The Residual Character of Finitely Generated Varieties of Unary Algebras and Groupoids, and Applications to the Restricted Quackenbush Problem

by

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Abstract

The crux of this thesis is to study the Restricted Quackenbush Problem: if **B** is a finite algebra of finite type and $\mathbb{V}(\mathbf{B})$ is residually finite, must $\mathbb{V}(\mathbf{B})$ be residually $< N$, for some positive integer N?

We show that the Restricted Quackenbush Problem is answered affirmatively with respect to unary algebras, a result first shown by Baldwin and Berman in 1975. Then we turn our attention to groupoids and show that all groupoids that generate residually finite varieties must satisfy an identity of the form $k(x) \approx x * x$, where $k(x)$ is an identity of type $\{*\}$ that is not equal to $x * x$. Due to this result, we focus on a particular class of idempotent groupoids, called absorbing groupoids. We show that if a certain property holds, with respect to absorbing groupoids, then the Restricted Quackenbush Problem is answered affirmatively.

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Introduction

1.1 The Problem

The *Restricted Quackenbush Problem* is a problem that originated in the seventies. An answer to this problem would yield a further understanding of the residual character of finitely generated varieties of finite type.

Problem (Restricted Quackenbush Problem). *Let* **B** *be a finite algebra of finite type. If all subdirectly irreducible algebras in the variety generated by* **B** *have a finite universe, must there exist a finite bound on the cardinality of the universe of each subdirectly irreducible algebra in this variety?*

In this chapter, we start by looking at some necessary background material in Universal Algebra to understand the problem. This chapter concludes with a section that discusses the origin and evolution of the Restricted Quackenbush Problem.

1.2 Background Material

To understand the problem, specific concepts and definitions are needed. The following background material is taken from either [2] or [13]. Unary algebras and groupoids will be used extensively throughout this document. For a list of common and exotic algebras, see §1 of Chapter 2, and §1 and §2 of Chapter 3 in [2] or see Chapter 1.1 in [13].

1.2.1 Algebras, Terms and Identities

A set $\mathcal{F}$ is called a *type of algebras* or a *language of algebras* if $\mathcal{F}$ is a family of finitary operation symbols such that to each operation symbol there corresponds a non-negative integer called the operation symbol's *arity* or *rank*. For each positive integer n, the symbol $\mathcal{F}_n$ denotes the set of n-ary operation symbols in $\mathcal{F}$.

Definition. An *algebra* $\mathbf{B}$ *of type* $\mathcal{F}$ is a non-empty set, called a *universe* or an *underlying set* B, together with a *family of finitary operations* F defined on the universe and indexed by members of $\mathcal{F}$ such that the arity of an operation agrees with the arity of the corresponding symbol in F. That is, for each n-ary symbol f in $\mathcal{F}$, there exists a unique finitary operation $f^{\mathbf{B}}$ in F such that the arity of $f^{\mathbf{B}}$ is n.

The algebra $\mathbf{B}$ described in the above definition is often written as $\mathbf{B} = \langle B; F\rangle$. If $|F|$ is small then, often, the finitary operations are listed in decreasing arity. Further, $\mathbf{B} = \langle B; \mathcal{F}\rangle$ is often used to define an algebra, with the implicit understanding that there exists a set F of finitary operations as defined above. Similarly, if $|\mathcal{F}|$ is small then, often, the finitary operations symbols are listed in decreasing arity.

Example. The group with four elements and addition and subtraction modulo 4,

$$\mathbf{Z}_4 = \left(\{0,1,2,3\}; \{+^{\mathbf{Z}_4}, -^{\mathbf{Z}_4}, 0^{\mathbf{Z}_4}\}\right) \qquad \text{or} \qquad \mathbf{Z}_4 = \left(\{0,1,2,3\}; +, -, 0\right),$$

is an algebra of type $\{+,-,0\}$, where $+$ is a binary operation symbol, $-$ is a unary operation symbol and 0 is a nullary operation symbol. The set $\{0,1,2,3\}$ is the universe of the group, while $\{+^{\mathbf{Z}_4}, -^{\mathbf{Z}_4}, 0^{\mathbf{Z}_4}\}$ is the family of fundamental operations. See Table 1.1 and Table 1.2

for the operation tables of $+^{\mathbf{Z}_4}$ and $-^{\mathbf{Z}_4}$, respectively. The nullary operation $0^{\mathbf{Z}_4}$ is defined to be 0.

$+^{\mathbf{Z}_4}$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 1.1: The Operation Table of $+^{\mathbf{Z}_4}$

	$-^{\mathbf{Z}_4}$
0	0
1	3
2	2
3	1

Table 1.2: The Operation Table of $-^{\mathbf{Z}_4}$

Definition. Let $\mathcal{F}$ be a language and X be a set of variables. Further, let $T(X)$ denote the smallest set that contains X and all nullary operation symbols such that if $p_1, p_2, \ldots p_n$ are in $T(X)$ and f is in $\mathcal{F}_n$, then $f(p_1, p_2, \ldots, p_n)$ is in $T(X)$. Each member of $T(X)$ is called a *term of type $\mathcal{F}$ over X*.

Alternatively, we can think of a term as a string that is a well defined concatenation of operation symbols and variables. When a term is realized in an algebra, it becomes a composition of operations.

Example. Let $\mathcal{F}$ be the language of Abelian groups. That is, let $\mathcal{F} = \{+,-,0\}$, where + is a binary operation symbol, - is a unary operation symbol and 0 is a nullary operation symbol. Further, let $X = \{x_1, x_2\}$. The strings

$$s_1(x_1,x_2) = (0+x_1)+(-x_2), \qquad s_2(x_1) = 0+(-0) \qquad \text{and} \qquad s_3(x_1,x_2,x_3) = x_1+x_1$$

are all terms of type $\mathcal{F}$ over X. Alone, they are no more than a concatenation of symbols.

However,

$$s_1^{\mathbf{Z}_4}(1,2) = (0^{\mathbf{Z}_4} +^{\mathbf{Z}_4} 1) +^{\mathbf{Z}_4} \left(-^{\mathbf{Z}_4}(2)\right) \qquad s_2^{\mathbf{Z}_4}(3) = 0^{\mathbf{Z}_4} +^{\mathbf{Z}_4} \left(-^{\mathbf{Z}_4}(0^{\mathbf{Z}_4})\right)$$

$$= (0 +^{\mathbf{Z}_4} 1) +^{\mathbf{Z}_4} 2 \qquad = 0 +^{\mathbf{Z}_4} \left(-^{\mathbf{Z}_4}(0)\right)$$

$$= 1 +^{\mathbf{Z}_4} 2 \qquad = 0 +^{\mathbf{Z}_4} 0$$

$$= 3, \qquad = 0$$

and

$$s_3^{\mathbf{Z}_4}(3,1,0) = 3 +^{\mathbf{Z}_4} 3$$

$$= 2.$$

Definition. Let $\mathcal{F}$ be a language of algebras and let n be a positive integer. Further, let $X = \{x_i\}_{i=1}^n$ denote an indexed set of n variables. An *n-ary identity of type $\mathcal{F}$ over X* is an expression of the form

$$\forall x_1 \forall x_2 \forall x_3 \cdots \forall x_n \left[t_1(x_1,x_2,x_3,\ldots,x_n) \approx t_2(x_1,x_2,x_3,\ldots,x_n)\right]. \tag{1.1}$$

where

$$t_1(x_1,x_2,x_3,\ldots,x_n) \qquad \text{and} \qquad t_2(x_1,x_2,x_3,\ldots,x_n)$$

are terms of type $\mathcal{F}$ over X.

Usually, an identity is written without the quantifiers. Sometimes, if the variables need not be explicitly defined, an identity is written without listing the variables. That is, often the identity in Statement (1.1) is written as $t_1 \approx t_2$. An algebra $\mathbf{B}$ of type $\mathcal{F}$ *satisfies* the n-ary identity given in Statement (1.1) if and only if

$$\forall \{b_i\}_{i=1}^n \subseteq B, \qquad t_1^{\mathbf{B}}(b_1,b_2,b_3,\ldots,b_n) = t_2^{\mathbf{B}}(b_1,b_2,b_3,\ldots,b_n).$$

This is denoted by

$$\mathbf{B} \models t_1(x_1,x_2,x_3,\ldots,x_n) \approx t_2(x_1,x_2,x_3,\ldots,x_n) \qquad \text{or} \qquad \mathbf{B} \models t_1 \approx t_2.$$

If $\mathbf{B}$ does not satisfy the identity given in Statement (1.1), then we write $\mathbf{B} \not\models t_1 \approx t_2$.

Example. Returning to the example on page 2, since $\mathbf{Z}_4$ is an Abelian group, it satisfies the Abelian group identities. That is,

$$\mathbf{B} \models (x+y)+z \approx x+(y+z), \qquad \mathbf{B} \models x+0 \approx x, \qquad \mathbf{B} \models x+\text{-}(x) \approx 0,$$

and

$$\mathbf{B} \models x+y \approx y+x.$$

When realized in $\mathbf{Z}_4$, the terms s_1 and s_2, defined in the example on page 3, correspond to compositions of the operations $+^{\mathbf{Z}_4}$, $\text{-}^{\mathbf{Z}_4}$ and $0^{\mathbf{Z}_4}$. That is,

$$s_1^{\mathbf{Z}_4}(x_1,x_2) = x_1 +^{\mathbf{Z}_4} \left(\text{-}^{\mathbf{Z}_4}(x_2)\right) \qquad \text{and} \qquad s_2^{\mathbf{Z}_4}(x_1) = 0^{\mathbf{Z}_4}.$$

In the previous definition, notice the use of $=$ and $\approx$. They both mean *equality* to some extent. They are similar in meaning; but, are different in their utilization. In general, when dealing only with the language of an algebra and other logical devices, the symbol $\approx$ is used; however, when a specific algebra is being looked at, with elements of the universe being used, the symbol $=$ is used because actual equality of elements is intended.

This distinction is very useful in switching from the class of all algebras of a similar type to a specific algebra in that class and vice versa, without formally stating the distinction. The next example illustrates the need for the distinction between $=$ and $\approx$.

Example. Consider the language of a monoid $\mathcal{F} = \{*, 1\}$, where $*$ is a binary operation symbol and 1 is a nullary operation symbol. The terms 1 and $1 * 1$ are not equal. Specifi-

$*^{\mathbf{E}_1}$	0	1
0	0	0
1	0	0

$*^{\mathbf{E}_2}$	0	1
0	1	1
1	1	1

Table 1.3: The Operation Tables of $\mathbf{E}_1$ and $\mathbf{E}_2$

cally, 1 is the concatenation of one symbol while $1 * 1$ is the concatenation of three. Therefore, the term equality $1 * 1 = 1$ is false. On the other hand, $1 * 1 \approx 1$ is an identity that may or may not hold in an algebra.

Consider the monoids $\mathbf{E}_1 = \langle \{0,1\}; *^{\mathbf{E}_1}, 1^{\mathbf{E}_1} \rangle$ and $\mathbf{E}_2 = \langle \{0,1\}; *^{\mathbf{E}_2}, 1^{\mathbf{E}_2} \rangle$ where $1^{\mathbf{E}_1} = 1$ and $1^{\mathbf{E}_2} = 1$. The operation table for the binary operation corresponding to each monoid is given in Table 1.3.

The monoid $\mathbf{E}_2$ satisfies the identity $1 * 1 \approx 1$, whereas the monoid $\mathbf{E}_1$ does not. Under these conditions, we may write

$$1^{\mathbf{E}_2} *^{\mathbf{E}_2} 1^{\mathbf{E}_2} = 1^{\mathbf{E}_2} \qquad \text{and} \qquad 1^{\mathbf{E}_1} *^{\mathbf{E}_1} 1^{\mathbf{E}_1} \neq 1^{\mathbf{E}_1}.$$

Let K denote a class of algebras of type $\mathcal{F}$. The set of identities of type $\mathcal{F}$ over some set of variables X that hold true in K is denoted by $\mathrm{Id}_K(X)$. Often, we are not too concerned with the set of variables and use Id_K instead of $\mathrm{Id}_K(X)$.

1.2.2 Obtaining New Algebras from Old Algebras

There are three general ways of constructing new algebras from old algebras: direct products, subalgebras and homomorphic images. Their definitions are described formally.

1.2.2.1 Direct Products

Let $\{\mathbf{B}_i\}_{i\in I}$ denote an indexed set of algebras of type $\mathcal{F}$. The algebra

$$\prod_{i\in I}\mathbf{B}_i = \left\langle \prod_{i\in I} B_i; \mathcal{F} \right\rangle$$

is called the *direct product* of the $\mathbf{B}_i$'s and each $\mathbf{B}_i$ is called a *factor* of the direct product. The universe of the direct product is the cartesian product of the universe of each factor. The finitary operations are defined coordinate wise. That is, for the positive integer n, if f is in $\mathcal{F}_n$ and $\{b_j\}_{j=1}^{n} \subseteq \prod_{i\in I} B_i$, then

$$\left(f^{\prod_{i\in I}\mathbf{B}_i}(b_1, b_2, \ldots, b_n)\right)(i) = f^{\mathbf{B}_i}\left(b_1(i), b_2(i), \ldots, b_n(i)\right).$$

For the remainder of this subsection, assume that $\mathbf{B}$ is an algebra of type $\mathcal{F}$.

If $|B| = 1$, then $\mathbf{B}$ is called a *trivial algebra* of type $\mathcal{F}$. Note that up to isomorphism, there exists only one trivial algebra of type $\mathcal{F}$.

Note that if the index I is empty, then $\prod_{i\in I}\mathbf{B}_i$ is a trivial algebra. The universe of this direct product is the set containing only the empty tuple.

1.2.2.2 Subuniverses and Subalgebras

A subset M of B is called a *subuniverse* of $\mathbf{B}$ if for all f in $\mathcal{F}$, the set M is closed under $f^{\mathbf{B}}$. The smallest subuniverse of B that contains M is denoted by $\mathrm{Sg}^{\mathbf{B}}(M)$.

Definition. Suppose that $S \subseteq B$ such that $\mathrm{Sg}^{\mathbf{B}}(S) = B$. Then S is called a *generating set* of B; the set S is said to *generate* B and the members of S are called *generators*. If for all s in S, the set $S\backslash\{s\}$ does not generate B, then S is said to be a *minimal generating set* or an *irredundant basis*.

Example. Consider the group

$$\mathbf{Z}_{10} = \langle \mathbb{Z}_{10}; +^{\mathbf{Z}_{10}}, -^{\mathbf{Z}_{10}}, 0^{\mathbf{Z}_{10}} \rangle,$$

where $+^{\mathbb{Z}_{10}}$ is addition modulo 10. The set $\{3,6,7\}$ is a generating set; but, it is not a minimal generating set as $\{7\}$ generates $\{0,1,\ldots,9\}$. Also, $\{2,5\}$ is an irredundant basis, as is $\{1\}$.

Definition. An algebra $\mathbf{M}$ of type $\mathcal{F}$ is called a *subalgebra* of $\mathbf{B}$ if M is a subuniverse of $\mathbf{B}$ and the operations of $\mathbf{M}$ are the restrictions of the operations of $\mathbf{B}$ to M. That is, if $\mathbf{M}$ is a subalgebra of $\mathbf{B}$, denoted by $\mathbf{M} \leq \mathbf{B}$, then M is a non-empty subuniverse of B and for all f in $\mathcal{F}$, we have $f^{\mathbf{M}} = f^{\mathbf{B}} \restriction_M$. If M is a non-empty proper subuniverse of $\mathbf{B}$, then $\mathbf{M}$ is a *strict subalgebra* of $\mathbf{B}$ and the notation changes to $\mathbf{M} < \mathbf{B}$.

The main difference between a subuniverse and a subalgebra, aside from the former being a set and the latter being an algebra, is a subuniverse can be empty, while the universe of a subalgebra cannot. If the language of an algebra $\mathbf{B}$ does not contain any nullary operation symbols, then the subuniverse generated by $\varnothing$ is $\varnothing$. If a nullary operation symbol is present, then this cannot occur as every subuniverse must contain at least one member, the constant corresponding to the nullary operation symbol.

1.2.2.3 Congruences and Homomorphic Images

Let θ denote a relation on B. The relation θ satisfies the *compatibility property* if for all $n > 0$, for all f in $\mathcal{F}_n$, and for all $\{a_i\}_{i=1}^n \subseteq B$ and $\{b_i\}_{i=1}^n \subseteq B$ such that

$$a_i \,\theta\, b_i \qquad \text{for all} \qquad i \in \{1,\ldots,n\}$$

we have

$$f^{\mathbf{B}}(a_1,a_2,\ldots,a_n) \,\theta\, f^{\mathbf{B}}(b_1,b_2,\ldots,b_n).$$

Definition. A *congruence* on $\mathbf{B}$ is an equivalence relation defined on B that satisfies the compatibility property.

Often, we want to discuss or use the smallest congruence that relates element b_1 to element b_2. This is denoted by $\mathrm{Cg}^{\mathbf{B}}\big(\{\langle b_1,b_2\rangle\}\big)$ or just $\mathrm{Cg}^{\mathbf{B}}\big(\langle b_1,b_2\rangle\big)$ and is read as *the congruence on* **B** *generated by relating* b_1 *to* b_2. A congruence generated by relating two elements is called a *principal congruence*. For S, a non-empty subset of B, the smallest congruence that relates every member of S to every member of S is denoted by $\mathrm{Cg}^{\mathbf{B}}(S^2)$.

There always exist at least two congruences on **B**. They are the congruence obtained by relating every element in B to every element in B and the congruence obtained by relating every element in B to only itself. Respectively, these congruence are denoted by

$$\Delta_B = \big\{\langle b,b\rangle \mid b \in B\big\} \qquad \text{and} \qquad \nabla_B = B^2.$$

These congruences are sometimes referred to as the *trivial congruences* on **B**.

Definition. A *first order formula* is a well defined concatenation of identities, relation symbols, propositional connectives and quantifiers. The propositional connectives are $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies) and $\leftrightarrow$ (if and only if). The quantifiers are $\exists$ (existential) and $\forall$ (universal).

Given a metric space $\langle B;d\rangle$, variables x, y and z, real addition $+$, and real less than or equal to $\leq$, the triangle inequality

$$\forall x \forall y \forall z[d(x,y)+d(y,z) \geq d(x,z)],$$

is a common first order formula found in the study of metric spaces. A principal congruence formula is an example of a first order formula that will be used in later sections and chapters of this document.

Definition. A *principal congruence formula* is a first order formula $\pi(w_1,w_2,w_3,w_4)$ of type $\mathcal{F}$ that is of the form

$$\exists \vec{k}\left(w_1 \approx t_1(x_1,\vec{k}) \wedge \bigwedge_{i=1}^{n-1} \big(t_i(y_i,\vec{k}) \approx t_{i+1}(x_{i+1},\vec{k})\big) \wedge t_n(y_n,\vec{k}) \approx w_2\right), \tag{1.2}$$

where n is a positive integer and the symbol $\vec{k}$ denotes a vector of variables distinct from w_1, w_2, w_3 and w_4. Further, for all $1 \le i \le n$, we have $\{x_i, y_i\} = \{w_3, w_4\}$ and the term t_i is of type $\mathcal{F}$ over $\{x_i, y_i\}$ and the variables in $\vec{k}$.

For b_1, b_2, b_3 and b_4 in B, we may represent $\pi(b_1, b_2, b_3, b_4)$ holding in $\mathbf{B}$ using the following staircase diagram:

$$
\begin{aligned}
b_1 &= t_1^{\mathbf{B}}(x_1, \vec{e}) \\
& t_1^{\mathbf{B}}(y_1, \vec{e}) = t_2^{\mathbf{B}}(x_2, \vec{e}) \\
& \qquad t_2^{\mathbf{B}}(y_2, \vec{e}) = t_3^{\mathbf{B}}(x_3, \vec{e}) \\
& \qquad\qquad \ddots \\
& \qquad\qquad\quad t_{n-1}^{\mathbf{B}}(y_{n-1}, \vec{e}) = t_n^{\mathbf{B}}(x_n, \vec{e}) \\
& \qquad\qquad\qquad\qquad t_n^{\mathbf{B}}(y_n, \vec{e}) = b_2,
\end{aligned}
$$

where for all $1 \le i \le n$, we have $\{x_i, y_i\} = \{b_3, b_4\}$ and $\vec{e}$ is a vector of members from B.

Theorem 1.2.1 (Mal'cev's Principal Congruence Formula Theorem). *Let* $\mathbf{B}$ *be an algebra and* $\{b_1, b_2, b_3, b_4\} \subseteq B$. *Then,* $\langle b_1, b_2 \rangle$ *is in* $Cg^{\mathbf{B}}(\langle b_3, b_4 \rangle)$ *if and only if, for some principal congruence formula* π, *the algebra* $\mathbf{B}$ *satisfies* $\pi(b_1, b_2, b_3, b_4)$.

Corollary 1.2.2. *Let* **B** *be a unary algebra and* b_1, b_2, b_3 *and* b_4 *be members in* B. *If* $\langle b_1, b_2\rangle$ *is in* $Cg^{\mathbf{B}}(\langle b_3, b_4\rangle)$ *then there exists some set of* n *unary terms realized in* **B**, *say* $\{t_i\}_{i=1}^n$, *such that*

$$
\begin{aligned}
b_1 &= t_1^{\mathbf{B}}(x_1) \\
t_1^{\mathbf{B}}(y_1) &= t_2^{\mathbf{B}}(x_2) \\
t_2^{\mathbf{B}}(y_2) &= t_3^{\mathbf{B}}(x_3) \\
&\ddots \\
t_{n-1}^{\mathbf{B}}(y_{n-1}) &= t_n^{\mathbf{B}}(x_n) \\
t_n^{\mathbf{B}}(y_n) &= b_2,
\end{aligned}
$$

where for all $1 \le i \le n$, *we have* $\{x_i, y_i\} = \{b_3, b_4\}$.

Proof. The principal congruence formula guaranteed to exist by Theorem 1.2.1 is of the form of Statement (1.2), where the terms are unary. □

Loosely speaking, if an equivalence relation is a congruence, then the fundamental operations respect the equivalence relation. In fact, this idea leads to a natural construction of an algebra based on the congruence.

Let **B** be an algebra of type $\mathcal{F}$ and θ be a congruence on **B**. Further, let

$$B/\theta = \{b/\theta \mid b \in B\} \qquad \text{where} \qquad b/\theta = \{b_1 \in B : b\,\theta\, b_1\}.$$

For all positive integers n and b_1 through b_n in B, define for all f in $\mathcal{F}_n$,

$$f^{\mathbf{B}/\theta}(b_1/\theta, b_2/\theta, \dots, b_n/\theta) = f^{\mathbf{B}}(b_1, b_2, \dots, b_n)/\theta.$$

The resulting algebra $\mathbf{B}/\theta = \langle B/\theta; \mathcal{F}\rangle$ is called the *factor algebra* or the *quotient algebra* of **B** by θ. The notation $\mathbf{B}/\theta$ is read as **B** *modulo* θ or just **B** *mod* θ.

We may think of $\mathbf{B}/\theta$ as a collapsed copy of $\mathbf{B}$. Further, we may think of θ as a set of instructions that describe how to collapse $\mathbf{B}$ without forcing the operations to become multivalued.

Definition. Let $\mathbf{B}$ and $\mathbf{M}$ be two algebras of the type $\mathcal{F}$. For all positive integers n, if the function $h{:}\,\mathbf{B} \to \mathbf{M}$ satisfies the property

$$h\big(f^{\mathbf{B}}(b_1,b_2,\ldots,b_n)\big) = f^{\mathbf{M}}\big(h(b_1),h(b_2),\ldots,h(b_n)\big),$$

for all f in $\mathcal{F}_n$ and $\{b_i\}_{i=1}^{n} \subseteq B$, then h is called a *homomorphism*.

Given a homomorphism $h{:}\,\mathbf{B} \to \mathbf{M}$, the relation defined on B^2,

$$\ker(h) = \big\{\langle b_1,b_2\rangle \in B^2 : h(b_1) = h(b_2)\big\},$$

is called the *kernel* of h. This relation is denoted by $\ker(h)$ and can be shown to be a congruence on $\mathbf{B}$.

Theorem 1.2.3 (First Isomorphism Theorem). *Let $h_1{:}\mathbf{B} \to \mathbf{M}$ be a surjective homomorphism. Then, there is an isomorphism h_2 from $\mathbf{B}/ker(h_1)$ to $\mathbf{M}$ defined by $h_1 = h_2 \circ h_3$, where h_3 is the homomorphism from $\mathbf{B}$ to $\mathbf{B}/ker(h_1)$ obtained by mapping each element in B to the congruence block that it is a member of in $\mathbf{B}/ker(h_1)$.*

If h is surjective, then $\mathbf{M}$ is called a *homomorphic image* of $\mathbf{B}$. If h is injective, then h is called an *embedding* of $\mathbf{B}$ and $\mathbf{B}$ is said to be *embedded* into $\mathbf{M}$. If h is bijective, then h is called an *isomorphism*. If h is an isomorphism and $\mathbf{B} = \mathbf{M}$ then h is called an *automorphism*. If $\mathbf{M}$ is a homomorphic image of $\mathbf{B}$, then by the First Isomorphism Theorem, there exists a congruence θ such that $\mathbf{M} \cong \mathbf{B}/\theta$. That is, all homomorphic images of $\mathbf{B}$ are collapsed copies of $\mathbf{B}$ with the elements in the universe of the collapsed copy relabelled.

To every congruence θ on $\mathbf{B}$ there corresponds a surjective homomorphism, $\nu_\theta: \mathbf{B} \to \mathbf{B}/\theta$ called the *natural homomorphism*, defined by $\nu_\theta(b) = b/\theta$. To every surjective homomorphism of $\mathbf{B}$ corresponds a congruence, namely, $\ker(h)$. Thus, there exists a one-to-one correspondence between the set of congruences on $\mathbf{B}$ and the set of homomorphic images of $\mathbf{B}$.

In general, *direct powers* of an algebra (direct products where each factor is the same algebra) yield an algebra with a larger universe, while subalgebras and homomorphic images of an algebra yield an algebra with a smaller universe.

1.2.3 Subdirectly Irreducible Algebras

Subdirectly irreducible algebras play a large role in the Restricted Quackenbush Problem. For the following few definitions, let n be a positive integer and I be some index set such that $\mathbf{B}$ and each member of $\{\mathbf{B}_i\}_{i \in I}$ are algebras of the same type.

Let X denote a non-empty set. For each positive integer n and $1 \leq i \leq n$, the i^{th} *n-ary projection map* is a function from X^n to X that maps each n-tuple in X^n to the value in the i^{th} coordinate of the n-tuple.

The i^{th} n-ary projection map is denoted by the symbol π_i. When the i^{th} n-ary projection map is restricted to some proper subset of its domain, ρ_i is used in place of π_i.

The algebra $\mathbf{B}$ is said to be a *subdirect product* of the $\mathbf{B}_i$'s if for some index set I,

$$\mathbf{B} \leq \prod_{i \in I} \mathbf{B}_i \qquad \text{and} \qquad \text{for all } i \text{ in } I, \quad \rho_i(\mathbf{B}) = \mathbf{B}_i.$$

An embedding h from $\mathbf{B}$ to $\prod_{i \in I} \mathbf{B}_i$ is a *subdirect embedding* if $h(\mathbf{B})$ is a subdirect product of the $\mathbf{B}_i$'s.

Definition. The algebra $\mathbf{B}$ is *subdirectly irreducible* if for every subdirect embedding

$$h: \mathbf{B} \to \prod_{i \in I} \mathbf{B}_i,$$

there exists j in I such that $\mathbf{B} \cong \mathbf{B}_j$.

Unfortunately, using the formal definition of subdirectly irreducible in practice is nettlesome. Luckily, a tool, given as an upcoming theorem, is available.

Definition. Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be an algebra and let Con **B** denote the set of congruences on **B**. For θ_1 and θ_2 in Con **B**, let

- $\theta_1 \wedge^{\mathrm{Con}\,\mathbf{B}} \theta_2$ denote $\theta_1 \cap \theta_2$ and let
- $\theta_1 \vee^{\mathrm{Con}\,\mathbf{B}} \theta_2$ denote the least congruence on **B** containing θ_1 and θ_2.

Then $\langle \mathrm{Con}\,\mathbf{B}; \wedge^{\mathrm{Con}\,\mathbf{B}}, \vee^{\mathrm{Con}\,\mathbf{B}} \rangle$ forms a lattice called the *congruence lattice of* **B**. The congruence lattice of **B** is denoted by **Con B**.

Theorem 1.2.4. *Let* $\mathbf{B} = \langle B; \mathcal{F} \rangle$ *be an algebra. The lattice* **Con B** *has a minimum non-*Δ_B *congruence or* **B** *is trivial if and only if* **B** *subdirectly irreducible.*

The minimum non-Δ_B congruence of a subdirectly irreducible is called the *monolith* of **B**.

The two following examples demonstrate how to apply the above theorem to show that an algebra is subdirectly irreducible.

Example. The commutative groupoid $\mathbf{E}_3$, whose single fundamental operation is described in Table 1.4, is subdirectly irreducible. The congruence lattice of $\mathbf{E}_3$, depicted in Figure 1.1, has a minimum non-Δ_{E_3} congruence, namely, $\{0,1\}^2 \cup \Delta_{E_3}$.

Example. The unary algebra $\mathbf{E}_4$, whose fundamental operations are described in Table 1.5, is not subdirectly irreducible. The congruence lattice of $\mathbf{E}_4$ is depicted in Figure 1.2. There are three minimal non-Δ_{E_4} congruences, namely,

$$\{0,3\}^2 \cup \Delta_{E_4}, \quad \{1,3\}^2 \cup \Delta_{E_4} \quad \text{and} \quad \{2,3\}^2 \cup \Delta_{E_4};$$

but, there does not exist a minimum non-Δ_{E_4} congruence.

$*^{\mathbf{E}_3}$	0	1	2	3
0	0	1	0	1
1	1	0	0	1
2	0	0	1	1
3	1	1	1	1

Table 1.4: The Operation Table of $*^{\mathbf{E}_3}$

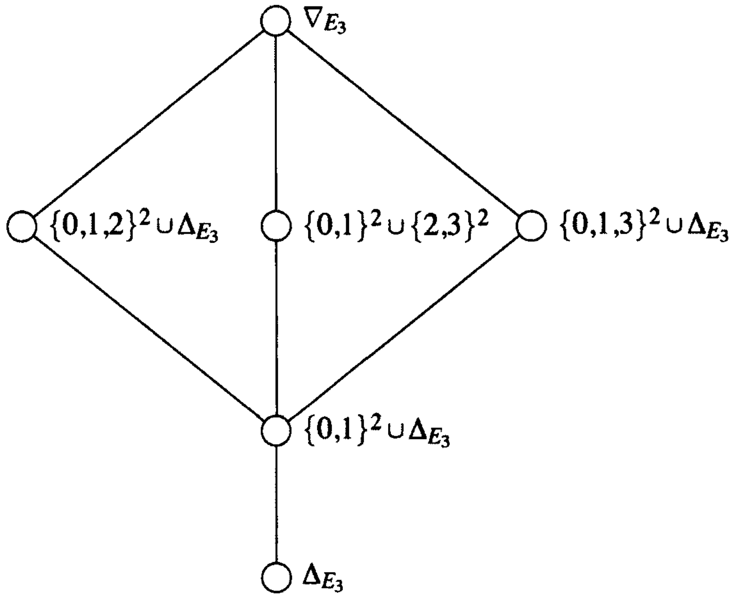

Figure 1.1: The Congruence Lattice of $\mathbf{E}_3$

	$f^{\mathbf{E}_4}$	$g^{\mathbf{E}_4}$	$h^{\mathbf{E}_4}$
0	3	0	0
1	1	1	3
2	2	3	2
3	3	3	3

Table 1.5: The Operation Table of $f^{\mathbf{E}_4}$, $g^{\mathbf{E}_4}$ and $h^{\mathbf{E}_4}$

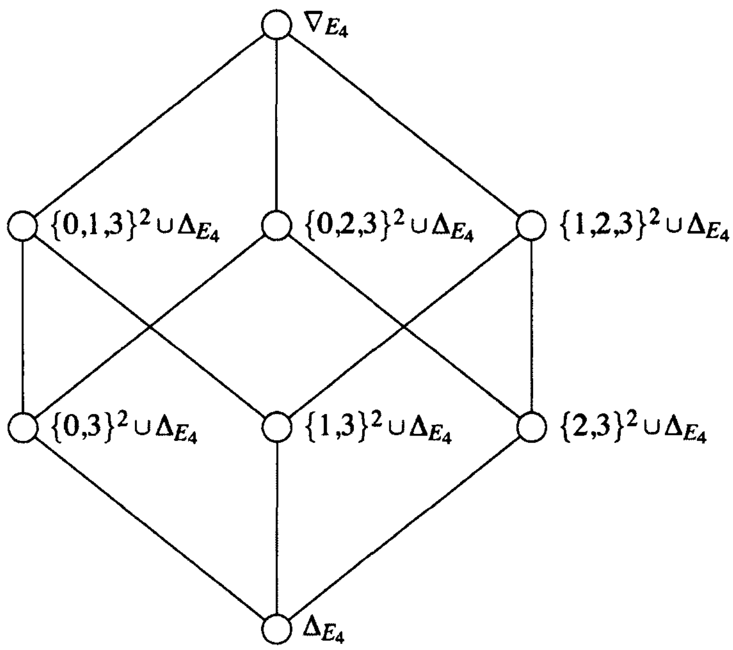

Figure 1.2: The Congruence Lattice of $\mathbf{E}_4$

Is the trivial algebra really subdirectly irreducible? The answer depends on whom this question is asked to. See §8 of Chapter 2 of [2]. For our purposes, we consider the trivial algebra to be subdirectly irreducible.

As we shall see in Theorem 1.2.8, in the next subsection, subdirectly irreducible algebras are a key component in understanding varieties.

1.2.4 Varieties

A class of objects that is of great interest in Universal Algebra is the class of varieties. In this subsection, the definition of a variety is given and the relationship between the variety and subdirectly irreducible algebras in the variety is explained. We then define many popular subclasses of varieties. This subsection is concluded with a tool that is useful in classifying varieties: the residual character of a variety.

1.2.4.1 Definition of a Variety

Due to Birkoff and Tarski, there exist two very different views of a variety. We present both as, under certain conditions, one is easier to apply than the other and vice versa.

Let K denote a class of algebras, all of the same type.

- $\mathbb{P}(K)$ to be the class of all direct products of non-empty families of members in K,
- $\mathbb{S}(K)$ to be the class of all subalgebras of each member in K,
- $\mathbb{H}(K)$ to be the class of all homomorphic images of each member in K,
- $\mathbb{P}_S(K)$ to be the class of all subdirect products of non-empty families of members in K and
- $\mathbb{P}_{\text{fin}}(K)$ to be the class of all direct products of non-empty finite families of members in K.

Here, $\mathbb{P}, \mathbb{S}, \mathbb{H}$, $\mathbb{P}_S$ and $\mathbb{P}_{\text{fin}}$ are operations acting on a class of algebras of the same type. In particular $\mathbb{P}$, $\mathbb{S}$ and $\mathbb{H}$ play a critical role in the construction of a variety.

Definition. A non-empty class of algebras, all of the same type, that is closed under $\mathbb{P}$, $\mathbb{S}$ and $\mathbb{H}$ is called a *variety*.

If K is a non-empty class of algebras, all of the same type, then denote $\mathbb{V}(K)$ as the smallest variety that contains K. The class of algebras $\mathbb{V}(K)$ is called the *variety generated by K*. The variety $\mathcal{V}$ is said to be *finitely generated* if $\mathcal{V} = \mathbb{V}(K)$, where K is a finite set of finite algebras. If K contains exactly one member, suppose $\mathbf{B}$, then $\mathbb{V}(\mathbf{B})$ is commonly used in place of $\mathbb{V}(K)$ or $\mathbb{V}(\{\mathbf{B}\})$ to denote the variety generated by K.

