RANK AND DUALITY OF ESCALATOR ALGEBRAS

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Abstract

We define a specific family of finite bi-unary algebras called escalator algebras. These algebras were introduced in the work of Hyndman and Willard [9] and Little [10]. We show that they have infinite rank, are dualizable but are not strongly dualizable.

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Chapter 1

Introduction

The birth of natural duality theory occurred in the 1930's when Birkhoff, Pontryagin and Stone developed representations for specific algebras using toplogical spaces [2]. By the 1980's many other algebras had been represented in a similar manner (Priestley duality for bounded distributive lattices [14] being the most well known). These were useful as they allowed algebraic problems to be solved using topological tools. At this time a general theory of duality emerged: dualities were defined in terms of quasi-varieties of finite algebras, and furthermore the concepts of full duality and strong duality were formalized. Subsequent research then focused on questions regarding which finite algebras are dualizable, which of the dualizable algebras are fully dualizable and which are strongly dualizable.

In 1998 Ross Willard [15] defined the rank function of an algebra homomorphism and of a finite algebra. He proved that if an algebra was dualizable and had finite rank then it was strongly dualizable. In 1999 Jennifer Hyndman and Ross Willard [9] constructed a three element algebra that was dualizable but not fully dualizable by any appropriate set of operations, partial operations and relations. This was the first example of such an algebra. In 2000 Richard Little [10], using Prolog, was able to compute an approximate rank of a homomorphism. Little's progam failed to approximate the rank of the algebra in [9]. This result was not surprising as the algebra is known to have infinite rank. He speculated that a similar four element algebra for which his program had failed also had infinite rank.

In this thesis we generalize the three element algebra of [9] into a family of algebras called escalator algebras and explore both the rank and duality of this family of algebras. To this end, Chapter 2 covers required definitions from universal algebra, topology, and natural duality theory. Particular attention is given to the definition of rank. Chapter 2 also presents some discussion of current research in the area of natural duality theory. Escalator algebras are carefully defined in Chapter 3 and several useful properties and constructions are presented. This leads to Chapter 4 where we show that escalator algebras have infinite rank, are dualizable but not strongly dualizable. We conclude in Chapter 5 with a brief discussion of questions about full duality of the escalator algebras that are not answered in this thesis.

Chapter 2

Preliminary Material

2.1 Definitions from Universal Algebra

To have an understanding of natural duality theory one must first be conversant in the languages of both universal algebra and topology. We start by defining the algebraic terminology that we require. These definitions are heavily influenced by [11] and [5] where they are discussed more fully. Sometimes the scope of the definition is restricted to the material needed here.

First we define an ***algebra*** as an ordered pair $\mathbf{A} = \langle A, F\rangle$, where A is a non-empty set, called the ***universe***, and F is a set of ***basic operations*** each taking a fixed number of elements of A as arguments and returning a single element of A. For simplicity we denote an algebra with universe A by $\mathbf{A}$ when it is clear which set of functions is intended. The ***arity*** of an operation is the number of arguments the operation takes. Thus an operation that takes n arguments is called an n-ary operation. For example a ***bi-unary algebra*** is an algebra that has exactly two operations both of which are unary, that is 1-ary. A ***nullary*** operation takes no arguments. Two algebras have the same ***type*** if for each $n \in \mathbb{N}$ the number of n-ary operations is the same for each algebra. If we have algebras $\mathbf{A}$ and $\mathbf{B}$, that use the same symbol f for an operation then for clarity we sometimes use the notation $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$ respectively and say that $f^{\mathbf{A}}$ is the ***interpretation*** of f in $\mathbf{A}$. If $S \subset A$ and $f : A \to B$ is a function then there is a function $h : S \to B$ given by $h(a) = f(a)$ for all $a \in S$ and it is called the ***restriction*** of f to S. It is denoted by $f|_S$.

A ***subalgebra*** of an algebra $\mathbf{A}$ is an algebra whose universe is a non-empty subset of A and the operations of the subalgebra are restrictions of the operations of $\mathbf{A}$ to the new universe. We use the notation $\mathbf{B} \leq \mathbf{A}$ to say that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. If $X \subseteq A$ is non-empty then the ***subalgebra generated by*** X is the smallest subalgebra of $\mathbf{A}$ that contains X. This is denoted by $\mathrm{Sg}_{\mathbf{A}}(X)$.

An algebra ***homomorphism*** is an operation preserving mapping from one algebra to another algebra of the same type. That is if $\mathbf{A}$, $\mathbf{B}$ are algebras, h is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, f is an n-ary operation and $a_1, a_2, \ldots, a_n$ are elements of A then

$$h(f^{\mathbf{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), h(a_2), \ldots, h(a_n)).$$

The set of all homomorphisms from $\mathbf{A}$ into $\mathbf{B}$ is denoted by $\mathrm{Hom}\,(\mathbf{A}, \mathbf{B})$. Notice that $\mathrm{Hom}\,(\mathbf{A}, \mathbf{B})$ is a subset of the set B^A of all mappings from A to B. A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is ***onto***, or ***surjective*** if for all $b \in B$ there exists an $a \in A$ such that $h(a) = b$, or more simply $h(A) = B$. And h is ***one-to-one***, or ***injective***, if $f(x) = f(y)$ implies that $x = y$. An ***isomorphism*** or ***bijection*** is a homomorphism that is both onto and one-to-one.

Given a non-empty set A and a positive integer n, an n-ary ***relation*** on A is a subset of A^n, the set of all n-tuples of A. The ***kernel*** of a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is the binary relation $\ker(h) = \{(a, a') : a, a' \in \mathbf{A} \text{ and } h(a) = h(a')\}$. Binary relations which occur as kernels of homomorphisms are called congruences, and the intersection of any family of congruences is again a congruence (see [1]). Given a congruence, Θ of $\mathbf{A}$, there is an algebra denoted $\mathbf{A}/\Theta$, and a homomorphism $\phi : \mathbf{A} \to \mathbf{A}/\Theta$ such that $\ker(\phi) = \Theta$.

Let $A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A\}$. If $\mathbf{A}$ is an algebra then $\mathbf{A}^n$ is called the **n-*th power*** of $\mathbf{A}$ and is the algebra $\langle A^n, F\rangle$ where F is the set of functions on A^n created by applying the operations of $\mathbf{A}$ coordinatewise. If $a \in \mathbf{A}^n$ then a is really an n-tuple $a = (a_1, a_2, \ldots, a_n)$. This long notation will only be used where context does not make the usage of a clear. For convenience the notation of tuples is sometimes truncated from $(a_1, a_2, \ldots, a_n)$ to $a_1a_2 \ldots a_n$. For an arbitrary set I let $A^I = \{< a_i : i \in I >\mid a_i \in A\}$. If $\mathbf{A}$ is an algebra then $\mathbf{A}^\mathbf{I}$ is also an algebra with the operations of $\mathbf{A}$ acting coordinatewise.

Example 2.1. Let $M = \{0, 1, 2, 3\}$ and f, g be unary operations on M given by $f(0) = 1, f(1) = 2, f(2) = f(3) = 3$ and $g(0) = g(1) = 0, g(2) = 1, g(3) = 2$. (See Figure 2.1.) Then $\mathbf{M} = \langle M; f, g\rangle$ is an algebra and so is $\mathbf{M}^2$. The pairs $(1, 0), (1, 3)$ are in M^2 so $g^{\mathbf{M}^2}((1, 0)) = (g^{\mathbf{M}}(1), g^{\mathbf{M}}(0)) = (0, 0)$ and $f^{\mathbf{M}^2}((1, 3)) = (f^{\mathbf{M}}(1), f^{\mathbf{M}}(3)) = (2, 3)$. Let $\mathbf{B} = \mathrm{Sg}_{\mathbf{M}^2}\{(1, 1)\}$, then $B = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$ and $f^{\mathbf{B}}$ equals $f^{\mathbf{M}^2}$ restricted to B, $g^{\mathbf{B}}$ is $g^{\mathbf{M}^2}$ restricted to B.

