} \right) e^{-\frac{b^2(\hat{x})}{2\sigma^2}} \right]. \quad (\text{B.6})$$

Since  $\hat{x}_\emptyset \leq \ell$  we have that  $\frac{\ell}{a(\hat{x})} \geq 1$ . Therefore,  $\mathcal{L}(\hat{x}) \leq 0$ , which implies that  $\mathcal{F}_\emptyset(\hat{x}) \leq \ell$ . Finally, since  $\int_{a(\hat{x})}^{b(\hat{x})} f_X(x)dx \approx 0$  and  $\mathcal{F}_\emptyset(\hat{x}) \leq \ell$ ,

$$\mathcal{F}_\mathfrak{C}(\hat{x}) = \frac{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x)dx}{1 - \int_{a(\hat{x})}^{b(\hat{x})} f_X(x)dx} \mathcal{F}_\emptyset(\hat{x}) \approx 0. \quad (\text{B.7})$$

The analysis can be repeated for  $\hat{x} \in \mathbb{Q}_2$ , which will lead to a set  $\mathbb{H}_2$  that can then be truncated to a compact set  $\mathbb{C}_2$ , also invariant to  $\mathcal{F}$ . Let

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{C}_1 \cup \mathbb{C}_2. \quad (\text{B.8})$$

This set is illustrated in Fig. B.1. □

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