For K, a non-empty class of algebras of the same type, we may use *Tarski's $\mathbb{HSP}$ Theorem* to interpret $\mathbb{V}(K)$ as the class of all homomorphic images of all subalgebras of all possible direct products of members in K.

Theorem 1.2.5 (Tarski's $\mathbb{HSP}$ Theorem). *Let K denote a non-empty class of algebras, all of the same type. Then* $\mathbb{V}(K) = \mathbb{HSP}(K)$.

We show that a finitely generated variety is equivalent to a variety generated by a single finite algebra, as these two constructions of a variety are often used interchangeably.

Lemma 1.2.6. *The variety $\mathcal{V}$ is finitely generated if and only if $\mathcal{V}$ is generated by a single finite algebra.*

Proof. For the forward implication, assume that $\mathcal{V}$ is a finitely generated algebra. That is, for a positive integer n, let $\{\mathbf{B}_i\}_{i=1}^n$ denote a finite set of finite algebras, all of the same type such that $\mathcal{V} = \mathbb{V}(\{\mathbf{B}_i\}_{i=1}^n)$. We show that

$$\mathbb{V}(\{\mathbf{B}_i\}_{i=1}^n) = \mathbb{V}\left(\prod_{i=1}^n \mathbf{B}_i\right) \tag{1.3}$$

to obtain the desired result: the variety $\mathcal{V}$ is generated by a single finite algebra.

Using Tarski's $\mathbb{HSP}$ Theorem, or Theorem 1.2.5,

$$\prod_{i=1}^n \mathbf{B}_i \in \mathbb{P}(\{\mathbf{B}_i\}_{i=1}^n) \subseteq \mathbb{HSP}(\{\mathbf{B}_i\}_{i=1}^n) = \mathbb{V}(\{\mathbf{B}_i\}_{i=1}^n).$$

Hence,

$$\mathbb{V}\left(\prod_{i=1}^{n} \mathbf{B}_i\right) \subseteq \mathbb{V}(\{\mathbf{B}_i\}_{i=1}^{n}). \tag{1.4}$$

Note that for all i between 1 and n, inclusive,

$$\pi_i : \left(\prod_{i=1}^{n} \mathbf{B}_i\right) \to \mathbf{B}_i$$

is a surjective homomorphism. That is,

$$\text{each} \quad \mathbf{B}_i \quad \text{is a homomorphic image of} \quad \prod_{i=1}^{n} \mathbf{B}_i.$$

Therefore,

$$\{\mathbf{B}_i\}_{i=1}^{n} \subseteq \mathbb{H}\left(\prod_{i=1}^{n} \mathbf{B}_i\right) \subseteq \mathbb{HSP}\left(\prod_{i=1}^{n} \mathbf{B}_i\right) = \mathbb{V}\left(\prod_{i=1}^{n} \mathbf{B}_i\right).$$

Thus,

$$\mathbb{V}(\{\mathbf{B}_i\}_{i=1}^{n}) \subseteq \mathbb{V}\left(\prod_{i=1}^{n} \mathbf{B}_i\right). \tag{1.5}$$

The subset relations in Statement (1.4) and Statement (1.5) yield the equality in Statement (1.3).

The reverse implication is immediate. □

Let K denote a class of algebras. If there exists a set of identities Σ, of type $\mathcal{F}$ over some set of variables, such that K is equal to the class of all algebras of type $\mathcal{F}$ that satisfy each identity in Σ, then K is called an *equational class*.

Theorem 1.2.7 (Birkoff). *Let K denote a class of algebras, all of the same type. The class K is a variety if and only if K is an equational class.*

Thus, there are two ways to interpret $\mathbb{V}(\mathbf{B})$:

1. the class of all homomorphic images of subalgebras of direct powers of $\mathbf{B}$ with a non-empty index, that is, $\mathbb{HSP}(\mathbf{B})$ or

2. the class of all algebras of type $\mathcal{F}$ that satisfy all of the identities in $\mathrm{Id}_{\{\mathbf{B}\}}$.

1.2.4.2 Essential Building Blocks of the Algebras in Varieties

The following Theorem explains the importance of determining what the subdirectly irreducible algebras in a given variety are.

Theorem 1.2.8 (Birkoff's Theorem). *Let $\mathcal{V}$ be a variety. Every member in $\mathcal{V}$ is isomorphic to a subdirect product of subdirectly irreducible members in $\mathcal{V}$.*

Therefore, the subdirectly irreducible members in a variety are, loosely speaking, the building blocks of the members in a variety. Their importance is that they yield a deeper understanding of the structure of varieties.

1.2.4.3 Flavours of Varieties

We give a brief outline of a few popular subclasses of varieties. The first few subclasses require that we look at the congruence lattice of the algebras in a given variety.

The algebra **B** is said to be *congruence-distributive* if **Con B** satisfies the *distributive identities*. That is, **B** is congruence distributive if **Con B** satisfies the identities

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \qquad \text{and} \qquad x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z).$$

This means that for all θ_1, θ_2 and θ_3 in Con **B**,

$$\theta_1 \vee^{\mathbf{Con\,B}} (\theta_2 \wedge^{\mathbf{Con\,B}} \theta_3) = (\theta_1 \vee^{\mathbf{Con\,B}} \theta_2) \wedge^{\mathbf{Con\,B}} (\theta_1 \vee^{\mathbf{Con\,B}} \theta_3)$$

and

$$\theta_1 \wedge^{\mathbf{Con\,B}} (\theta_2 \vee^{\mathbf{Con\,B}} \theta_3) = (\theta_1 \wedge^{\mathbf{Con\,B}} \theta_2) \vee^{\mathbf{Con\,B}} (\theta_1 \wedge^{\mathbf{Con\,B}} \theta_3).$$

If **Con B** satisfies the *meet-semidistributive formula*

$$\forall x \forall y \forall z \left[x \wedge y \approx x \wedge z \rightarrow x \wedge y \approx x \wedge (y \vee z) \right],$$

then **B** is said to be *congruence-meet-semidistributive*. Note that **Con B** satisfies the above first order formula if and only if for all θ_1, θ_2 and θ_3 in Con **B**,

$$\theta_1 \wedge \theta_2 \neq \theta_1 \wedge \theta_3 \qquad \text{or} \qquad \theta_1 \wedge \theta_2 = \theta_1 \wedge (\theta_2 \vee \theta_3)$$

Let σ_1 and σ_2 be two binary relations on the set B. The *relational product* of σ_1 and σ_2 is defined as

$$\{\langle a,c\rangle \mid \exists b \in B \text{ such that } \langle a,b\rangle \in \sigma_1 \text{ and } \langle b,c\rangle \in \sigma_2\}$$

and is denoted by $\sigma_1 \circ \sigma_2$. The congruence lattice of **B** is said to be *permutable* if for all θ_1 and θ_2 in Con **B**

$$\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 .$$

If the congruence lattice of **B** is *permutable*, then **B** is said to be *congruence-permutable*.

If the congruence lattice of **B** satisfies

$$\forall x \forall y \forall z \left[x \wedge y \approx x \rightarrow x \vee (y \wedge z) \approx y \wedge (x \vee z)\right]$$

then **Con B** is said to be *modular* and **B** is said to be *congruence-modular*.

Both congruence-distributivity and congruence-permutability imply congruence-modularity. Hence, if a variety is not congruence-modular, then that variety is neither congruence-distributive nor congruence-permutable.

One can determine if a finite lattice, and hence a finite congruence lattice, is not modular by looking at the Hasse diagram that corresponds to the lattice. Let $\mathbf{N}_5$ be the lattice depicted by the Hasse diagram in Figure 1.3.

Theorem 1.2.9 (Dedekind's Theorem). *A lattice is not modular if and only if* $\mathbf{N}_5$ *can be embedded into the lattice.*

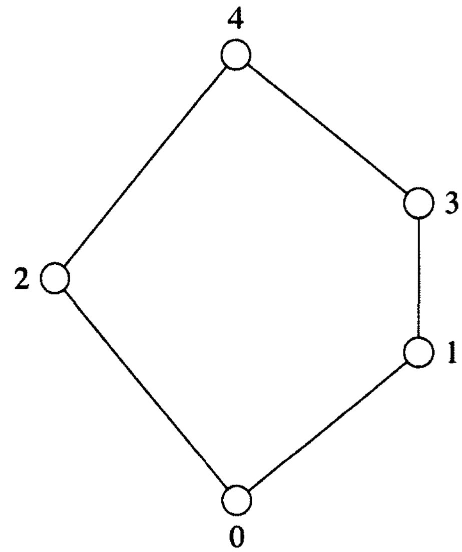

Figure 1.3: The Lattice $\mathbf{N}_5$

The following definition is taken from [16]. An algebra **B** is said to be *congruence-regular* if and only if every congruence of the algebra is completely and uniquely determined by any equivalence class. Groups and rings are examples of congruence-regular algebras.

Definition. If all algebras in a variety $\mathcal{V}$ are congruence-distributive, congruence-meet-semidistributive, congruence-permutable, congruence-modular or congruence-regular then $\mathcal{V}$ is said to be congruence-distributive, congruence-meet-semidistributive, congruence-permutable, congruence-modular or congruence-regular, respectively.

A *Mal'cev condition* is an identity or set of identities, that involves a particular term, such that when satisfied by all algebras in a given variety, implies a particular property must hold for the variety. For example, there exist Mal'cev conditions that can be used to determine if a variety is congruence-distributive, if a variety is congruence-permutable or if a variety is congruence-modular. The next example shows how a Mal'cev condition can be used to show that all varieties of lattices are congruence-distributive.

Example. All varieties of lattices are congruence-distributive. It is known that for a variety $\mathcal{V}$ and ternary term $t(x,y,z)$, if $\mathcal{V}$ satisfies

$$t(x,x,y) \approx t(x,y,x) \approx t(y,x,x) \approx x, \tag{1.6}$$

then $\mathcal{V}$ is congruence-distributive. The identities above form an example of a Mal'cev condition. If such a term exists, it is called a *majority term for* $\mathcal{V}$.

Let $\mathcal{V}$ be a variety of lattices, $\mathbf{L}$ be a lattice in $\mathcal{V}$ and

$$t(x,y,z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

It can be shown, using the absorption identity, that for all ℓ_1 and ℓ_2 in L,

$$t^{\mathbf{L}}(\ell_1,\ell_1,\ell_2) \approx t^{\mathbf{L}}(\ell_1,\ell_2,\ell_1) \approx t^{\mathbf{L}}(\ell_2,\ell_1,\ell_1) \approx \ell_1.$$

Therefore, $\mathbf{L}$ satisfies the identities listed in Statement (1.6) and hence $\mathcal{V}$ satisfies the same identities. Thus, $\mathcal{V}$ is congruence-distributive.

An algebra $\mathbf{B}$ is said to be *locally finite* if every finitely generated subalgebra of $\mathbf{B}$ is finite. A variety $\mathcal{V}$ is *locally finite* if every algebra in $\mathcal{V}$ is locally finite. As the next theorem states, every finitely generated variety is locally finite.

Theorem 1.2.10. *If* $\mathbf{B}$ *is a finite algebra, then* $\mathbb{V}(\mathbf{B})$ *is a locally finite variety.*

1.2.4.4 Residual Character of a Variety

A method of classifying varieties is by the existence of or size of the minimum bound on the cardinalities of the subdirectly irreducible algebras in the variety.

Definition. Let $\mathcal{V}$ be a variety. If there exists a least cardinal number κ such that the cardinality of the universe of every subdirectly irreducible member in the variety is less than κ, then κ is called the *residual character* or *residual bound* of $\mathcal{V}$.

There also exist more general classifications of varieties that involve the residual bound. A variety $\mathcal{V}$ is said to be

- *residually large* if no such cardinal number κ exists,
- *residually small* if the cardinal number κ exists,
- *residually finite* if the cardinality of the universe of each subdirectly irreducible member in $\mathcal{V}$ is finite and
- *residually* $< N$ if, for some fixed positive integer N, the cardinality of the universe of each subdirectly irreducible member in $\mathcal{V}$ is less than N.

Determining the residual character of a given variety, or deciding if a given variety is residually finite, is generally extremely difficult.

Example. Consider the groupoid $\mathbf{E}_5$, whose binary operation table is given in Table 1.6. The binary operation $*^{\mathbf{E}_5}$ is the first projection map on $\{0,1\}^2$. What is the residual character of $\mathbb{V}(\mathbf{E}_5)$?

$*^{\mathbf{E}_5}$	0	1
0	0	0
1	1	1

Table 1.6: The Operation Table of $*^{\mathbf{E}_5}$

$*^{\mathbf{B}}$	0	1	2	$\cdots$	$n-1$	n
0	0	0	0	$\cdots$	0	0
1	1	1	1	$\cdots$	1	1
2	2	2	2	$\cdots$	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$n-1$	$n-1$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$
n	n	n	n	$\cdots$	n	n

Table 1.7: The Operation Table of $*^{\mathbf{B}}$, for a Finite Algebra $\mathbf{B}$ in $\mathbb{V}(\mathbf{E}_5)$

As $\mathbf{E}_5$ satisfies the identity $x * y \approx x$, every member in $\mathbb{V}(\mathbf{E}_5)$ satisfies the same identity. Therefore, for all finite $\mathbf{B}$ in $\mathbb{V}(\mathbf{E}_5)$, the operation table of $*^{\mathbf{B}}$ looks like that given in Table 1.7.

Let b_1 and b_2 in B and define

$$\theta_{\langle b_1,b_2\rangle} = \mathrm{Cg}^{\mathbf{B}}(\langle b_1,b_2\rangle).$$

Further, let

$$\tau = \Delta_B \cup \{\langle b_1,b_2\rangle, \langle b_2,b_1\rangle\}.$$

Certainly, $\tau \subseteq \theta_{\langle b_1,b_2\rangle}$. Is there something in $\theta_{\langle b_1,b_2\rangle}$ that is not in τ? To answer this, we need to note that τ is the smallest reflexive, symmetric and transitive set containing $\langle b_1,b_2\rangle$. We verify that it is a subuniverse of $\mathbf{B}^2$ as this is tantamount to τ satisfying the compatibility property. To see that τ is a subuniverse of $\mathbf{B}^2$, note that if $\langle t_1,t_2\rangle$ and $\langle t_3,t_4\rangle$ is in τ then

$$\begin{aligned}\langle t_1,t_2\rangle *^{\mathbf{B}\times\mathbf{B}} \langle t_3,t_4\rangle &= \langle t_1 *^{\mathbf{B}} t_3, t_2 *^{\mathbf{B}} t_4\rangle \\ &= \langle t_1,t_2\rangle \\ &\in \Delta_B \cup \{\langle b_1,b_2\rangle, \langle b_2,b_1\rangle\}.\end{aligned}$$

Hence, τ is the smallest congruence on $\mathbf{B}$ that contains $\langle b_1,b_2\rangle$. That is,

$$\text{for all } b_1 \text{ and } b_2 \text{ in } B, \text{ we have } \theta_{\langle b_1,b_2\rangle} = \Delta_B \cup \{\langle b_1,b_2\rangle, \langle b_2,b_1\rangle\}. \tag{1.7}$$

Suppose that $\mathbf{B}$ is in $\mathbb{V}(\mathbf{E}_5)$ and $|B| > 2$. Without loss of generality, assume that $\{b_1,b_2,b_3\}$ is a three element subset of B. Under this assumption, and using Statement (1.7), we have

$$\begin{aligned}\theta_{\langle b_1,b_2\rangle} \wedge^{\mathbf{Con\,B}} \theta_{\langle b_2,b_3\rangle} &= \left(\Delta_B \cup \{\langle b_1,b_2\rangle, \langle b_2,b_1\rangle\}\right) \cap \left(\Delta_B \cup \{\langle b_2,b_3\rangle, \langle b_3,b_2\rangle\}\right) \\ &= \Delta_B \cup \left(\{\langle b_1,b_2\rangle, \langle b_2,b_1\rangle\} \cap \{\langle b_2,b_3\rangle, \langle b_3,b_2\rangle\}\right) \\ &= \Delta_B \cup \varnothing \\ &= \Delta_B.\end{aligned}$$

Neither $\theta_{\langle b_1,b_2\rangle}$ nor $\theta_{\langle b_2,b_3\rangle}$ equals Δ_B, but, their meet is Δ_B. Thus, the algebra $\mathbf{B}$ cannot possible have a monolith and therefore is not subdirectly irreducible. We have just shown that if $\mathbf{B}$ is in $\mathbb{V}(\mathbf{E}_5)$ such that $|B| > 2$ then $\mathbf{B}$ is not subdirectly irreducible. In fact, the only subdirectly irreducible algebras in $\mathbb{V}(\mathbf{E}_5)$, up to isomorphism, are $\mathbf{E}_5$ and the trivial algebra. That is, $\mathbb{V}(\mathbf{E}_5)$ is residually < 3.

Must all residually finite varieties be residually $< N$, for some positive integer N? The following example, taken from a footnote in [8], will show that the answer is no.

A *Pixley variety* is a variety of finite type that is residually finite, but, is not residually $<$ N, for any positive integer N. Note that a Pixley variety is not required to be finitely generated.

Example. Consider the variety $\mathcal{V}$ of all bi-unary algebras over the language $\{f, g\}$ that satisfy the following identities:

$$(f \circ g)(x) \approx x \qquad \text{and} \qquad (g \circ f)(x) \approx x. \tag{1.8}$$

In the remainder of this section, we show that $\mathcal{V}$ is a residually finite variety that is not residually $< N$, for any positive integer N, and that $\mathcal{V}$ is not finitely generated. Throughout the proofs of this claim, we make use of the fact, shown in the next lemma, that all unary algebras in $\mathcal{V}$ have at least one irredundant basis.

Lemma 1.2.11. *Every member of $\mathcal{V}$ has at least one irredundant basis.*

Proof. Let $\mathbf{B}$ be a member of $\mathcal{V}$. Let $\mathbf{G_B}$ denote the undirected multi-graph obtained by removing the labels from the edges and their direction, in the graph that corresponds to $\mathbf{B}$. For all r_1 and r_2 in B, say that $\langle r_1, r_2\rangle$ is in the relation R if and only if r_1 and r_2 are in the same component of $\mathbf{G_B}$. Thus, there exists a path from r_1 to r_2 in $\mathbf{G_B}$ if and only if $\langle r_1, r_2\rangle$ is in R. That is, there exists a finite sequence $\{b_i'\}_{i=0}^{n}$, of members in B such that $b_0' = r_1$

and $b'_n = r_2$ and

$$f^{\mathbf{B}}(b'_j) = b'_{j+1}, \qquad f^{\mathbf{B}}(b'_{j+1}) = b'_j, \qquad g^{\mathbf{B}}(b'_j) = b'_{j+1}, \qquad \text{or} \qquad g^{\mathbf{B}}(b'_{j+1}) = b'_j$$

for all $0 \leq j < n$. Note that the path from r_1 to r_2 must be finite.

Since **B** satisfies the identities in Statement (1.8),

$$\text{for all} \qquad 0 \leq j < n \qquad \text{we have} \qquad f^{\mathbf{B}}(b'_j) = b'_{j+1} \qquad \text{or} \qquad g^{\mathbf{B}}(b'_j) = b'_{j+1}. \tag{1.9}$$

Without loss of generality, assume that $f^{\mathbf{B}}(b'_0) = b'_1$. We show that $f^{\mathbf{B}}(b'_j) = b'_{j+1}$ for all $0 \leq j < n$, using weak induction. We have already dealt with the base case. For the induction hypothesis, assume that $f^{\mathbf{B}}(b'_{k-1}) = b'_k$, for some k less than n. Using Statement (1.9), we have

$$f^{\mathbf{B}}(b'_k) = b'_{k+1} \qquad \text{or} \qquad g^{\mathbf{B}}(b'_k) = b'_{k+1}.$$

If the former of the two outcomes occurs then we are done. Suppose, for a contradiction, that the latter of the two outcomes occurs. That is, assume that $g^{\mathbf{B}}(b'_k) = b'_{k+1}$. Since **B** satisfies the identities in Statement (1.8), we apply $g^{\mathbf{B}}$ to $f^{\mathbf{B}}(b'_{k-1}) = b'_k$ to obtain $g^{\mathbf{B}}(b'_k) = b'_{k-1}$, by the induction hypothesis. This is impossible as $g^{\mathbf{B}}$ is not a multi-valued operation. Since a contradiction occurred under the assumption that $g^{\mathbf{B}}(b'_k) = b'_{k+1}$, we must have that $f^{\mathbf{B}}(b'_k) = b'_{k+1}$. This completes the induction process. Thus, $f^{\mathbf{B}}(b'_j) = b'_{j+1}$ for all $0 \leq j < n$.

We have just shown that either

$$(f^{\mathbf{B}})^n(b'_0) = b'_n \qquad \text{or} \qquad (g^{\mathbf{B}})^n(b'_0) = b'_n.$$

Thus, the subuniverse generated by any member in B generates the component that that member belongs to in $\mathbf{G_B}$. Therefore, picking a representative from each equivalence class modulo R yields an irredundant basis. □

Note that in the proof of the previous lemma, producing an irredundant basis required the use of a choice function when picking one member from each equivalence class modulo R. That is, the Axiom of Choice is implicitly being applied.

Lemma 1.2.12. *Let* $\mathbf{B}$ *be member of* $\mathcal{V}$ *and* S *be an irredundant basis of* B*. If* $|S| > 3$*, then* $\mathbf{B}$ *is not subdirectly irreducible.*

Sketch of Proof. For a positive integer m, given a term $t = t_1 \circ t_2 \circ \cdots \circ t_m$, with t_i in $\{f, g\}$, set $\bar{t} = \bar{t}_m \circ \bar{t}_{m-1} \circ \cdots \circ \bar{t}_1$ where $\bar{t}_i$ is in $\{f, g\} \backslash \{t_i\}$. For example,

$$\text{if} \qquad t = f \circ g \circ g \circ f \circ f \circ f \circ g \qquad \text{then} \qquad \bar{t} = f \circ g \circ g \circ g \circ f \circ f \circ g.$$

Note that $(\bar{t} \circ t)(x) \approx x$ by Statement (1.8).

Let s_1 and s_2 be in S such that $s_1 \neq s_2$. Note that for all terms t_1 and t_2 of type $\{f, g\}$, we must have $t_1^{\mathbf{B}}(s_1) \neq t_2^{\mathbf{B}}(s_2)$ as otherwise, there exits a term t of type $\{f, g\}$ such that $t^{\mathbf{B}}(s_1) = s_2$ and hence S is not an irredundant basis. To see this, assume that $t_1^{\mathbf{B}}(s_1) = t_2^{\mathbf{B}}(s_2)$. Let $t = \overline{t_2} \circ t_1$ and

$$t^{\mathbf{B}}(s_1) = \left((\overline{t_2})^{\mathbf{B}} \circ t_1^{\mathbf{B}}\right)(s_1) = \left((\overline{t_2})^{\mathbf{B}} \circ t_2^{\mathbf{B}}\right)(s_2) = s_2.$$

Therefore,

$$\text{for all } s_1 \text{ and } s_2 \text{ in } S, \text{if } s_1 \neq s_2 \text{ then } \mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cap \mathrm{Sg}^{\mathbf{B}}(\{s_2\}) = \varnothing. \qquad (1.10)$$

We require Lemma 2.2.2 on page 57, that will be proved in a later chapter. The lemma states that, for unary algebras, if $\mathbf{M}_1$ is a subalgebra of $\mathbf{B}_1$ and θ is a congruence on $\mathbf{M}_1$, then

$$\mathrm{Cg}^{\mathbf{B}_1}(\theta) = \theta \cup \Delta_{B_1}.$$

3 By hypothesis $|S| > 3$. Without loss of generality, let $\{s_1, s_2, s_3, s_4\}$ be a four element subset of S. Then, acknowledging that

$$\left(\mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_2\})\right)^2 \quad \text{and} \quad \left(\mathrm{Sg}^{\mathbf{B}}(\{s_3\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_4\})\right)^2$$

are congruences on

$$\mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_2\}) \quad \text{and} \quad \mathrm{Sg}^{\mathbf{B}}(\{s_3\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_4\}),$$

respectively, and that both are subalgebras of **B**, we can apply Lemma 2.2.2 and utilize Statement (1.10) to obtain the following:

$$\begin{aligned}
&\mathrm{Cg}^{\mathbf{B}}\left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_2\})\right)^2\right) \wedge^{\mathbf{Con\,B}} \mathrm{Cg}^{\mathbf{B}}\left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_3\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_4\})\right)^2\right) \\
&= \left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_2\})\right)^2 \cup \Delta_B\right) \cap \left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_3\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_4\})\right)^2 \cup \Delta_B\right) \\
&= \left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_1\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_2\})\right)^2 \cap \left(\mathrm{Sg}^{\mathbf{B}}(\{s_3\}) \cup \mathrm{Sg}^{\mathbf{B}}(\{s_4\})\right)^2\right) \cup \Delta_B \\
&= \varnothing \cup \Delta_B \\
&= \Delta_B.
\end{aligned}$$

Since we obtained a meet of two non-trivial congruences that equate to Δ_B, the congruence lattice of **B** cannot possibly have a minimum non-trivial element. That is, **B** is not subdirectly irreducible, under the assumption that $|S| > 3$. □

We define an algebra called **Q** in $\mathcal{V}$ that plays a crucial role in the proof of an upcoming lemma. Let $\mathbf{Q} = \langle Q; f^{\mathbf{Q}}, g^{\mathbf{Q}} \rangle$, where Q is the set of integers and for all q in Q,

$$f^{\mathbf{Q}}(q) = q + 1 \quad \text{and} \quad g^{\mathbf{Q}}(q) = q - 1, \tag{1.11}$$

using regular integer addition and subtraction.

Lemma 1.2.13. *The algebra* $\mathbf{Q}$ *is not subdirectly irreducible.*

Proof. For each prime p, the equivalence relation modulo p, denoted by $\equiv_p$, is an equivalence relation on $\mathbf{Q}$ because it is an equivalence relation on the integers with the usual operations. Since for all q_1 and q_2 in Q and any prime p,

$$q_1 \equiv_p q_2 \qquad \text{if and only if} \qquad q_1 + 1 \equiv_p q_2 + 1 \qquad \text{if and only if} \qquad q_1 - 1 \equiv_p q_2 - 1,$$

the relation $\equiv_p$ is a congruence on $\mathbf{Q}$.

Consider the congruence

$$\theta = \bigcap_{p \text{ is prime}} \equiv_p .$$

Suppose that $\langle q_1, q_2 \rangle$ is in θ. Then for any prime p, we have $p \mid (q_1 - q_2)$ and $q_1 - q_2$ is an integer. This can only occur if $q_1 = q_2$. Therefore, $\theta = \Delta_Q$. That is, θ is a meet of non-Δ_Q congruences that is equal to Δ_Q. Hence, $\mathbf{Q}$ cannot have a monolith and is therefore not subdirectly irreducible. □

Lemma 1.2.14. *Let* $\mathbf{B}$ *be in* $\mathcal{V}$ *and* S *be an irredundant basis of* B. *If* $|B| \geq \omega$ *and* $|S| \leq 3$, *then* $\mathbf{Q}$ *can be embedded into* $\mathbf{B}$.

Proof. As $|S| \leq 3$ and $|B| \geq \omega$, there exists s in S such that $\left|\mathrm{Sg}^{\mathbf{B}}(\{s\})\right| \geq \omega$. Note that since f undoes g and vice versa,

$$\mathrm{Sg}^{\mathbf{B}}(\{s\}) = \{h^n(s) \mid h \text{ is in } \{f^{\mathbf{B}}, g^{\mathbf{B}}\} \text{ and } n \text{ is a positive integer}\} \cup \{s\}.$$

Since $\mathcal{V}$ satisfies the identities in Statement (1.8), for all h_1 and h_2 in $\{f^{\mathbf{B}}, g^{\mathbf{B}}\}$ and positive integers n_1 and n_2, we have

$$\begin{gathered} h_1^{n_1}(s) \neq s \qquad \text{and} \\ \text{if } h_1^{n_1}(s) = h_2^{n_2}(s) \text{ then } h_1 = h_2 \text{ and } n_1 = n_2 \end{gathered} \tag{1.12}$$

as otherwise $\left|\mathrm{Sg}^{\mathbf{B}}(\{s\})\right| < \omega$.

Let

$$\mathbf{D} = \left\langle \mathrm{Sg}^{\mathbf{B}}(\{s\}); f^{\mathbf{D}}, g^{\mathbf{D}} \right\rangle,$$

where $f^{\mathbf{D}} = f^{\mathbf{B}}\restriction_D$ and $g^{\mathbf{D}} = g^{\mathbf{B}}\restriction_D$. It is easier to work with **Q**, defined on page 28, than it is to work with **D**.

Since f undoes g and vice versa, for all positive integers n, define f^{-n} to be g^n. We show that $\alpha : \mathbf{Q} \to \mathbf{D}$, defined by

$$\alpha(q) = \begin{cases} (f^{\mathbf{D}})^q(s) & \text{if } q \neq 0 \\ s & \text{if } q = 0 \end{cases}$$

is an isomorphism and continue onward using **Q** in place of **D**. Statement (1.12) implies that α is injective. From its definition, α is surjective. To show that α is a homomorphism, we look at 3 cases, dependent upon what q is.

Case 1 Assume that q is in $Q/\{-1,0,1\}$.

$$\begin{aligned} \alpha\left(f^{\mathbf{Q}}(q)\right) &= \alpha(q+1) \\ &= (f^{\mathbf{D}})^{q+1}(s) \\ &= f^{\mathbf{D}}\left((f^{\mathbf{D}})^q(s)\right) \\ &= f^{\mathbf{D}}\left(\alpha(q)\right) \end{aligned} \qquad \begin{aligned} \alpha\left(g^{\mathbf{Q}}(q)\right) &= \alpha(q-1) \\ &= (f^{\mathbf{D}})^{q-1}(s) \\ &= g^{\mathbf{D}}\left((f^{\mathbf{D}})^q(s)\right) \\ &= g^{\mathbf{D}}\left(\alpha(q)\right) \end{aligned}$$

Case 2 Assume that q is in $\{1,-1\}$.

$$\begin{aligned} \alpha\left(f^{\mathbf{Q}}(1)\right) &= \alpha(2) \\ &= (f^{\mathbf{D}})^2(s) \\ &= f^{\mathbf{D}}\left(f^{\mathbf{D}}(s)\right) \\ &= f^{\mathbf{D}}(\alpha(1)) \end{aligned} \qquad \begin{aligned} \alpha\left(g^{\mathbf{Q}}(1)\right) &= \alpha(0) \\ &= s \\ &= g^{\mathbf{D}}\left(f^{\mathbf{D}}(s)\right) \\ &= g^{\mathbf{D}}\left(\alpha(1)\right) \end{aligned}$$

Use a similar idea when $q = -1$.

Case 3 Assume that $q = 0$.

$$\alpha(f^{\mathbf{Q}}(0)) = \alpha(1) \qquad \alpha(g^{\mathbf{Q}}(0)) = \alpha(-1)$$
$$= f^{\mathbf{D}}(s) \qquad = g^{\mathbf{D}}(s)$$
$$= f^{\mathbf{D}}(\alpha(0)) \qquad = g^{\mathbf{D}}(\alpha(0))$$

As, **D** is a subalgebra of **B** and **Q** is isomorphic to **D**, we have just shown that **Q** can be embedded into **B**. □

Theorem 1.2.15. *The variety $\mathcal{V}$ of all bi-unary algebras satisfying the identities $(f \circ g)(x) \approx x$ and $(g \circ f)(x) \approx x$ is residually finite.*

Sketch of Proof. Let **B** be an algebra in $\mathcal{V}$ and S be an irredundant basis of B. Recall Lemma 1.2.12: If $|S| > 3$, then **B** is not subdirectly irreducible. Further, recall Lemma 1.2.14: If $|B| \geq \omega$ and $|S| \leq 3$, then **Q** can be embedded into **B** and **Q** is not subdirectly irreducible.

To continue, we use Lemma 2.2.6 on page 59, that will be proved in a later chapter. The lemma states that if $\mathbf{M}_1$ is a subalgebra of $\mathbf{B}_1$ and $\mathbf{B}_1$ is a subdirectly irreducible unary algebra then $\mathbf{M}_1$ is subdirectly irreducible. The contrapositive of this Lemma applied to Lemma 1.2.14 yields the following: if $|B| \geq \omega$ and $|S| \leq 3$ then **B** is not subdirectly irreducible. This implication together with Lemma 1.2.12 imply that if $|B| \geq \omega$, then **B** is not subdirectly irreducible. Finally, we have shown that $\mathcal{V}$ is residually finite. □

Theorem 1.2.16. *The variety $\mathcal{V}$ of all bi-unary algebras satisfying the identities $(f \circ g)(x) \approx x$ and $(g \circ f)(x) \approx x$ is not residually $< N$, for any positive integer N.*

Proof. Suppose that **B** is a singly generated finite member in $\mathcal{V}$. Under these assumptions, the graph of **B** can best be described as a cycle. In particular, for a prime number p, the *prime cycles*

$$\mathbf{P}_p = \langle \{0, 1, 2, \ldots, p-1\}; f^{\mathbf{P}_p}, g^{\mathbf{P}_p} \rangle,$$

where

$$f^{\mathbf{P}_p}(x) = x + 1 \pmod p \qquad \text{and} \qquad g^{\mathbf{P}_p}(x) = x - 1 \pmod p,$$

are of interest. See Figure 1.4 for the graphs of $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_5$. The prime cycles are of interest because they are simple and hence subdirectly irreducible. Thus, since there exist infinitely many primes and $\mathcal{V}$ contains all prime cycles, $\mathcal{V}$ is not residually $< N$, for any positive integer N.

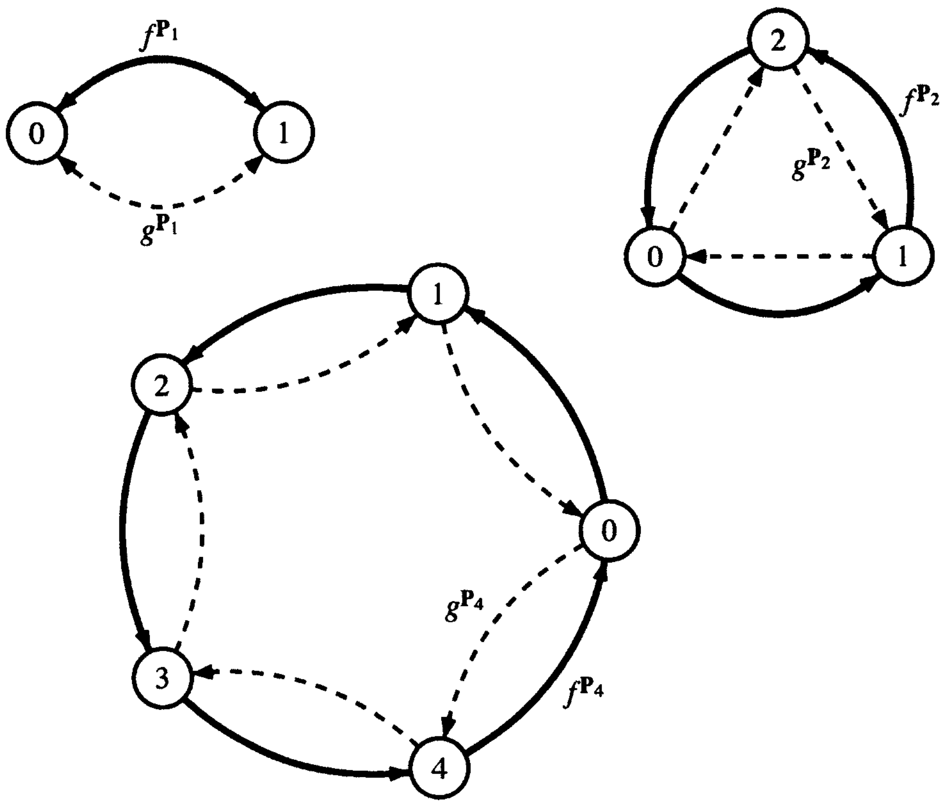

Figure 1.4: The Graphs of the Prime Cycles $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{P}_4$

□

Theorem 1.2.17. *The variety $\mathcal{V}$ of all bi-unary algebras satisfying the identities $(f \circ g)(x) \approx x$ and $(g \circ f)(x) \approx x$ is not finitely generated.*

Sketch of Proof. Notice that all of the prime cycles are singly generated. We can use this fact to show that $\mathcal{V}$ is not finitely generated. To do this, we use Lemma 1.3.3 on page 37, that will be proved in a later chapter. The Lemma can be used to show the following: for the finite unary algebra $\mathbf{W}$, where $|W| = w$, if $\mathbf{U}$ is a singly generated algebra in $\mathbb{V}(\mathbf{W})$, then $|U| \leq w^w$. Since $\mathcal{V}$ contains all prime cycles, there can exist no finite algebra that generates $\mathcal{V}$. Therefore, $\mathcal{V}$ is not finitely generated. □

Thus, $\mathcal{V}$ is a residually finite variety; however, $\mathcal{V}$ is neither residually $< N$, for any positive integer N, nor is $\mathcal{V}$ finitely generated.