Assume R^M is a k-ary relation on a set M. Then R^{M^n} is the k-ary relation on M^n defined by $(a^1, a^2, \ldots, a^k) \in R^{M^n}$ if and only if $(a_i^1, a_i^2, \ldots, a_i^k) \in R$ for all

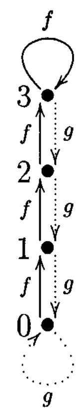

Figure 2.1: The algebra $\mathbf{M} = \langle \{0, 1, 2, 3\}; f, g \rangle$ of Example 2.1

$i \leq n$. We almost always omit the superscript on R^{M^n}. In Example 2.1 the standard integer relations $\leq$ and $<$ restricted to M are binary relations on M. Therefore we can consider the corresponding binary relations $\leq$ and $<$ on M^2. It should be clear that $((0, 2), (1, 2))$ is in $\leq$ but $((2, 0), (1, 2))$ is not in $\leq$.

Let $\mathbf{A}$ be an algebra. For each $i \leq n$ there is an n-ary ***projection operation*** π_i on A given by $\pi_i(a_1, a_2, \ldots, a_n) = a_i$. If f is an n-ary operation on A and $g_1, g_2 \ldots, g_n$ are all k-ary operations on A then $h = f(g_1, g_2, \ldots, g_n)$ is a k-ary operation on A given by $h(a) = f(g_1(a), g_2(a), \ldots, g_n(a))$. This is called ***composition of operations***.

Given a set F of operation symbols and an index set S let $X = \{x_s \mid s \in S\}$ be a set of ***variables***. Then an ***S-ary term of type F*** is defined recursively by

1. every $x \in X$ and every nullary $f \in F$ is an S-ary term, and

2. if $t_1, t_2, \ldots t_n$ are S-ary terms and f is an n-ary operation symbol of F for some $n \geq 1$ then the string $f(t_1, t_2, \ldots t_n)$ is an S-ary term.

The ***length of a term*** is defined recursively with length of $x_i = 0$ and length of $f(g_1, \ldots, g_n) = 1 + \max(\text{length } g_i)$. The ***term operations***, or ***term functions***, of $\mathbf{A}$ are all the operations constructed by composition of operations using the basic operations of $\mathbf{A}$, the projection operations on A and all operations that are constructed in this manner. For an example if f is a unary basic operation of $\mathbf{A}$ then $f \circ \pi_3$ is a term operation and so is $ff\pi_3$. If S is a non-empty set and X is a subset of A^S then X is ***term-closed*** if for all $y \in A^S \setminus X$ there exists S-ary term functions $\sigma, \tau : A^S \to A$ such that $\sigma|_X = \tau|_X$ but $\sigma(y) \neq \tau(y)$.

A set P is called a ***partially ordered set*** if there is a binary relation $\leq$ on P such that for any $a, b, c \in P$ we have

1. $a \leq a$,
2. $a \leq b$ and $b \leq a$ implies $a = b$, and
3. $a \leq b$ and $b \leq c$ implies $a \leq c$.

A ***chain*** is a partially ordered set in which any two elements are related. That is, for all $a, b \in P$ either $a \leq b$ or $b \leq a$. Other names for a chain are ***totally ordered set*** or ***linearly ordered set***. An intuitive example of a chain is the set of non-negative integers where the $\leq$ relation on this set is the every-day definition of less than or equal to.

We define the operations, ***meet*** ($\wedge$) and ***join*** ($\vee$), on a partially ordered set P. For all $a, b, c, d \in P$ we say $a \wedge b = c$ if for every $p \in P$ such that $p \leq a$, and $p \leq b$ then $p \leq c$. Also $a \vee b = d$ if for every $q \in P$ such that $a \leq q$, and $b \leq q$ then $d \leq q$. Note that $\wedge$ and $\vee$ may not be defined for all pairs on a partially ordered set P. If P is a chain and $a \leq b$ then $a \wedge b = a$ and $a \vee b = b$.

A ***lattice*** is an algebra where the universe is a partially ordered set and the operations are $\wedge$ and $\vee$ and these are defined on every pair of elements (a chain is one example of a lattice). A ***lattice homomorphism*** is an algebra homomorphism where the two algebras are lattices.

Let $\mathbf{M}$ be some algebra and $\mathcal{K}$ be a class of algebras. Then $\mathbb{P}(\mathbf{M})$ is the class of all algebras that are isomorphic to a power of $\mathbf{M}$, $\mathbb{S}(\mathcal{K})$ is the class of all algebras that are isomorphic to a subalgebra of a member of $\mathcal{K}$, and $\mathbb{I}(\mathcal{K})$ is the class of all algebras that are isomorphic to a member of $\mathcal{K}$. The ***quasi-variety generated by*** $\mathbf{M}$ is $\mathbb{ISP}(\mathbf{M})$, the class of all algebras that are isomorphic to a subalgebra of a power of $\mathbf{M}$.

2.2 Rank

In this section we provide the definition of the rank of a homomorphism and the rank of an algebra. These were originally defined in [15]. As this is a fairly complex concept we start with some simpler definitions and build up to rank.

A ***diagram*** is a directed graph in which the nodes represent sets and the edges represent functions mapping one set to another. A ***commuting diagram*** is a diagram where for every path between a pair of sets the composition of the functions represented by the paths yields the same result. In a commuting diagram we use $A \overset{\alpha}{\hookrightarrow} B$ to denote that α is injective.

Example 2.2. The diagram in Figure 2.2 has nodes representing the sets A, B, C and D. The edge from A to B labelled α indicates that α is a map from A to B. This diagram commutes if $\beta \circ \alpha = g \circ f = h$.

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dr, "h"] & B \arrow[d, "\beta"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
$$

Figure 2.2: A commuting diagram

Let $\mathbf{M}$ be a finite algebra and n and k be finite positive integers. Let $\mathbf{B}$ be a subalgebra of $\mathbf{M}^n$ and let $h : \mathbf{B} \to \mathbf{M}$ be a homomorphism. We say a map σ from $\mathbf{B}$ to $\mathbf{M}^{n+k}$ uses ***repetition of coordinates*** when there is a surjective map $\tau : \{1, 2, \ldots, n+k\} \to \{1, 2, \ldots, n\}$ such that for all $a \in \mathbf{B}$ and for $i \in \{1, 2, \ldots, n+k\}$ we have $\sigma(a)_i = a_{\tau(i)}$. Let σ be such a map and let $\mathbf{B}'$ be the subalgebra of $\mathbf{M}^{n+k}$ obtained by applying σ to $\mathbf{B}$. Denote this embedding by $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$ and note that $\mathbf{B}$ is isomorphic to $\mathbf{B}'$. Let $h' = h \circ \sigma^{-1}$, the homomorphism that is the the natural extension of h from $\mathbf{B}$ to $\mathbf{B}'$.