1.3 Origin and Evolution of the Problem

The *Restricted Quackenbush Problem* started as a more general question posed to others in an article by Quackenbush. The crux of the problem is to determine if residually finite implies residually $< N$, for some positive integer N. Around the time of the problem's conception, "very little was known ... about [the] residual smallness of specific varieties" and posing the problem "nicely exposed our [everyones] ignorance and caused a lot of work to be done" (McKenzie, [10]).

1.3.1 Quackenbush's Problem

The earliest precursor of the Restricted Quackenbush Problem appeared in 1971:

> *" The example given below shows that an equational class $\mathcal{K}$ can have the following property: $(*)$ $\mathcal{K}$ has infinitely many finite subdirectly irreducible algebras but no infinite ones. In the example below, $\mathcal{K}$ is not generated by a finite algebra. Does there exist a finite algebra such that the equational class it generates has $(*)$? Can the algebra be of finite type; can it be a groupoid, semigroup, or group? "*
>
> *(Quackenbush, [15])*

Note that when discussing the number or amount of subdirectly irreducible algebras, Quackenbush implicitly means up to isomorphism.

Although written in 1969, the article [14], by A. Ju. Ol'šanskiĭ, can be used to answer Quackenbush's problem, with respect to groups. Specifically, Ol'šanskiĭ proves the following Theorem,

Theorem 1.3.1 (Theorem 2, [14]). *If $\mathcal{V}$ is a variety of finitely approximable groups, it contains a finite group* **B** *such that any other group in $\mathcal{V}$ is embedded in some full direct power of* **B**.

As is described in his paper, a finitely approximable group is a residually finite group. Thus, Ol'šanskiĭ's result can be restated as follows: If $\mathcal{V}$ is a residually finite variety, then there exists a finite group **B** such that $\mathcal{V} = \mathbb{ISP}(\mathbf{B})$. Projection maps can be used to show that, up to isomorphism, the subdirectly irreducible algebras in $\mathbb{ISP}(\mathbf{B})$ are all in $\mathbb{S}(\mathbf{B})$. Thus, up to isomorphism, there exist only a finite number of finite subdirectly irreducible algebras in $\mathbb{ISP}(\mathbf{B})$. Therefore, Ol'šanskiĭ showed that if $\mathcal{V}$ is a variety of groups, and hence an equational class of groups, that contains no infinite subdirectly irreducible algebras then $\mathcal{V}$ contains, up to isomorphism, a finite number of finite subdirectly irreducible algebras. That is, Ol'šanskiĭ's result can be used to answer Quackenbush's question negatively for groups.

Not long after the Quackenbush's problem was posed to the public, a more succinct version appeared in [17]. We will call this version of Quackenbush's problem the *Quackenbush Problem*.

Problem (Quackenbush Problem). *Does there exist a finite algebra* **B** *such that* $\mathbb{V}(\mathbf{B})$ *contains infinitely many finite subdirectly irreducible members but no infinite subdirectly irreducible members?*

We will show in the next section that the Quackenbush Problem can be rephrased as follows: Does there exist a finite algebra **B** such that $\mathbb{V}(\mathbf{B})$ is residually finite and not residually $< N$, for any positive integer N?

In 1975, Baldwin and Berman partially answered the Quackenbush problem. They assume that the language of **B** is finitary and countable. That is, the arity of the operation symbols in the language of **B** is finite and there can exist at most countably many operation symbols. To describe what Baldwin and Berman did, we need to describe what is meant by a variety having definable principal congruences.

Let Π denote the set of principal congruence formulas of type $\mathcal{F}$ over some infinite set of variables K. A variety $\mathcal{V}$ is said to have *definable principal congruences* if there exists a finite set $\Pi_0 \subseteq \Pi$ such that for all **B** in $\mathcal{V}$ and all b_1, b_2, b_3 and b_4 in B, the 2-tuple $\langle b_1, b_2 \rangle$ is in $\mathrm{Cg}^{\mathbf{B}}(\langle b_3, b_4 \rangle)$ if and only if, for some π in Π_0, the algebra **B** satisfies $\pi(b_1, b_2, b_3, b_4)$.

With regard to the Quackenbush Problem, Baldwin and Berman show "that there is no such variety with definable principal congruences" (Baldwin and Berman, [1]). Specifically, they proved that the Quackenbush Problem is answered negatively if the variety has definable principal congruences (Theorem 4). They prove this under the assumption that the variety is only residually small and not necessarily residually finite. Further, they do not assume that the variety is finitely generated.

As McKenzie showed that all directly representable varieties have definable principal congruences, the Quackenbush Problem is settled negatively for Boolean algebras. See Chapter 3 §13 of [2] for a definition of a directly representable variety. See Chapter 5 §3 of [2] for a proof showing that a given variety being directly representable implies that the same variety has definable principal congruences.

Further, Baldwin and Berman prove that if a variety is locally finite and has the congruence extension property, then that variety has definable principal congruences (Theorem 3).

See the set of exercises in Chapter 2 §5 of [2] for a definition of the congruence extension property. Hence, if **B** is a unary algebra, then the Quackenbush Problem is answered negatively, as all finitely generated varieties are locally finite and all classes of unary algebras have the congruence extension property.

A few years later, in 1979, Taylor answered the Quackenbush problem negatively when $\mathbb{V}(\mathbf{B})$ is congruence-permutable and congruence-regular. The central theorem in Taylor's article [17] is that if **B** is a finite algebra, with $\mathbb{V}(\mathbf{B})$ being congruence-regular and congruence-permutable, and $\mathbb{V}(\mathbf{B})$ containing arbitrarily large finite subdirectly irreducible algebras, then it contains an infinite subdirectly irreducible algebra. As all rings generate congruence-permutable and congruence-regular varieties, the Quackenbush Problem is answered negatively if **B** is a finite ring. That is, there does not exist a finite ring **B** such that $\mathbb{V}(\mathbf{B})$ contains infinitely many finite subdirectly irreducible members and no infinite subdirectly irreducible members.

1.3.2 The Reworded Quackenbush Problem

In 1981, the wording of the problem changed; but, the crux of the problem remained exactly the same. The following rewording of the Quackenbush Problem appeared in [4]:

Problem (Reworded Quackenbush Problem). *Does there exist a finite algebra* **B** *such that* $\mathbb{V}(\mathbf{B})$ *is residually finite and not residually* $< N$*, for any positive integer* N*?*

We show, in an upcoming theorem, that the Quackenbush Problem and the Reworded Quackenbush Problem are equivalent problems. First, we require a few technical lemmas.

1.3.2.1 The Equivalence of the Quackenbush Problem and the Reworded Quackenbush Problem

The following Lemma makes use of the *free algebra of type $\mathcal{F}$ over the set of variables $\overline{X} = \{\overline{x}\}_{i=1}^{n}$ in the variety $\mathcal{V}$*. This special algebra is denoted by $\mathbf{F}_{\mathcal{V}}(\overline{X})$. Though much can be said of the free algebra, all we need to know is that $\mathbf{F}_{\mathcal{V}}(\overline{X})$ is in $\mathcal{V}$ and that $\mathbf{F}_{\mathcal{V}}(\overline{X})$ has the *universal mapping property for $\mathcal{V}$ over $\overline{X}$*. That is, for all $\mathbf{B}$ in $\mathcal{V}$ and every map

$$\alpha{:}\overline{X} \to B, \qquad \text{there exists a homomorphism} \qquad \beta{:}\mathbf{F}_{\mathcal{V}}(\overline{X}) \to \mathbf{B}$$

such that for all $\overline{x}$ in $\overline{X}$, we have $\beta(\overline{x}) = \alpha(\overline{x})$. Note that if $\alpha(\overline{X})$ generates B, then $\mathbf{B}$ is a homomorphic image of $\mathbf{F}_{\mathcal{V}}(\overline{X})$. Another interesting property of the free algebra is stated as the following Lemma.

Lemma 1.3.2. *Let $\mathbf{B}$ be an algebra of type $\mathcal{F}$. For any positive integer n, the free algebra of type $\mathcal{F}$ over a set $\overline{X}$ of n variables $\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{X})$ can be embedded into $\mathbf{B}^{|B|^{|\overline{X}|}}$.*

The previous Lemma is posed as an exercise on page 85 of §12, chapter 2 of [2]. For a complete description of the free algebra, see §10 and §11 of Chapter 2 in [2] or see Chapter 4.11 in [13].

Lemma 1.3.3. *Let $\mathbf{B} = \langle B;\mathcal{F}\rangle$ be a finite algebra, where $|B| = b$, and let $\mathbf{M}$ be an algebra in $\mathbb{V}(\mathbf{B})$. If M has a finite irredundant basis of size n then $|M| \le b^{b^n}$.*

Proof. Note that if $\mathbf{B}$ is trivial then every member in $\mathbb{V}(\mathbf{B})$ is trivial and the claim is true. Hence, we may assume that $\mathbb{V}(\mathbf{B})$ contains a non-trivial member. That is, we may assume that $b > 1$.

For some positive integer n, let $S = \{s_i\}_{i=1}^{n}$ denote an irredundant basis of M and let $\overline{Y} = \{y_i\}_{i=1}^{n}$ be a set of n variables. Recall that $\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y})$ has the universal mapping property.

Define $\alpha:\overline{Y} \to S$ by $\alpha(\overline{y}_i) = s_i$. By the universal mapping property, there exists a homomorphism

$$\beta:\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y}) \to \mathbf{M}$$

such that $\beta(\overline{y}_i) = s_i$ for all $1 \le i \le n$.

Note that as M is generated by S, every member of M is equal to some term realized in $\mathbf{M}$ and applied to members of S. That is, for all m in M, there exists at least one term $t_m(y_1, y_2, \ldots, y_n)$ of type $\mathcal{F}$ over $Y = \{y_i\}_{i=1}^{n}$ such that $t_m^{\mathbf{M}}(s_1, s_2, \ldots, s_n) = m$. Therefore, for all m in M, we have

$$\begin{aligned}\beta\left(t_m^{\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y})}(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)\right) &= t_m^{\mathbf{M}}(\beta(\overline{y}_1), \beta(\overline{y}_2), \ldots, \beta(\overline{y}_n)) \\ &= t_m^{\mathbf{M}}(s_1, s_2, \ldots, s_n) \\ &= m.\end{aligned}$$

Thus, β is a surjective homomorphism.

Using Lemma 1.3.2, we obtain the following sequence of inequalities:

$$|M| \le |F_{\mathbb{V}(\mathbf{B})}(\overline{Y})| \le |B|^{|B|^{|\overline{Y}|}} = |B|^{|B|^n} = b^{b^n}.$$

This proves the lemma. □

Corollary 1.3.4. *Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a finite algebra of finite type. For any positive integer n, there exists only a finite number of members in $\mathbb{V}(\mathbf{B})$, up to isomorphism, that are minimally generated by n elements.*

Proof. Let $|B| = b$. By hypothesis, we have $|\mathcal{F}| < \omega$ and $b < \omega$. For each f in $\mathcal{F}$, let $r(f)$ denote the arity of f. Note that if K is a finite set that contains exactly k elements, then there exist k^{k^p} distinct p-ary operations that can be defined on K. Thus, up to isomorphism,

$$\text{there can exist at most } \prod_{f \in \mathcal{F}} k^{k^{r(f)}} \text{ algebras of type } \mathcal{F} \text{ that have a universe with cardinality } k. \tag{1.13}$$

Using Lemma 1.3.3, if **M** in $\mathbb{V}(\mathbf{B})$ is minimally generated by n elements then $|M| \leq b^{b^n}$. Therefore, using Statement (1.13), there can exist, up to isomorphism, at most

$$\sum_{k=1}^{b^{b^n}} \prod_{f \in \mathcal{F}} k^{k^{r(f)}}$$

members of $\mathbb{V}(\mathbf{B})$ that are minimally generated by n elements. As all constants are finite, the claim is proven. □

The next Lemma yields some insight into a finitely generated variety being residually $< N$, for some positive integer N.

Lemma 1.3.5. *Let* **B** *be a finite algebra of type* $\mathcal{F}$ *that generates a residually finite variety. The variety generated by* **B** *contains finitely many finite subdirectly irreducible members, up to isomorphism, if and only if* $\mathbb{V}(\mathbf{B})$ *is residually* $< N$*, for some positive integer* N.

Proof. Since **B** generates a residually finite variety, all subdirectly irreducible algebras in $\mathbb{V}(\mathbf{B})$ are finite. If $\mathbb{V}(\mathbf{B})$ contains finitely many finite subdirectly irreducible members, up to isomorphism, then take the subdirectly irreducible member with maximum cardinality and add 1 to determine the residual character of the variety.

Now suppose that $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N. Let $|B| = b$ and $\overline{Y}$ denote a set of N variables. Since $\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y})$ has the universal mapping property, every algebra in $\mathbb{V}(\mathbf{B})$ with a universe of cardinality less than or equal to N is a homomorphic image of $\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y})$. Thus, by the assumption that $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N, all subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$ are members in $\mathbb{H}(\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y}))$. By Lemma 1.3.3, we know that $|F_{\mathbb{V}(\mathbf{B})}(\overline{Y})| \leq b^{b^N}$. Hence, up to isomorphism, $\mathbb{H}(\mathbf{F}_{\mathbb{V}(\mathbf{B})}(\overline{Y}))$ contains a finite number of algebras. That is, $\mathbb{V}(\mathbf{B})$ contains finitely many finite subdirectly irreducible members, up to isomorphism, □

We now have all that is needed to show that answering the Reworded Quackenbush Problem answers the Quackenbush Problem and vice versa.

Theorem 1.3.6. *The Reworded Quackenbush Problem and the Quackenbush Problem are equivalent.*

Proof. If the Reworded Quackenbush Problem is answered *no* then there does not exist a finite algebra $\mathbf{B}$ that generates a variety that is residually finite and not residually $< N$, for any positive integer N. Therefore, for any finite algebra $\mathbf{M}$, if the variety $\mathbb{V}(\mathbf{M})$ is residually finite, then $\mathbb{V}(\mathbf{M})$ residually $< N$, for some finite N. Thus, using Lemma 1.3.5, if $\mathbb{V}(\mathbf{B})$ contains no infinite subdirectly irreducible members, then it contains finitely many finite subdirectly irreducible algebras. That is, the Quackenbush Problem is answered *no*.

If the Reworded Quackenbush Problem is answered *yes*, then there exists a finite algebra $\mathbf{B}$ such that $\mathbb{V}(\mathbf{B})$ is residually finite and not residually $< N$. Using Lemma 1.3.5, there exists infinitely many finite subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$, but, no infinite ones. That is, the Quackenbush Problem is answered *yes*. □

Therefore, since the Quackenbush Problem and the Reworded Quackenbush Problem are equivalent, we may state them interchangeably.

1.3.2.2 Before the Reworded Quackenbush Problem

Although the Reworded Quackenbush Problem was asked in 1981, a paper published in 1964 by Foster and Pixley [3] can be used to answer the problem if $\mathbb{V}(\mathbf{B})$ is congruence-distributive, as pointed out by Kearnes and Willard in [8]. Specifically, Theorem 2.5 of [3] can be used. We state Theorem 2.5 without proof and proceed to show that this theorem can be used to answer the Reworded Quackenbush Problem if $\mathbb{V}(\mathbf{B})$ is congruence distributive.

Theorem 1.3.7 (Theorem 2.5 of [3]). *If* $\mathbf{B}$ *is congruence distributive and* $\mathbf{B}$ *is isomorphic to a subdirect product of algebras* $\mathbf{B}_1, \ldots, \mathbf{B}_n$*, where* n *is a positive integer, then every homomorphic image of* $\mathbf{B}$ *is isomorphic to a subdirect product of homomorphic images of* *the* $\mathbf{B}_i$.

Notice that Foster and Pixley's theorem is only applicable to congruence-distributive algebras that are isomorphic to a subdirect product of a *finite* number of algebras. The following well known Lemma, will help to deal with this problem.

Lemma 1.3.8. *Suppose that* $\mathbf{B}$ *is a finite algebra. If* $\mathbf{M}$ *is a member in* $\mathbb{V}(\mathbf{B})$ *and* $|M|$ *is finite, then* $\mathbf{M}$ *is a member in* $\mathbb{HSP}_{fin}(\mathbf{B})$.

For the proof of the upcoming theorem, we will need to know what a directly indecomposable algebra is. A *directly indecomposable* algebra is one that is not isomorphic to a direct product of two non-trivial algebras. They are related to subdirectly irreducible algebras in that all subdirectly irreducible algebras are directly indecomposable; but, the converse is not necessarily true.

In the proof of the following Theorem, we make use of Foster and Pixley's Theorem 2.5 in [3], or Theorem 1.3.7, to show that Reworded Quackenbush Problem is answered negatively with respect to congruence-distributive algebras.

Theorem 1.3.9. *The Reworded Quackenbush Problem is answered negatively with respect to congruence-distributive algebras.*

Proof. Let $\mathbf{B}$ be a finite algebra and assume that $\mathbb{V}(\mathbf{B})$ is residually finite and congruence-distributive. We show that, under this assumption, the variety generated by $\mathbf{B}$ is residually $< N$, for some positive integer N. Doing this implies that the Reworded Quackenbush Problem is answered negatively.

If $\mathbf{B}$ is the trivial algebra then $\mathbb{V}(\mathbf{B})$ contains only trivial algebras. Thus, we may assume that $\mathbf{B}$ is not the trivial algebra.

By Tarski's $\mathbb{HSP}$ Theorem, on page 17, we have

$$\mathbb{V}(\mathbf{B}) = \mathbb{HSP}(\mathbf{B}) = \mathbb{H}\Big(\mathbb{S}\big(\mathbb{P}(\mathbf{B})\big)\Big).$$

As $\mathbf{B}$ is not trivial, the only algebra in $\mathbb{P}(\mathbf{B})$ that is possibly directly indecomposable is $\mathbf{B}$. The contrapositive of subdirectly irreducible implies directly indecomposable yields the following: with the exception of possibly $\mathbf{B}$, all algebras in $\mathbb{P}(\mathbf{B})$ are not subdirectly irreducible.

Suppose that $\mathbf{M}$ is a subdirectly irreducible member in $\mathbb{S}(\mathbb{P}(\mathbf{B}))$ and that $\mathbf{M}$ is not a member in $\mathbb{P}(\mathbf{B})$. For some index set I, the algebra $\mathbf{M}$ is a subalgebra of $\mathbf{B}^I$. As homomorphisms preserve subalgebras, for all i in I, we have $\rho_i(\mathbf{M})$ is a subalgebra of $\mathbf{B}$, where ρ_i is the i^{th} projection map on $\mathbf{M}$. Note that

$$\mathbf{M} \leq \prod_{i \in I} \rho_i(\mathbf{M}) \leq \mathbf{B}^I.$$

Thus, $\mathbf{M}$ is a subdirect product of the members in $\{\rho_i(\mathbf{M})\}_{i \in I}$.

As $\mathbf{M}$ is subdirectly irreducible and the identity map from $\mathbf{M}$ to $\prod_{i \in I} \pi_i(\mathbf{M})$ is an embedding, there must exist some i in I such that $\mathbf{M} \cong \rho_i(\mathbf{M})$. As $\rho_i(\mathbf{M})$ is a subalgebra of $\mathbf{B}$, we have $|M| < |B|$. Hence, the subdirectly irreducible members in $\mathbb{S}(\mathbb{P}(\mathbf{B}))$ all have a universe with cardinality less than $|B|$.

Now suppose that $\mathbf{M}$ is a subdirectly irreducible member in $\mathbb{H}(\mathbb{S}(\mathbb{P}(\mathbf{B})))$ and that $\mathbf{M}$ is not isomorphic to a member in $\mathbb{S}(\mathbb{P}(\mathbf{B}))$. Since $\mathbb{V}(\mathbf{B})$ is residually finite, the algebra $\mathbf{M}$ is finite. There must be an index set J such that $\mathbf{M}$ is a homomorphic image of $\mathbf{S}$ where $\mathbf{S}$ is a subalgebra of $\mathbf{B}^J$. By Lemma 1.3.8, the index set J can be assumed to be finite. Like before, for all j in J, the homomorphic image $\rho_j(\mathbf{S})$ is a subalgebra of $\mathbf{B}$, where ρ_j is the j^{th} projection map on $\mathbf{S}$. Further, $\mathbf{S}$ is a subdirect product of the members in $\{\rho_j(\mathbf{S})\}_{j \in J}$, that is

$$\mathbf{S} \leq \prod_{j \in J} \rho_j(\mathbf{S}) \leq \mathbf{B}^I.$$

By applying Foster and Pixley's Theorem 2.5 in [3], or Theorem 1.3.7, we obtain the following: the algebra $\mathbf{M}$ is isomorphic to a subdirect product of homomorphic images of the members in $\{\rho_j(\mathbf{S})\}_{j\in J}$. As **M** is subdirectly irreducible, there exists j in J such that **M** is isomorphic to a homomorphic image of $\rho_j(\mathbf{S})$. Thus, **M** is a homomorphic image of $\rho_j(\mathbf{S})$. Let $\alpha{:}\,\rho_j(\mathbf{S}) \to \mathbf{M}$ be this homomorphism. We have just shown that

$$|M| = \left|\alpha\left(\rho_j(S)\right)\right| \leq \left|\rho_j(S)\right| \leq |B|.$$

Thus, all subdirectly irreducible members in $\mathbb{H}\left(\mathbb{S}\left(\mathbb{P}(\mathbf{B})\right)\right)$ have universes with cardinality no greater than $|B|$. That is, $\mathbb{V}(\mathbf{B})$ is residually $< |B|$. □

As all lattices can be shown to generate congruence-distributive varieties, using a Mal'cev condition, the Reworded Quackenbush Problem is answered negatively if **B** is a lattice. Further, as all n-Post algebras, Boolean algebras and Heyting algebras can be shown to generate arithmetical varieties and hence congruence-distributive varieties, using a Mal'cev like condition, Quackenbush's problem is answered negatively if **B** is a n-Post algebra, Boolean algebra or a Heyting algebra.

Even though the Quackenbush Problem has been dealt with by Foster and Pixley in 1964, a paper by Jónsson [7], written in 1967, can also be used to answer the Reworded Quackenbush Problem, with respect to algebras that generate congruence-distributive varieties. In Jónsson's paper, Corollary 3.4 states that if **B** is finite and $\mathbb{V}(\mathbf{B})$ is congruence-distributive, then every subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ belongs to $\mathbb{HS}(\mathbf{B})$. Therefore, $\mathbb{V}(\mathbf{B})$ is residually less than $|B|$. This is a stronger result than that obtained by answering the Reworded Quackenbush Problem. Specifically, Jónsson's result reveals that all congruence-distributive varieties that are generated by a finite algebra are residually $< N$, for some positive integer N. Not only this, Jónsson's result gives the location, with respect to the class operations $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$, of the subdirectly irreducible members.

It is interesting to note that in [1], Berman and Baldwin define and utilize the residual character of a variety. For example, they define and use the terms residually $< N$ and residually small. However, they do not use these terms when stating Quackenbush's Problem. That is, the tools to state the Reworded Quackenbush Problem were around in 1975; but, were not used to do so until 1981 in an article by Freese and McKenzie, [4].

1.3.2.3 After the Reworded Quackenbush Problem

Theorem 8 of [4] shows that for an algebra **B** of size m, if **B** is in a congruence-modular variety then $\mathbb{V}(\mathbf{B})$ is residually small if and only if $\mathbb{V}(\mathbf{B})$ is

$$\text{residually} < (\ell+1)!\cdot m+1, \qquad \text{where} \qquad \ell = m^{m^{m+1}}.$$

In other words, the Reworded Quackenbush Problem is answered negatively if **B** generates a congruence-modular variety. As congruence-permutable implies congruence-modular and all varieties of quasigroups are congruence-permutable, the Reworded Quackenbush Problem is answered negatively for quasigroups.

During the same year, McKenzie answered the Reworded Quackenbush Problem negatively when **B** is a semigroup. See Theorem 30 of [9]. In this article, McKenzie does more than just answer the Reworded Quackenbush Problem, he determines conditions for a variety of semigroups to be residually small.

In 1982, McKenzie proved that any locally finite and residually small variety of K-algebras is residually $< N$, for some positive integer N. See Theorem 8.1 in [10]. For a definition of a K-algebra, see [10].

An example of a finite algebra that generated a residually finite variety that was *not* residually $< N$, for any positive integer N, was found in 1996 by McKenzie in [12]. That is, McKenzie found an example of an algebra that could be used to answer the Reworded Quackenbush Problem affirmatively. In his own words, he "destroy[ed] the Quackenbush Conjecture" (McKenzie, [12]).

1.3.3 Dealing with the Destruction of Quackenbush's Problem

Though the Reworded Quackenbush Problem, and hence the Quackenbush Problem, was answered by McKenzie, the examples used to achieve this answer were algebras of infinite type and "there is no obvious way to convert them into algebras of finite type with the same residual bound" (McKenzie [12]). Hence, a new problem can be asked. The following problem is taken from [8]:

Problem (Restricted Quackenbush Problem). *Let* $\mathbf{B}$ *be a finite algebra of finite type. If* $\mathbb{V}(\mathbf{B})$ *is residually finite, must* $\mathbb{V}(\mathbf{B})$ *be residually* $< N$*, for some positive integer* N*?*

How is the Restricted Quackenbush Problem related to the Reworded Quackenbush Problem and hence the Quackenbush Problem? The following Lemma answers this question.

Lemma 1.3.10. *If the Reworded Quackenbush Problem is answered negatively for a class of algebras then the Restricted Quackenbush Problem is answered affirmatively for that class of algebras.*

Proof. Suppose that the Reworded Quackenbush Problem is answered *no* for a class of algebras $\mathcal{K}$. That is, there does not exist a finite algebra $\mathbf{B}$ in $\mathcal{K}$ such that $\mathbb{V}(\mathbf{B})$ is residually finite and not residually $< N$, for any positive integer N. Hence, for all $\mathbf{B}$ in $\mathcal{K}$, if $\mathbb{V}(\mathbf{B})$ is residually finite then $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N. That is, the Restricted Quackenbush Problem is answered *yes* for the class of algebras $\mathcal{K}$. □

Note the that the inverse of the previous Lemma is not necessarily true. Suppose that the Reworded Quackenbush Problem is answered *yes* for a class of algebras $\mathcal{K}$. Under this assumption, there exists a finite algebra $\mathbf{B}$ in $\mathcal{K}$ such that $\mathbb{V}(\mathbf{B})$ is residually finite and not residually $< N$, for any positive integer N. This finite algebra, like McKenzie's, may not be of finite type. That is, **B** may not satisfy the hypothesis of the Restricted Quackenbush Problem and hence, an answer to the Restricted Quackenbush Problem with respect to the algebras in $\mathcal{K}$ is not immediate.

During 1999, in the article [8], Kearnes and Willard proved that the Restricted Quackenbush Problem is answered affirmatively when $\mathbb{V}(\mathbf{B})$ is congruence-meet-semidistributive. Specifically, they answer *Pixley's Problem* (Theorem 4.1). Pixley's Problem is similar to the Restricted Quackenbush Problem and can be used to answer it. The authors point out that due to meet-semidistributivity, the Restricted Quackenbush Problem is answered affirmatively with respect to algebras that include a semilattice operation.

1.4 Restatement of the Problem

The problem of interest is the Restricted Quackenbush Problem.

Problem (Restricted Quackenbush Problem). *Let* **B** *be a finite algebra of finite type. If* $\mathbb{V}(\mathbf{B})$ *is residually finite, must* $\mathbb{V}(\mathbf{B})$ *be residually* $< N$*, for some positive integer* N*?*

This problem has been answered with respect to many popular algebras: groups, Heyting algebras, lattices, quasigroups, rings and semigroups. The problem has also been answered when $\mathbb{V}(\mathbf{B})$ satisfies certain properties: being congruence-distributive, being congruence-modular, being congruence-permutable or having definable principal congruences. Even with all of this work done, the problem, to date, remains open when considering an arbitrary algebra.

In Chapter 2, we give an explicit proof showing that the Restricted Quackenbush Problem is answered affirmatively with respect to unary algebras. Then our attention turns to groupoids. Specifically, in Chapter 3, we show that groupoids that generate a residually finite variety must satisfy an identity of a particular form. Then, in Chapter 4, we look at absorbing groupoids and show that if a certain property holds then the Restricted Quackenbush Problem is answered affirmatively. Lastly, in Chapter 5, a summary of what was done in this thesis is given and some questions are posed to the reader.

Chapter 2

Unary Algebras and the Restricted Quackenbush Problem

Amongst other results, Baldwin and Berman, in [1], show that for $\mathbf{B}$, a finite unary algebra of finite type, if $\mathbb{V}(\mathbf{B})$ is residually finite then $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N. That is, they show that the Restricted Quackenbush Problem is answered affirmatively, with respect to unary algebras. In *Subsection 1.3.1 Quackenbush's Problem*, on page 35, are details as to how Baldwin and Berman showed their result. In particular, Baldwin and Berman make use of all unary algebras having the Congruence Extension Property.

In this chapter, we show that the Restricted Quackenbush Problem is answered affirmatively, with respect to unary algebras without the explicit use of the Congruence Extension Property.

As an introduction to unary algebras, the first section of this chapter will present Yoeli's result on connected subdirectly irreducible mono-unary algebras. These algebras are easy to describe and have a nice visual representation.

Since there are subdirectly irreducible mono-unary algebras that are not connected, Yoeli's result is not sufficient to answer the Restricted Quackenbush Problem for mono-unary algebras. The remaining sections provide an explicit proof to show that the Restricted Quackenbush Problem is answered affirmatively for arbitrary unary algebras.

2.1 Yoeli's Result

A 1967 article [19], by Yoeli, can be used to visually classify all connected subdirectly irreducible mono-unary algebras. To state Yoeli's result, we need to make use of his definitions.

Definition (Yoeli, [19]). A finite mono-unary algebra $\mathbf{B}$ is *connected* if the corresponding graph is connected.

Alternatively, a finite mono-unary algebra $\mathbf{B} = \langle B; f^{\mathbf{B}} \rangle$ is connected if for all b_1 and b_2 in B there exists positive integers n_1 and n_2 such that

$$(f^{\mathbf{B}})^{n_1}(b_1) = (f^{\mathbf{B}})^{n_2}(b_2).$$

Definition (Yoeli, [19]). A finite mono-unary algebra $\mathbf{B}$ is *irreducible* if whenever $\mathbf{B}$ can be embedded into a direct product of two other mono-unary algebras, suppose $\mathbf{B}_1$ and $\mathbf{B}_2$, then either $\mathbf{B}$ can be embedded into $\mathbf{B}_1$ or $\mathbf{B}$ can be embedded $\mathbf{B}_2$.

We show that Yoeli's definition of irreducible and our definition of subdirectly irreducible are equivalent properties of a finite mono-unary algebra. We need the following technical Lemma.

Lemma 2.1.1. *Let $\mathbf{B}$ and $\mathbf{M}$ be finite algebras of the same type. Suppose that $\mathbf{B}$ can be embedded into $\mathbf{M}^I$, for some infinite index set I. Then $\mathbf{B}$ can be embedded into $\mathbf{M}^J$ for some finite index J.*

Proof. Suppose that $|B| = b$ and $|M| = m$. Let α be the embedding of $\mathbf{B}$ into $\mathbf{M}^I$. Since $\mathbf{B}$ is finite, the algebra $\alpha(\mathbf{B})$ is also finite. Thus, $|\alpha(B)| = b$. Enumerate the elements in $\alpha(B)$ by $\{a_j\}_{j=1}^b$ and consider the relation

$$\tau = \left\{ \langle a_j(i) \rangle_{j=1}^b \mid i \in I \right\}.$$

Note that $|\tau| < m^b < \omega$. Now define the relation σ on I as follow: for all i_1 and i_2 in I, let $\langle i_1, i_2 \rangle$ be in σ if and only if $\langle a_j(i_1) \rangle_{j=1}^b = \langle a_j(i_2) \rangle_{j=1}^b$. The relation τ can be shown to be an equivalence relation. Then, since τ is a finite relation, the collection I/σ must also be finite.

We will show that $\alpha(\mathbf{B})$ can be embedded into $\mathbf{M}^{I/\sigma}$ using the mapping $\beta : \alpha(\mathbf{B}) \to \mathbf{M}^{I/\sigma}$ where for all a_j in $\alpha(B)$, we have $(\beta(a_j))(i/\sigma) = a_j(i)$. Then, since $\mathbf{B} \cong \alpha(\mathbf{B})$ and I/σ is a finite collection, the claim will be proved. Thus, to complete the proof, we verify that β is well-defined and injective and is a homomorphism.

Note that if for some a_j in $\alpha(B)$, and i_1/σ and i_2/σ in I/σ such that $i_1/\sigma = i_2/\sigma$, then $a_j(i_1) = a_j(i_2)$ and hence,

$$(\beta(a_j))(i_1/\sigma) = a_j(i_1) = a_j(i_2) = (\beta(a_j))(i_2/\sigma).$$

Thus, β is a well-defined map.

Let a_{j_1} and a_{j_2} be in $\alpha(B)$ such that $\beta(a_{j_1}) = \beta(a_{j_2})$. Hence, for all i/σ in I/σ, we have

$$(\beta(a_{j_1}))(i/\sigma) = (\beta(a_{j_2}))(i/\sigma).$$

Thus, for all i/σ in I/σ and hence all i in I, we have $a_{j_1}(i) = a_{j_2}(i)$. That is, $a_{j_1} = a_{j_2}$. Therefore, β is injective.

Let f be an n-ary operation symbol in the language of $\mathbf{B}$ and $\mathbf{M}$. Further, let $a_{j_1}, \ldots, a_{j_n}$

be n-elements from $\alpha(B)$. For all i/σ in I/σ, we have

$$\begin{aligned}
\Big(\beta\big(f^{\alpha(\mathbf{B})}(a_{j_1},\ldots,a_{j_n})\big)\Big)(i/\sigma) &= \big(f^{\alpha(\mathbf{B})}(a_{j_1},\ldots,a_{j_n})\big)(i) \\
&= \big(f^{\mathbf{M}^I}(a_{j_1},\ldots,a_{j_n})\big)(i) \\
&= f^{\mathbf{M}}\big(a_{j_1}(i),\ldots,a_{j_n}(i)\big) \\
&= f^{\mathbf{M}}\Big(\big(\beta(a_{j_1})\big)(i/\sigma),\ldots,\big(\beta(a_{j_n})\big)(i/\sigma)\Big) \\
&= \Big(f^{\mathbf{M}^{I/\sigma}}\big(\beta(a_{j_1}),\ldots,\beta(a_{j_n})\big)\Big)(i/\sigma)
\end{aligned}$$

Thus, β is a homomorphism.

□

Lemma 2.1.2. *Let $\mathbf{B} = \langle B; f^{\mathbf{B}}\rangle$ denote a finite non-trivial connected mono-unary algebra. The algebra $\mathbf{B}$ is subdirectly irreducible if and only if $\mathbf{B}$ is irreducible.*

Proof. We start by showing that if $\mathbf{B}$ is subdirectly irreducible then $\mathbf{B}$ is irreducible. We do this by proving the contrapositive. Suppose that $\mathbf{B}$ is not irreducible. That is, there exists mono-unary algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ such that $\mathbf{B}$ can be embedded into $\mathbf{M}_1 \times \mathbf{M}_2$; but, $\mathbf{B}$ cannot be embedded into either $\mathbf{M}_1$ or $\mathbf{M}_2$. Assume that $\mathbf{B}$ is isomorphic to $\mathbf{M}$, where $\mathbf{M}$ is a subalgebra of $\mathbf{M}_1 \times \mathbf{M}_2$.

For the first and second projection maps on $\mathbf{M}$, denoted by ρ_1 and ρ_2 respectively, let

$$\theta = \ker(\rho_1) \cap \ker(\rho_2)$$

and note that θ is in Con $\mathbf{M}$ as θ is the meet of two congruences.

As $\mathbf{M}$ cannot be embedded into $\mathbf{M}_1$ or into $\mathbf{M}_2$, there does not exist an injective homomorphism from $\mathbf{M}$ to $\mathbf{M}_1$ or from $\mathbf{M}$ to $\mathbf{M}_2$. Therefore, ρ_1 and ρ_2 are not injective. Hence, $\ker(\rho_1)$ does not equal Δ_M and $\ker(\rho_2)$ does not equal Δ_M.

Let

$$t = \big\langle\langle m_1, m_2\rangle, \langle m_3, m_4\rangle\big\rangle.$$

be an element in θ. As t is an element in $\ker(\rho_1)$, we have $m_1 = m_3$. Similarly, as t is an element in $\ker(\rho_2)$, we have $m_2 = m_4$. Therefore, $\theta \subseteq \Delta_M$ and hence $\theta = \Delta_M$. Thus, θ is a meet of two non-Δ_M congruences that equates to Δ_M. That is, no monolith can exist. Therefore, **B** is not subdirectly irreducible as **M** is not subdirectly irreducible.

For the reverse implication, assume that **B** is irreducible. We want to show that **B** is subdirectly irreducible. That is, we want to show that, if for some index set I, if $\alpha: \mathbf{B} \to \prod_{i \in I} \mathbf{B}_i$ is a subdirect embedding then there exists i in I such that $\mathbf{B} \cong \mathbf{B}_i$. Since α is a subdirect embedding, for all i in I, the image of the i^{th} projection map applied to $\alpha(\mathbf{B})$ is $\mathbf{B}_i$. Thus, each $\mathbf{B}_i$ is a homomorphic image of $\alpha(\mathbf{B})$. As $\alpha(\mathbf{B}) \cong \mathbf{B}$ and **B** is a finite algebra, there are at most a finite number of homomorphic images of $\alpha(\mathbf{B})$. For the positive integer n, let the members of $\{\mathbf{D}_j\}_{j=1}^{n}$ denote the distinct homomorphic images of $\alpha(\mathbf{B})$. We have implicitly just shown that

$$\prod_{i \in I} \mathbf{B}_i \cong \prod_{j=1}^{n} (\mathbf{D}_j)^{I_j} \qquad \text{where} \qquad I_j = \{i \in I : \mathbf{B}_i \cong \mathbf{D}_j\}.$$

Hence, there exists an embedding from **B** to $\prod_{j=1}^{n} (\mathbf{D}_j)^{I_j}$. As n is finite and **B** is irreducible, by repeated application of the definition of irreducible, we must have an embedding from **B** into $(\mathbf{D}_j)^{I_j}$ for some j between 1 and n, inclusive. By Lemma 2.1.1, there exists a finite subset I'_j of I_j such that **B** can be embedded into $(\mathbf{D}_j)^{I'_j}$. Again by repeated application of the definition of irreducible, there must exist an embedding from **B** into $\mathbf{D}_j$. Therefore, $|B| \leq |D_j|$. Since $\mathbf{D}_j$ is a homomorphic image of **B**, we must have $|B| \geq |D_j|$. Hence, the embedding from **B** into $\mathbf{D}_j$ is an isomorphism, as $|B| = |D_j|$. Thus, **B** is subdirectly irreducible. □

Yoeli proved that the graph of a connected and irreducible mono-unary algebra is limited to a finite number of possible general shapes. To state Yoeli's result, we need to identify two classes of mono-unary algebras. Denote the mono-unary algebra of size p^n

whose graph is in Figure 2.1 by $\mathbf{H}_{p^n}$, where p is a prime number and n is a positive integer, and denote the mono-unary algebra of size h whose graph is in Figure 2.2 by $\mathbf{J}_h$, where h is a positive integer greater than 1.

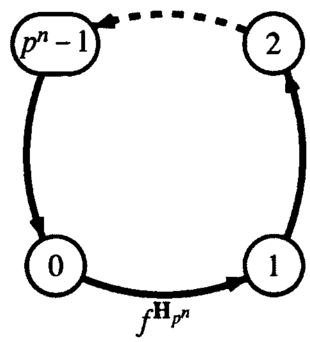

Figure 2.1: The Graph of $\mathbf{H}_{p^n}$

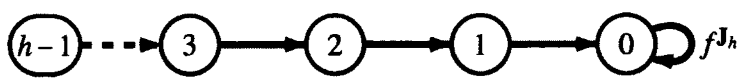

Figure 2.2: The Graph of $\mathbf{J}_h$

Recall, from the previous lemma, that a finite mono-unary algebra is subdirectly irreducible if and only if it is irreducible. We now state Yoeli's result.

Theorem 2.1.3 (Yoeli, [19]). *A finite connected non-trivial mono-unary algebra* $\mathbf{B}$ *is irreducible if and only if* $\mathbf{B}$ *is isomorphic to* $\mathbf{H}_{p^n}$ *or to* $\mathbf{J}_h$*, where* p *is a prime number,* n *is a positive integer and* h *is a positive integer greater than 1.*

Notice that Yoeli's result only deals with connected mono-unary algebras. With respect to mono-unary algebras, Yoeli's result needs to be generalized to include algebras that are not connected, to be helpful in answering the Restricted Quackenbush Problem.

In the following sections, we construct a model of what finite subdirectly irreducible unary algebras look like. Loosely, their corresponding graphs look like part of a circulatory system. This model is then used to answer the Restricted Quackenbush Problem.

2.2 Connected, Pseudoconnected and Disconnected Unary Algebras

The class of all finite unary algebras are partitioned into three subclasses, based on the appearance of each unary algebra's corresponding graph, to assist in determining which finite unary algebras are subdirectly irreducible. In this section, we define three subclasses of unary algebras, that union to the class of all unary algebras, and show that only one class needs to be explicitly considered, when answering the Restricted Quackenbush Problem.

For the remainder of this subsection assume that $\mathbf{B} = \langle B; \mathcal{F} \rangle$ is a unary algebra of finite type.

A unary algebra $\mathbf{B}$ can be viewed as a directed multi-labelled graph. Let $G(\mathbf{B})$ denote the undirected multi-graph that is obtained by removing the direction and labels from the directed multi-labelled graph corresponding to $\mathbf{B}$. Say that there is a *walk* from b_1 to b_2 in $\mathbf{B}$ if and only if there is a path from b_1 to b_2 in $G(\mathbf{B})$.

Example. Looking at the undirected multi-graph $G(\mathbf{E}_6)$ that corresponds to $\mathbf{E}_6$, displayed in Figure 2.3, there exists a walk from 1 to 2. In fact, for all e_1 and e_2 in E_6, there exists a walk from e_1 to e_2.

We can use a walk, from one member of B to another, to define an equivalence relation that will later help in the classification of all finite subdirectly irreducible unary algebras of finite type. Define the equivalence relation $\mathcal{R}_B$ on B by relating b_1 to b_2 if and only if there exists a walk from b_1 to b_2.

Lemma 2.2.1. *Each equivalence class of $\mathcal{R}_B$ is a subuniverse of* $\mathbf{B}$.

Proof. Let T be an equivalence class of $\mathcal{R}_B$. Further, let f be any operation symbol in $\mathcal{F}$. To prove the Lemma, we must show that for all t in T, the element $f^{\mathbf{B}}(t)$ is in T as well. Since there exists a path from t to $f^{\mathbf{B}}(t)$ in $G(\mathbf{B})$, the elements t and $f^{\mathbf{B}}(t)$ must belong to the same equivalence class. Hence, $f^{\mathbf{B}}(t)$ is an element in T. □

Definition. If there is only one distinct equivalence class of $\mathcal{R}_B$, then $\mathbf{B}$ is said to be *connected*. If there are exactly two distinct equivalence classes on $\mathcal{R}_B$, such that at least one equivalence class has unit cardinality, then $\mathbf{B}$ is said to be *pseudoconnected*. Otherwise $\mathbf{B}$ is said to be *disconnected*.

Applying Lemma 2.2.1, if $\mathbf{B}$ pseudoconnected, then there exist two proper connected subalgebras, $\mathbf{B}_1$ and $\mathbf{B}_2$, such that $B_1 \uplus B_2 = B$ and at least one of $\mathbf{B}_1$ and $\mathbf{B}_2$ is trivial.

Example. See Figure 2.3 for $\mathbf{E}_6$, an example of a connected algebra. See Figure 2.4 for $\mathbf{E}_7$, an example of a pseudoconnected algebra. See Figure 2.5 for $\mathbf{E}_8$, an example of a disconnected algebra.

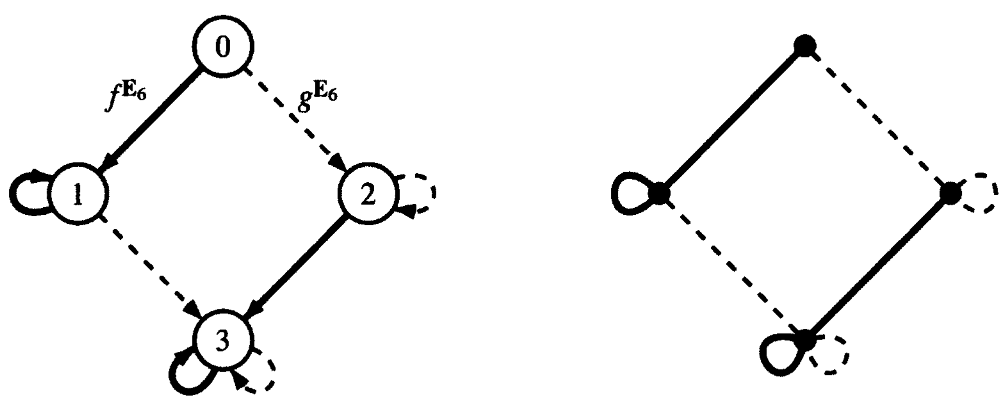

Figure 2.3: A Connected Unary Algebra $\mathbf{E}_6$ and Associated Undirected Multi-Graph $G(\mathbf{E}_6)$

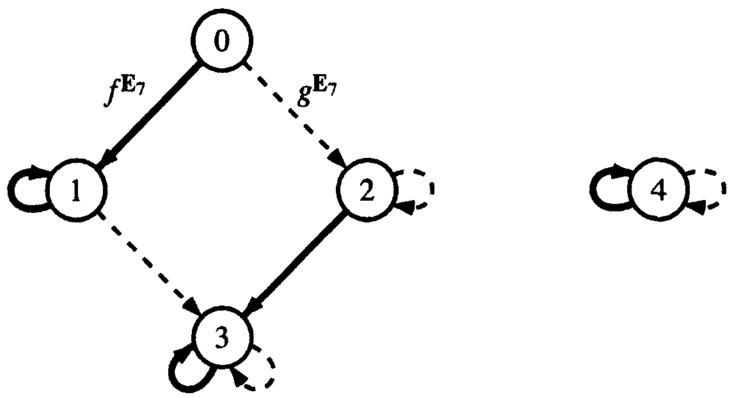

Figure 2.4: A Pseudoconnected Unary Algebra: $\mathbf{E}_7$

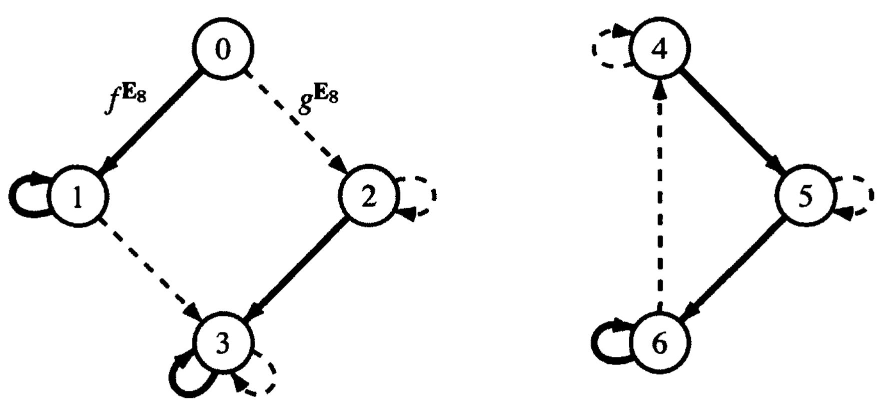

Figure 2.5: A Disconnected Unary Algebra: $\mathbf{E}_8$

2.2.1 Subdirectly Irreducible Implies Connected or Pseudoconnected

Of connected, pseudoconnected and disconnected unary algebras, which class contains subdirectly irreducible algebras? We show that all subdirectly irreducible unary algebras must either be connected or pseudoconnected. An understanding of congruences that are present in a disconnected unary algebra is necessary.

Lemma 2.2.2. *If* **M** *is a subalgebra of* **B** *and* θ *is a congruence on* **M** *then* $Cg^{\mathbf{B}}(\theta) = \theta \cup \Delta_B$.

Proof. We show that $\theta \cup \Delta_B$ is a congruence on **B**. Then, since $\theta \cup \Delta_B$ is the the smallest binary relation on **B** that contains θ, we obtain the desired result.

The binary relation $\theta \cup \Delta_B$ is certainly reflexive and symmetric. To complete the proof, we must show that $\theta \cup \Delta_B$ satisfies the transitive property and the compatibility property.

Let $\langle b_1, b_2\rangle$ and $\langle b_2, b_3\rangle$ be elements in $\theta \cup \Delta_B$. Suppose that both $\langle b_1, b_2\rangle$ and $\langle b_2, b_3\rangle$ are in θ. Since θ is transitive, we obtain the following:

$$\langle b_1, b_3\rangle \in \theta \subseteq \theta \cup \Delta_B.$$

Now, without loss of generality, assume that $\langle b_1, b_2\rangle$ is in Δ_B. Then $b_1 = b_2$ and hence $\langle b_1, b_3\rangle$ is in $\theta \cup \Delta_B$. Therefore, $\theta \cup \Delta_B$ is transitive.

Let f be an operation symbol in $\mathcal{F}$. Suppose that $\langle b_1, b_2\rangle$ is in θ, which implies that b_1 and b_2 are in M and hence B. As θ satisfies the compatibility property, we have

$$\left\langle f^{\mathbf{B}}(b_1), f^{\mathbf{B}}(b_2)\right\rangle = \left\langle f^{\mathbf{M}}(b_1), f^{\mathbf{M}}(b_2)\right\rangle \in \theta \subseteq \theta \cup \Delta_B.$$

Now suppose that $\langle b_1, b_2\rangle$ is in Δ_B. As $b_1 = b_2$, we must have $f^{\mathbf{B}}(b_1) = f^{\mathbf{B}}(b_2)$. Hence,

$$\left\langle f^{\mathbf{B}}(b_1), f^{\mathbf{B}}(b_2)\right\rangle \in \Delta_B \subseteq \theta \cup \Delta_B.$$

Therefore, $\theta \cup \Delta_B$ satisfies the compatibility property. □

In the proofs of the next few lemmas, we use the following set equality: for the sets A, B, C and D,

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D). \tag{2.1}$$

Lemma 2.2.3. *If there exist at least two distinct equivalence classes in $\mathcal{R}_{\mathbf{B}}$ such that neither have unit cardinality then* **B** *is not subdirectly irreducible.*

Proof. By Lemma 2.2.1, there exist two distinct connected subalgebras, $\mathbf{B}_1$ and $\mathbf{B}_2$, of $\mathbf{B}$ such that $B_1 \cap B_2 = \varnothing$. Further, $|B_1| > 1$ and $|B_2| > 1$.

By Lemma 2.2.2, we have

$$\begin{aligned} \mathrm{Cg}^{\mathbf{B}}({B_1}^2) \wedge^{\mathbf{Con\,B}} \mathrm{Cg}^{\mathbf{B}}({B_2}^2) &= ({B_1}^2 \cup \Delta_B) \cap ({B_2}^2 \cup \Delta_B) \\ &= ({B_1}^2 \cap {B_2}^2) \cup \Delta_B \\ &= (B_1 \cap B_2)^2 \cup \Delta_B \\ &= \varnothing^2 \cup \Delta_B \\ &= \Delta_B \end{aligned}$$

and neither $\mathrm{Cg}^{\mathbf{B}}({B_1}^2)$ nor $\mathrm{Cg}^{\mathbf{B}}({B_2}^2)$ is Δ_B. Hence, $\mathbf{B}$ is not subdirectly irreducible. □

Lemma 2.2.4. *If there exists at least three distinct equivalence classes of $\mathcal{R}_{\mathbf{B}}$ such that two of them have unit cardinality then* **B** *is not subdirectly irreducible.*

Proof. By hypothesis, there exist at least three distinct connected subalgebras, $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$, of $\mathbf{B}$ such that $\{B_1, B_2, B_3\}$ is a mutually disjoint collection of subuniverses of $\mathbf{B}$ and two of the subalgebras are trivial. Without loss of generality, assume that $\mathbf{B}_1$ and $\mathbf{B}_2$ are trivial. Let $B_1 = \{b_1\}$ and $B_2 = \{b_2\}$.

By Lemma 2.2.2, we obtain the following:

$$\mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_2)^2\big) \wedge^{\mathbf{Con\,B}} \mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_3)^2\big) = \big((B_1 \cup B_2)^2 \cap (B_1 \cup B_3)^2\big) \cup \Delta_B.$$

By Statement (2.1),

$$\begin{aligned}(B_1 \cup B_2)^2 \cap (B_1 \cup B_3)^2 &= \big((B_1 \cup B_2) \cap (B_1 \cup B_3)\big)^2 \\ &= {B_1}^2 \\ &= \{\langle b_1, b_1 \rangle\}.\end{aligned}$$

Thus,

$$\mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_2)^2\big) \wedge^{\mathbf{Con\,B}} \mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_3)^2\big) = \Delta_B.$$

Since both $\mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_2)^2\big)$ and $\mathrm{Cg}^{\mathbf{B}}\big((B_1 \cup B_3)^2\big)$ are non-Δ_B congruences, but, meet to Δ_B, we have shown that $\mathbf{B}$ is not subdirectly irreducible. □

Theorem 2.2.5. *If* $\mathbf{B}$ *is subdirectly irreducible, then* $\mathbf{B}$ *is connected or pseudoconnected.*

Proof. We prove the contrapositive of the claim. Assume that $\mathbf{B}$ is disconnected. We must show that, under this assumption, $\mathbf{B}$ is not subdirectly irreducible. As $\mathbf{B}$ is disconnected, there exist two possible cases. Either there exist at least two distinct equivalence classes in $\mathcal{R}_{\mathbf{B}}$, such that neither have unit cardinality, or there exist at least three distinct equivalence classes in $\mathcal{R}_{\mathbf{B}}$, such that two of them have unit cardinality. Lemma 2.2.3 implies that the former case does not yield a subdirectly irreducible algebra while Lemma 2.2.4 implies that the latter case does not yield a subdirectly irreducible algebra either. □

2.2.2 Correspondence of Connected and Pseudoconnected Unary Algebras

We show that every pseudoconnected subdirectly irreducible unary algebra is the disjoint union of the trivial unary algebra and a connected subdirectly irreducible unary algebra.

Lemma 2.2.6. *Let* $\mathbf{M}$ *be a subalgebra of* $\mathbf{B}$. *If* $\mathbf{B}$ *is subdirectly irreducible, then* $\mathbf{M}$ *is subdirectly irreducible.*

Proof. We prove the contrapositive. Suppose that **M** is not subdirectly irreducible. We want to show that **B** is not subdirectly irreducible. Let $T = (\text{Con } \mathbf{M}) \setminus \{\Delta_M\}$.

As **M** is not subdirectly irreducible, no monolith can exist and hence

$$\bigwedge_{\theta \in T} \theta = \bigcap_{\theta \in T} \theta = \Delta_M. \tag{2.2}$$

Further, since **M** is not subdirectly irreducible,

$$|T| > 2. \tag{2.3}$$

Notice that for all θ in T, we have $\theta \neq \Delta_M$. Thus, applying Lemma 2.2.2, we obtain the following:

$$\text{Cg}^{\mathbf{B}}(\theta) \neq \Delta_B. \tag{2.4}$$

By Lemma 2.2.2 and Statement (2.2),

$$\begin{aligned}
\bigwedge_{\theta \in T} \text{Cg}^{\mathbf{B}}(\theta) &= \bigwedge_{\theta \in T} (\theta \cup \Delta_B) \\
&= \bigcap_{\theta \in T} (\theta \cup \Delta_B) \\
&= \left(\bigcap_{\theta \in T} \theta \right) \cup \Delta_B \\
&= \Delta_M \cup \Delta_B \\
&= \Delta_B.
\end{aligned}$$

Thus, the above meet of congruences equates to Δ_B. By Statement (2.3), the above meet of congruences involves at least two non-trivial congruences. By Statement (2.4), none of these congruences are Δ_B. Thus, **B** cannot possibly have a monolith. Therefore, **B** is not subdirectly irreducible. □

Suppose that **B** is pseudoconnected. Recall that under this assumption, there exist two proper connected subalgebras, $\mathbf{B}_1$ and $\mathbf{B}_2$, such that $B_1 \cup B_2 = B$ and at least one of $\mathbf{B}_1$ and $\mathbf{B}_2$ is trivial. Without loss of generality, assume that $\mathbf{B}_1$ is trivial.

Lemma 2.2.6 implies that if **B** is subdirectly irreducible, then $\mathbf{B}_2$ is a connected subdirectly irreducible subalgebra. Thus, if there exist, up to isomorphism, a finite number of subdirectly irreducible connected algebras in a variety, then there exist, up to isomorphism, a finite number of subdirectly irreducible pseudoconnected algebras in the variety. Hence, from here, we focus on connected unary algebras.

2.3 Every Unary Algebra in a Finitely Generated Variety Has an Irredundant Basis

To continue looking at subdirectly irreducible unary algebras, we look at their irredundant bases. A problem arises: which unary algebras have at least one irredundant basis, finite or infinite?

Example. Consider the mono-unary algebra $\mathbf{E}_9 = \langle E_9; f^{\mathbf{E}_9}\rangle$, where E_9 is the set of non-positive integers and

$$f^{\mathbf{E}_9}(e) = \begin{cases} e+1 & \text{if } e \neq 0 \\ 0 & \text{if } e = 0. \end{cases}$$

The algebra $\mathbf{E}_9$ is one of the subdirectly irreducible mono unary algebras looked at, by Wenzel, in [18]. The algebra $\mathbf{E}_9$ does not have an irredundant basis, finite or infinite. See Figure 2.6 for the graph of $\mathbf{E}_9$.

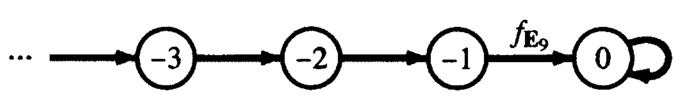

Figure 2.6: A Unary Algebra Without an Irredundant Basis: $\mathbf{E}_9$

Showing that each member in a finitely generated variety has at least one irredundant basis will be enough for our purposes.

Suppose that **B** is an algebra in some finitely generated variety. We show that each member in the universe of **B** belongs to at least one maximal singly-generated subuniverse, in the following sense: if b_1 generates a maximal singly-generated subuniverse of **B** and b_2 is in B such that b_1 is in $\mathrm{Sg}^{\mathbf{B}}(\{b_2\})$ then $\mathrm{Sg}^{\mathbf{B}}(\{b_1\}) = \mathrm{Sg}^{\mathbf{B}}(\{b_2\})$. We do this by using Zorn's Lemma, which requires the definition of a chain.

Definition (Burris and Sankappanavar, [2]). A *chain* of sets C is a family of sets such that for each set C_1 and C_2 in C either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

We state Zorn's Lemma as it appears in [2], without proof.

Lemma 2.3.1 (Zorn's Lemma). *If F is a non-empty family of sets such that for each chain C of members of F there is a member of F containing $\bigcup C$ then F has a maximal member.*

Note that Zorn's Lemma is equivalent to the Axiom of Choice.

Lemma 2.3.2. *Let* **B** *be a unary algebra contained in some finitely generated variety of type $\mathcal{F}$. For all members b in B, there exists a maximal singly-generated subuniverse of* **B** *that contains b.*

Proof. Let $\mathcal{K}$ denote the set of all singly-generated subuniverses of **B** that contain b. The collection $\mathcal{K}$ is non-empty as $\mathrm{Sg}^{\mathbf{B}}(\{b\})$ is in $\mathcal{K}$. To complete the proof, we will show that $\mathcal{K}$ has a maximal element, using Zorn's Lemma.

Let $\mathcal{C}$ be a chain of elements from $\mathcal{K}$. By hypothesis, there exists an algebra **M** such that **B** is in $\mathbb{V}(\mathbf{M})$ and $|M| = m$, for some positive integer m. Note that for all C in $\mathcal{C}$, the unary algebra $\langle C; \mathcal{F}\rangle$ is a member in the variety generated by **B** and hence is a member in the variety generated by **M**. Hence, by Lemma 1.3.3, on page 37,

$$\text{all singly-generated subuniverses in } \mathcal{C} \text{ can contain at most } m^m \text{ elements.} \tag{2.5}$$

To complete the proof, we show that $\bigcup_{C \in \mathcal{C}} C$ is equal to one of the members in $\mathcal{C}$. By Statement (2.5), we obtain the following:

$$\max\{|C| \mid C \in \mathcal{C}\} \leq m^m.$$

Pick a C in $\mathcal{C}$ that has maximum cardinality. We must have $C \subseteq \bigcup \mathcal{C}$. Pick a c in $\bigcup \mathcal{C}$. There must exist a C' in $\mathcal{C}$ such that c is in C'. As $\mathcal{C}$ is a chain, either $C \subseteq C'$ or $C' \subseteq C$. Since C was chosen to have maximum cardinality, we must have $C' \subseteq C$. Hence, c is in C and thus, $C = \bigcup \mathcal{C}$.

Therefore, by Zorn's Lemma or Lemma 2.3.1, we have shown that $\mathcal{K}$ has a maximal element. □

Theorem 2.3.3. *Let* **B** *be a unary algebra that is a member of a finitely generated variety. There exists at least one irredundant basis of B.*

Proof. Let S' denote the set of all members of B that generate a maximal singly-generated subuniverse. Enumerate the elements of S' using $\{s_i'\}_{i \in I}$, for some index set I. By Lemma 2.3.2,

$$\bigcup_{i \in I} \mathrm{Sg}^{\mathbf{B}}(\{s_i'\}) = B. \tag{2.6}$$

Define the equivalence relation μ on S' by

$$\text{relating} \quad s_1' \quad \text{to} \quad s_2' \qquad \text{if and only if} \qquad \mathrm{Sg}^{\mathbf{B}}(\{s_1'\}) = \mathrm{Sg}^{\mathbf{B}}(\{s_2'\}).$$

Use a choice function to pick a member from each equivalence class modulo μ and enumerate these members by $\{s_j\}_{j \in J}$ for some index set J. We have

$$\bigcup_{j \in J} \mathrm{Sg}^{\mathbf{B}}(\{s_j\}) = \bigcup_{j \in J} \left(\bigcup_{s' \in s_j/\mu} \mathrm{Sg}^{\mathbf{B}}(\{s'\}) \right) = \bigcup_{i \in I} \mathrm{Sg}^{\mathbf{B}}(\{s_i'\}).$$

Thus, by the previous equations and Statement (2.6) we obtain that $\{s_j\}_{j \in J}$ generates B. Let $S = \{s_j\}_{j \in J}$.

To complete the proof, we show that S is irredundant. Each member of S generates a maximal singly-generated subuniverse. Hence, by the the definition of the equivalence relation μ, for each s in S we have for all s' in $S\setminus\{s\}$, we have the following:

$$s \quad \text{is not in} \quad \mathrm{Sg}^{\mathbf{B}}(\{s'\}).$$

Thus,

$$s \quad \text{is not in} \quad \bigcup_{s' \in S\setminus\{s\}} \mathrm{Sg}^{\mathbf{B}}(\{s'\}) \quad \text{or} \quad \mathrm{Sg}^{\mathbf{B}}(S\setminus\{s\}).$$

□

Due to the previous Theorem, when we are working with an algebra that is a member of a finitely generated variety, we implicitly assume that an irredundant basis exists.

2.4 The Appearance of Connected Subdirectly Irreducible Unary Algebras

The graph of a finite subdirectly irreducible unary algebra looks similar to a circulatory system. In this subsection, we define the heart of a unary algebra, assuming it exists, and define the veins of an algebra.

2.4.1 The Heart of a Unary Algebra

Let $\mathbf{B} = \langle B; \mathcal{F}\rangle$ be a unary algebra. Let $H^{\mathbf{B}}$ be the intersection of all maximal singly-generated subuniverses of $\mathbf{B}$. If $H^{\mathbf{B}}$ is empty then say that $\mathbf{B}$ is *heartless*. If $H^{\mathbf{B}}$ is not empty then call $\mathbf{H}^{\mathbf{B}} = \langle H^{\mathbf{B}}; \mathcal{F}\rangle$ the *heart* of $\mathbf{B}$.

Example. Consider the algebra $\mathbf{E}_{10}$, depicted in Figure 2.7. Let $\mathcal{F}$ denote the language of $\mathbf{E}_{10}$. The subalgebra

$$\mathbf{H}^{\mathbf{E}_{10}} = \langle\{1,2,3,4,5,6,7\};\mathcal{F}\rangle$$

is the heart of $\mathbf{E}_{10}$.

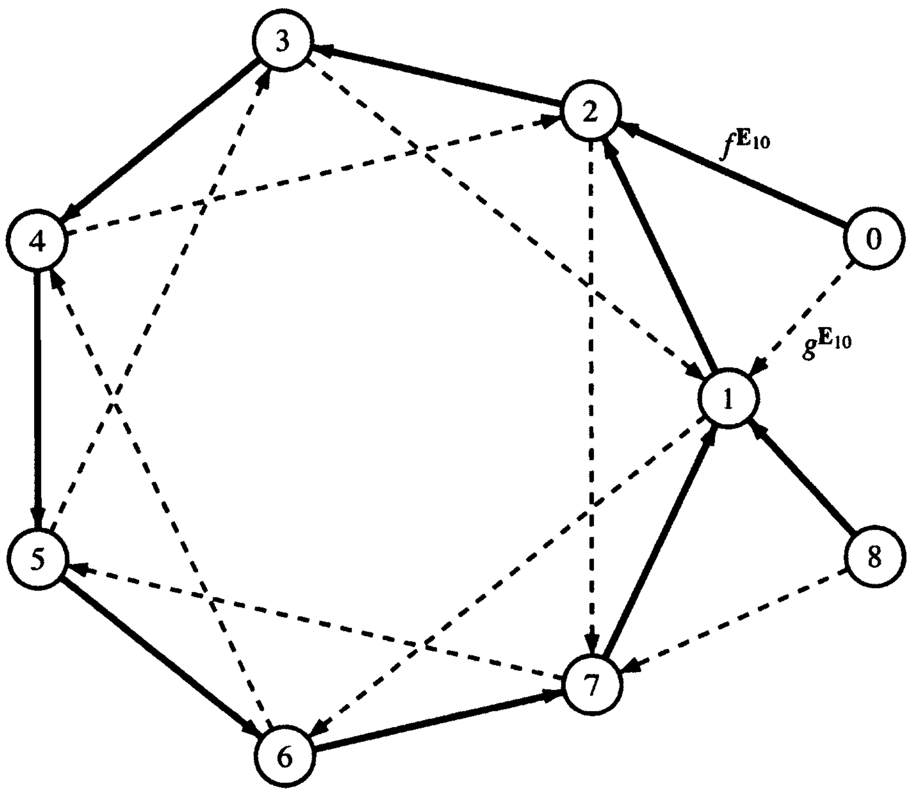

Figure 2.7: A Connected Unary Algebra with a Heart: $\mathbf{E}_{10}$

As we show in the following theorem, to determine what the heart of a unary algebra is, rather than finding all of the maximal singly-generated subuniverses, we may look at the subuniverses generated by each member in any irredundant basis.

Theorem 2.4.1. *Let* $\mathbf{B}$ *be a unary algebra that has a heart and for some index set* I, *let* $\{u_i\}_{i\in I}$ *denote the set of members of* B *that generate maximal singly-generated sub-*

universes. Further, for some index set J, let $S = \{s_j\}_{j\in J}$ be an irredundant basis of B. Then

$$H^{\mathbf{B}} = \bigcap_{i\in I} Sg^{\mathbf{B}}(\{u_i\}) = \bigcap_{j\in J} Sg^{\mathbf{B}}(\{s_j\}).$$

Also, each member of S generates a maximal-singly generated subuniverse of B.

Proof. We show that each member in S generates a maximal singly-generated subuniverse of B. By doing this, we are showing that

$$\bigcap_{i\in I} \mathrm{Sg}^{\mathbf{B}}(\{u_i\}) \subseteq \bigcap_{j\in J} \mathrm{Sg}^{\mathbf{B}}(\{s_j\}). \tag{2.7}$$

Suppose that there exist b in B and j_1 in J such that

$$\mathrm{Sg}^{\mathbf{B}}(\{s_{j_1}\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{b\}).$$

As S is an irredundant basis, there exists j_2 in J such that

$$\mathrm{Sg}^{\mathbf{B}}(\{b\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{s_{j_2}\}).$$

Therefore,

$$\mathrm{Sg}^{\mathbf{B}}(\{s_{j_1}\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{s_{j_2}\}).$$

As S is an irredundant basis, $s_{j_1} = s_{j_2}$. Hence,

$$\mathrm{Sg}^{\mathbf{B}}(\{s_{j_1}\}) = \mathrm{Sg}^{\mathbf{B}}(\{b\}).$$

That is, s_{j_1} generates a maximal singly-generated subuniverse. We have just shown that Statement (2.7) holds.

Suppose now that u_i generates a maximal singly-generated subuniverse of B, for some i in I. We show that

$$\mathrm{Sg}^{\mathbf{B}}(\{u_i\}) = \mathrm{Sg}^{\mathbf{B}}(\{s_j\})$$

for some j in J. Doing this proves that

$$\bigcap_{j \in J} \mathrm{Sg}^{\mathbf{B}}(\{s_j\}) \subseteq \bigcap_{i \in I} \mathrm{Sg}^{\mathbf{B}}(\{u_i\}). \tag{2.8}$$

Since S is an irredundant basis of B, there must exist j in J such that

$$\mathrm{Sg}^{\mathbf{B}}(\{u_i\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{s_j\}).$$

As u_i generates a maximal singly-generated subuniverse, we must have equality. We have just shown that the subset relation in Statement (2.8) holds.