Let $\mathbf{C}, \mathbf{D}$ be subalgebras of $\mathbf{M}^{n+k}$ where $\mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \mathbf{M}^{n+k}$ and let $Y \subseteq \mathrm{Hom}\,(\mathbf{D}, \mathbf{M})$. Denote the algebra $\mathbf{D}/\bigcap\{\ker g \mid g \in Y\}$ by $\mathbf{D}/Y$. Note that $\mathbf{D}/Y$ is isomorphic to $\Pi Y(\mathbf{D}) = \{\langle g(a) : g \in Y\rangle \mid a \in D\}$. The notation $\Pi Y(\mathbf{D})$ is used in [8]. Similarly denote the algebra $\mathbf{C}/\bigcap\{\ker g|_{\mathbf{C}} \mid g \in Y\}$ by $\mathbf{C}/Y$. The set Y ***separates*** $\mathbf{B}'$ if $\bigcap\{\ker(g|_{\mathbf{B}'}) \mid g \in Y\} = \{(x, x) \mid x \in \mathbf{B}'\}$. A map $h' : \mathbf{B}' \to \mathbf{M}$ ***lifts*** to $\mathbf{C}/Y$ if Y separates $\mathbf{B}'$ and there exists a homomorphism γ such that the diagram in Figure 2.3 commutes.

Example 2.3. Let $\mathbf{M}$ be the same algebra given in Example 2.1. (See Figure 2.1.) Let $\mathbf{B}$ be the subalgebra of $\mathbf{M}^3$ depicted in Figure 2.4 and $\sigma : \mathbf{B} \to \mathbf{M}^6$ be the

$$\begin{tikzcd}
\mathbf{B}' \arrow[r, hook, "\leq"] \arrow[d, "h'"] \arrow[dr, hook] & \mathbf{C} \arrow[d] \\
\mathbf{M} & \mathbf{C}/Y \arrow[l, dashed, "\gamma"]
\end{tikzcd}$$

Figure 2.3: The commuting diagram for lifting

repetition of coordinates map given by $\sigma(xyz) = (xyyyzz)$. Applying σ to $\mathbf{B}$ gives us $\mathbf{B}'$. (See Figure 2.5.) Finally let $\mathbf{C} = \mathbf{D} = \mathrm{Sg}_{\mathbf{M}^6}\{000011, 000111, 011111\}$, the subalgebra of $\mathbf{M}^6$ shown in Figure 2.6, and let $Y = \{\pi_1, \pi_4, \pi_6\} \subseteq \mathrm{Hom}\,(\mathbf{D}, \mathbf{M})$. We use the table in Figure 2.7 to convince ourselves that

$$\begin{aligned}
\bigcap\{\ker g \mid g \in Y\} = \{(x,x) \mid x \in D\} \\
\cup\{(000111, 011111), (111222, 122222), (222333, 233333), \\
(011111, 000111), (122222, 111222), (233333, 222333)\}.
\end{aligned}$$

The elements $000111, 111222, 222333 \notin \mathbf{B}'$ so

$$\bigcap\{\ker(g|_{\mathbf{B}'}) \mid g \in Y\} = \{(x,x) \mid x \in D\}$$

and the set Y separates $\mathbf{B}'$.

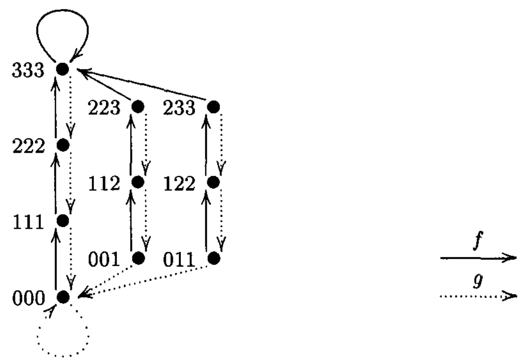

Figure 2.4: An algebra $\mathbf{B} \leq \mathbf{M}^3$ for Example 2.3

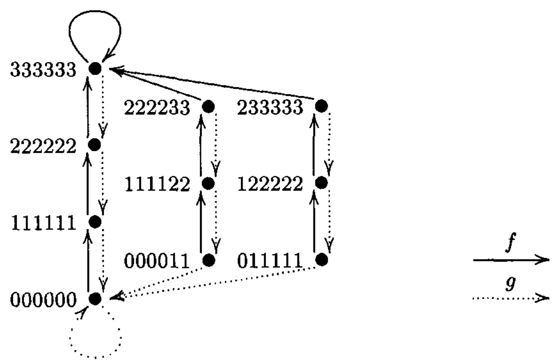

Figure 2.5: An algebra $\mathbf{B}' \leq \mathbf{M}^6$ obtained by repetition of coordinates

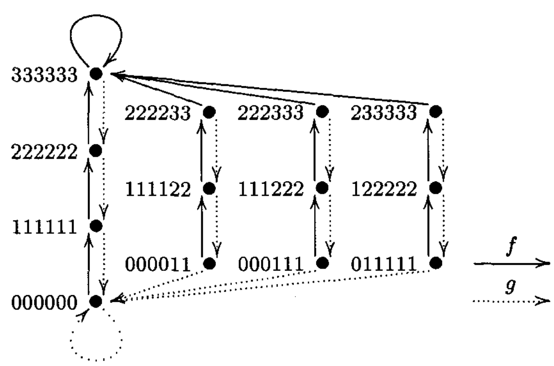

Figure 2.6: A diagram showing $\mathbf{C} = \mathbf{D} \leq \mathbf{M}^6$ of Example 2.3

x	$\pi_1(x)$	$\pi_4(x)$	$\pi_6(x)$
000000	0	0	0
111111	1	1	1
222222	2	2	2
333333	3	3	3
000011	0	0	1
111122	1	1	2
222233	2	2	3
000111	0	1	1
111222	1	2	2
222333	2	3	3
011111	0	1	1
122222	1	2	2
233333	2	3	3

Figure 2.7: Table of values of π_1, π_4 and π_6 for Example 2.3

Given a homomorphism $h : \mathbf{B} \to \mathbf{M}$ where $\mathbf{B} \leq \mathbf{M}^n$ define the ***rank*** of h recursively as:

1. $\operatorname{rank}(h) \leq 0$ if and only if h is a projection.

2. For a countable ordinal α, $\operatorname{rank}(h) \leq \alpha$ if and only if there exists an integer $N \geq 1$ such that for all integers $k \geq 0$, for all $\mathbf{D} \leq \mathbf{M}^{n+k}$, and for all commuting diagrams as in Figure 2.8 where there exists a homomorphism $h^+ : \mathbf{D} \to \mathbf{M}$, there exists $Y \subseteq \operatorname{Hom}(\mathbf{D}, \mathbf{M})$ such that

   (a) $|Y| \leq N$, and

   (b) h' lifts to C/Y, and

   (c) for all $g \in Y$, $\operatorname{rank}(g|_C) < \alpha$.

$$
\begin{tikzcd}
\mathbf{B} \arrow[r, phantom, "\Rrightarrow_\sigma"] \arrow[dr, "h"'] & \mathbf{B}' \arrow[r, phantom, "\leq"] \arrow[d, "h'"] & \mathbf{C} \arrow[r, phantom, "\leq"] & \mathbf{D} \arrow[r, phantom, "\leq"] \arrow[dll, dashed, "h^+"'] & \mathbf{M}^{n+k} \\
 & \mathbf{M} & & &
\end{tikzcd}
$$

Figure 2.8: The commuting diagram for rank

If $\operatorname{rank}(h) \leq \alpha$ and it is not true that $\operatorname{rank}(h) < \alpha$ then $\operatorname{rank}(h) = \alpha$. If for all homomorphisms $h : \mathbf{B} \to \mathbf{M}$ where $\mathbf{B}$ is a subalgebra of a finite power of $\mathbf{M}$ we have $\operatorname{rank}(h) \leq \alpha$ but at least one of these homomorphisms does not have rank strictly less than α then $\operatorname{rank}(\mathbf{M}) = \alpha$.