Together, Statement (2.8) and Statement (2.7) yield the desired result. □

Theorem 2.4.1 makes the heart, assuming that one exists, easier to find. When we wish to identify the heart of a unary algebra, we often make use of this Theorem implicitly.

Example. Consider the algebra $\mathbf{E}_{11}$, depicted in Figure 2.8. The elements 1 and 5 must be in every irredundant basis. Further, $\mathrm{Sg}^{\mathbf{B}}(\{1\})$ and $\mathrm{Sg}^{\mathbf{B}}(\{5\})$ are maximal singly-generated subuniverses of $\mathbf{E}_{11}$. As their intersection is empty, Theorem 2.4.1 implies that $\mathbf{E}_{11}$ is heartless.

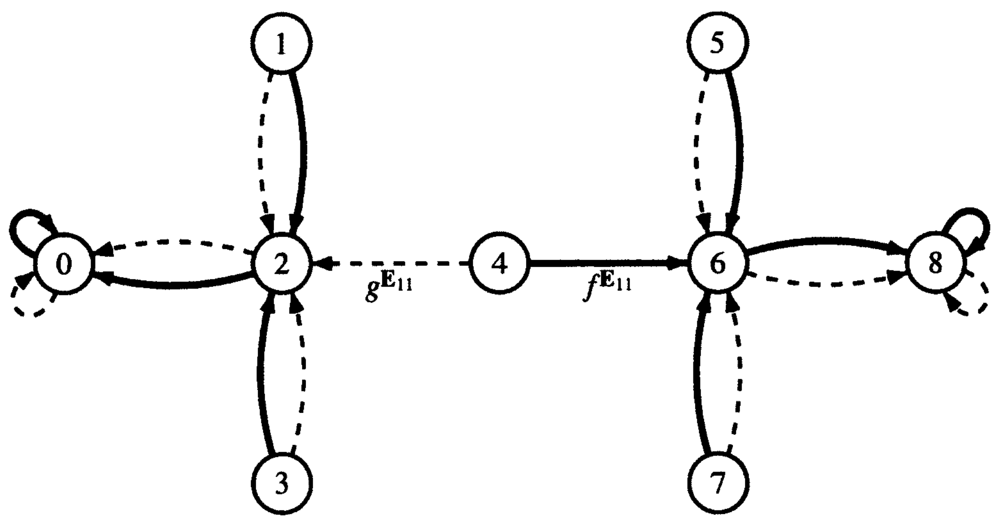

Figure 2.8: A Connected Unary Algebra that is Heartless: $\mathbf{E}_{11}$

In the previous two examples, on page 64 and page 67, the algebra $\mathbf{E}_{10}$ is subdirectly irreducible and has a heart; whereas, $\mathbf{E}_{11}$ is not subdirectly irreducible and does not have a heart. The next theorem shows that all connected subdirectly irreducible unary algebras that belong to a finitely generated variety have a heart.

Theorem 2.4.2. *Let* $\mathbf{B} = \langle B; \mathcal{F} \rangle$ *be a connected unary algebra that is a member of a finitely generated variety. If* $\mathbf{B}$ *is subdirectly irreducible, then* $\mathbf{B}$ *has a heart.*

Proof. Note that if $\mathbf{B}$ is generated by a single element, then $\mathbf{H}^{\mathbf{B}} = \mathbf{B}$ and the claim is true. Assume that $\mathbf{B}$ is not singly generated. That is, assume that each irredundant basis contains at least 2 elements.

We prove the contrapositive. For some index set I that contains at least 2 elements, let $\{s_i\}_{i \in I}$ be an irredundant basis of B. We are guaranteed the existence of an irredundant basis by Theorem 2.3.3. Further, suppose that $\mathbf{B}$ does not have a heart. To complete the proof, we show that $\mathbf{B}$ is not subdirectly irreducible.

As $\mathbf{B}$ is heartless, for each b in B, there must be some generator s_{j_b}, such that b is not in $\mathrm{Sg}^{\mathbf{B}}(\{s_{j_b}\})$. For all b in B, define

$$\theta_b = \mathrm{Cg}^{\mathbf{B}}\left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s_{j_b}\})\right)^2\right).$$

Thus

$$\text{for all } b \text{ in } B, \qquad \text{we have} \qquad b/\theta_b = \{b\}. \tag{2.9}$$

Notice that if $\theta_b = \Delta_B$, for some b in B, then $\mathrm{Sg}^{\mathbf{B}}(\{s_{j_b}\}) = \{s_{j_b}\}$. Hence, if $\theta_b = \Delta_B$ then as s_{j_b} is a member of an irredundant basis of B, this means that $\mathbf{B}$ is not connected and a contradiction occurs. As $\mathbf{B}$ is connected, we may assume that

$$\text{for all } b \text{ in } B, \qquad \theta_b \neq \Delta_B. \tag{2.10}$$

Since $\mathbf{B}$ is minimally generated by at least two elements,

$$\text{there exists } b_1 \text{ and } b_2 \text{ in } B \qquad \text{such that} \qquad \theta_{b_1} \neq \theta_{b_2}. \tag{2.11}$$

By Statement (2.9), we obtain the following:

$$\bigwedge_{b \in B} \theta_b = \Delta_B.$$

Statement (2.10) implies that none of the congruences are Δ_B. Statement (2.11) implies that this meet must involve at least two distinct congruences. Thus, $\mathbf{B}$ cannot possibly have a monolith. Finally, $\mathbf{B}$ is not subdirectly irreducible. □

The next example shows that a connected unary algebra that is not subdirectly irreducible may or may not have a heart. Thus, the inverse of the previous Theorem is not necessarily true.

Example. Consider the unary algebras $\mathbf{E}_{11}$ and $\mathbf{E}_{12}$ seen in Figure 2.8 and Figure 2.9, on page 67 and page 70, respectively. It can be determined that both algebras are connected and not subdirectly irreducible. Further, it can be determined that $\mathbf{E}_{11}$ is heartless and

$$\mathbf{H}^{\mathbf{E}_{12}} = \mathbf{Sg}^{\mathbf{E}_{12}}(\{1,2,3\}).$$

Lemma 2.4.2 states that a connected subdirectly irreducible unary algebra has a heart. The next lemma states that the heart has a universe of at least size 2.

Lemma 2.4.3. *Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a non-trivial unary algebra that has a heart and is a member of some finitely generated variety. If $\mathbf{B}$ is subdirectly irreducible, then $\mathbf{H}^{\mathbf{B}}$ is not trivial.*

Proof. We prove the contrapositive. Assume that $\mathbf{H}^{\mathbf{B}}$ is trivial. That is, assume that $H^{\mathbf{B}} = \{h\}$, and let S be an irredundant basis of B. We are guaranteed the existence of an irredundant

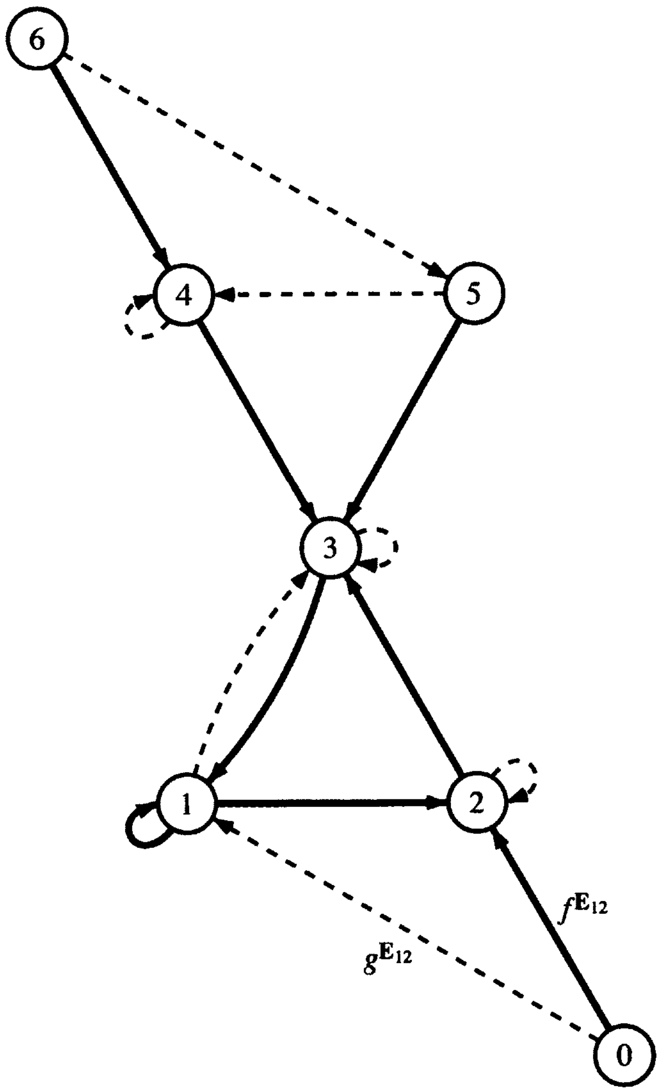

Figure 2.9: A Unary Algebra that has a Heart and is not Subdirectly Irreducible: $\mathbf{E}_{12}$

basis by Theorem 2.3.3. By Theorem 2.4.1, the universe of the heart of **B** is equal to the intersection of the subuniverses generated by each element of S. Thus,

$$H^{\mathbf{B}} = \bigcap_{s \in S} \mathrm{Sg}^{\mathbf{B}}(\{s\}) = \{h\}. \tag{2.12}$$

The objective is to show that **B** is not subdirectly irreducible.

If $|S| = 1$ then B is singly generated and $B = H^{\mathbf{B}} = \{h\}$. As **B** is not trivial, we may assume that $|S| > 1$.

For all t in S, we must have $\left|\mathrm{Sg}^{\mathbf{B}}(\{t\})\right| > 1$ as otherwise, using Statement (2.12),

$$\{h\} = \bigcap_{s \in S} \mathrm{Sg}^{\mathbf{B}}(\{s\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{t\}) = \{t\}$$

and hence t would be generated by each individual member of S, contradicting the assumption that S is an irredundant basis that contains at least 2 elements.

By Lemma 2.2.2 and Statement (2.12), we have

$$\begin{aligned}
\bigwedge_{s \in S} \mathrm{Cg}^{\mathbf{B}}\left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s\})\right)^2\right) &= \bigcap_{s \in S}\left(\left(\mathrm{Sg}^{\mathbf{B}}(\{s\})\right)^2 \cup \Delta_B\right) \\
&= \left(\bigcap_{s \in S}\left(\mathrm{Sg}^{\mathbf{B}}(\{s\})\right)^2\right) \cup \Delta_B \\
&= \left(\bigcap_{s \in S} \mathrm{Sg}^{\mathbf{B}}(\{s\})\right)^2 \cup \Delta_B \\
&= \{h\}^2 \cup \Delta_B \\
&= \Delta_B.
\end{aligned}$$

Therefore, since $|S| > 1$ and, for all s in S, since $\left|\mathrm{Sg}^{\mathbf{B}}(\{s\})\right| > 1$, the algebra **B** cannot possibly have a monolith and hence is not subdirectly irreducible. □

2.4.2 Veins of a Unary Algebra

The subalgebra generated by a single member of an irredundant basis, used in the proof of Theorem 2.4.2, turned out to be helpful. We generalize this concept in the following definition.

Definition. Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a unary algebra and $\mathbf{M}$ be a member of $\mathbb{V}(\mathbf{B})$. Call each maximal singly-generated subalgebra of $\mathbf{M}$ a *vein*. Further, define a *vein* of $\mathbb{V}(\mathbf{B})$ to be a vein of some member of $\mathbb{V}(\mathbf{B})$.

Example. In the algebra $\mathbf{E}_{10}$ depicted in Figure 2.7 on page 65, the subalgebra

$$\left\langle \mathrm{Sg}^{\mathbf{E}_{10}}(\{0\}); \mathcal{F} \right\rangle$$

is a vein of $\mathbf{E}_{10}$. In fact, up to isomorphism, it is the only vein of $\mathbf{E}_{10}$.

Recall that by the proof of Theorem 2.4.1, the subuniverse generated by any member of any irredundant basis is maximal, with respect to singly generated subuniverses. We show that each member in the universe of a unary algebra that is a member of a finitely generated variety is a member of some irredundant basis if and only if it generates a maximal subuniverse.

Lemma 2.4.4. *Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a unary algebra that is a member of a finitely generated variety. For all b in B, we have $\left\langle Sg^{\mathbf{B}}(\{b\}); \mathcal{F} \right\rangle$ is a vein of $\mathbf{B}$ if and only if b is a member of some irredundant basis of B.*

Proof. We start with the forward implication. Let $\left\langle \mathrm{Sg}^{\mathbf{B}}(\{b\}); \mathcal{F} \right\rangle$ be a vein of $\mathbf{B}$. Further, let S be an irredundant basis of B. We are guaranteed the existence of an irredundant basis by Theorem 2.3.3. There exists an s in S such that

$$\mathrm{Sg}^{\mathbf{B}}(\{b\}) \subseteq \mathrm{Sg}^{\mathbf{B}}(\{s\}).$$

By assumption, $\mathrm{Sg}^{\mathbf{B}}(\{b\})$ is a maximal subuniverse of $\mathbf{B}$, with respect to singly generated subuniverses. Hence,

$$\mathrm{Sg}^{\mathbf{B}}(\{b\}) = \mathrm{Sg}^{\mathbf{B}}(\{s\}).$$

Thus, $(S\backslash\{s\}) \cup \{b\}$ is an irredundant basis of B.

The reverse implication was proved in Theorem 2.4.1. □

The following Corollary gives information regarding the size of veins in a finitely generated variety.

Corollary 2.4.5. *Let* $\mathbf{B}$ *be a finite unary algebra of finite type,* $|B| = b$ *and* $\mathbf{M}$ *be a member of* $\mathbb{V}(\mathbf{B})$. *The following statements are true.*

1. *Every vein of* $\mathbf{M}$ *has a universe with a cardinality less than or equal to* b^b.

2. *There exists only a finite number of veins, up to isomorphism, of* $\mathbb{V}(\mathbf{B})$.

Proof. Recall that Lemma 1.3.3, on page 37, states that for all $\mathbf{M}_1$ in $\mathbb{V}(\mathbf{B})$, if M_1 has an irredundant basis of size n, then $|M_1| < b^{b^n}$. Thus, to prove Part 1 of the Corollary, apply Lemma 1.3.3 and note that every vein of $\mathbf{M}$ is a subalgebra of $\mathbf{M}$ and is singly generated.

Recall that Corollary 1.3.4, on page 38 states that for any positive integer n, there exists only a finite number of members in $\mathbb{V}(\mathbf{B})$ that are minimally generated by n elements, up to isomorphism. Thus, for Part 2 of the Corollary, apply Corollary 1.3.4 and let $n = 1$. □

2.5 Orientation

What happens when two isomorphic veins of a unary algebra *overlap* each other? In this subsection, we define what it means for two veins to overlap each other and show that, if the overlap is in some sense *nice*, then the resulting unary algebra is not subdirectly irreducible.

Definition. Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a unary algebra. Suppose that distinct elements b_i and b_j generate isomorphic veins of $\mathbf{B}$ and the intersection $\mathbf{N}$ of the veins is non-empty. Let $\mathbf{Q}$ be a subalgebra of $\mathbf{N}$. If for each q in Q and each term $t(x)$ of type $\mathcal{F}$,

$$t^{\mathbf{B}}(b_i) = q \qquad \text{if and only if} \qquad t^{\mathbf{B}}(b_j) = q,$$

then the two veins of $\mathbf{B}$,

$$\left\langle \mathrm{Sg}^{\mathbf{B}}(\{b_i\}); \mathcal{F} \right\rangle \qquad \text{and} \qquad \left\langle \mathrm{Sg}^{\mathbf{B}}(\{b_j\}); \mathcal{F} \right\rangle,$$

are said to have the *same orientation with respect to* $\mathbf{Q}$ *in* $\mathbf{B}$.

Example. Consider $\mathbf{E}_{10}$, depicted in Figure 2.7 on page 65, the subalgebra generated by $\{0\}$ and $\{8\}$ do not have the same orientation with respect to the subalgebra generated by $\{1,2,3,4,5,6,7\}$ because, for example,

$$f^{\mathbf{E}_{10}}(0) = 2 \qquad \text{and} \qquad f^{\mathbf{E}_{10}}(8) = 1.$$

Example. In $\mathbf{E}_{13}$, depicted in Figure 2.10, the subalgebra generated by $\{0\}$ and $\{1\}$ have the same orientation with respect to the subalgebra generated by $\{2,3,4\}$. This algebra is not subdirectly irreducible as the congruence generated by collapsing the elements 0 and 1 has nothing in common, outside of elements in $\Delta_{E_{13}}$, with the congruence generated by collapsing everything in $\{2,3,4\}$.

We show, in the next lemma, that if a unary algebra with a heart has 2 distinct veins that have the same orientation with respect to the heart of the algebra, then the algebra is not subdirectly irreducible.

Lemma 2.5.1. *Let* $\mathbf{B} = \langle B; \mathcal{F} \rangle$ *be a unary algebra. Further, suppose that* $\mathbf{M}$ *is in* $\mathbb{V}(\mathbf{B})$ *and that* $\mathbf{H}^{\mathbf{M}}$ *exists and is non-trivial. If* m_1 *and* m_2 *generate distinct veins of* $\mathbf{M}$ *that have the same orientation with respect to* $\mathbf{H}^{\mathbf{M}}$ *in* $\mathbf{M}$*, then* $\mathbf{M}$ *is not subdirectly irreducible.*

Proof. By hypothesis, $|H^{\mathbf{M}}| > 1$. Therefore,

$$\mathrm{Cg}^{\mathbf{M}}\left((H^{\mathbf{M}})^2\right) \neq \Delta_M. \tag{2.13}$$

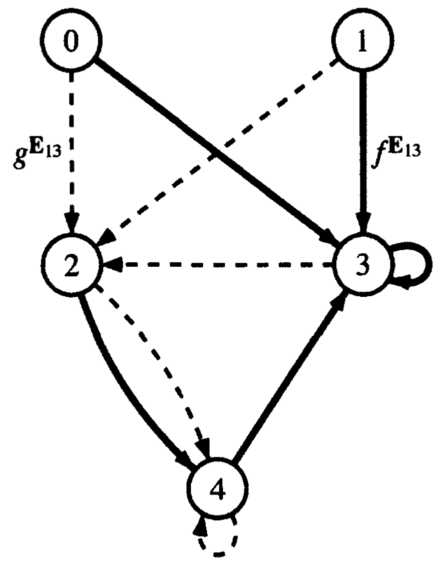

Figure 2.10: A Connected Unary Algebra with a Heart that has Two Veins that have the Same Orientation with Respect to the Heart: $\mathbf{E}_{13}$

By hypothesis, m_1 is not equal to m_2. Hence,

$$\mathrm{Cg}^{\mathbf{M}}\Big(\big\{\langle m_1, m_2\rangle\big\}\Big) \neq \Delta_M. \tag{2.14}$$

Suppose that $\langle m_3, m_4\rangle$ is in $\mathrm{Cg}^{\mathbf{M}}\Big(\big\{\langle m_1, m_2\rangle\big\}\Big)$. Then, by the Corollary 1.2.2 on page 11, for some positive integer n, there exist terms $\{t_i\}_{i=1}^{n}$ of type $\mathcal{F}$ such that

$$\begin{aligned}
m_3 &= t_1^{\mathbf{M}}(x_1)\\
t_1^{\mathbf{M}}(y_1) &= t_2^{\mathbf{M}}(x_2)\\
t_2^{\mathbf{M}}(y_2) &= t_3^{\mathbf{M}}(x_3)\\
&\ddots\\
t_{n-1}^{\mathbf{M}}(y_{n-1}) &= t_n^{\mathbf{M}}(x_n)\\
t_n^{\mathbf{M}}(y_n) &= m_4,
\end{aligned}$$

where for all $1 \leq i \leq n$, we have $\{x_i, y_i\} = \{m_1, m_2\}$.

Suppose that m_3 is in $H^{\mathbf{M}}$. Thus, $m_3 = t_1^{\mathbf{M}}(x_1)$ where x_1 is in $\{m_1, m_2\}$. As m_1 and m_2 generate subalgebras of $\mathbf{M}$ that have the same orientation with respect to $\mathbf{H}^{\mathbf{M}}$, the following equation is derived: $t_1^{\mathbf{M}}(m_1) = t_1^{\mathbf{M}}(m_2)$. Therefore, $m_3 = t_1^{\mathbf{M}}(y)$. Using induction, for all $1 \le i \le n$, we can show that $t_i^{\mathbf{M}}(x_i) = t_i^{\mathbf{M}}(y_i)$. Therefore, under the assumption that m_3 is in $H^{\mathbf{M}}$, we obtain $m_3 = m_4$. Using a similar idea, we can show that if m_4 is in $H^{\mathbf{M}}$ then $m_3 = m_4$. That is, if one of m_3 and m_4 is a member of $H^{\mathbf{M}}$, then $m_3 = m_4$. Therefore,

$$\mathrm{Cg}^{\mathbf{M}}\left(\{\langle m_1, m_2\rangle\}\right) \cap \left(H^{\mathbf{M}}\right)^2 = \Delta_{H^{\mathbf{M}}}.$$

Thus, using Lemma 2.2.2, we obtain the following:

$$\begin{aligned}
&\mathrm{Cg}^{\mathbf{M}}\left(\{\langle m_1, m_2\rangle\}\right) \wedge^{\mathbf{Con\,M}} \mathrm{Cg}^{\mathbf{M}}\left(\left(H^{\mathbf{M}}\right)^2\right) \\
&\quad = \mathrm{Cg}^{\mathbf{M}}\left(\{\langle m_1, m_2\rangle\}\right) \cap \left(\left(H^{\mathbf{M}}\right)^2 \cup \Delta_B\right) \\
&\quad = \left(\mathrm{Cg}^{\mathbf{M}}\left(\{\langle m_1, m_2\rangle\}\right) \cap \left(H^{\mathbf{M}}\right)^2\right) \cup \left(\mathrm{Cg}^{\mathbf{M}}\left(\{\langle m_1, m_2\rangle\}\right) \cap \Delta_B\right) \\
&\quad = \Delta_{H^{\mathbf{M}}} \cup \Delta_M \\
&\quad = \Delta_M.
\end{aligned}$$

Statement (2.13) and Statement (2.14) imply that the above expression is a meet of 2 distinct congruences on $\mathbf{M}$ that equates to Δ_B. Thus, the algebra $\mathbf{M}$ cannot possibly have a monolith. Finally, $\mathbf{M}$ is not subdirectly irreducible. □

We have shown that the orientation of veins in a given unary algebra can affect whether or not the unary algebra is subdirectly irreducible. The orientation of veins play a crucial role in answering the Restricted Quackenbush Problem.

2.6 Subdirectly Irreducible Mono-Unary Algebras

We have enough tools available to describe all finite subdirectly irreducible mono-unary algebras. Recall the mono-unary algebras $\mathbf{H}_{p^n}$ and $\mathbf{J}_h$, whose graphs are depicted in Figure 2.1 and Figure 2.2, respectively. Further, recall Yoeli's result regarding subdirectly irreducible mono-unary algebras:

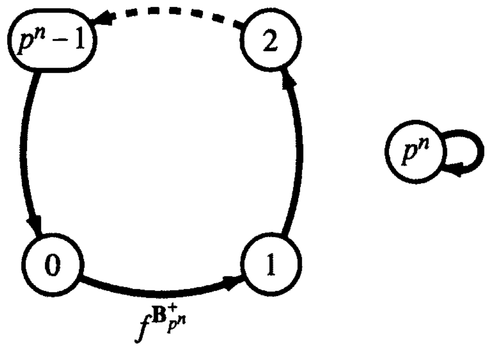

Figure 2.11: The Graph of $\mathbf{H}^+_{p^n}$

Figure 2.12: The Graph of $\mathbf{T}^+$.

Theorem (Yoeli, [19]). *A finite connected non-trivial mono-unary algebra* $\mathbf{B}$ *is irreducible if and only if* $\mathbf{B}$ *is isomorphic to* $\mathbf{H}_{p^n}$ *or to* $\mathbf{J}_h$, *where* p *is a prime number,* n *is a positive integer and* h *is a positive integer greater than 1.*

To extend Yoeli's results, we introduce notation for additional mono-unary algebras. Define $\mathbf{T}$ to be the trivial mono-unary algebra. Further, define $\mathbf{H}^+_{p^n}$ to be $\mathbf{H}_{p^n}$ with the addition of a single element to the universe, such that this element is mapped to itself, and define $\mathbf{T}^+$ to be $\mathbf{T}$ with the addition of a single element to the universe, such that this element is mapped to itself. See Figure 2.11 for the graph of $\mathbf{H}^+_{p^n}$ and Figure 2.12 for the graph of $\mathbf{T}^+$.

Recall, by Theorem 2.2.5, on page 59, that all subdirectly irreducible unary algebras are either connected or pseudoconnected. Hence, to extend Yoeli's result, we must determine what the pseudoconnected subdirectly irreducible mono-unary algebras are.

Lemma 2.2.6, on page 59, can be stated as follows: if a unary algebra is subdirectly irreducible, then every subalgebra must be subdirectly irreducible. Hence, to identify the finite pseudoconnected subdirectly irreducible mono-unary algebras, we need only take disjoint unions of the trivial mono-unary algebra and finite connected subdirectly irreducible mono-unary algebras. The disjoint union of the trivial mono-unary algebra and $\mathbf{H}_{p^n}$ yields $\mathbf{H}^+_{p^n}$

and the disjoint union of the trivial mono-unary algebra and $\mathbf{T}$ yields $\mathbf{T}^+$. The disjoint union of the trivial mono-unary algebra and $\mathbf{J}_h$ does not yield a subdirectly irreducible algebra. The restriction of Theorem 1 in [18] to finite algebras gives the following Theorem, which is an extension of Theorem 2.1.3.

Theorem 2.6.1 (Wenzel). *Let $\mathbf{B}$ be a finite mono-unary algebra. The algebra $\mathbf{B}$ is subdirectly irreducible if and only $\mathbf{B}$ is isomorphic to $\mathbf{H}_{p^n}$, to $\mathbf{H}^+_{p^n}$, to $\mathbf{J}_h$, to $\mathbf{T}$ or to $\mathbf{T}^+$, where p is a prime number, n is a positive integer and h is a positive integer greater than 1.*

Notice, in the above Theorem, that all finite connected subdirectly irreducible mono-unary algebras are veins or veins with the addition of the trivial mono-unary algebra. By Part 2 of Corollary 2.4.5, on page 73, if $\mathbf{B}$ is a finite unary algebra of finite type, then there exists a finite number of veins, up to isomorphism, of $\mathbb{V}(\mathbf{B})$. Therefore, if $\mathbb{V}(\mathbf{B})$ is residually finite, then $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N. That is, the Restricted Quackenbush Problem is answered affirmatively, with respect to mono-unary algebras.

2.7 Answering the Restricted Quackenbush Problem with Respect to Unary Algebras

We are now ready to answer the Restricted Quackenbush Problem, with respect to unary algebras. The answer is obtained by showing that within the variety generated by a finite unary algebra, there exists a finite bound on the number of generators a subdirectly irreducible algebra can have. Specifically, we show that when a unary algebra has too many distinct veins, all isomorphic to one another, the algebra is not subdirectly irreducible.

Let D denote a class of algebras, all of the same type. For all $\mathbf{K}_1$ and $\mathbf{K}_2$ in D, define the relation σ on D by

$$\text{relating } \mathbf{K}_1 \text{ to } \mathbf{K}_2 \qquad \text{if and only if} \qquad \mathbf{K}_1 \cong \mathbf{K}_2.$$

The relation σ can be shown to be an equivalence relation. Each member of $\{\mathbf{K}/\sigma \mid \mathbf{K} \in D\}$ is called an *isomorphism class*.

Assume that $\mathbf{B}$ is a finite unary algebra of finite type. By Part 2 of Corollary 2.4.5, on page 73, there exist only finitely many veins, up to isomorphism, of $\mathbb{V}(\mathbf{B})$. Let v denote this number. Choose a set of distinct representatives from each isomorphism class in the class of all veins of $\mathbb{V}(\mathbf{B})$ and label the representatives using the indexed set $\{\mathbf{V}_i\}_{i=1}^{v}$. By Part 1 of Corollary 2.4.5,

$$\text{for all } i \text{ in } \{1,2,3,\ldots,v\} \qquad |V_i| < |B|^{|B|} < \omega.$$

By Theorem 2.4.2, on page 68, every connected subdirectly irreducible member of $\mathbb{V}(\mathbf{B})$ has a heart. The heart of a unary algebra, assuming that it exists, is a subalgebra of every vein of that algebra. Thus, the heart of every connected subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ must be isomorphic to a subalgebra of at least one of the members in $\{\mathbf{V}_i\}_{i=1}^{v}$. Noting that, up to isomorphism, there exist a finite number of veins and each vein is finite, there exist, up to isomorphism, a finite number of possible hearts of connected subdirectly irreducible elements in $\mathbb{V}(\mathbf{B})$. For the finite integer h, let $\{\mathbf{H}_j\}_{j=1}^{h}$ be a set of distinct representatives from each isomorphism class of $\mathbb{S}(\{\mathbf{V}_i\}_{i=1}^{v})$.

For i in $\{1,2,3,\ldots,v\}$ and j in $\{1,2,3,\ldots,h\}$, let $m_{i,j}^{*}$ denote the number of distinct embeddings of $\mathbf{H}_j$ into $\mathbf{V}_i$. Note that since both H_j and V_i are finite, $m_{i,j}^{*}$ must also be finite. Define $m_{i,j}$ to be $m_{i,j}^{*}+2$.

Lemma 2.7.1. *For all i in $\{1,2,3,\ldots,v\}$ and j in $\{1,2,3,\ldots,h\}$, if $\mathbf{M}$ is a connected non-trivial algebra in $\mathbb{V}(\mathbf{B})$ that has a heart isomorphic to $\mathbf{H}_j$ and $\mathbf{M}$ has $m_{i,j}$ distinct veins* *isomorphic to* $\mathbf{V}_i$ *then there exist two distinct veins of* $\mathbf{M}$ *that have the same orientation with respect to* $\mathbf{H^M}$.

Proof. Let S be an irredundant basis of $\mathbf{M}$. By hypothesis, there exists $m_{i,j}$ distinct veins of $\mathbf{M}$ isomorphic to $\mathbf{V}_i$. Let $\{s_k\}_{k=1}^{m_{i,j}}$ be a subset of distinct generators from S such that for all $1 \le k \le m_{i,j}$, the vein $\langle \mathrm{Sg}^{\mathbf{M}}(\{s_k\}); \mathcal{F} \rangle$ is isomorphic to $\mathbf{V}_i$.

Without loss of generality, focus on the generator s_1. For $2 \le \ell \le m_{i,j}$, let α_ℓ denote an isomorphism from

$$\langle \mathrm{Sg}^{\mathbf{M}}(\{s_\ell\}); \mathcal{F} \rangle \qquad \text{to} \qquad \langle \mathrm{Sg}^{\mathbf{M}}(\{s_1\}); \mathcal{F} \rangle.$$

Consequently, there are $m_{i,j} - 1 = m^*_{i,j} + 1$ isomorphisms in

$$\{\alpha_\ell \mid 2 \le \ell \le m_{i,j}\}$$

Hence, by the Pigeon Hole Principle, at least two of the maps in

$$\{\alpha_\ell \restriction_{H^{\mathbf{M}}} \mid 2 \le \ell \le m_{i,j}\}$$

are the same embeddings of $\mathbf{H^M}$ into $\langle \mathrm{Sg}^{\mathbf{M}}(\{s_1\}); \mathcal{F} \rangle$. Assume that

$$\alpha_{t_1} \restriction_{H^{\mathbf{M}}} = \alpha_{t_2} \restriction_{H^{\mathbf{M}}}$$

for some t_1 and t_2 in $\{s_\ell\}_{\ell=2}^{m_{i,j}}$. Consider

$$\alpha = \alpha_{t_2}^{-1} \circ \alpha_{t_1},$$

an isomorphism from

$$\langle \mathrm{Sg}^{\mathbf{M}}(\{t_1\}); \mathcal{F} \rangle \qquad \text{to} \qquad \langle \mathrm{Sg}^{\mathbf{M}}(\{t_2\}); \mathcal{F} \rangle. \tag{2.15}$$

and note that

$$\alpha \restriction_{H^{\mathbf{M}}} = \mathrm{id}_{H^{\mathbf{M}}}, \tag{2.16}$$

where $\mathrm{id}_{H^{\mathbf{M}}}$ is the identity automorphism on $H^{\mathbf{M}}$.

Note that $\alpha(t_1)$ might not equal t_2. There could be many members of $\mathrm{Sg}^{\mathbf{M}}(\{t_2\})$ that singly generate $\mathrm{Sg}^{\mathbf{M}}(\{t_2\})$. As α is an isomorphism, $\alpha(t_1)$ must singly generate $\mathrm{Sg}^{\mathbf{M}}(\{t_2\})$. Hence, we may replace t_2 with $\alpha(t_1)$ in the irredundant basis S and in $\{s_k\}_{k=1}^{m_{i,j}}$. We are permitted to change irredundant bases in this manner because we are working with unary operations and

$$\left\langle \mathrm{Sg}^{\mathbf{M}}(\{t_2\}); \mathcal{F} \right\rangle = \left\langle \mathrm{Sg}^{\mathbf{M}}\big(\{\alpha(t_1)\}\big); \mathcal{F} \right\rangle. \tag{2.17}$$

Therefore, we may assume that $t_2 = \alpha(t_1)$. Define

$$\mathbf{C}_1 = \left\langle \mathrm{Sg}^{\mathbf{M}}(\{t_1\}); \mathcal{F} \right\rangle \qquad \text{and} \qquad \mathbf{C}_2 = \left\langle \mathrm{Sg}^{\mathbf{M}}(\{t_2\}); \mathcal{F} \right\rangle.$$

We show that $\mathbf{C}_1$ and $\mathbf{C}_2$ have the same orientation with respect to $\mathbf{H}^{\mathbf{M}}$ in $\mathbf{M}$. Note that using Statement (2.15) and Statement (2.17), the map α is an isomorphism from $\mathbf{C}_1$ to $\mathbf{C}_2$. Let h be a member of $H^{\mathbf{M}}$ and suppose that $r(x)$ is a term of type $\mathcal{F}$, such that $r^{\mathbf{C}_1}(t_1) = h$. Then, using Statement (2.16),

$$\begin{aligned} h &= \alpha(h) \\ &= \alpha\big(r^{\mathbf{C}_1}(t_1)\big) \\ &= r^{\mathbf{C}_2}\big(\alpha(t_1)\big) \\ &= r^{\mathbf{C}_2}(t_2). \end{aligned}$$

That is, for all h in $H^{\mathbf{M}}$, if $r^{\mathbf{C}_1}(t_1) = h$, then $r^{\mathbf{C}_2}(t_2) = h$. We can use α^{-1} to get the converse. Therefore, $\mathbf{C}_1$ and $\mathbf{C}_2$ have the same orientation with respect to $\mathbf{H}^{\mathbf{M}}$ in $\mathbf{M}$. □

Theorem 2.7.2. *Let $\mathbf{B} = \langle B; \mathcal{F} \rangle$ be a finite unary algebra of finite type. If $\mathbb{V}(\mathbf{B})$ is residually finite then $\mathbb{V}(\mathbf{B})$ is residually $< N$, for some positive integer N.*

Proof. To prove the theorem, we show that, for some positive integer m, every connected subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ must have less than m generators. We then show that there exist, up to isomorphism, a finite number of subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$. To complete the proof, we take the cardinality of the universe of a subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ that contains the most elements, add one, and call the resulting integer N.