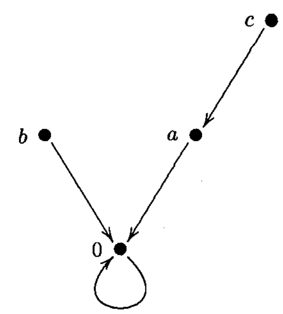

Figure 2.9: An algebra $\mathbf{A}$ that has rank 2

Example 2.4. Let $\mathbf{A} = \langle\{0, a, b, c\}, f\rangle$ where $f(a) = f(b) = f(0) = 0$ and $f(c) = a$. See Figure 2.9. This example, found by Ross Willard, was the first known algebra to have rank 2.

2.3 Definitions from Topology

Next we need some basic definitions from topology. These are either motivated by, or come directly from, [12] and [2]. As with the algebra definitions, the scope is often limited to what is needed for this thesis.

Given a set X, a ***topology*** on X is a collection, $\mathcal{T}$, of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{T}$.

2. Given any subcollection of $\mathcal{T}$ the union of its elements is also in $\mathcal{T}$.

3. Given a finite subcollection of $\mathcal{T}$ the intersection of its elements is also in $\mathcal{T}$.

The pair $\langle X; \mathcal{T}\rangle$ is a ***topological space***, often referred to simply as the space X. Operations, or functions, that map one topological space to another can be labelled as injective, surjective and bijective similar to algebra homomorphisms. We will also consider ***partial operations***, that is, functions $h : X \to Y$ where $\mathrm{dom}(h)$ is a proper subset of X. Relations on the underlying set of a topological space are defined similarly to those on the universe of an algebra. Let G, H, R be fixed sets of operations, partial operations and relations on the set X, respectively. Then $\mathbb{X} = \langle X; G, H, R, \mathcal{T} \rangle$ is a ***structured topological space of type*** $\langle G, H, R \rangle$. The structure $\mathbb{X}$ is a ***total structure*** if it contains no partial operations, that is if $\mathbb{X} = \langle X; G, R, \mathcal{T} \rangle$.

For Y a subset of X we say Y is ***open*** if $Y \in \mathcal{T}$, we say Y is closed if $X \setminus Y \in \mathcal{T}$ and Y is ***clopen*** if it is both closed and open. The ***discrete topology*** on a set X is the collection of all subsets of X. All sets in this topology are clopen.

A ***basis*** for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$ there is at least one $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

The ***topology generated by a basis*** $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$. A ***subbasis*** for a topology on a set X is a collection of subsets of X whose union equals X. The ***topology generated by a subbasis*** is the collection of all unions of finite intersections of elements of the subbasis.

Let $X = \langle X; \mathcal{T}_X \rangle$ be a topological space and $Y \subset X$. The ***subspace topology induced on*** Y ***by*** $\mathcal{T}_X$ is the set $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}$. So $Y = \langle Y; \mathcal{T}_Y \rangle$ is a topological ***subspace*** of X. If $\mathbb{X} = \left\langle X; G^X, H^X, R^X, \mathcal{T}_X \right\rangle$ and $\mathbb{Y} = \left\langle Y; G^Y, H^Y, R^Y, \mathcal{T}_Y \right\rangle$ are structured topological spaces of the same type then $\mathbb{Y}$ is a ***substructure*** of $\mathbb{X}$

if Y is a subset of X and

1. If g is an n-ary operation then for all $y_1, y_2 \ldots, y_n \in Y$ we have

$$g^X(y_1, y_2, \ldots, y_n) = g^Y(y_1, y_2, \ldots, y_n),$$

2. If h is an n-ary partial operation then $\mathrm{dom}(h^Y) = \mathrm{dom}(h^X) \cap Y^n$ and for each $(y_1, y_2, \ldots, y_n) \in \mathrm{dom}(h^Y)$ we have

$$h^X(y_1, y_2, \ldots, y_n) = h^Y(y_1, y_2, \ldots, y_n),$$

3. If r is an n-ary relation then $r^Y = r^X \cap Y^n$, and

4. $\mathcal{T}_Y$ is the subspace topology induced on Y by $\mathcal{T}_X$.

Let S be a set and $\{X_\alpha\}_{\alpha \in S}$ be an indexed family of topological spaces then $\prod_{\alpha \in S} X_\alpha$ is a topological space whose topology is the ***product topology***. The basis for the product topology on $\prod_{\alpha \in S} X_\alpha$ is all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ except for finitely many values of α. If $X_\alpha = X_\beta$ for all $\alpha, \beta \in S$ then we define $X^S = \prod_{\alpha \in S} X_\alpha$.

If X and Y are both topological spaces then $f : X \to Y$ is a ***continuous function*** if for all open subsets $V \in Y$ the set $f^{-1}(V)$ is an open subset of X. If f is a bijection and both f and f^{-1} are continuous then f is a ***homeomorphism***. If $\mathbb{X}$ and $\mathbb{Y}$ are topological structures of the same type then $f : \mathbb{X} \to \mathbb{Y}$ is a ***morphism*** if f is continuous and

1. if $g \in G$ is an n-ary operation then for all $x_1, x_2 \ldots, x_n \in X$ we have

$$f(g^X(x_1, x_2, \ldots, x_n)) = g^Y(f(x_1), f(x_2), \ldots, f(x_n));$$

2. if h is an n-ary partial operation then for all $(x_1, x_2, \ldots, x_n) \in \mathrm{dom}(h^X)$ we have $(f(x_1), f(x_2), \ldots, f(x_n)) \in \mathrm{dom}(h^Y)$ and $f(h^X(x_1, x_2, \ldots, x_n)) = h^Y(f(x_1), f(x_2), \ldots, f(x_n))$; and

3. if R is an n-ary relation then for all $(x_1, x_2, \ldots, x_n) \in r^X$ we have

$$(f(x_1), f(x_2), \ldots, f(x_n)) \in r^Y.$$

The set of all morphisms from $\mathbb{X}$ to $\mathbb{Y}$ is denoted Hom $(\mathbb{X}, \mathbb{Y})$. A morphism $f : \mathbb{X} \to \mathbb{Y}$ is an ***isomorphism*** if there exists a morphism $g : \mathbb{Y} \to \mathbb{X}$ such that for all $x \in X$ we have $(g \circ f)(x) = x$ and similarly for all $y \in Y$ we have $(f \circ g)(y) = y$. If $f : \mathbb{X} \to \mathbb{Y}$ is a morphism, $f(\mathbb{X})$ is a substructure of $\mathbb{Y}$ and $f : \mathbb{X} \to f(\mathbb{X})$ is an isomorphism we call f an ***embedding***.

Let $\mathbb{M}$ be a topological structure and $\mathcal{K}$ be a class of topological structures of the same type. Then $\mathbb{P}(\mathbb{M})$ is the class of all topological structures that are isomorphic to a power of $\mathbb{M}$, $\mathbb{S}_c(\mathcal{K})$ is the class of all topological structures that are isomorphic to a closed substructure of a member of $\mathcal{K}$, and $\mathbb{I}(\mathcal{K})$ is the class of all topological structures that are isomorphic to a member of $\mathcal{K}$. So $\mathfrak{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}(\mathbb{M})$ is the class of all topological substructures of the same type as $\mathbb{M}$ which are isomorphic and homeomorphic to a closed substructure of a power of $\mathbb{M}$.

2.4 Definitions from Natural Duality Theory

The definitions given in this section are specific to natural duality theory and are given a full treatment in both [2] and [4]. Let $\mathbf{M}$ be any non-trivial finite algebra. An ***alter ego*** of $\mathbf{M}$ is any structured topological space $\mathbb{M} = \langle M; G, H, R, \mathcal{T} \rangle$ where

1. G is a set of total operations on M such that if $g \in G$ is nullary then $\{g\}$ is a subalgebra of $\mathbf{M}$ and if $g \in G$ is n-ary for $n \geq 1$ then $g : \mathbf{M}^n \to \mathbf{M}$ is a homomorphism.