Recall that by Theorem 2.4.2, on page 68, every connected subdirectly irreducible member of $\mathbb{V}(\mathbf{B})$ has a heart. The heart of a unary algebra, assuming that it exists, is a subalgebra of every vein of that algebra. By Lemma 2.7.1, if $\mathbf{M}$ is a connected non-trivial algebra in $\mathbb{V}(\mathbf{B})$ that has a heart isomorphic to $\mathbf{H}_j$ and, for some i in I, the algebra $\mathbf{M}$ has $m_{i,j}$ distinct veins isomorphic to $\mathbf{V}_i$, then there exist two distinct veins of $\mathbf{M}$ that have the same orientation with respect to $\mathbf{H}^{\mathbf{M}}$.

If $\mathbf{H}^{\mathbf{M}}$ is non-trivial then by Lemma 2.5.1, on page 74, $\mathbf{M}$ is not subdirectly irreducible. If $\mathbf{H}^{\mathbf{M}}$ is trivial, then by the contrapositive of Lemma 2.4.3, on page 69, $\mathbf{M}$ is not subdirectly irreducible. Thus, if $\mathbf{M}$ is a subdirectly irreducible connected non-trivial algebra in $\mathbb{V}(\mathbf{B})$ that has a heart isomorphic to $\mathbf{H}_j$, then $\mathbf{M}$ can have at most $m_{i,j}-1$ veins isomorphic to $\mathbf{V}_i$.

Now let

$$m^* = \max\{m_{i,j}-1 \mid 1 \le i \le v,\ 1 \le j \le h\} \qquad \text{and} \qquad m = v\cdot m^* + 1.$$

Note that m is finite, due to there existing a finite number of veins and hearts and each vein and heart is finite. Recall that by Lemma 2.4.4, on page 72, for a unary algebra $\mathbf{M}_1$ that is a member of a finitely generated variety, every member of every irredundant basis of M_1 generates a vein of $\mathbf{M}_1$. Thus, given any connected non-trivial algebra $\mathbf{M}$ in $\mathbb{V}(\mathbf{B})$, by the Extended Pigeon Hole Principle, if there are at least m members in an irredundant basis of M, then at least m^* of them generate veins all isomorphic to $\mathbf{V}$, for some $\mathbf{V}$ in $\{\mathbf{V}_i\}_{i=1}^{v}$. By the argument in the preceding paragraphs, $\mathbf{M}$ is not subdirectly irreducible. Therefore, every connected subdirectly irreducible non-trivial member in $\mathbb{V}(\mathbf{B})$ must have an irredundant basis with a cardinality less than m.

By Corollary 1.3.4, on page 38, there exist a finite number of algebras in $\mathbb{V}(\mathbf{B})$, up to isomorphism, that are minimally generated by k elements, where k is a finite integer. As m is a finite integer, there exists, up to isomorphism, a finite number of connected subdirectly irreducible non-trivial members in $\mathbb{V}(\mathbf{B})$.

Every pseudoconnected subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ must have a connected subdirectly irreducible component and a trivial component that make up the entire pseudoconnected unary algebra. By Lemma 2.2.6, on page 59, as there exists, up to isomorphism, a finite number of connected subdirectly irreducible members in $\mathbb{V}(\mathbf{M})$, there exists, up to isomorphism, a finite number of pseudoconnected subdirectly irreducible members in $\mathbb{V}(\mathbf{M})$.

By Theorem 2.2.5, on page 59, each subdirectly irreducible member in $\mathbb{V}(\mathbf{B})$ is either connected or pseudoconnected. Thus, there exists, up to isomorphism, a finite number of subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$. By hypothesis, the universe of each subdirectly irreducible member is finite. Take the universe with the largest cardinality, add one to the obtained finite cardinal number and call the resulting finite cardinal N. Then $\mathbb{V}(\mathbf{B})$ is residually $< N$. □

We have just shown that the Restricted Quackenbush Problem is answered affirmatively for unary algebras.

Chapter 3

A Property of Finite Groupoids that Generate Residually Finite Varieties

The Restricted Quackenbush Problem has been answered affirmatively with respect to groups in [17] and semigroups in [9]. So far, the Restricted Quackenbush Problem has not been answered with respect to arbitrary groupoids. In this chapter we look at groupoids that are influenced by a partial order relation. Groupoids are of importance because Quackenbush specifically asked about them in [15], the article where the earliest form of the Restricted Quackenbush Problem appeared.

Since the language of any groupoid $\mathbf{B} = \langle B, *^{\mathbf{B}} \rangle$ consists of a single binary operation symbol, we refer to this operation as *multiplication*. Thus, if b_1 and b_2 are in B, we may call $*^{\mathbf{B}}(b_1, b_2)$ or $b_1 *^{\mathbf{B}} b_2$, the *product of* b_1 *and* b_2 in $\mathbf{B}$.

Groupoids are versatile in the sense that for many properties, there often exists a groupoid that does not satisfy such a property. For example, not all groupoids are associative or commutative or idempotent or congruence-regular or congruence-modular.

Example. To see that not all idempotent groupoids are congruence-modular or associative, consider the groupoid $\mathbf{E}_{14}$. The operation table of the binary operation of $\mathbf{E}_{14}$ is presented in Table 3.1 and the congruence lattice of $\mathbf{E}_{14}$ is pictured in Figure 3.1. Since $\mathbf{Con\ E}_{14} \cong \mathbf{N}_5$, we may apply Theorem 1.2.9, on page 20, to show that the groupoid $\mathbf{E}_{14}$ is not congruence-modular. To see why $\mathbf{E}_{14}$ is not associative, note the following:

$$(0 *^{\mathbf{E}_{14}} 1) *^{\mathbf{E}_{14}} 2 = 2 *^{\mathbf{E}_{14}} 2 = 2 \qquad \text{and} \qquad 0 *^{\mathbf{E}_{14}} (1 *^{\mathbf{E}_{14}} 2) = 0 *^{\mathbf{E}_{14}} 0 = 0.$$

$*^{\mathbf{E}_{14}}$	0	1	2	3	4	5
0	0	2	2	0	2	2
1	2	1	0	2	1	0
2	1	1	2	1	1	2
3	0	2	2	3	5	5
4	2	1	0	5	4	3
5	1	1	2	4	4	5

Table 3.1: The Operation Table of $*^{\mathbf{E}_{14}}$

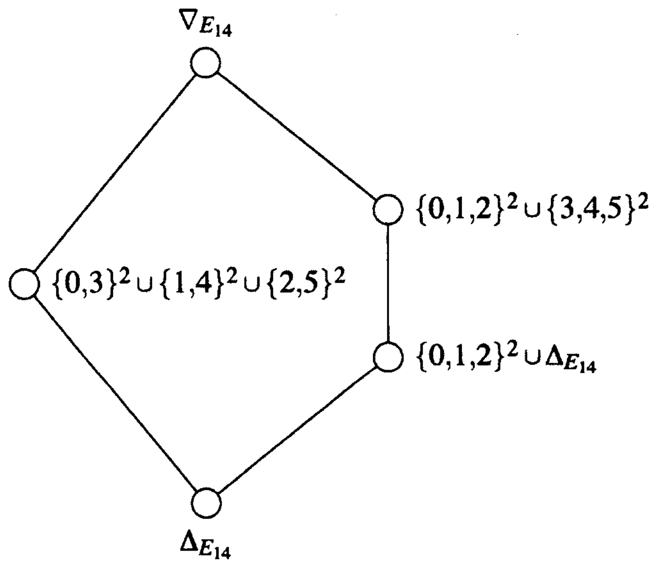

Figure 3.1: The Congruence Lattice of $\mathbf{E}_{14}$

In this chapter we determine a property that the generator of a variety of groupoids must satisfy to be residually finite. Specifically, we show that every residually finite variety, generated by a finite groupoid, must satisfy a particular kind of identity. These kinds of identities help to decide what subclass of groupoids to look at, with respect to the Restricted Quackenbush Problem.

We start by looking at a particular class of finite groupoids whose elements, in a loose sense, *converge* to a particular element. These groupoids are said to be *influenced by a partial order relation*. We show that each groupoid in a subclass of these groupoids generates a residually large variety. To conclude this section, we derive an identity that every residually finite variety generated by a finite groupoid must satisfy.

3.1 Groupoids that are Influenced by a Partial Order Relation

The formal definition a groupoid influenced by a partial order relation follows after two examples.

Example. The groupoid $\mathbf{E}_{15}$ is influenced by the partial order relation $\leq$ of height 4. The operation table of $*^{\mathbf{E}_{15}}$ is given in Table 3.2 and $\leq$ is displayed as a Hasse diagram in Figure 3.2. Note that the product of e_1 and e_2 in $E_{15}\backslash\{7\}$ is greater than both e_1 and e_2 with respect to the partial order relation $\leq$. Further, $\mathbf{E}_{15}$ is not a semigroup due to

$$(0 *^{\mathbf{E}_{15}} 1) *^{\mathbf{E}_{15}} 2 = 3 *^{\mathbf{E}_{15}} 2 = 6 \qquad \text{and} \qquad 0 *^{\mathbf{E}_{15}} (1 *^{\mathbf{E}_{15}} 2) = 0 *^{\mathbf{E}_{15}} 4 = 5.$$

Example. The operation table for the groupoid $\mathbf{E}_{16}$'s operator $*^{\mathbf{E}_{16}}$ is given in Table 3.3. The groupoid $\mathbf{E}_{16}$ is influenced by the partial order relation $\leq$ of height 3, displayed as a Hasse diagram in Figure 3.3. The groupoid $\mathbf{E}_{16}$ is associative due to the property that the product of any three elements is 11.

$*E_{15}$	0	1	2	3	4	5	6	7
0	7	3	7	7	5	7	7	7
1	7	7	4	7	7	7	7	7
2	7	7	7	7	7	7	7	7
3	7	7	6	7	7	7	7	7
4	7	7	7	7	7	7	7	7
5	7	7	7	7	7	7	7	7
6	7	7	7	7	7	7	7	7
7	7	7	7	7	7	7	7	7

Table 3.2: The Operation Table of $*E_{15}$

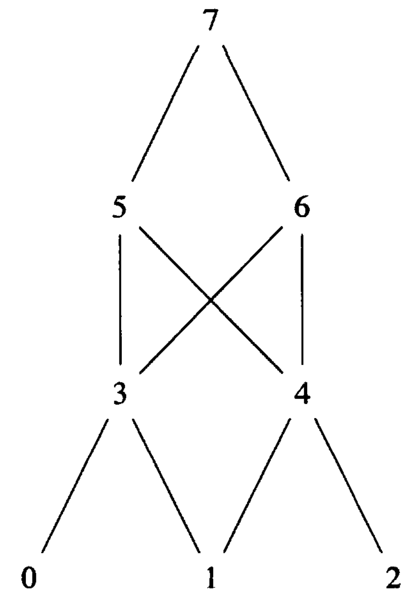

Figure 3.2: The Partial Order Relation $\leq$ that Corresponds to $*E_{15}$

$*E_{16}$	0	1	2	3	4	5	6	$\cdots$	11
0	6	8	8	6	8	8	11	$\cdots$	11
1	8	7	6	8	7	6	11	$\cdots$	11
2	7	7	8	7	7	8	11	$\cdots$	11
3	6	8	8	9	11	11	11	$\cdots$	11
4	8	7	6	11	10	9	11	$\cdots$	11
5	7	7	8	10	10	11	11	$\cdots$	11
6	11	11	11	11	11	11	11	$\cdots$	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
11	11	11	11	11	11	11	11	$\cdots$	11

Table 3.3: The Operation Table of $*E_{16}$

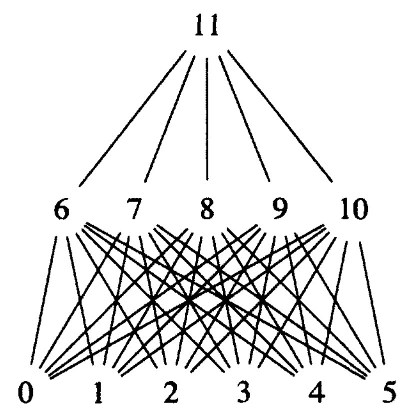

Figure 3.3: The Partial Order Relation $\leq$ that Corresponds to $*E_{16}$

Definition. Let $\mathbf{B} = \langle B; *^{\mathbf{B}}, \leq^{\mathbf{B}} \rangle$ be a finite first order structure such that $*^{\mathbf{B}}$ is a binary operation and $\leq^{\mathbf{B}}$ is a partial order relation. Suppose that the following conditions are satisfied:

1. (Maximum Element) There exists a unique maximum element λ in B such that for all b in B, we have $b \leq^{\mathbf{B}} \lambda$.

2. (Upward Multiplication) For all b_1 in $B \backslash \{\lambda\}$ and b_2 in B,

$$b_1 <^{\mathbf{B}} b_2 *^{\mathbf{B}} b_1 \qquad \text{and} \qquad b_1 <^{\mathbf{B}} b_1 *^{\mathbf{B}} b_2.$$

3. (Operation Respects Partial Order) For all b_1, b_2, b_3 and b_4 in B,

$$\text{if} \quad b_1 \leq^{\mathbf{B}} b_2 \quad \text{and} \quad b_3 \leq^{\mathbf{B}} b_4 \quad \text{then} \quad b_1 *^{\mathbf{B}} b_3 \leq^{\mathbf{B}} b_2 *^{\mathbf{B}} b_4.$$

Let $\mathbf{M} = \langle M; *^{\mathbf{M}} \rangle$ be a reduct of $\mathbf{B}$ to a groupoid. That is, let $M = B$ and $*^{\mathbf{M}} = *^{\mathbf{B}}$. Lastly, let

$$h = \max\left\{|C| : \langle C; \leq^{\mathbf{B}} \rangle \text{ is a chain such that } C \subseteq B\right\}.$$

If a groupoid $\mathbf{M}$ can be constructed in this way, say that $\mathbf{M}$ is *influenced by a partial order relation of height h* and use λ_M to denote the maximum element.

Remarks. A few comments regarding the previous definition.

1. The partial order $\leq$, and subsequently $<$, are not relation symbols in the language of $\mathbf{M}$. Hence, $\leq^{\mathbf{M}}$ and $<^{\mathbf{M}}$ do not have any meaning. We make use of $\leq$ and $<$, with respect to members of M, with the understanding that $\mathbf{M}$ is a reduct of a structure with an appropriate language.

2. As $\mathbf{M} = \langle M, *^{\mathbf{M}} \rangle$ is influenced by a partial order relation, for any given m in M, if m is multiplied by itself enough times, the result is λ_M. In fact, h instances of any element in M multiplied by itself yields λ_M.

3. As the set M is finite, the height of the partial order relation h must be a positive integer.

3.1.1 Subdirectly Irreducible Groupoids that are Influenced by a Partial Order Relation

What do the subdirectly irreducible algebras, that belong to a variety generated by a groupoid that is influenced by a partial order relation, look like? In this subsection, we describe two special elements in the universe of such a groupoid and how they interact with the other elements in the universe. Further, we give a complete description of its monolith.

For the remainder of this chapter, assume that $\mathbf{B}$ is a groupoid that is influenced by a partial order operation of height h. Further, assume that $\mathbf{S}$ is a subalgebra of $\mathbf{B}^I$, for some index set I, and assume that θ is a congruence on $\mathbf{S}$. Let $T_\theta = \{s/\theta \mid s \in S\}$, the set of congruence blocks of θ on S. Recall that since $\mathbf{B}$ is a groupoid that is influenced by a partial order relation, there exists a partial order relation on B, denoted by $\leq$. For b_1 and b_2 in B^I, define $b_1 \leq b_2$ if and only if, for all i in I, we have $b_1(i) \leq b_2(i)$.

For b in B^I, define the relation ϕ_b, on T_θ, as follows: for all t_1 and t_2 in T_θ, relate t_1 to t_2 if and only if

1. for all j in $\{1,2\}$, there exists an element s_j in t_j such that $b \leq s_j$; or

2. t_1 is equal to t_2.

Lemma 3.1.1. *For b in B^I, the relation ϕ_b is a congruence on $\mathbf{S}/\theta$.*

Proof. We start by showing that ϕ_b is an equivalence relation. The reflexive property of ϕ_b follows from Part 2 the definition of ϕ_b. The symmetry property follows from the lack of the use of the order of t_1 and t_2 in the definition of ϕ_b. For the transitive property, suppose that $t_1 \ \phi_b \ t_2$ and $t_2 \ \phi_b \ t_3$. We consider 3 cases.

Case 1 Suppose that $t_1 \neq t_2$ and $t_2 \neq t_3$. By Part 1 of the definition of ϕ_b, for j in $\{1,2,3\}$, there exists s_j in t_j such that $b \leq s_j$. There may be two such elements in t_2. Regardless, we have $t_1 \ \phi_b \ t_3$.

Case 2 Without loss of generality, suppose that $t_1 \neq t_2$ and $t_2 = t_3$. Then

$$\langle t_1, t_3 \rangle = \langle t_1, t_2 \rangle \in \phi_b.$$

Case 3 Suppose that $t_1 = t_2$ and $t_2 = t_3$. Then $t_1 = t_3$, by the transitivity of equality, and hence $t_1 \ \phi_b \ t_3$.

Thus, ϕ_b is an equivalence relation.

To complete the proof, we show that ϕ_b satisfies the compatibility property. Suppose that $t_1 \ \phi_b \ t_2$ and $t_3 \ \phi_b \ t_4$. We want to show that

$$(t_1 *^{\mathbf{S}/\theta} t_3) \ \phi_b \ (t_2 *^{\mathbf{S}/\theta} t_4).$$

If $t_1 = t_2$ and $t_3 = t_4$, then $t_1 *^{\mathbf{S}/\theta} t_3 = t_2 *^{\mathbf{S}/\theta} t_4$. Now, without loss of generality, assume that $t_1 \neq t_2$. That is, assume that there exists elements s_1 and s_2 in t_1 and t_2, respectively, such that $b \leq s_1$ and $b \leq s_2$. Pick s_3 in t_3 and s_4 in t_4. Thus, for all j in $\{1,2,3,4\}$, the set t_j is equal to s_j/θ. Due to upward multiplication of $\mathbf{B}$, we obtain the following:

$$\begin{aligned} b \leq s_1 \leq s_1 *^{\mathbf{S}} s_3 \in (s_1 *^{\mathbf{S}} s_3)/\theta & \\ &= s_1/\theta *^{\mathbf{S}/\theta} s_3/\theta \\ &= t_1 *^{\mathbf{S}/\theta} t_3 \end{aligned}$$

and using a similar idea

$$b \leq s_2 *^{\mathbf{S}} s_4 \in t_2 *^{\mathbf{S}/\theta} t_4.$$

Thus, the relation ϕ_b is a congruence on $\mathbf{S}/\theta$. □

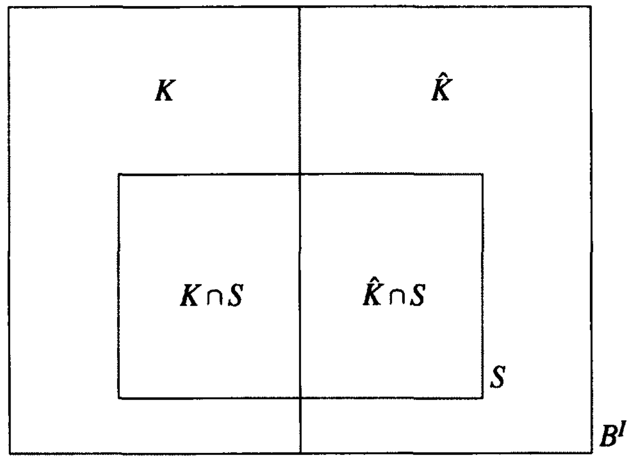

Figure 3.4: How B^I, S, K and $\hat{K}$ Interact

For b in B, knowing whether or not ϕ_b is $\Delta_{S/\theta}$ will play a role in describing the subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$. We use the following notations throughout the remainder of this subsection. Define

$$K = \{b \in B^I : \phi_b \neq \Delta_{S/\theta}\} \qquad \text{and} \qquad \hat{K} = B^I \backslash K.$$

Further, let

$$\Phi = \bigwedge_{k \in K} \phi_k.$$

See Figure 3.4 for a visualization of how B^I, S, K and $\hat{K}$ interact. The constant mapping from I to $\{\lambda_B\}$, in B^I, plays an important role in the following proofs. Denote this element by $\overline{\lambda_B}$.

Lemma 3.1.2. *The following are true:*

1. *The element $\overline{\lambda_B}$ is a member of S.*

2. *For all b in B^I, such that $\phi_b = \Delta_{S/\theta}$, we have*

$$\{s \in S : b \leq s\} \subseteq \overline{\lambda_B}/\theta.$$

3. *The set* $\hat{K} \cap S$ *is contained in* $\overline{\lambda_B}/\theta$.

4. *For all* t *in* T_θ, *if* $t \cap \hat{K} \neq \varnothing$, *then* $t = \overline{\lambda_B}/\theta$.

5. *If* t_1 *and* t_2 *are in* T_θ *such that* $t_1 \cap \hat{K} \neq \varnothing$ *and* $t_2 \cap \hat{K} \neq \varnothing$, *then* $t_1 = t_2$.

6. *If* t *is in* T_θ *such that* $t \cap \hat{K} = \varnothing$ *then* $t \neq \overline{\lambda_B}/\theta$.

Proof. The Lemma's Parts are proved in sequential order.

By Remark 2 concerning groupoids influenced by a partial order relation, on page 88, for all b in B^I, we have $b^h = \overline{\lambda_B}$, where b^h is some h-fold product of b. Therefore, as S is a non-empty subuniverse of $\mathbf{B}^I$, the element $\overline{\lambda_B}$ is in S. This proves Part 1 of the Lemma.

Note that for all s in S, we must have $s \leq \overline{\lambda_B}$. That is, $\overline{\lambda_B}$ is the maximum element in S. With regard to Part 2 of the Lemma, note that if, for some b in B, we have $\phi_b = \Delta_{S/\theta}$ then the elements in $\{s \in S : b \leq s\}$ must belong to the same θ congruence block, specifically

$$\{s \in S : b \leq s\} \subseteq \overline{\lambda_B}/\theta,$$

otherwise at least two distinct θ congruence blocks must be related in ϕ_b.

Since $\hat{K} = B^I \backslash K$, Part 3 of the Lemma follows from Part 2 of the Lemma.

Assume that for some t in T_θ we have $t \cap \hat{K} \neq \varnothing$. By Part 3 of the Lemma, since there exists an element s in $t \cap \hat{K}$, the element s is in $\overline{\lambda_B}/\theta$. That is,

$$t = s/\theta = \overline{\lambda_B}/\theta.$$

This proves Part 4 of the Lemma.

Part 5 of the Lemma is a consequence of Part 4 of the Lemma.

To prove Part 6 of the Lemma, assume that $t \cap \hat{K} = \varnothing$. Hence, $t \subseteq K$. Suppose, for a contradiction, that

$$t = \overline{\lambda_B}/\theta.$$

Thus,

$$\overline{\lambda_B} \in \overline{\lambda_B}/\theta = t \subseteq K$$

and hence $\phi_{\overline{\lambda_B}} \neq \Delta_{S/\theta}$. Hence, there must exist an element in S that is greater than $\overline{\lambda_B}$. A contradiction has occurred, as $\overline{\lambda_B}$ is the maximum element in S. □

For the remainder of this chapter, we acknowledge $\overline{\lambda_B}$ as the maximum element in S, without statement.

Lemma 3.1.3. *Suppose that for distinct elements t_1 and t_2 in T_θ, we have $\langle t_1, t_2 \rangle$ in Φ. If $t_1 \cap \hat{K} = \varnothing$, then for all t in T_θ,*

$$t_1 *^{S/\theta} t = \overline{\lambda_B}/\theta \qquad \textit{and} \qquad t *^{S/\theta} t_1 = \overline{\lambda_B}/\theta.$$

Proof. By assumption $t_1 \cap \hat{K} = \varnothing$. Hence, by Part 6 of Lemma 3.1.2,

$$t_1 \neq \overline{\lambda_B}/\theta. \tag{3.1}$$

By hypothesis, $\langle t_1, t_2 \rangle$ is in Φ. Thus, for all k in K, the pair $\langle t_1, t_2 \rangle$ is in ϕ_k. Therefore,

$$\text{for all} \quad k \in K \quad \text{there exists} \quad u_k \in t_1 \quad \text{such that} \quad k \leq u_k. \tag{3.2}$$

Pick s_1 in t_1 and r in t. We will show that $s_1 *^{S} r$ is in $\overline{\lambda_B}/\theta$. By doing this, we are showing that

$$t_1 *^{S/\theta} t = \overline{\lambda_B}/\theta.$$

Using a similar idea, we will also obtain

$$t *^{S/\theta} t_1 = \overline{\lambda_B}/\theta$$

and thus the proof will be complete.

Now to see that $s_1 *^{\mathbf{S}} r$ is in $\overline{\lambda_B}/\theta$ suppose, for a contradiction, that

$$s_1 *^{\mathbf{S}} r \notin \overline{\lambda_B}/\theta. \tag{3.3}$$

Recall from Part 3 of Lemma 3.1.2 that $\hat{K} \cap S \subseteq \overline{\lambda_B}/\theta$. Thus

$$s_1 *^{\mathbf{S}} r \notin \hat{K} \cap S;$$

As both s_1 and r are in S, we have $s_1 *^{\mathbf{S}} r$ in S and hence, $s_1 *^{\mathbf{S}} r$ is in K.

By upward multiplication, for all i in I, we have $s_1(i) < s_1(i) *^{\mathbf{B}} r(i)$. Thus, $s_1 < s_1 *^{\mathbf{S}} r$. By Statement (3.2), there exists r_1 in t_1 such that $s_1 *^{\mathbf{S}} r \leq r_1$. Thus, for all i in I,

$$s_1(i) < (s_1 *^{\mathbf{S}} r)(i) \leq r_1(i).$$

By Statement (3.1), the element r_1 cannot be equal to $\overline{\lambda_B}$. Further, $r_1 *^{\mathbf{S}} r$ cannot be in $\overline{\lambda_B}/\theta$ as otherwise $(r_1 *^{\mathbf{S}} r)/\theta = \overline{\lambda_B}/\theta$ and, as $r_1/\theta = t_1$, we obtain the following:

$$\begin{aligned}(s_1 *^{\mathbf{S}} r)/\theta &= s_1/\theta *^{\mathbf{S}/\theta} r/\theta \\ &= t_1 *^{\mathbf{S}/\theta} r/\theta \\ &= r_1/\theta *^{\mathbf{S}/\theta} r/\theta \\ &= (r_1 *^{\mathbf{S}} r)/\theta \\ &= \overline{\lambda_B}/\theta.\end{aligned}$$

Hence, if $r_1 *^{\mathbf{S}} r$ is in $\overline{\lambda_B}/\theta$, then a contradiction occurs from the assumption made in Statement (3.3).

Since $t_1 \subseteq K$, we may repeat the above process to find r_2 in t_1 such that for all i in I,

$$r_1(i) < (r_1 *^{\mathbf{S}} r)(i) \leq r_2(i).$$

Repeating this process h times in total, where h denotes the height of the partial order relation that influences $\mathbf{B}$, we obtain $r_h(i) = \lambda_B$ for all $i \in I$. Therefore, $r_h = \overline{\lambda_B}$ and r_h is a member in t_1. A contradiction has occurred due to Statement (3.1). Thus, $s_1 *^{\mathbf{S}} r$ is in $\overline{\lambda_B}/\theta$ and the proof is complete. $\square$

Lemma 3.1.4. *Let* $\mathbf{M}$ *denote a groupoid. If there exists* m_1, m_2 *and* m_3 *in* M *such that for all* m *in* M *and* j *in* $\{1,2\}$,

$$m *^{\mathbf{M}} m_j = m_3 \qquad \text{and} \qquad m_j *^{\mathbf{M}} m = m_3$$

then

$$Cg^{\mathbf{M}}\left(\langle m_1,m_2\rangle\right) = \left\{\langle m_1,m_2\rangle,\langle m_2,m_1\rangle\right\} \cup \Delta_M.$$

Proof. Denote $\left\{\langle m_1,m_2\rangle,\langle m_2,m_1\rangle\right\} \cup \Delta_M$ by τ. Clearly, τ is an equivalence relation on $\mathbf{M}$.

Let $\langle t_1,t_2\rangle$ and $\langle t_3,t_4\rangle$ be in τ. If $t_1 = t_2$ and $t_3 = t_4$, then $t_1 *^{\mathbf{M}} t_3 = t_2 *^{\mathbf{M}} t_4$ and hence

$$\langle t_1 *^{\mathbf{M}} t_3, t_2 *^{\mathbf{M}} t_4\rangle \in \tau.$$

Without loss of generality, assume that $t_1 \neq t_2$. Then $\{t_1,t_2\} = \{m_1,m_2\}$. Again, without loss of generality, assume that $t_1 = m_1$ and $t_2 = m_2$. By hypothesis,

$$\langle t_1 *^{\mathbf{M}} t_3, t_2 *^{\mathbf{M}} t_4\rangle = \langle m_1 *^{\mathbf{M}} t_3, m_2 *^{\mathbf{M}} t_4\rangle = \langle m_3,m_3\rangle \in \tau.$$

Thus, τ satisfies the compatibility property and hence is a congruence on $\mathbf{M}$.

As τ is the smallest congruence on $\mathbf{M}$ that relates m_1 to m_2, we obtain the desired result. $\square$

Corollary 3.1.5. *Let* t_1 *and* t_2 *be elements in* T_θ *such that* $\langle t_1,t_2\rangle$ *is in* Φ. *If* $t_1 \cap \hat{K} = \varnothing$ *and* $t_2 \cap \hat{K} = \varnothing$, *then either* $t_1 = t_2$ *or* $\mathbf{S}/\theta$ *is not subdirectly irreducible.*

Proof. By assumption $t_j \cap \hat{K} = \varnothing$ for j in $\{1,2\}$. By Part 6 of Lemma 3.1.2, we have

$$t_j \neq \overline{\lambda_B}/\theta. \tag{3.4}$$

Lemma 3.1.3 and Lemma 3.1.4 imply that both τ_1 and τ_2, defined below, are congruences:

$$\tau_j = \left\{ \langle t_j, \overline{\lambda_B}/\theta \rangle, \langle \overline{\lambda_B}/\theta, t_j \rangle \right\} \cup \Delta_{S/\theta}.$$

Note that by Statement (3.4), neither τ_1 nor τ_2 are equal to $\Delta_{S/\theta}$. Thus, two possibilities emerge. Either the congruence blocks t_1 and t_2 are equal or they are not. With regard to the latter scenario, since $\tau_1 \wedge \tau_2 = \Delta_{S/\theta}$ and neither $\tau_1 = \Delta_{S/\theta}$ nor $\tau_2 = \Delta_{S/\theta}$, the algebra $\mathbf{S}/\theta$ is not subdirectly irreducible. □

Lemma 3.1.6. *Suppose that* $\mathbf{S}/\theta$ *is subdirectly irreducible and that there exist elements* t_1 *and* t_2 *in* T_θ *such that* $\langle t_1, t_2 \rangle$ *is in* Φ. *If* $t_1 \cap \hat{K} \neq \varnothing$ *and* $t_2 \cap \hat{K} = \varnothing$, *then there exists a unique element* α_θ *in* T_θ *such that for all* t *in* T_θ, *we have*

$$t *^{\mathbf{S}/\theta} \alpha_\theta = \alpha_\theta \qquad \text{and} \qquad \alpha_\theta *^{\mathbf{S}/\theta} t = \alpha_\theta$$

and there exists a unique element β_θ *in* $T_\theta \backslash \{\alpha_\theta\}$ *such that for all* t *in* T_θ, *we have*

$$t *^{\mathbf{S}/\theta} \beta_\theta = \alpha_\theta \qquad \text{and} \qquad \beta_\theta *^{\mathbf{S}/\theta} t = \alpha_\theta.$$

Further, $\alpha_\theta = t_1$ *and* $\beta_\theta = t_2$.

Proof. By Part 4 of Lemma 3.1.2, we have $t_1 = \overline{\lambda_B}/\theta$ and by Part 6 of Lemma 3.1.2 we have $t_2 \neq \overline{\lambda_B}/\theta$. Further, by Lemma 3.1.3, we have for all t in T_θ,

$$t_2 *^{\mathbf{S}/\theta} t = \overline{\lambda_B}/\theta \qquad \text{and} \qquad t *^{\mathbf{S}/\theta} t_2 = \overline{\lambda_B}/\theta.$$

Let $\alpha_\theta = t_1$ and $\beta_\theta = t_2$.

The element α_θ is unique due to $\overline{\lambda_M}/\theta$ being the only member in S/θ, and hence T_θ, that contains the maximal element in S.

Suppose that there exists an element r in T_θ, distinct from β_θ and α_θ such that for all t in T_θ,

$$r *^{\mathbf{S}/\theta} t = \overline{\lambda_B}/\theta \qquad \text{and} \qquad t *^{\mathbf{S}/\theta} r = \overline{\lambda_B}/\theta.$$

Thus, we have three distinct elements, α_θ, β_θ and r in T_θ, such that any product that involves any one of them is α_θ. By Lemma 3.1.4,

$$\{\langle\alpha_\theta,\beta_\theta\rangle,\langle\beta_\theta,\alpha_\theta\rangle\}\cup\Delta_{S/\theta} \qquad \text{and} \qquad \{\langle\alpha_\theta,r\rangle,\langle r,\alpha_\theta\rangle\}\cup\Delta_{S/\theta}$$

are congruences on $\mathbf{S}/\theta$. Therefore, as r is distinct from α_θ and β_θ then $\mathbf{S}/\theta$ cannot be subdirectly irreducible due to the presence of two non-trivial congruences that meet to $\Delta_{S/\theta}$. Hence, for $\mathbf{S}/\theta$ to be subdirectly irreducible, we must have $\beta_\theta = t$. Uniqueness of β_θ has been achieved. □

We have all of the ingredients to state the Theorem central to this subsection. That is, we are able to provide a partial description of the subdirectly irreducible members in varieties generated by groupoids that are influenced by a partial order relation.

Theorem 3.1.7. *Let* $\mathbf{B} = \langle B; *^{\mathbf{B}}\rangle$ *be a non-trivial groupoid that is influenced by a partial order relation and suppose that* $\mathbf{M}$ *is in* $\mathbb{V}(\mathbf{B})$. *If* $\mathbf{M}$ *is subdirectly irreducible, then the following statements are true.*

1. There exists a unique element α_M *in* M *such that for all* m *in* M,

$$m *^{\mathbf{M}} \alpha_M = \alpha_M \qquad \textit{and} \qquad \alpha_M *^{\mathbf{M}} m = \alpha_M$$

and there exists a unique element β_M *in* $M\backslash\{\alpha_M\}$ *such that for all* m *in* M,

$$m *^{\mathbf{M}} \beta_M = \alpha_M \qquad \textit{and} \qquad \beta_M *^{\mathbf{M}} m = \alpha_M.$$

$*^{\mathbf{M}}$	m_1	m_2	m_3	$\cdots$	$m_{\lvert M\rvert-2}$	β_M	α_M
m_1	?	?	?	$\cdots$	?	α_M	α_M
m_2	?	?	?	$\cdots$	?	α_M	α_M
m_3	?	?	?	$\cdots$	?	α_M	α_M
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	?	α_M	α_M
$m_{\lvert M\rvert-2}$	?	?	?	?	?	α_M	α_M
β_M	α_M	α_M	α_M	α_M	α_M	α_M	α_M
α_M	α_M	α_M	α_M	α_M	α_M	α_M	α_M

Table 3.4: The Operation Table of a Finite Subdirectly Irreducible Member in a Variety Generated by a Groupoid Whose Binary Operation is Influenced by a Partial Order Relation

2. *The monolith of* **M** *is*

$$\mu_M = \{\langle\alpha_M,\beta_M\rangle,\langle\beta_M,\alpha_M\rangle\}\cup\Delta_M.$$

Before the proof is given, note that Part 1 of Theorem 3.1.7 states that, for $1 < |M| < \omega$, the operation table of $*^{\mathbf{M}}$ must look like the operation table given in Table 3.4.

Proof Of Lemma 3.1.7. We start with Part 1 of the Theorem.

By Tarski's $\mathbb{HSP}$ Theorem, Theorem 1.2.5 on page 17, for each **M** in $\mathbb{V}(\mathbf{B})$, there exists an index I and subalgebra **S**, such that $\mathbf{S} \leq \mathbf{B}^I$, and there exists a θ in Con **S** such that $\mathbf{M} \cong \mathbf{S}/\theta$. Hence, for the remainder of the proof we focus explicitly on $\mathbf{S}/\theta$. To simplify notation, let $T_\theta = S/\theta$.

Recall from Part 3 of Lemma 3.1.2 that $\hat{K}\cap S \subseteq \overline{\lambda_B}/\theta$. Note that if $K = \varnothing$ then $S = \hat{K}\cap S$. Thus, $S \subseteq \overline{\lambda_B}/\theta$. This implies that $S = \overline{\lambda_B}/\theta$. Therefore, if $K = \varnothing$ then $\mathbf{S}/\theta$ is the trivial algebra. Since we are assuming that $\mathbf{S}/\theta$ is not the trivial algebra, this means that $K \neq \varnothing$ and hence, the congruence Φ exists.

Suppose that for t_1 and t_2 in S/θ, we have $\langle t_1, t_2 \rangle$ in Φ. There exist a few possibilities regarding how t_1 and t_2 interact with $\hat{K}$. They are as follows.

Possibility 1: Suppose that $t_1 \cap \hat{K} \neq \varnothing$ and $t_2 \cap \hat{K} \neq \varnothing$. By Part 5 of Lemma 3.1.2, we have $t_1 = t_2$.

Possibility 2: Suppose that $t_1 \cap \hat{K} = \varnothing$ and $t_2 \cap \hat{K} = \varnothing$. By Corollary 3.1.5, we have $t_1 = t_2$ or $\mathbf{S}/\theta$ is not subdirectly irreducible.

Possibility 3: Without loss of generality suppose that $t_1 \cap \hat{K} \neq \varnothing$ and $t_2 \cap \hat{K} = \varnothing$. By Lemma 3.1.6, there exists a unique element α_θ in S/θ such that for all t in T_θ, we have

$$t *^{\mathbf{S}/\theta} \alpha_\theta = \alpha_\theta \qquad \text{and} \qquad \alpha_\theta *^{\mathbf{S}/\theta} t = \alpha_\theta$$

and there exists a unique element β_θ in $T_\theta \setminus \{\alpha_\theta\}$ such that for all t in T_θ, we have

$$t *^{\mathbf{S}/\theta} \beta_\theta = \alpha_\theta \qquad \text{and} \qquad \beta_\theta *^{\mathbf{S}/\theta} t = \alpha_\theta.$$

With respect to all $\langle t_1, t_2 \rangle$ in Φ, if only Possibility 1 and Possibility 2 occur then either $\Phi = \Delta_{S/\theta}$ or $\mathbf{S}/\theta$ is not subdirectly irreducible.

Note that if $\Phi = \Delta_{S/\theta}$ then $|K| \neq 1$ as by definition, for all k in K, we have $\phi_k \neq \Delta_{S/\theta}$. Thus, if $\Phi = \Delta_{S/\theta}$ then Φ is a meet of at least two non-$\Delta_{S/\theta}$ congruences that equates to $\Delta_{S/\theta}$. That is, $\mathbf{S}/\theta$ is not subdirectly irreducible.

For $\mathbf{S}/\theta$ to be subdirectly irreducible and non-trivial, there must exist a pair $\langle t_1, t_2 \rangle$ in Φ such that, with respect to t_1 and t_2, Possibility 3 occurs. This proves Part 1.

For Part 2, assume that $\mathbf{S}/\theta$ is subdirectly irreducible and not trivial. Part 1 of the Theorem and Lemma 3.1.4 imply that τ, defined below, is a congruence on $\mathbf{S}/\theta$:

$$\tau = \{\langle\alpha_\theta,\beta_\theta\rangle,\langle\beta_\theta,\alpha_\theta\rangle\} \cup \Delta_{S/\theta}.$$

Since $\mathbf{S}/\theta$ is subdirectly irreducible, we must have $\mu_{S/\theta} \subseteq \tau$, where $\mu_{S/\theta}$ is the monolith of $\mathbf{S}/\theta$. As τ covers $\Delta_{S/\theta}$, this implies that $\tau = \mu_{S/\theta}$. □

Can any more be stated, regarding a description of subdirectly irreducible algebras in varieties generated by groupoids that are influenced by a partial order relation? In the next subsection, we show that, under certain circumstances, the answer is yes.

3.1.2 Subdirectly Irreducible Groupoids that are Influenced by a Partial Order Relation of Height 3

If $\mathbf{B}$ is a groupoid influenced by a partial order relation of height 3, then we can obtain an even clearer picture of what the subdirectly irreducible members in $\mathbb{V}(\mathbf{B})$ look like than that presented in Theorem 3.1.7. Note that such a groupoid is associative, as the product of any three elements must be the maximal element in the universe of that groupoid. We conclude this subsection by showing that the varieties generated by some groupoids, that are influenced by a partial order relation of height 3, are residually large.

Corollary 3.1.8. *Let $\mathbf{B} = \langle B; *^{\mathbf{B}}\rangle$ be a non-trivial groupoid that is influenced by a partial order of height 3 and $\mathbf{M}$ in $\mathbb{V}(\mathbf{B})$. If $\mathbf{M}$ is subdirectly irreducible, then the following statements are true.*

1. *There exists a unique element α_M in M such that for all m in M,*

$$m *^{\mathbf{M}} \alpha_M = \alpha_M \qquad \textit{and} \qquad \alpha_M *^{\mathbf{M}} m = \alpha_M$$

and there exists a unique element β_M in $M \backslash \{\alpha_M\}$ such that for all m in M,

$$m *^{\mathbf{M}} \beta_M = \alpha_M \qquad \textit{and} \qquad \beta_M *^{\mathbf{M}} m = \alpha_M.$$

2. *The monolith of* **M** *is*

$$\mu_M = \{\langle \alpha_M, \beta_M \rangle, \langle \beta_M, \alpha_M \rangle\} \cup \Delta_M.$$

3. *For all m_1 and m_2 in M,*

$$m_1 *^{\mathbf{M}} m_2 \in \{\alpha_M, \beta_M\}.$$

Proof. By Tarski's $\mathbb{HSP}$ Theorem, Theorem 1.2.5 on page 17, for some index I such that $\mathbf{S} \leq \mathbf{B}^I$ and for some θ in Con $\mathbf{S}$, we may assume that $\mathbf{M} \cong \mathbf{S}/\theta$.

Together, Part 1 and Part 2 are a restatement of Theorem 3.1.7. Hence, only Part 3 of the Corollary must be proved.

Recall from the Part 1 of Lemma 3.1.2, on page 91, that the maximum element in S exists and is the constant function from I to $\{\lambda_B\}$. This element is denoted by $\overline{\lambda_B}$. The congruence block in S/θ that contains $\overline{\lambda_B}$ is $\overline{\lambda_B}/\theta$.

Since **B** is a groupoid influenced by a partial order relation of height 3, the product of any three elements in B is λ_B. Thus, the product of any three elements in B^I, or any three elements in S, is $\overline{\lambda_B}$, while the product of any three elements in S/θ is $\overline{\lambda_B}/\theta$. Note that α_M is the element in M that corresponds to $\overline{\lambda_B}/\theta$, with respect to the isomorphism that relates **M** to $\mathbf{S}/\theta$. Thus, for all m_1, m_2 and m_3 in M,

$$(m_1 *^{\mathbf{M}} m_2) *^{\mathbf{M}} m_3 = \alpha_M \qquad \text{and} \qquad m_1 *^{\mathbf{M}} (m_2 *^{\mathbf{M}} m_3) = \alpha_M.$$

Suppose that for m_1 and m_2 in M, we have $m_3 = m_1 *^{\mathbf{M}} m_2$. Therefore,

$$\text{for all } m_4 \text{ in } M \qquad m_3 *^{\mathbf{M}} m_4 = \alpha_M \qquad \text{and} \qquad m_4 *^{\mathbf{M}} m_3 = \alpha_M.$$

Lemma 3.1.4, on page 95, yields that the following relation is a congruence:

$$\tau = \{\langle \alpha_M, m_3 \rangle, \langle m_3, \alpha_M \rangle\} \cup \Delta_M.$$

By Part 2, the monolith is

$$\mu_M = \{\langle \alpha_M, \beta_M \rangle, \langle \beta_M, \alpha_M \rangle\} \cup \Delta_M.$$

Thus, τ can either be μ_M or Δ_M. That is, m_3 is in $\{\alpha_M, \beta_M\}$. □

The previous Corollary yields the construction of a specific family of groupoids, namely, the $\mathbf{D}_\kappa$'s. For a non-zero cardinal number κ, define

$$\mathbf{D}_\kappa = \langle \{\gamma \mid \gamma \text{ is a cardinal number and } \gamma < \kappa\} \cup \{\alpha, \beta\}; *^{\mathbf{D}_\kappa} \rangle$$

be the groupoid where for all d_1 and d_2 in D_κ,

$$d_1 *^{\mathbf{D}_\kappa} d_2 = \begin{cases} \beta & \text{if } d_1 \neq d_2 \text{ and for all } j \text{ in } \{1,2\}, \text{the element } d_j \text{ is not in } \{\alpha, \beta\} \\ \alpha & \text{otherwise.} \end{cases}$$

See Table 3.5 for the operation table of $*^{\mathbf{D}_\kappa}$, if κ is a finite cardinal number.

Lemma 3.1.9. *Let κ be a non-zero cardinal number. The groupoid $\mathbf{D}_\kappa$ is subdirectly irreducible.*

Proof. We show that every non-trivial congruence on $\mathbf{D}_\kappa$ contains the element $\langle \alpha, \beta \rangle$. This will imply that $\mathbf{D}_\kappa$'s monolith is $\mathrm{Cg}^{\mathbf{D}_\kappa}(\langle \alpha, \beta \rangle)$.

If there do not exist any non-trivial congruences, then $\mathbf{D}_\kappa$ is simple and hence, subdirectly irreducible. Hence, we may assume that there exists at least one non-trivial congruence on $\mathbf{D}_\kappa$. For θ in $(\mathrm{Con}\ \mathbf{D}_\kappa) \setminus \{\Delta_{D_\kappa}, \nabla_{D_\kappa}\}$, let $\langle d_1, d_2 \rangle$ in θ such that $d_1 \neq d_2$. We examine 3 cases.

$*^{\mathbf{D}_\kappa}$	0	1	2	$\cdots$	$\kappa-1$	β	α
0	α	β	β	$\cdots$	β	α	α
1	β	α	β	$\cdots$	β	α	α
2	β	β	α	$\cdots$	β	α	α
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	β	α	α
$\kappa-1$	β	β	β	β	α	α	α
β	α	α	α	α	α	α	α
α	α	α	α	α	α	α	α

Table 3.5: The Operation Table of $*^{\mathbf{D}_\kappa}$, Assuming κ is a Finite Cardinal Number

Case 1 Suppose that d_1 and d_2 are not in $\{\alpha,\beta\}$. Since $\langle d_1,d_1\rangle$ is in θ, by the compatibility property we have

$$\langle \alpha,\beta\rangle = \langle d_1 *^{\mathbf{D}_\kappa} d_1, d_2 *^{\mathbf{D}_\kappa} d_1\rangle \in \theta.$$

Case 2 Without loss of generality, suppose that d_1 is not in $\{\alpha,\beta\}$ and d_2 is in $\{\alpha,\beta\}$. Pick a d_3 in $D_\kappa \backslash \{d_1,\alpha,\beta\}$. We are guaranteed such an element, as $|D_\kappa| > 3$. Therefore, as $\langle d_3,d_3\rangle$ in θ, we obtain the following

$$\langle \beta,\alpha\rangle = \langle d_1 *^{\mathbf{D}_\kappa} d_3, d_2 *^{\mathbf{D}_\kappa} d_3\rangle \in \theta.$$

By symmetry of a congruence, we obtain $\langle \alpha,\beta\rangle$ is in θ.

Case 3 Suppose that $\{d_1,d_2\} = \{\alpha,\beta\}$. Without loss of generality, assume that $d_1 = \alpha$ and $d_2 = \beta$. Then,

$$\langle \alpha,\beta\rangle = \langle d_1,d_2\rangle \in \theta.$$

Thus, $\mathbf{D}_\kappa$ is subdirectly irreducible. □

The $\mathbf{D}_\kappa$'s each satisfy many identities. We use this property to show that a groupoid that is influenced by a partial order relation of height 3 generates a residually large variety.

For the binary operation symbol $*$, let t be a term of type $\{*\}$. Define the *length of the term* t as the number of explicit occurrences of the symbol $*$ in t.

Let t_1 and t_2 be terms of type $\{*\}$ such that the length of t_1 is n_1 and the length of t_2 is n_2. Define the *length of the identity* $t_1 \approx t_2$ as the two-tuple $\langle n_1, n_2 \rangle$.

Example. Let

$$t_1(x,y) \approx x, \qquad t_2(x) \approx x * x \qquad \text{and} \qquad t_3(x,y,z) \approx (x*y)*(x*z)$$

be terms of type $\{*\}$. The lengths of $t_1(x,y)$, of $t_2(x)$ and of $t_3(x,y,z)$ are 0, 1 and 3, respectively. The lengths of the identities

$$t_1(x,y) \approx t_2(x) \qquad \text{and} \qquad t_2(x) \approx t_3(x,y,z)$$

are $\langle 0,1 \rangle$ and $\langle 1,3 \rangle$, respectively.

Lemma 3.1.10. *Let $*$ denote a binary operation symbol. Using the variables w, x, y and z, the following are the only identities of type $\{*\}$ and of length $\langle 1,1 \rangle$, up to relabelling the variables or swapping sides of the identity.*

1. $w*w \approx w*w$	*5.* $w*x \approx w*x$	*9.* $w*x \approx y*x$
2. $w*w \approx w*x$	*6.* $w*x \approx x*w$	*10.* $w*x \approx y*w$
3. $w*w \approx x*w$	*7.* $w*w \approx x*y$	*11.* $w*x \approx y*z$
4. $w*w \approx x*x$	*8.* $w*x \approx w*y$	

Proof. We build all of the possible identities, up to relabelling and swapping, of length $\langle 1,1 \rangle$ using the variables w, x, y and z.

There is only one identity of length $\langle 1,1 \rangle$ where one variable appears four times, namely equation 1.

Of all identities, up to relabelling and swapping, of length $\langle 1,1 \rangle$ where one variable appears three times, only two are distinct, namely equations 2 and 3.

The following are all identities, up to relabelling and swapping, of length $\langle 1,1 \rangle$ where two distinct variables appear two times:

4. $w*w \approx x*x$ 5. $w*x \approx w*x$ 6. $w*x \approx x*w$

The following are all identities, up to relabelling and swapping, of length $\langle 1,1 \rangle$ where exactly one variable appears two times:

7. $w*w \approx x*y$ 9. $w*x \approx y*x$ 10. $w*x \approx y*w$

8. $w*x \approx w*y$

There is only one identity, up to relabelling, of length $\langle 1,1 \rangle$, where four distinct variables appear, namely Equation 11. □

Lemma 3.1.11. *For each non-zero cardinal number κ and $p_1 > 1$ and $p_2 > 1$, the algebra $\mathbf{D}_\kappa$ satisfies every identity of length $\langle p_1, p_2 \rangle$.*

Proof. Note that a product of any three or more elements in D_κ is α. Thus, for any integers $p_1 > 1$ and $p_2 > 1$, each $\mathbf{D}_\kappa$ satisfies every identity of type $\{*\}$ and of length $\langle p_1, p_2 \rangle$. □

Lemma 3.1.12. *Let κ be any non-zero cardinal number. The algebra $\mathbf{D}_\kappa$ satisfies Identity 1, Identity 4, Identity 5 and Identity 6, defined in Lemma 3.1.10.*

Proof. Recall the following identities listed from Lemma 3.1.10.

Identity 1:	$w * w \approx w * w$
Identity 4:	$w * w \approx x * x$
Identity 5:	$w * x \approx w * x$
Identity 6:	$w * x \approx x * w$

Every groupoid satisfies Identity 1 and Identity 5. As the product of any element in D_κ with itself is α, the algebra $\mathbf{D}_\kappa$ satisfies Identity 4. As the product of two distinct elements in $D_\kappa \backslash \{\alpha, \beta\}$ is β and any product involving α or β is α, the algebra $\mathbf{D}_\kappa$ satisfies Identity 6. □

Lemma 3.1.13. *Let $X = \{w, x, y, z\} \cup \{x_1, x_2, x_3, \ldots\}$ be a set of distinct variables. Let κ be a non-zero cardinal number. Assume that $\{y_1, y_2, y_3, \ldots, y_m\} \subseteq X$. For every $n > 1$ and every term $t(y_1, y_2, y_3, \ldots, y_m)$ of length n, the algebra $\mathbf{D}_\kappa$ does not satisfy any of the following identities:*

Length $\langle 0,1 \rangle$:	*Length* $\langle 1,1 \rangle$:	*Length* $\langle 0,0 \rangle$:
$w \approx w * w$	$w * w \approx w * x$	$w \approx x$
$w \approx w * x$	$w * w \approx x * w$	*Length* $\langle 0,n \rangle$:
$w \approx x * w$	$w * w \approx x * y$	$x \approx t(y_1, y_2, y_3, \ldots, y_m)$
$w \approx x * x$	$w * x \approx w * y$	*Length* $\langle 1,n \rangle$:
$w \approx x * y$	$w * x \approx y * w$	$x * y \approx t(y_1, y_2, y_3, \ldots, y_m)$
	$w * x \approx y * x$	
	$w * x \approx y * z$	

Proof. Each $\mathbf{D}_\kappa$ cannot satisfy the length $\langle 0,0\rangle$ identity listed in the statement of the Lemma, as α and β are in each D_κ and $\alpha \neq \beta$.

For each identity of length $\langle 0,1\rangle$, Table 3.6 shows that appropriate choices of a, b and c in D_κ yield an instance that falsify that identity. For example, if $f(w,x,y) \approx g(w,x,y)$ is the identity $w \approx w * x$ then letting $a = \beta$, $b = \alpha$ and c be any element in D_κ yields $f^{\mathbf{D}_\kappa}(a,b,c) = \beta$ and $g^{\mathbf{D}_\kappa}(a,b,c) = \alpha$. Thus, $\mathbf{D}_\kappa$ cannot satisfy the identity $w \approx w * x$, otherwise $\beta = \alpha$ and a contradiction occurs. Therefore, each $\mathbf{D}_\kappa$ does not satisfy any identities of length $\langle 0,1\rangle$.

$f(w,x,y) \approx g(w,x,y)$	a	b	c	$f^{\mathbf{D}_\kappa}(a,b,c)$	$g^{\mathbf{D}_\kappa}(a,b,c)$
$w \approx w * w$	0	-	-	0	α
$w \approx w * x$	β	α	-	β	α
$w \approx x * w$	0	0	-	0	α
$w \approx x * x$	0	0	-	0	α
$w \approx x * y$	β	α	α	β	α

Table 3.6: Identities of Length $\langle 0,1\rangle$ that each $\mathbf{D}_\kappa$ does not Satisfy

For each identity of length $\langle 1,1\rangle$ listed in the statement of the Lemma, Table 3.7 shows that appropriate choices of a, b, c and d in D_κ yield an instance that falsify that identity. Therefore, each $\mathbf{D}_\kappa$ does not satisfy any identities of length $\langle 1,1\rangle$ listed in the statement of the Lemma.

When the length of $t(x_1,x_2,x_3,\ldots,x_m)$ is at least 2, as the product of any three elements in D_κ is α, we have the following:

$$\text{for all } \langle d_1,d_2,d_3,\ldots,d_m\rangle \text{ in } D_\kappa^m, \qquad t^{\mathbf{D}_\kappa}(d_1,d_2,d_3,\ldots,d_m) = \alpha.$$

Thus, each $\mathbf{D}_\kappa$ does not satisfy any identity of the form $x \approx t(y_1,y_2,y_3,\ldots,y_m)$, as otherwise replacing x with β and the y_i's with 0 yields $\beta = \alpha$. Similarly, each $\mathbf{D}_\kappa$ does not satisfy any

$f(w,x,y,z) \approx g(w,x,y,z)$	a	b	c	d	$f^{\mathbf{D}_\kappa}(a,b,c,d)$	$g^{\mathbf{D}_\kappa}(a,b,c,d)$
$w*w \approx w*x$	0	1	-	-	α	β
$w*w \approx x*w$	0	1	-	-	α	β
$w*w \approx x*y$	0	0	1	-	α	β
$w*x \approx w*y$	0	0	1	-	α	β
$w*x \approx y*w$	0	0	1	-	α	β
$w*x \approx y*x$	0	0	1	-	α	β
$w*x \approx y*z$	0	0	0	1	α	β

Table 3.7: Identities of Length $\langle 1,1\rangle$ that each $\mathbf{D}_\kappa$ does not Satisfy

identity of the form $x*y \approx t(y_1,y_2,y_3,\dots,y_m)$, as otherwise replacing x with 0, y with 1 and the y_i's with 0 yields $\beta = \alpha$. Thus, each $\mathbf{D}_\kappa$ does not satisfy any identities of lengths $\langle 0,n\rangle$ or $\langle 1,n\rangle$, where $n > 1$. □

Lemma 3.1.14. *For every non-zero cardinal number κ, the algebra $\mathbf{D}_\kappa$ satisfies every identity not listed in Lemma 3.1.13.*

Proof. Every identity not listed in Lemma 3.1.13 is dealt with in either Lemma 3.1.11 or Lemma 3.1.12. □

Theorem 3.1.15. *If $\mathbf{B} = \langle B; *^{\mathbf{B}}\rangle$ is a non-trivial groupoid that is influenced by a partial order relation of height 3 and there exists an element b in B such that $b *^{\mathbf{B}} b \neq \lambda_B$ then $\mathbb{V}(\mathbf{B})$ is residually large.*

Proof. For any non-zero cardinal number κ and for any set of variables X, we will show that

$$\mathrm{Id}_{\{\mathbf{B}\}}(X) \subseteq \mathrm{Id}_{\{\mathbf{D}_\kappa\}}(X). \tag{3.5}$$

Since $\mathbb{V}(\mathbf{B})$ is an equational class, this will imply that $\mathbf{D}_\kappa$ is in $\mathbb{V}(\mathbf{B})$. Therefore, by Lemma 3.1.9, the class of algebras $\mathbb{V}(\mathbf{B})$ is residually large. Explicitly, we will show that for an identity $t_1 \approx t_2$ of type $\{*\}$,

$$\text{if} \qquad \mathbf{D}_\kappa \not\models t_1 \approx t_2 \qquad \text{then} \qquad \mathbf{B} \not\models t_1 \approx t_2. \tag{3.6}$$

The contrapositive of this implication yields Statement (3.5). It follows that $\mathbf{B}$ does not satisfy any of the identities listed in Lemma 3.1.13.

The only algebra that satisfies a non-trivial length $\langle 0,0\rangle$ identity is the trivial algebra. Thus, $\mathbf{B}$ does not satisfy the length $\langle 0,0\rangle$ identity.

Since there exists a b' in B such that $b' *^{\mathbf{B}} b' \neq \lambda_B$, we must have

$$b' < b' *^{\mathbf{B}} b' < \lambda_B.$$

For each identity of length $\langle 0,1\rangle$, Table 3.8 shows that appropriate choices of a, b and c in B yield an instance that falsify that identity. For example, if $f(w,x,y) \approx g(w,x,y)$ is the identity $w \approx x*x$ then letting $a = b'$, $b = \lambda_B$ and c be any element in B yields $f^{\mathbf{D}_\kappa}(a,b,c) = b'$ and $g^{\mathbf{D}_\kappa}(a,b,c) = \lambda_B$. Thus, $\mathbf{B}$ cannot satisfy the identity $w \approx x*x$, otherwise $b' = \lambda_B$ and a contradiction occurs. Therefore, $\mathbf{B}$ does not satisfy any identities of length $\langle 0,1\rangle$, that each $\mathbf{D}_\kappa$ does not satisfy. Similarly, for each identity of length $\langle 1,1\rangle$, that each $\mathbf{D}_\kappa$ does not satisfy, Table 3.9 shows that appropriate choices of a, b and c in B yield an instance that falsify that identity. Thus, $\mathbf{B}$ does not satisfy any length $\langle 1,1\rangle$ identities that each $\mathbf{D}_\kappa$ does not satisfy.

When the length of $t(y_1,y_2,y_3,\ldots,y_m)$ is at least 2 and $\mathbf{B}$ is influenced by a partial order relation of height 3, we have the following:

$$\text{for all } \langle b_1,b_2,b_3,\ldots,b_m\rangle \text{ in } B^m, \qquad t^{\mathbf{B}}(b_1,b_2,b_3,\ldots,b_m) = \lambda_B.$$

If $\mathbf{B}$ satisfied any identity of the form $x \approx t(y_1,y_2,y_3,\ldots,y_m)$ then replacing each variable

$f(w,x,y) \approx g(w,x,y)$	a	b	c	$f^{\mathbf{B}}(a,b,c)$	$g^{\mathbf{B}}(a,b,c)$
$w \approx w * w$	b'	-	-	b'	$b' *^{\mathbf{B}} b'$
$w \approx w * x$	b'	λ_B	-	b'	λ_B
$w \approx x * w$	b'	λ_B	-	b'	λ_B
$w \approx x * x$	b'	λ_B	-	b'	λ_B
$w \approx x * y$	b'	λ_B	λ_B	b'	λ_B

Table 3.8: Identities of Length $\langle 0,1 \rangle$ that $\mathbf{B}$ and each $\mathbf{D}_\kappa$ do not Satisfy

$f(w,x,y,z) \approx g(w,x,y,z)$	a	b	c	d	$f^{\mathbf{B}}(a,b,c,d)$	$g^{\mathbf{B}}(a,b,c,d)$
$w * w \approx w * x$	b	λ_B	-	-	$b *^B b$	λ_B
$w * w \approx x * w$	b	λ_B	-	-	$b *^B b$	λ_B
$w * w \approx x * y$	b	λ_B	λ_B	-	$b *^B b$	λ_B
$w * x \approx w * y$	b	b	λ_B	-	$b *^B b$	λ_B
$w * x \approx y * w$	b	b	λ_B	-	$b *^B b$	λ_B
$w * x \approx y * x$	b	b	λ_B	-	$b *^B b$	λ_B
$w * x \approx y * z$	b	b	λ_B	λ_B	$b *^B b$	λ_B

Table 3.9: Identities of Length $\langle 1,1 \rangle$ that $\mathbf{B}$ and each $\mathbf{D}_\kappa$ do not Satisfy

with b', yields $b' = \lambda_B$. Hence, for $n > 1$, the algebra **B** does not satisfy any identities of length $\langle 0,n\rangle$ that each $\mathbf{D}_\kappa$ does not satisfy.

If **B** satisfied any identity of the form $x * y \approx t(y_1, y_2, y_3, \ldots, y_m)$ then replacing each variable with b' yields $b' *^{\mathbf{B}} b' = \lambda_B$. Thus, for $n > 1$, the algebra **B** does not satisfy any identities of length $\langle 1,n\rangle$ that each $\mathbf{D}_\kappa$ does not satisfy.

Finally, we have shown that Statement (3.6) holds. □

Although the previous Theorem does not answer the Restricted Quackenbush Problem, it can be used to tell us what groupoids to avoid. In the next section we alter the previous Theorem to derive an identity that all finite groupoids, that generate residually finite varieties, must satisfy.

3.2 An Identity that Residually Finite Varieties Generated by Finite Groupoids Must Satisfy

We generalize Theorem 3.1.15 to show that the variety generated by a finite non-trivial groupoid is residually finite if the generating groupoid satisfies a particular kind of identity that is expressed in one variable.

Theorem 3.2.1. *Let* $\mathbf{B} = \langle B; *^{\mathbf{B}}\rangle$ *be a non-trivial finite groupoid. For the variable* x*, if* $\mathbb{V}(\mathbf{B})$ *is not residually large then there exists some term* $k(x)$ *of type* $\{*\}$*, such that* $k(x) \neq x * x$ *and* **B** *satisfies* $k(x) \approx x * x$.

Proof. We prove the contrapositive. Specifically, we show that if

$$\text{for all terms} \quad k(x) \neq x * x, \quad \text{we have} \quad \mathbf{B} \not\models k(x) \approx x * x \tag{3.7}$$

then $\mathbb{V}(\mathbf{B})$ contains all of the $\mathbf{D}_\kappa$'s. We do this by showing that **B** does not satisfy any of the identities that each $\mathbf{D}_\kappa$ does not satisfy. Namely, **B** does not satisfy any of the identities of length $\langle 0,0\rangle$, $\langle 0,1\rangle$, $\langle 0,n\rangle$, $\langle 1,1\rangle$ or $\langle 1,n\rangle$ described in Lemma 3.1.13. In other words, each of the $\mathbf{D}_\kappa$'s satisfy all of the identities that $\mathbf{B}$ satisfies. Thus, for any non-zero cardinal κ and any set of variables X,

$$\mathrm{Id}_{\{\mathbf{B}\}}(X) \subseteq \mathrm{Id}_{\{\mathbf{D}_\kappa\}}(X)$$

and hence, all of the $\mathbf{D}_\kappa$ are in $\mathbb{V}(\mathbf{B})$. That is, $\mathbb{V}(\mathbf{B})$ is residually large.

The generating algebra $\mathbf{B}$ cannot satisfy the identity of length $\langle 0,0\rangle$ as $\mathbf{B}$ is non-trivial.

Under the assumption made at the beginning of the proof, $\mathbf{B}$ cannot be idempotent. Hence, there must exist an element b in B such that $b *^{\mathbf{B}} b \neq b$. Hence, $\mathbf{B}$ cannot satisfy any identity of length $\langle 0,1\rangle$ that each $\mathbf{D}_\kappa$ does not satisfy as otherwise for each such identity, each variable could be replaced with b yielding $b = b *^{\mathbf{B}} b$.

By Statement (3.7), the algebra $\mathbf{B}$ does not satisfy the following identities:

$$x * x \approx x * (x * x) \qquad \text{and} \qquad x * x \approx (x * x) * x.$$

Thus, there must exist elements b_1 and b_2 in B such that

$$b_1 *^{\mathbf{B}} b_1 \neq b_1 *^{\mathbf{B}} (b_1 *^{\mathbf{B}} b_1) \qquad \text{and} \qquad b_2 *^{\mathbf{B}} b_2 \neq (b_2 *^{\mathbf{B}} b_2) *^{\mathbf{B}} b_2. \tag{3.8}$$

For each identity of length $\langle 1,1\rangle$ that each $\mathbf{D}_\kappa$ does not satisfy, Table 3.10 shows that appropriate choices of a, b, c and d in B yield an instance that falsify that identity. Therefore, $\mathbf{B}$ does not satisfy any of the identities of length $\langle 1,1\rangle$ that each $\mathbf{D}_\kappa$ does not satisfy.

Consider a term $t(y_1, y_2, y_3, \ldots, y_m)$ of length n, where $n > 1$. If $\mathbf{B}$ did satisfy an identity of length $\langle 0,n\rangle$, that each $\mathbf{D}_\kappa$ does not satisfy, then we may replace every variable with the variable x to obtain

$$\mathbf{B} \models x \approx t(x,x,x,\ldots,x)$$

and hence

$$\mathbf{B} \models x * x \approx t(x,x,x,\ldots,x) * t(x,x,x,\ldots,x).$$

$f(w,x,y,z) \approx g(w,x,y,z)$	a	b	c	d	$f^{\mathbf{B}}(a,b,c,d)$	$g^{\mathbf{B}}(a,b,c,d)$
$w*w \approx w*x$	b_1	$b_1 *^{\mathbf{B}} b_1$	-	-	$b_1 *^{\mathbf{B}} b_1$	$b_1 *^{\mathbf{B}} (b_1 *^{\mathbf{B}} b_1)$
$w*w \approx x*w$	b_2	$b_2 *^{\mathbf{B}} b_2$	-	-	$b_2 *^{\mathbf{B}} b_2$	$(b_2 *^{\mathbf{B}} b_2) *^{\mathbf{B}} b_2$
$w*w \approx x*y$	b_1	b_1	$b_1 *^{\mathbf{B}} b_1$	-	$b_1 *^{\mathbf{B}} b_1$	$b_1 *^{\mathbf{B}} (b_1 *^{\mathbf{B}} b_1)$
$w*x \approx w*y$	b_1	b_1	$b_1 *^{\mathbf{B}} b_1$	-	$b_1 *^{\mathbf{B}} b_1$	$b_1 *^{\mathbf{B}} (b_1 *^{\mathbf{B}} b_1)$
$w*x \approx y*w$	b_2	b_2	$b_2 *^{\mathbf{B}} b_2$	-	$b_2 *^{\mathbf{B}} b_2$	$(b_2 *^{\mathbf{B}} b_2) *^{\mathbf{B}} b_2$
$w*x \approx y*x$	b_2	b_2	$b_2 *^{\mathbf{B}} b_2$	-	$b_2 *^{\mathbf{B}} b_2$	$(b_2 *^{\mathbf{B}} b_2) *^{\mathbf{B}} b_2$
$w*x \approx y*z$	b_2	b_2	$b_2 *^{\mathbf{B}} b_2$	b_2	$b_2 *^{\mathbf{B}} b_2$	$(b_2 *^{\mathbf{B}} b_2) *^{\mathbf{B}} b_2$

Table 3.10: Identities of Length $\langle 1,1 \rangle$ that $\mathbf{B}$ and each $\mathbf{D}_\kappa$ do not Satisfy

By the assumption described in Statement (3.7), the algebra $\mathbf{B}$ cannot satisfy an identity of length $\langle 0,n \rangle$ that each $\mathbf{D}_\kappa$ does not satisfy. Similarly, $\mathbf{B}$ cannot satisfy an identity of $\langle 1,n \rangle$, that each $\mathbf{D}_\kappa$ does not satisfy, as otherwise we may replace each variable with the variable x to obtain

$$\mathbf{B} \models x*x \approx t(x,x,x,\ldots,x).$$

Thus, the algebra $\mathbf{B}$ does not satisfy any of the identities that each $\mathbf{D}_\kappa$ does not satisfy.

□

Note that in the proof of Theorem 3.2.1, we showed that $\mathbf{D}_\kappa$ is in $\mathbb{V}(\mathbf{B})$, for any non-zero cardinal number κ. This leads into the following corollary.

Corollary 3.2.2. *Let $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ be a finite groupoid and x be a variable. If for every term $k(x)$ of type $\{*\}$ such that $k(x) \neq x*x$, the algebra $\mathbf{B}$ does not satisfy $k(x) \approx x*x$ then*

1. *the class of algebras $\mathbb{V}(\mathbf{B})$ is residually large; and*

$*^{\mathbf{E}_{17}}$	0	1	2	3	$\cdots$	n
0	1	2	2	?	$\cdots$	?
1	2	2	2	?	$\cdots$	?
2	2	2	2	?	$\cdots$	?
3	?	?	?	?	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	?
n	?	?	?	?	?	?