2. H is a set of partial operations on M of arity at least 1 such that if $h \in H$ is n-ary then the domain, $\mathrm{dom}(h)$, of h is a non-empty subalgebra of $\mathbf{M}^n$ and $h : \mathrm{dom}(h) \to \mathbf{M}$ is a homomorphism.

3. R is a set of finitary relations on M of arity at least 1 such that if $r \in R$ is n-ary then r is a subalgebra of $\mathbf{M}^n$.

4. $\mathcal{T}$ is the discrete topology.

For the remainder of this thesis $\mathcal{A}$ is defined to be $\mathbb{ISP}(\mathbf{M})$, the quasi-variety generated by $\mathbf{M}$. For any $\mathbf{A} \in \mathcal{A}$ define the ***dual*** of $\mathbf{A}$ to be $D(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathbf{M})$ seen as a substructure of $\mathbb{M}^{\mathbf{A}}$. Then for any $X \in \mathcal{X}$ the ***dual*** of X is $E(X) = \mathrm{Hom}(\mathbb{X}, \mathbb{M})$, seen as a subalgebra of $\mathbf{M}^X$.

For each $\mathbf{A} \in \mathcal{A}$ there is a natural embedding $e_{\mathbf{A}} : \mathbf{A} \to E(D(\mathbf{A}))$ defined for all $a \in A$ and for all $x \in D(\mathbf{A})$ by $e_{\mathbf{A}}(a)(x) = x(a)$. There is a similar natural embedding for all $\mathbb{X} \in \mathcal{X}$. The map $\varepsilon_{\mathbb{X}} : \mathbb{X} \to D(E(\mathbb{X}))$ is defined for all $x \in \mathbb{X}$

and for all $a \in E(\mathbb{X})$ by $\varepsilon_{\mathbb{X}}(x)(a) = a(x)$. These natural embeddings are called ***evaluation maps***.

We say that $\mathbb{M}$ ***yields a duality on*** $\mathcal{A}$ if for every $\mathbf{A} \in \mathcal{A}$ the evaluation map $e_{\mathbf{A}}$ is an isomorphism. We say that $\mathbb{M}$ ***yields a full duality on*** $\mathcal{A}$ if the *additional* condition that for all $\mathbb{X} \in \mathcal{X}$ the evaluation map $\varepsilon_{\mathbb{X}}$ is *also* an isomorphism is met. If $\mathbb{M}$ yields a (full) duality on $\mathcal{A}$ we may alternately say that $\mathbb{M}$ ***(fully) dualizes*** $\mathbf{M}$. An algebra $\mathbf{M}$ is ***dualizable*** if there is some alter ego $\mathbb{M}$ that dualizes $\mathbf{M}$. Not surprisingly an algebra $\mathbf{M}$ is ***fully dualizable*** if there is some alter ego $\mathbb{M}$ that fully dualizes $\mathbf{M}$.

The Full Duality Theorem (see [2] or [4]) says, in part, that if $\mathbb{M}$ yields a full duality on $\mathcal{A}$ then every closed substructure of a non-zero power of $\mathbb{M}$ *is isomorphic to* a term-closed substructure of a power of $\mathbb{M}$. If in fact every closed substructure of a non-zero power of $\mathbb{M}$ *is* a term-closed substructure of a power of $\mathbb{M}$ we say that $\mathbb{M}$ ***yields a strong duality on*** $\mathcal{A}$, or that $\mathbb{M}$ ***strongly dualizes*** $\mathbf{M}$. An algebra $\mathbf{M}$ is ***strongly dualizable*** if there is some alter ego $\mathbb{M}$ that strongly dualizes $\mathbf{M}$. Clearly every strong duality is a full duality.

Example 2.5. Although bounded distributive lattices are not directly relevant to this thesis they provide the example that, in many ways, started study in the area of natural duality theory.

Let $\mathbf{M} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$ be the two element bounded distributed lattice. Then $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ is the class of all bounded distributed lattices. If $\mathbb{M} = \langle \{0, 1\}; \leq, \mathcal{T} \rangle$ then $\mathbb{M}$ yields a duality on $\mathcal{A}$. This duality was first found by Hilary Priestley and was one of the first non-trivial dualities ever found. In fact it can be shown that $\mathbb{M}$ strongly dualizes $\mathbf{M}$. See [4] and [14].

We say that the ***interpolation condition*** or (IC) holds if for each $n \in \mathbb{N}$ and each substructure $\mathbb{X}$ of $\mathbb{M}^n$, every morphism $\alpha : \mathbb{X} \to \mathbb{M}$ extends to a term function $\tau : M^n \to M$ of the algebra $\mathbf{M}$.

The next theorem is a portion of the Second Duality Theorem (see [2], Theorem 2.7) and is useful for showing that an algebra is dualizable.

Theorem 2.1. *Assume that the alter ego* $\mathbb{M} = \langle M; G, R, \mathcal{T} \rangle$ *is a total structure with* R *finite. If* (IC) *holds then* $\mathbb{M}$ *yields a duality on* $\mathcal{A}$.

2.5 Work Leading to this Thesis

Which finite algebras are dualizable? This is called the ***dualizability problem***. What is the relationship between dualizability, full dualizability and strong dualizability? These are the main questions of natural duality theory as of the publication of [2] in 1998. Three years later these are still the main questions although some advances have been made on both fronts.

Strong dualizability implies full dualizability which requires dualizability. (See [2]). Also, if an algebra has finite rank and is dualizable, then it is strongly dualizable [15]. Two questions naturally arise from these statements.

1. Are all dualizable algebras also fully dualizable?

2. Are all fully dualizable algebras also strongly dualizable?

If the answer to these questions is no then not only are we interested in which algebras can be dualized but also in what type of dualizability the algebra possesses.

In [9] Hyndman and Willard provide an example of a bi-unary three-element algebra that is dualizable but not fully dualizable, thus answering no to Question 1. Although rank does not appear explicitly in that paper they were able to use rank to determine that their algebra was a good candidate for further study. The fact that it had infinite rank meant that it might not be strongly dualizable. If their algebra had been shown to be fully dualizable then they would have known it was a good candidate for getting a no answer to question 2. This is a fine example of how determining the rank of an algebra before proceeding with research into its dualizability can be quite useful. This will certainly guide the direction the research takes and may prove the strong dualizability of the algebra being studied.

Little's work [10] provides a tool for approximating the rank of a function. When he ran his program on algebras of known rank one of three things happened: the program correctly approximated the rank of a function; it failed to make an approximation due to memory issues; or it failed due to what he called separation issues. The last case occurred when the algebra being tested was that of [9] which is known to have infinite rank. When Little ran his program on the corresponding four-element algebra he got a similar failure due to separation issues, which suggested that this algebra might also have infinite rank. This thesis shows that not only was his prediction correct, but that a whole class of such algebras have infinite rank.

The rank function was used again by Hyndman in [6] to show that mono-unary algebras are strongly dualizable. In [3], Clark, Davey and Pitkethly looked at all three-element unary algebras and solved the dualizability problem for these algebras. In the companion paper [13] Pitkethly looks at the nature of strong and full duality within the class of three-element unary algebras. She was able to show that for these algebras the answer to Question 2 is yes. In [8] Hyndman and Pitkethly further explore the area of finite rank in relation to three-element unary algebras.

Hyndman and Willard [9] describe a bi-unary three-element algebra that is dualizable but not fully dualizable. As noted above Pitkethly et al then classified three-element unary algebras. We generalize Hyndman and Willard in a different direction, by looking at all bi-unary n-element algebras that are similar to their algebra. We carefully define this family of bi-unary algebras; determine their rank; and study their dualizability.

Chapter 3

Escalator Algebras of Length μ

3.1 Escalator Algebras

In extending the work of Hyndman and Willard in [9] we need to generalize their algebra. To this end we define an ***escalator algebra of length μ*** as an algebra $\mathrm{M} = \langle \{0, 1, \ldots, \mu\}; f, g \rangle$ where $\mu \geq 2$ and f and g are the unary functions

$$f(x) = \begin{cases} x+1 & \text{if } \; x \neq \mu \\ \mu & \text{if } \; x = \mu, \end{cases} \qquad \textit{and} \qquad g(x) = \begin{cases} x-1 & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0. \end{cases}$$

where the symbols $+$ and $-$ are used to represent normal integer addition and subtraction. (See Figure 3.1). Thus $f(x) = \min(x+1, \mu)$ and $g(x) = \max(x-1, 0)$. The algebra of [9] is an escalator algebra of length 2.

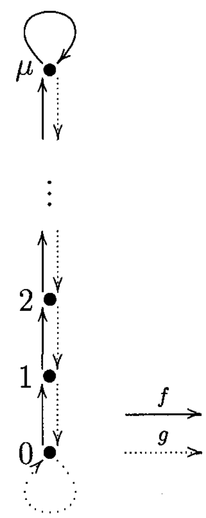

Figure 3.1: An escalator algebra of length μ

3.2 Basic Properties and Definitions

In looking at escalator algebras and the quasi-variety generated by an escalator algebra there are many things that are useful to note or define ahead of time. For the rest of this chapter let $\mathbf{M}$ be an escalator algebra of length μ. If there is a need to specify the value of μ then we write $\mathbf{M}_\mu$. Since the universe M is the set of integers from 0 to μ the normal integer relations $\leq$ and $<$ behave exactly as expected on M and M^n, that is if $x \leq y$ then $f(x) \leq f(y)$ and $g(x) \leq g(y)$.

Let x be an element of $\mathbf{M}$. Then $f^1 = f$, $g^1 = g$, and for $n > 1$ the term operations $f^n = f \circ f^{n-1}$, and $g^n = g \circ g^{n-1}$ satisfy

$$f^n(x) = \begin{cases} x + n & \text{if} \quad x + n \leq \mu \\ \mu & \text{if} \quad x + n \geq \mu, \end{cases} \quad \textit{and} \quad g^n(x) = \begin{cases} x - n & \text{if} \quad x - n \geq 0 \\ 0 & \text{if} \quad x - n \leq 0. \end{cases}$$

That is, $f^n(x) = \min(x+n, \mu)$ and $g^n(x) = \max(x-n, 0)$.

Recall that if $x \in \mathbf{M}^n$ then x is really an n-tuple $x = (x_1, x_2, \ldots, x_n)$. The ***form*** of x is the $(n-1)$-tuple $(x_2-x_1, x_3-x_2, \ldots, x_n-x_{n-1})$. For an algebra $\mathbf{A}$ denote the tuple $(x, \ldots, x) \in \mathbf{A}$ as $c_{\mathbf{A}}(x)$ or $c(x)$ if the context is clear. If $\mathbf{B} \leq \mathbf{M}^n$ the ***centre*** of $\mathbf{B}$ is $C_B = \{c_{\mathbf{B}}(m) | m \in M\}$. If $x \in \mathbf{M}^I$ such that $x \in \{0,1\}^n \setminus \{c_{\mathbf{M}^I}(0), c_{\mathbf{M}^I}(1)\}$ we call x a ***{0,1}-element.***

For any $x \in \mathbf{M}^n$ define $\min(x), \max(x), s_x, t_x \in \mathbb{Z}$ as follows:
$\min(x) = \min(x_1, x_2, \ldots, x_n)$, $\max x = \max(x_1, x_2, \ldots, x_n)$, $s_x = \mu - \min(x)$, and $t_x = \max(x)$.

Lemma 3.1. *If $x \in \mathbf{M}^I$ then s_x is the smallest integer such that $f^{s_x}(x) = c_{\mathbf{M}^I}(\mu)$ and t_x is the smallest integer such that $g^{t_x}(x) = c_{\mathbf{M}^I}(0)$.*

Proof. Let k_1, k_2 be integers such that $f^{k_1}(x) = c_{\mathbf{M}^I}(\mu)$ and $g^{k_2}(x) = c_{\mathbf{M}^I}(0)$. For all i we have $x_i + k_1 \geq \mu$. So $k_1 \geq \mu - \min(x) \geq \mu - x_i$ as $x_i \geq \min(x)$. Hence s_x is the smallest integer such that $f^{s_x}(x) = c_{\mathbf{M}^I}(\mu)$. Similarly for all i we have $x_i - k_2 \leq 0$ and $x_i \leq \max(x)$. So $k_1 \geq \max(x) \geq x_i$. Hence t_x is the smallest integer such that$g^{t_x}(x) = c_{\mathbf{M}^I}(0)$. □

Lemma 3.2. *Let $\mathbf{A} \leq \mathbf{M}^I$ and $h : \mathbf{A} \to \mathbf{M}$ be a homomorphism. For all $c_{M^I}(m)$ in the centre of $\mathbf{A}$, $h(c_{M^I}(m)) = m$.*

Proof. Let $a = h(c_{M^n}(0))$. Therefore $0 = g^\mu(a) = g^\mu(h(c_{M^n}(0))) = h(g^\mu(c_{M^n}(0))) = h(c_{M^n}(0))$ as h is a homomorphism. So $h(c_{M^n}(0)) = 0$. For all $m \in \mathbf{M}$ we have $f^m(0) = m$ so $h(c_{M^n}(m)) = h(f^m(c_{M^n}(0))) = f^m(h(c_{M^n}(0))) = f^m(0) = m$. □

Lemma 3.3. *Let $\mathbf{A} \leq \mathbf{M}^I$ and $h : \mathbf{A} \to \mathbf{M}$ be a homomorphism. For all $x \in A$ such that x is a $\{0,1\}$-element we have $h(x) = 0$ or $h(x) = 1$.*

Proof. If x is a {0,1}-element then $g(x) = c_{M^n}(0)$. So by Lemma 3.2 we have $g(h(x)) = h(g(x)) = h(c_{M^n}(0)) = 0$ which implies that $h(x) = 0$ or $h(x) = 1$. □

If $\mathbf{A} \leq \mathbf{M}^n$ then a homomorphism h from $\mathbf{A}$ into $\mathbf{M}$ is called ***irresponsible*** if there exist {0,1}-elements a and $b \in \mathbf{A}$ such that $a < b$ and $h(a) = 1$ and $h(b) = 0$. Conversely h is ***responsible*** if for all {0,1}-elements a and b with $a < b$ if $h(a) = 1$ then $h(b) = 1$.

Example 3.1. Let $\mu = 3$ and $\mathbf{B}$ be the $\mathrm{Sg}_{\mathbf{M}^3}(001, 011)$. (See Figure 2.4 page 11). For $i \in \{1, 2, 3, 4\}$ let $h_i(001)$ and $h_i(011)$ be given by the following table:

	h_1	h_2	h_3	h_4
001	0	0	1	1
011	0	1	0	1

By inspection of the table it is easy to determine that each h_i extends to a homomorphism $h_i : \mathbf{B} \to \mathbf{M}$ and h_1, h_2, and h_4 are responsible while h_3 is irresponsible.