Table 3.11: The Operation Table of $*^{\mathbf{E}_{17}}$

2. *for any cardinal number γ greater than 3, there exists a subdirectly irreducible member $\mathbf{M}$ in $\mathbb{V}(\mathbf{B})$ such that $|M| = \gamma$.*

Proof. Part 1 of the Corollary is the contrapositive of Theorem 3.2.1. With regard to Part 2 of the Corollary, if $3 < \gamma < \omega$ then let $\mathbf{M} = \mathbf{D}_{\gamma-3}$ and if $\gamma \geq \omega$ then let $\mathbf{M} = \mathbf{D}_{\gamma}$. □

We can use the previous Corollary to rig the construction of groupoids to yield groupoids that generate residually large varieties.

Example. Let $\mathbf{E}_{17} = \langle E_{17}; *^{\mathbf{E}_{17}} \rangle$ be a groupoid whose partial operation table is described in Table 3.11. Even though a partial description is given, $\mathbb{V}(\mathbf{E}_{17})$ is residually large by Corollary 3.2.2. To see why, note that $0 *^{\mathbf{E}_{17}} 0 = 1$ and $t^{\mathbf{E}_{17}}(0) = 2$, for any term $t(x)$ of type $\{*\}$ that is at least of length 2. Thus, for all terms $t(x) \neq x * x$ of type $\{*\}$, we have $\mathbf{E}_{17} \not\models t(x) \approx x * x$.

Corollary 3.2.2 yields an immediate question: If $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ is a non-trivial finite groupoid such that $\mathbb{V}(\mathbf{B})$ is residually finite, must $*^{\mathbf{B}}$ be idempotent? If we consider constant groupoids, that is, groupoids where the lone binary operation maps everything in the universe to one particular element, then the answer is no. The following example demonstrates this.

Example. Let $\mathbf{E}_{18}$ be a constant non-trivial groupoid. One can verify that $\mathbb{V}(\mathbf{E}_{18})$ is the class of all constant groupoids. This variety is residually finite as the only non-trivial constant groupoid that is subdirectly irreducible is the two element constant groupoid.

If $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ is a non-trivial finite groupoid such that $\mathbb{V}(\mathbf{B})$ is residually finite, must $*^{\mathbf{B}}$ be idempotent? If we do not consider constant groupoids, then the answer to this question becomes less obvious. We conclude this section with the following question.

Question. *With regards to Corollary 3.2.2, does there exist a non-trivial and non-constant groupoid* $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ *such that* $*^{\mathbf{B}}$ *is not idempotent and* $\mathbb{V}(\mathbf{B})$ *is residually finite? Must such a groupoid be a reduct of a group?*

3.3 Application: RS-Conjecture

In 1988, Hobby and McKenzie posed the following problem in [6]:

Problem (Problem 12). *Prove or disprove: If* $\mathbf{B}$ *is a finite algebra such that* $\mathbb{V}(\mathbf{B})$ *admits no finite bound for the cardinals of its subdirectly irreducible algebras, then this class of cardinals is not bounded by any cardinal.*

In the same article [12] that McKenzie answered the Quackenbush Problem, he also answered Problem 12 and the *RS-Conjecture*. Recall the definition of the residual character of a variety on page 22. For a given variety $\mathcal{V}$, if there exists a least cardinal number κ such that the cardinality of the universe of every subdirectly irreducible member in $\mathcal{V}$ is less than κ, then κ is called the residual character of $\mathcal{V}$.

Conjecture (RS-Conjecture). *Let* **B** *be an algebra. If the residual character of* $\mathbb{V}(\mathbf{B})$ *is greater than or equal to* ω, *then* $\mathbb{V}(\mathbf{B})$ *is residually large.*

Specifically, McKenzie disproved Problem 12 and showed that the RS-Conjecture is false in general. He did this by constructing an eight-element algebra of residual bound ω_1 with just eight basic operations.

From the work in the previous section, specifically Corollary 3.2.2 on page 113, we have shown that Problem 12 is proved and the RS-Conjecture is true when **B** is a finite groupoid that satisfies the following condition: for the variable x and all terms $k(x)$ of type $\{*\}$ such that $k(x) \neq x * x$, the algebra **B** does not satisfy $k(x) \approx x * x$.

3.4 A Strategy Concerning Groupoids and the Restricted Quackenbush Problem

Recall Theorem 3.2.1 on page 111.

Theorem. *Let* $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ *be a non-trivial finite groupoid. For the variable* x, *if* $\mathbb{V}(\mathbf{B})$ *is not residually large then there exists some term* $k(x)$ *of type* $\{*\}$, *such that* $k(x) \neq x * x$ *and* **B** *satisfies* $k(x) \approx x * x$.

This theorem implies that, with respect to the Restricted Quackenbush Problem, only groupoids that are almost idempotent need to be looked at, that is, groupoids that satisfy an identity of the form $k(x) \approx x * x$, where $k(x)$ is a term of type $\{*\}$ such that $k(x) \neq x * x$. The class of strictly idempotent groupoids is a reasonable place to start. In the next chapter, we look at idempotent groupoids and attempt to answer the Restricted Quackenbush Problem.

Chapter 4

Groupoids and the Restricted Quackenbush Problem

With respect to the Restricted Quackenbush Problem, one class of algebras that was specifically asked about by Quackenbush, and still remains unanswered today, is the class of groupoids.

Groupoids are of interest because, using the tools developed by McKenzie in [11], answering the Restricted Quackenbush Problem with respect to groupoids answers the Restricted Quackenbush Problem with respect to arbitrary algebras. Specifically, McKenzie constructed an isomorphism F from the variety of all algebras of an arbitrary type $\mathcal{F}$ to a particular variety of groupoids, dependent on $\mathcal{F}$. In Theorem 2.18, he shows that, for each algebra $\mathbf{B}$ of type $\mathcal{F}$, the congruence lattice of $F(\mathbf{B})$ is isomorphic to the congruence lattice of $\mathbf{B}$, with the addition of a new maximum element. Thus, $\mathbf{B}$ is subdirectly irreducible if and only if $F(\mathbf{B})$ is subdirectly irreducible. Since F is an isomorphism, $F(V(\mathbf{B})) = V(F(\mathbf{B}))$. That is, F preserves subvarieties. Lastly, by the first Lemma in the paper, isomorphisms between varieties preserve finite algebras. That is, $\mathbf{B}$ is residually finite if and only if $F(\mathbf{B})$ is residually finite. Using all of these ingredients yields the following: answering the Restricted Quackenbush Problem with respect to $F(\mathbf{B})$ answers the Restricted Quackenbush Problem with respect to $\mathbf{B}$.

In this chapter, we define a particular kind of idempotent groupoid and identify some properties that could be useful in showing that the Restricted Quackenbush Problem can be answered with respect to these kinds of groupoids.

4.1 A Property of Idempotent Groupoids

Idempotent groupoids have a curious property; for an idempotent groupoid $\mathbf{B}$, the congruence blocks of any congruence on $\mathbf{B}$, are subuniverses of $\mathbf{B}$. In the following Lemma, we formalize this property and prove that it holds.

Lemma 4.1.1. *Let $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ be an idempotent groupoid and θ be a congruence on $\mathbf{B}$. There exists a family $\{\mathbf{S}_i\}_{i \in I}$ of subalgebras of $\mathbf{B}$, such that $\dot{\bigcup}_{i \in I} S_i = B$ and the congruence blocks of θ are $\{S_i\}_{i \in I}$. That is,*

$$B/\theta = \{b/\theta \mid b \in B\} = \{S_i\}_{i \in I}.$$

Proof. We will show that each congruence block of θ is a subuniverse of $\mathbf{B}$. Then, because each member of B must belong to exactly one congruence block, B is equal to a disjoint union of subuniverses. Each block cannot be empty. Hence, each block together with $*^{\mathbf{B}}$ yields a subalgebra of $\mathbf{B}$.

For b in B, let b_1 and b_2 be elements in b/θ. To show that b/θ is a subalgebra, we must show that $b_1 *^{\mathbf{B}} b_2$ in b/θ. Note that $\langle b_1, b_1 \rangle$ and $\langle b_1, b_2 \rangle$ are members of θ. Therefore, due to the compatibility property of congruences,

$$\langle b_1 *^{\mathbf{B}} b_1, b_1 *^{\mathbf{B}} b_2 \rangle \in \theta \qquad \text{and thus,} \qquad \langle b_1, b_1 *^{\mathbf{B}} b_2 \rangle \in \theta.$$

That is, b_1 and $b_1 *^{\mathbf{B}} b_2$ are in the same congruence block. Hence, $b_1 *^{\mathbf{B}} b_2$ is in b/θ. □

$*^{\mathbf{E}_{19}}$	0	1	2	3	4
0	0	2	4	0	4
1	4	1	3	0	4
2	1	1	2	4	4
3	1	4	0	3	4
4	4	4	4	4	4

Table 4.1: The Operation Table of $*^{\mathbf{E}_{19}}$

4.2 Absorbing Groupoids

We start this section with an example of an absorbing groupoid and then give its definition.

Example. Consider the groupoid $\mathbf{E}_{19}$. The operation table for $*^{\mathbf{E}_{19}}$ is given in Table 4.1 and the lattice of subuniverses of $\mathbf{E}_{19}$ is given in Figure 4.1. Notice that every subuniverse of $\mathbf{E}_{19}$ that contains at least two elements also contains 4. Further, notice that $\mathbf{E}_{19}$ is not associative as

$$0 *^{\mathbf{E}_{19}} (1 *^{\mathbf{E}_{19}} 2) = 0 \qquad \text{and} \qquad (0 *^{\mathbf{E}_{19}} 1) *^{\mathbf{E}_{19}} 2 = 2.$$

Let $\mathbf{B}$ be a groupoid. If for some δ_B in B and all b in B we have

$$\delta_B *^{\mathbf{B}} b = \delta_B \qquad \text{and} \qquad b *^{\mathbf{B}} \delta_B = \delta_B$$

then call δ_B the *zero of* $\mathbf{B}$ or just the *zero* and say that $\mathbf{B}$ *has a zero*. Note that if a groupoid has a zero, then the zero of the groupoid is unique.

Definition. Say that a finite groupoid $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ is *absorbing* if

1. the operation $*^{\mathbf{B}}$ is idempotent,

2. the algebra $\mathbf{B}$ has a zero, and

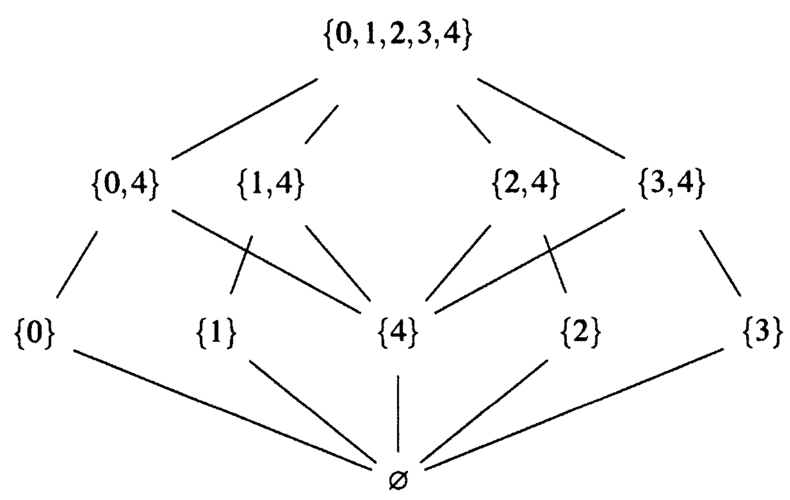

Figure 4.1: The Lattice of Subuniverses of $\mathbf{E}_{19}$

3. the universe of every subalgebra of **B** that contains at least 2 elements also contains the zero of **B**.

Note that not all absorbing groupoids are semigroups as the binary operation of an absorbing groupoid need not be associative.

4.2.1 Absorbing Groupoids Do Not Necessarily Generate Congruence-Modular Varieties

Is it worth looking at absorbing groupoids? That is, has the Restricted Quackenbush Problem been answered with respect to absorbing groupoids? The groupoid $\mathbf{E}_{19}$, defined on page 119, is an example of an absorbing groupoid that is not associative. Hence, $\mathbf{E}_{19}$ is not a semigroup. We show that $\mathbf{E}_{19}$ does not generate a congruence-modular variety. Hence, there exist varieties of absorbing groupoids that are not congruence-modular and thus neither congruence-distributive nor congruence-permutable.

It is unknown whether all absorbing groupoids generate varieties that either have definable principal congruences or are congruence-meet-semidistributive or are residually large. This means previous techniques can not yet be used to answer the Restricted Quackenbush Problem for absorbing groupoids.

To show that $\mathbf{E}_{19}$ does not generate a congruence-modular variety, we use tools developed by Freese and Valeriote in [5]. Call a quadruple $\langle b_1, b_2, b_3, b_4\rangle$ in an algebra $\mathbf{B}$ a *Day quadruple* if in the subalgebra $\mathbf{M}$ generated by $\{b_1, b_2, b_3, b_4\}$, we have the following:

$$\langle b_1, b_2\rangle \notin \mathrm{Cg}^{\mathbf{M}}(\langle b_3, b_4\rangle) \vee \left(\mathrm{Cg}^{\mathbf{M}}(\langle b_1, b_2\rangle, \langle b_3, b_4\rangle) \wedge \mathrm{Cg}^{\mathbf{M}}(\langle b_1, b_3\rangle, \langle b_2, b_4\rangle)\right).$$

In the language of a groupoid, Theorem 3.6 in [5] becomes the following:

Theorem 4.2.1. *Let* $\mathbf{B}$ *be an idempotent groupoid. Then* $\mathbb{V}(\mathbf{B})$ *fails to be congruence-modular if and only if there is a Day quadruple in* $\mathbf{B}^2$.

For the remainder of this subsection, we denote the 2-tuple $\langle a_1, a_2\rangle$ by a_1a_2. We explicitly show that

$$\langle b_1, b_2, b_3, b_4\rangle = \langle 00, 01, 40, 41\rangle \tag{4.1}$$

is a Day quadruple in $\mathbf{E}_{19}{}^2$. To show this, we must determine the following congruences:

$$\mathrm{Cg}^{\mathbf{M}}(\langle 40, 41\rangle), \qquad \mathrm{Cg}^{\mathbf{M}}(\langle 00, 01\rangle, \langle 40, 41\rangle) \qquad \text{and} \qquad \mathrm{Cg}^{\mathbf{M}}(\langle 00, 40\rangle, \langle 01, 41\rangle).$$

Note that

$$\mathrm{Sg}^{\mathbf{B}^2}(\{00, 01, 40, 41\}) = \{00, 01, 02, 03, 04, 40, 41, 42, 43, 44\}.$$

Denote this set by M and define $\mathbf{M} = \langle M; *^{\mathbf{M}}\rangle$, where $*^{\mathbf{M}} = *^{\mathbf{E}_{19}{}^2}\restriction_M$. See Table 4.2 for the operation table of $*^{\mathbf{M}}$.

Recall that for an equivalence relation σ, defined on the universe of a groupoid $\mathbf{G}$ to satisfy the compatibility property, the following condition must be satisfied: for all $\langle g_1, g_2\rangle$ and $\langle g_3, g_4\rangle$ in σ, the element $\langle g_1 *^{\mathbf{G}} g_3, g_2 *^{\mathbf{G}} g_4\rangle$ is in σ.

$*^{\mathbf{M}}$	00	01	02	03	04	40	41	42	43	44
00	00	02	04	00	04	40	42	44	40	44
01	01	01	03	00	04	44	41	43	40	44
02	01	01	02	04	04	41	41	42	44	44
03	01	04	00	03	04	41	44	40	43	44
04	04	04	04	04	04	44	44	44	44	44
40	40	42	44	40	44	40	42	44	40	44
41	44	41	43	40	44	44	41	43	40	44
42	41	41	42	44	44	41	41	42	44	44
43	41	44	40	43	44	41	44	40	43	44
44	44	44	44	44	44	44	44	44	44	44

Table 4.2: The Operation Table of $*^M$

Looking at Table 4.2, we see that the relation

$$\tau_1 = \{40,41,42,43,44\}^2 \cup \Delta_M$$

is a congruence on **M**. Since $\langle 40,41\rangle$ is a member of τ_1, we must have

$$\mathrm{Cg}^{\mathbf{M}}(\langle 40,41\rangle) \subseteq \tau_1. \tag{4.2}$$

Further, since

$$\langle 40,41\rangle *^{\mathbf{M}} \langle 00,00\rangle = \langle 40,44\rangle, \qquad \langle 40,41\rangle *^{\mathbf{M}} \langle 01,01\rangle = \langle 42,41\rangle \qquad \text{and}$$
$$\langle 40,41\rangle *^{\mathbf{M}} \langle 02,02\rangle = \langle 44,43\rangle,$$

the elements $\langle 40,44\rangle$, $\langle 42,41\rangle$ and $\langle 44,43\rangle$ are all in $\mathrm{Cg}^{\mathbf{M}}(\langle 40,41\rangle)$. Thus, 40, 41, 42, 43, 44 are all related to each other, with respect to $\mathrm{Cg}^{\mathbf{M}}(\langle 40,41\rangle)$. No other distinct elements can be related to each other, due to Statement (4.2). Hence,

$$\mathrm{Cg}^{\mathbf{M}}(\langle b_3, b_4\rangle) = \mathrm{Cg}^{\mathbf{M}}(\langle 40, 41\rangle) = \tau_1. \tag{4.3}$$

The relation

$$\tau_2 = \{00, 01, 02, 03, 04\}^2 \cup \{40, 41, 42, 43, 44\}^2$$

is the congruence on **M** that is obtained from the first projection map on **M**. Since $\langle 00, 01\rangle$ and $\langle 40, 41\rangle$ are elements in τ_2, we obtain the following:

$$\mathrm{Cg}^{\mathbf{M}}(\langle 00, 01\rangle, \langle 40, 41\rangle) \subseteq \tau_2. \tag{4.4}$$

By Statement (4.3), the elements 40, 41, 42, 43 and 44 must all be related to each other, in $\mathrm{Cg}^{\mathbf{M}}(\langle 00, 01\rangle, \langle 40, 41\rangle)$. Further, since

$$\langle 00, 01\rangle *^{\mathbf{M}} \langle 00, 00\rangle = \langle 01, 04\rangle, \qquad \langle 00, 01\rangle *^{\mathbf{M}} \langle 01, 01\rangle = \langle 02, 01\rangle \qquad \text{and}$$
$$\langle 00, 01\rangle *^{\mathbf{M}} \langle 02, 02\rangle = \langle 04, 03\rangle,$$

the elements 00, 01, 02, 03 and 04 must all be related to each other, in $\mathrm{Cg}^{\mathbf{M}}(\langle 00, 01\rangle, \langle 40, 41\rangle)$. Therefore, by Statement (4.4),

$$\mathrm{Cg}^{\mathbf{M}}(\langle b_1, b_2\rangle, \langle b_3, b_4\rangle) = \mathrm{Cg}^{\mathbf{M}}(\langle 00, 01\rangle, \langle 40, 41\rangle) = \tau_2.$$

The relation

$$\tau_3 = \{00, 40\}^2 \cup \{01, 41\}^2 \cup \{02, 42\}^2 \cup \{03, 43\}^2 \cup \{04, 44\}^2$$

is the congruence on **M** that is obtained from the second projection map on **M**. Since $\langle 00, 40\rangle$ and $\langle 01, 41\rangle$ is contained in τ_3, the following is true:

$$\mathrm{Cg}^{\mathbf{M}}(\langle 00, 40\rangle, \langle 01, 41\rangle) \subseteq \tau_3. \tag{4.5}$$

As

$$\langle 00,40\rangle *^{\mathbf{M}} \langle 02,02\rangle = \langle 02,42\rangle, \qquad \langle 00,40\rangle *^{\mathbf{M}} \langle 03,03\rangle = \langle 04,44\rangle \qquad \text{and}$$

$$\langle 01,41\rangle *^{\mathbf{M}} \langle 02,02\rangle = \langle 03,43\rangle,$$

the elements 00, 01, 02, 03 and 04 are related to 40, 41, 42, 43 and 44, respectively, in $\mathrm{Cg}^{\mathbf{M}}(\langle 00,40\rangle, \langle 01,41\rangle)$. Therefore, due to Statement (4.5),

$$\mathrm{Cg}^{\mathbf{M}}(\langle b_1,b_3\rangle, \langle b_2,b_4\rangle) = \mathrm{Cg}^{\mathbf{M}}(\langle 00,40\rangle, \langle 01,41\rangle) = \tau_3.$$

Let

$$\langle b_1,b_2,b_3,b_4\rangle = \langle 00,01,40,41\rangle.$$

Using τ_1, τ_2 and τ_3, defined earlier, we obtain the following:

$$\begin{aligned}
&\mathrm{Cg}^{\mathbf{M}}(\langle b_3,b_4\rangle) \vee \left(\mathrm{Cg}^{\mathbf{M}}(\langle b_1,b_2\rangle, \langle b_3,b_4\rangle) \wedge \mathrm{Cg}^{\mathbf{M}}(\langle b_1,b_3\rangle, \langle b_2,b_4\rangle)\right) \\
&= \tau_1 \vee (\tau_2 \wedge \tau_3) \\
&= \tau_1 \vee \Delta_M \\
&= \tau_1.
\end{aligned}$$

Thus, since

$$\langle 00,01\rangle \notin \{40,41,42,43,44\}^2 \cup \Delta_M = \tau_1,$$

the quadruple listed in Statement (4.1) is a Day quadruple. Hence, by Theorem 4.2.1, the variety generated by $\mathbf{E}_{19}$ is not congruence-modular.

4.2.2 A Possible Approach to Deal with Absorbing Groupoids and the Restricted Quackenbush Problem

For the remainder of this chapter, assume that $\mathbf{B}$ is an absorbing groupoid and $\mathbf{S}$ is a subalgebra of $\mathbf{B}^I$, for some index set I. Further, assume that θ is a congruence on $\mathbf{S}$.

Recall that the i^{th} projection map on B^I restricted to S is denoted by ρ_i instead of π_i. Further recall, from Lemma 4.1.1, that each member of S/θ is a subuniverse of $\mathbf{S}$. Hence, we denote the elements in S/θ by the elements $\{S_i\}_{j\in J}$, where J is an index set. For all i in I, define the binary relation ϕ_i on S/θ as follows: for all S_1 and S_2 in S/θ, the 2-tuple $\langle S_1, S_2\rangle$ is in ϕ_i if and only if

1. the element δ_B is in $\rho_i(S_1) \cap \rho_i(S_2)$ or

2. the congruence blocks S_1 and S_2 are equal.

Lemma 4.2.2. *For all i in I, the relation ϕ_i is a congruence on* $\mathbf{S}/\theta$.

Proof. To prove the Lemma, we show that ϕ_i is an equivalence relation on S/θ that satisfies the compatibility property.

The relation ϕ_i is reflexive by definition and symmetry follows from the commutativity of set intersection and symmetry of equality. Suppose that S_1, S_2, S_3 and S_4 is in S/θ. To show that ϕ_i is transitive, assume that $\langle S_1, S_2\rangle$ and $\langle S_2, S_3\rangle$ is in ϕ_i. Three cases emerge.

Case 1 Suppose that δ_B is in both $\rho_i(S_1) \cap \rho_i(S_2)$ and $\rho_i(S_3) \cap \rho_i(S_3)$. Then δ_B is in $\rho_i(S_1) \cap \rho_i(S_3)$ and hence $\langle S_1, S_3\rangle$ is in ϕ_i.

Case 2 Without loss of generality, suppose that δ_B is in $\rho_i(S_1) \cap \rho_i(S_2)$ and that $S_2 = S_3$. Then

$$\rho_i(S_1) \cap \rho_i(S_2) = \rho_i(S_1) \cap \rho_i(S_3)$$

and hence $\langle S_1, S_3\rangle$ is in ϕ_i.

Case 3 If $S_1 = S_2$ and $S_2 = S_3$, then $S_1 = S_3$ and hence $\langle S_1, S_3\rangle$ is in ϕ_i.

Thus, ϕ_i is an equivalence relation.

To show that ϕ_i satisfies the compatibility property, assume that $\langle S_1, S_2 \rangle$ and $\langle S_3, S_4 \rangle$ are in ϕ_i. Either $S_1 = S_2$ and $S_3 = S_4$ or one of these equalities is not true. With respect to the former scenario, if $S_1 = S_2$ and $S_3 = S_4$ then

$$S_1 *^{\mathbf{S}/\theta} S_3 = S_2 *^{\mathbf{S}/\theta} S_4 \qquad \text{and hence} \qquad \langle S_1 *^{\mathbf{S}/\theta} S_3, S_2 *^{\mathbf{S}/\theta} S_4 \rangle \in \phi_i.$$

With respect to the latter scenario, assume that $S_1 \neq S_2$, without loss of generality. Thus, δ_B is in $\rho_i(S_1) \cap \rho_i(S_2)$. There exists s_1 in S_1 and s_2 in S_2 such that

$$s_1(i) = s_2(i) = \delta_B.$$

Pick arbitrary members s_3 in S_3 and s_4 in S_4. Then,

$$\begin{aligned}(s_1 *^{\mathbf{S}} s_3)(i) &= s_1(i) *^{\mathbf{B}} s_3(i) \\ &= \delta_B *^{\mathbf{B}} s_3(i) \\ &= \delta_B.\end{aligned}$$

Similarly, $(s_2 *^{\mathbf{S}} s_4)(i) = \delta_B$. As $s_1/\theta = S_1$ and $s_3/\theta = S_3$, we have

$$\begin{aligned}s_1 *^{\mathbf{S}} s_3 &\in (s_1 *^{\mathbf{S}} s_3)/\theta \\ &= s_1/\theta *^{\mathbf{S}/\theta} s_3/\theta \\ &= S_1 *^{\mathbf{S}/\theta} S_3.\end{aligned}$$

Similarly, $s_2 *^{\mathbf{S}} s_4$ is in $S_2 *^{\mathbf{S}/\theta} S_4$. Thus,

$$\delta_B \in \rho_i(S_1 *^{\mathbf{S}/\theta} S_3) \cap \rho_i(S_2 *^{\mathbf{S}/\theta} S_4) \qquad \text{and hence} \qquad \langle S_1 *^{\mathbf{S}/\theta} S_3, S_2 *^{\mathbf{S}/\theta} S_4 \rangle \in \phi_i.$$

Therefore, ϕ_i satisfies the compatibility property. □

Lemma 4.2.3. *Suppose that I is a finite set that contains at least two elements. If for all i in I, we have ϕ_i not equal to $\Delta_{S/\theta}$, then $\mathbf{S}/\theta$ is not subdirectly irreducible.*

Proof. By Lemma 4.2.2, the relation ϕ_i is a congruence on $\mathbf{S}/\theta$. By hypothesis, for some positive integer n, we have $1 < |I| < n$ and for all i in I, the congruence ϕ_i is not equal to $\Delta_{S/\theta}$.

For S_1 and S_2 in S/θ, suppose that $\langle S_1, S_2\rangle$ is in $\bigwedge_{i\in I}\phi_i$. Therefore, for all i in I, we have $\langle S_1, S_2\rangle$ in ϕ_i. Thus, either $S_1 = S_2$ or, for all i in I,

$$\delta_B \in \rho_i(S_1) \cap \rho_i(S_2).$$

Assume that the latter is true. Thus, for all i in I, we may pick t_1^i in S_1 and t_2^i in S_2 such that

$$t_1^i(i) = t_2^i(i) = \delta_B. \tag{4.6}$$

For j in $\{1,2\}$, define

$$T_j = \left(\cdots\left((t_j^1 *^{\mathbf{S}} t_j^2) *^{\mathbf{S}} t_j^3\right) *^{\mathbf{S}} \cdots *^{\mathbf{S}} t_j^{n-1}\right) *^{\mathbf{S}} t_j^n.$$

Lemma 4.1.1 yields S_1 and S_2 being subuniverses of $\mathbf{S}$. Hence, both S_1 and S_2 are closed under $*^{\mathbf{S}}$. Thus, for all j in $\{1,2\}$, we have T_j in S_j. By Statement (4.6), we have both T_1 and T_2 equal to the n-tuple in B^I where every coordinate is δ_B. Hence, $S_1 \cap S_2 \neq \varnothing$ and thus $S_1 = S_2$.

Therefore, $\bigwedge_{i\in I}\phi_i = \Delta_{S/\theta}$. We have just shown that $\mathbf{S}/\theta$ is not subdirectly irreducible as there exists a family of congruences on $\mathbf{S}/\theta$ that are all not $\Delta_{S/\theta}$, but, their meet is $\Delta_{S/\theta}$. □

Define a $(\theta, I, \mathbf{S}, \mathbf{B})$*-algebra* to be an algebra isomorphic to $\mathbf{S}/\theta$, where $\mathbf{S}$ is a subalgebra of $\mathbf{B}^I$, for some finite index set I, and θ is a congruence on $\mathbf{S}$. Notice that all $(\theta, I, \mathbf{S}, \mathbf{B})$-algebras are in $\mathbb{V}(\mathbf{B})$. We are now ready to state and prove the main result of this section.

Theorem 4.2.4. *Let* $\mathbf{B}$ *be an absorbing groupoid that generates a residually finite variety and* N *be an integer greater than* $|B|$. *If all subdirectly irreducible* $(\theta, I, \mathbf{S}, \mathbf{B})$*-algebras with* $|I| \geq 2$ *satisfy the implication*

there exists i *in* I *such that* $\phi_i = \Delta_{S/\theta}$ *implies* $|S/\theta| < N$

then $\mathbb{V}(\mathbf{B})$ *is residually* $< N$.

Proof. Recall Lemma 1.3.8, on page 41: if $\mathbf{B}$ is a finite algebra then all finite members in $\mathbb{V}(\mathbf{B})$ are in $\mathbb{HSP}_{\text{fin}}(\mathbf{B})$. Thus, each finite algebra in $\mathbb{V}(\mathbf{B})$ is a $(\theta, I, \mathbf{S}, \mathbf{B})$-algebra. As $\mathbb{V}(\mathbf{B})$ is residually finite, each subdirectly irreducible algebra in $\mathbb{V}(\mathbf{B})$ is a $(\theta, I, \mathbf{S}, \mathbf{B})$-algebra.

Let N be a fixed integer greater than $|B|$ and let $\mathbf{M}$ be a subdirectly irreducible $(\theta, I, \mathbf{S}, \mathbf{B})$-algebra. If $|I| = 1$, then $\mathbf{M}$ is in $\mathbb{HS}(\mathbf{B})$ and hence $|M| < |B|$. Now assume $|I| \geq 2$. By Lemma 4.2.3, there must exist i in I such that $\phi_i = \Delta_{S/\theta}$, as otherwise $\mathbf{S}/\theta$, and hence $\mathbf{M}$, is not subdirectly irreducible and a contradiction occurs. By the hypothesis, if there exists an i in I such that $\phi_i = \Delta_{S/\theta}$ then $|S/\theta| < N$ and hence $|M| < N$. Therefore, $\mathbb{V}(\mathbf{B})$ is residually less than $\max\{|B|, N\} = N$. □

The chapter is concluded with a question that asks if it is possible to use Theorem 4.2.4 to answer the Restricted Quackenbush Problem, with respect to absorbing groupoids.

Question. *Let* $\mathbf{B}$ *be an absorbing groupoid. What additional conditions can be imposed on* $\mathbf{B}$ *to force the existence of an integer* N *greater than* $|B|$ *such that all subdirectly irreducible* $(\theta, I, \mathbf{S}, \mathbf{B})$*-algebras with* $|I| \geq 2$ *satisfy the following implication: if there exists an* i *in* I *such that* $\phi_i = \Delta_{S/\theta}$ *then* $|S/\theta| < N$*?*

Chapter 5

Future Work

Recall the Restricted Quackenbush Problem.

Problem (Restricted Quackenbush Problem). *Let* $\mathbf{B}$ *be a finite algebra of finite type. If* $\mathbb{V}(\mathbf{B})$ *is residually finite, must* $\mathbb{V}(\mathbf{B})$ *be residually* $< N$*, for some positive integer* N*?*

The problem looks at the number of subdirectly irreducible algebras in a finitely generated variety of finite type. An answer to the problem would yield a deeper understanding of varieties and their construction, as the subdirectly irreducible algebras in a given variety act as the building blocks of the algebras in that variety. Though the problem appeared in [8], the origins of the problem are in the 1970's.

Over the past 40 years, a great deal of work has been done on answering the Restricted Quackenbush Problem and its precursors by the following individuals: Baldwin, Berman, Freese, Kearnes, McKenzie, Taylor, Willard and others. The problem has been answered with respect to many familiar algebras, with the exception of groupoids.

5.1 Summary

In this thesis, we have given an explicit proof of the Restricted Quackenbush Problem with respect to unary algebras, a result that was initially discovered by Baldwin and Berman in [1]. The proof devised here did not explicitly use the Congruence Extension Theorem, whereas Baldwin and Berman explicitly utilized the Congruence Extension Property.

In the latter half of the thesis, we turned our attention to groupoids. The subdirectly irreducible members in varieties generated by groupoids that are influenced by a partial order relation were then analysed. We were able to show that some groupoids influenced by a partial order relation of height 3 generate varieties that are residually large. From this result, the following result was derived:

Theorem. *Let* $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ *be a non-trivial finite groupoid. For the variable* x*, if* $\mathbb{V}(\mathbf{B})$ *is not residually large then there exists some term* $k(x)$ *of type* $\{*\}$*, such that* $k(x) \neq x * x$ *and* $\mathbf{B}$ *satisfies* $k(x) \approx x * x$*.*

This theorem yields some insight in terms of what groupoids need to be looked at. Using the ideas that lead up to this Theorem, we were also able to answer the RS-Conjecture, a conjecture similar to the Restricted Quackenbush Problem, with respect to groupoids that do not satisfy any of the following identities: $k(x) \approx x * x$, where $k(x)$ is some term of type $\{*\}$ such that $k(x) \neq x * x$.

Lastly, we looked at absorbing groupoids and tried to answer the Restricted Quackenbush Problem. Difficulties occurred when dealing with $(\theta, I, \mathbf{S}, \mathbf{B})$-algebras where there exists i in I such that $\phi_i = \Delta_{S/\theta}$. The outcome of this attempt yielded the following theorem:

Theorem. *Let* $\mathbf{B}$ *be an absorbing groupoid that generates a residually finite variety and* n *be an integer greater than* $|B|$*. If all subdirectly irreducible* $(\theta, I, \mathbf{S}, \mathbf{B})$*-algebras with* $|I| > 2$

satisfy the implication

$$\textit{there exists } i \textit{ in } I \textit{ such that } \phi_i = \Delta_{S/\theta} \textit{ implies } |S/\theta| < N$$

then $\mathbb{V}(\mathbf{B})$ *is residually* $< N$.

The above theorem may be a step towards answering the Restricted Quackenbush Problem, with respect to absorbing groupoids.

5.2 Questions

The following is a list of questions that were asked throughout this thesis.

Question. *With regards to Corollary 3.2.2, does there exist a non-trivial and non-constant groupoid* $\mathbf{B} = \langle B; *^{\mathbf{B}} \rangle$ *such that* $*^{\mathbf{B}}$ *is not idempotent and* $\mathbb{V}(\mathbf{B})$ *is residually finite? Must such a groupoid be a reduct of a group?*

Question. *Let* $\mathbf{B}$ *be an absorbing groupoid. What additional conditions can be imposed on* $\mathbf{B}$ *to force the existence of an integer* N *greater than* $|B|$ *such that all subdirectly irreducible* $(\theta, I, \mathbf{S}, \mathbf{B})$*-algebras with* $|I| \geq 2$ *satisfy the following implication: if there exists an* i *in* I *such that* $\phi_i = \Delta_{S/\theta}$ *then* $|S/\theta| < N$*?*

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