3.3 Ladders

A ***ladder of*** $\mathbf{M^I}$ is a set $\{x_1, x_2, \ldots, x_r\} \subseteq M^I$ with the following properties:

1. For all $t < r$, $f(x_t) = x_{t+1}$,

2. For all $t < r$, $g(x_{t+1}) = x_t$,

3. If $x_1 \neq c_{\mathbf{M}^I}(0)$, $f(g(x_1)) \neq x_1$, and

4. If $x_r \neq c_{\mathbf{M}^I}(\mu)$, $g(f(x_r)) \neq x_r$.

The element x_1 is the ***ladder foot***, and x_r is the ***ladder head***. Notice that for all ladder elements x_i except the foot and the head we have $f(g(x_i)) = g(f(x_i))$. The conditions on the foot and the head elements mean precisely that x_1 must have at least one coordinate equal to 0 and that x_r must have at least one coordinate equal to μ (see Lemma 3.4). Moreover, any ladder element with an occurance of μ must be the head of its ladder, and it is this property that gives uniqueness of a ladder with foot x. It is sometimes useful to use L_x to denote the ladder that has x as the foot element. If y is the head of L_x then there is an integer k such that $f^k(x) = y$ and $g^k(y) = x$. We say the ***length of the ladder*** is k. Note that $\mathbf{M}$ itself is a ladder of length μ with 0 as its foot.

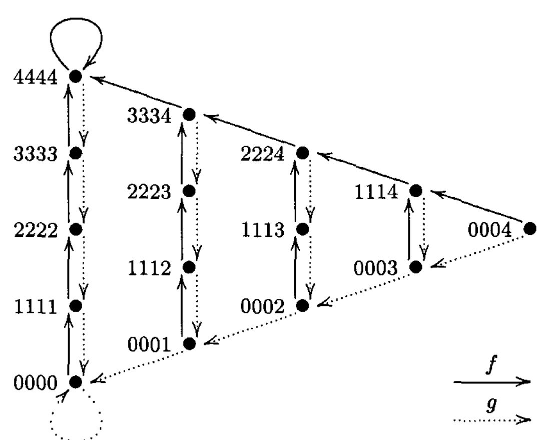

Figure 3.2: A subalgebra of $\mathbf{M}_4{}^4$:

Recall that the form of x is $(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$ which is an $(n-1)$-tuple. All elements of a ladder have the same form. The behaviour of a homomorphism on a ladder is completely determined by the behaviour of the head (or foot) element. This is illustrated in Lemma 3.6

Example 3.2. In Figure 3.2 There are 5 ladders. The feet of the ladders are 0000, 0001, 0002, 0003 and 0004 and the heads of the ladders are 4444, 3334, 2224, 1114 and 0004 respectively. The form of 0001 and 3334 is 001.

Lemma 3.4. *Let $x \in \mathbf{M}^I$ be an element of a ladder. Then $x_i = 0$ for some i if and only if x is the foot of the ladder, and $x_j = \mu$ for some j if and only if x is the head of the ladder.*

Proof. If $x_i = 0$ then $fg(x_i) = fg(0) = f(0) = 1 \neq x_i$ so $fg(x) \neq x$ and therefore x is the foot of the ladder. Now assume x is the foot of the ladder. If $x = c_{\mathbf{M}^I}(0)$ we are done so we assume $x \neq c_{\mathbf{M}^I}(0)$. By property (3) $f(g(x)) \neq x$ so $(fg(x_1), fg(x_2), \ldots, fg(x_n)) \neq (x_1, x_2, \ldots, x_n)$ which implies that for some i we have $fg(x_i) \neq x_i$. As $\mathbf{M}$ is a ladder the only element that satisfies this is 0.

If $x_j = \mu$ then $gf(x_j) = gf(\mu) = g(\mu) = \mu - 1 \neq x_j$ so $gf(x) \neq x$ and therefore x is the head of the ladder. Now assume that x is the head of the ladder. If $x = c_{\mathbf{M}^I}(\mu)$ we are done so assume $x \neq c_{\mathbf{M}^I}(\mu)$. Then $g(f(x)) = (gf(x_1), gf(x_2), \ldots, gf(x_n)) \neq (x_1, x_2, \ldots, x_n)$ which implies that for some i we have $gf(x_i) \neq x_i$. Again $\mathbf{M}$ is a ladder so $x_i = \mu$. □

Lemma 3.5. *Let x be the foot of a ladder. Then the ladder L_x has length $\mu - t_x$ and the head of the ladder is $f^{\mu - t_x}(x)$.*

Proof. Recall that $t_x = \max(x)$. There is some $j \leq n$ such that $x_j = t_x$, so $f^{\mu-t_x}(x_j) = f^{\mu-t_x}(t_x) = \mu$ and $f^{\mu-t_x-1}(x_j) = f^{\mu-t_x-1}(t_x) = \mu - 1$. This means that $\mu - t_x$ is the smallest integer k such that $f^k(x)$ has μ as a coordinate. By Lemma 3.4 and the definition of a ladder $f^{\mu-t_x}(x)$ is the head of the ladder L_x and the length of the L_x is $\mu - t_x$. □

Lemma 3.6. *Let $x \in \mathbf{A} \leq \mathbf{M}^I$ and $h : \mathbf{A} \to \mathbf{M}$ be a homomorphism. Assume x is the foot of a ladder L_x. For all y in L_x there exists $j \leq \mu$ such that $y = f^j(x)$ and if $h(x) = \pi_i(x)$ then $h(y) = \pi_i(y)$.*

Proof. Let $L_x = \{b_1, \ldots, b_r\}$ and assume $y \in L_x$ and $h(x) = x_i$. So $y \in \{b_1, b_2, \ldots, b_r\}$ and by property (1) we may assume $f(b_1) = b_2$, $f^2(b_1) = f(b_2) = b_3$, $\ldots$, $f^{r-1}(b_1) = f(b_{r-1}) = b_r$. If $y = b_i$ let $j = i-1$. The cardinality of any ladder is at most $\mu+1$ so $j \leq \mu$ and $f^j(x) = f^j(b_1) = y$. From this we see that $h(y) = h(f^j(x)) = f^j(h(x)) = f^j(x_i) = y_i = \pi_i(y)$. □

For $1 \leq j < n$ define the {0,1}-element v^j of $\mathbf{M}^n$ by $v^j = (v^j_1, v^j_2, \ldots, v^j_n)$ where $v^j_1 = v^j_2 = \ldots = v^j_j = 0$ and $v^j_{j+1} = \ldots = v^j_n = 1$. It is important to note that if $i < j$ then $v^j < v^i$. So $v^n < v^{n-1} < \ldots < v^2 < v^1$. Define $u^j = f^{\mu-1}(v^j)$. For future reference note that

$$v^j_i = \begin{cases} 0 & \text{if } \; i \leq j \\ 1 & \text{if } \; i > j, \end{cases} \qquad \text{and} \qquad u^j_i = \begin{cases} \mu - 1 & \text{if } \; i \leq j \\ \mu & \text{if } \; i > j. \end{cases}$$

Clearly each v^j is a foot element of a ladder that has u^j as the head element. In the case that $v^j \in \mathbf{M}^n$ and $v^{j'} \in \mathbf{M}^{n'}$ we indicate the difference with the notation $v^{j,n}$ and $v^{j',n'}$.

If $a, b \in \mathbf{M}^n$ are {0,1}-elements and $a < b$, define the w-element $w(a,b) = (w_1, w_2, \ldots, w_n)$ as follows: $w_i = \begin{cases} 0 & \text{if } \; a_i = b_i = 0 \\ 1 & \text{if } \; a_i \neq b_i \\ 2 & \text{if } \; a_i = b_i = 1 \end{cases}$

Or more concisely $w_i = a_i + b_i$. The w-elements are the foot elements of ladders that connect the ladders of a and b. This can be seen in Lemma 3.7. We can think of a and b as being connected by $w(a,b)$. (See Figure 3.3).

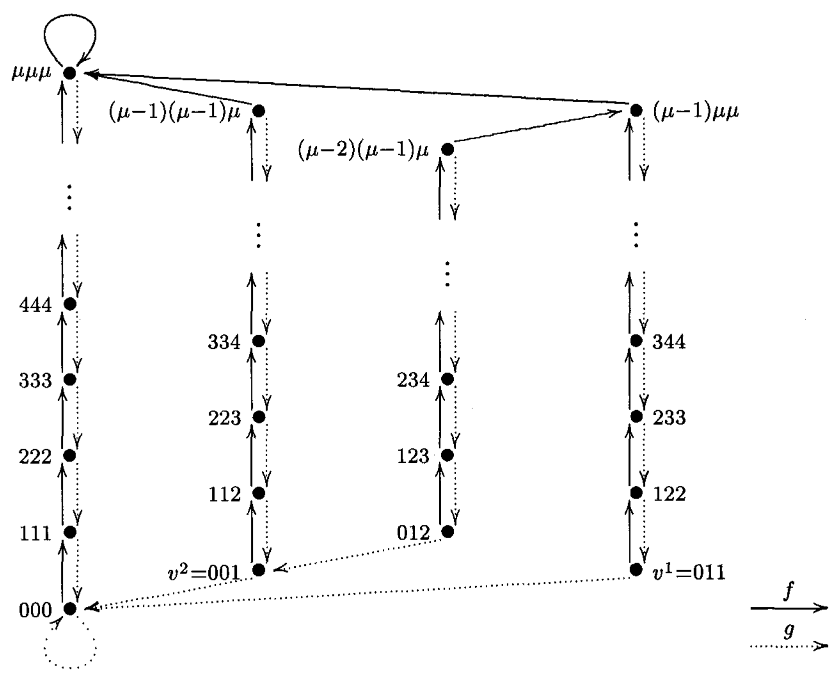

Figure 3.3: A subalgebra of $\mathbf{M}^3$ that has a w-element, $w(v^2, v^1)$

Lemma 3.7. *If a, b are $\{0,1\}$-elements such that $a < b$ and $a, b, w(a,b)$ are in $\mathbf{A} \leq \mathbf{M}^I$ then $L_{w(a,b)}$ has length $\mu - 2$ and $g(w(a,b)) = a$ and $f^{\mu-1}(w(a,b)) = f^{\mu-1}(b)$.*

Proof. Since a, b are $\{0,1\}$-elements and $a < b$ there is some i such that $a_i = b_i = 1$ and $w_i = 2$. Because $\max(a) = \max(b) = 1$, we have $\max(w(a,b)) = 2$. So length $L_{w(a,b)} = \mu - t_{w(a,b)} = \mu - 2$. If $w_i = 0$ or $w_i = 1$ then $a_i = 0$ and $b_i = w_i$, so $g(w_i) = 0 = a_i$ and $f^{\mu-1}(w_i) = f^{\mu-1}(b_i)$. If $w_i = 2$ then $a_i = b_i = 1$ and $g(w_i) = g(2) = 1 = a_i$ and $f^{\mu-1}(w_i) = f^{\mu-1}(2) = \mu = f^{\mu-1}(1) = f^{\mu-1}(b_i)$. So

$g(w(a,b)) = a$ and $f^{\mu-1}(w(a,b)) = f^{\mu-1}(b)$. □

Lemmas 3.8, 3.9, and 3.10 illustrate how homomorphisms behave on w-elements.

Lemma 3.8. *Let a, b be {0,1}-elements such that $a < b$ and $a, b, w(a,b)$ are in $\mathbf{A} \leq \mathbf{M}^I$. If $h : \mathbf{A} \to \mathbf{M}$ is a homomorphism and $h(a) = 1$ then $h(b) = 1$.*

Proof. Assume $h(a) = 1$ then $g(h(w(a,b))) = h(g(w(a,b))) = h(a) = 1$ and so $h(w(a,b)) = 2$. Applying $f^{\mu-1}$ we get $f^{\mu-1}(h(w(a,b))) = f^{\mu-1}(2)$ or we get $h(f^{\mu-1}(w(a,b))) = \mu$. By Lemma 3.7 $f^{\mu-1}(w(a,b)) = f^{\mu-1}(b)$ and it follows that $h(f^{\mu-1}(b)) = \mu$. As the element b is a {0,1}-element, it is the foot of a ladder of length $\mu - 1$ and $g^{\mu-1}(f^{\mu-1}(b)) = b$. Apply h to obtain $h(b) = h(g^{\mu-1}f^{\mu-1}(b)) = g^{\mu-1}(h(f^{\mu-1}(b))) = g^{\mu-1}(\mu) = 1$. Thus $h(b) = 1$. □

Lemma 3.9. *Given {0,1}-elements a, b such that $a < b$ and $a, b, w(a,b)$ are in $\mathbf{A} \leq \mathbf{M}^I$ and $m_1, m_2 \in \{0,1\}$, $m_1 \leq m_2$ and $h : \mathbf{A} \to \mathbf{M}$ a homomorphism, then $h(a) = m_1$ and $h(b) = m_2$ if and only if $h(w(a,b)) = m_1 + m_2$.*

Proof. Assume $h(a) = m_1$ and $h(b) = m_2$. Then by Lemmas 3.3 and 3.8 there are three cases.

First if $h(a) = h(b) = 0$ then $h(f^{\mu-1}(w(a,b))) = h(f^{\mu-1}(b)) = f^{\mu-1}(h(b)) = f^{\mu-1}(0) = \mu - 1$. A ladder of length $\mu - 1$ has its head mapped to $\mu - 1$ only when its foot, in this case $w(a,b)$, is mapped to 0. That is $h(w(a,b)) = 0 = m_1 + m_2$.

Next if $h(a) = h(b) = 1$ then by Lemma 3.7 $g(h(w(a,b)) = h(g(w(a,b))) = h(a) = 1$. So $h(w(a,b)) = 2 = m_1 + m_2$.

Finally if $h(a) = 0$ and $h(b) = 1$ then again by Lemma 3.7, we have

$$h(f^{\mu-1}(w(a,b))) = h(f^{\mu-1}(b)) = f^{\mu-1}(h(b)) = f^{\mu-1}(1) = \mu$$

so $h(w(a,b)) \geq 1$. Since

$$g(h(w(a,b))) = h(g(w(a,b))) = h(a) = 0$$

we have $h(w(a,b)) \leq 1$. So $h(w(a,b)) = 1 = m_1 + m_2$. So for all possible cases we have $h(w(a,b)) = m_1 + m_2$.

Now assume $h(w(a,b)) = m_1 + m_2$. By Lemmas 3.3 and 3.7 there are three cases. Since $f^{\mu-1}(b) = f^{\mu-1}(w(a,b))$ we have

$$\begin{aligned}
h(b) &= h(g^{\mu-1} f^{\mu-1}(w(a,b))) \\
&= g^{\mu-1} f^{\mu-1}(h(w(a,b))) = g^{\mu-1} f^{\mu-1}(m_1 + m_2) \\
&= \begin{cases} g^{\mu-1} f^{\mu-1}(0) & \text{if} \quad m_1 = m_2 = 0, \\ g^{\mu-1} f^{\mu-1}(1) & \text{if} \quad m_1 = 0, m_2 = 1, \\ g^{\mu-1} f^{\mu-1}(2) & \text{if} \quad m_1 = m_2 = 1, \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad m_1 = m_2 = 0, \\ 1 & \text{if} \quad m_1 = 0, m_2 = 1, \\ 1 & \text{if} \quad m_1 = m_2 = 1, \end{cases} \\
&= m_2.
\end{aligned}$$

Thus $h(b) = m_2$ as required. To see that $h(a) = m_1$ is simpler:

$$\begin{aligned} h(a) &= h(g(w(a,b))) \\ &= g(h(w(a,b))) \\ &= g(m_1 + m_2) \\ &= \begin{cases} g(0) & \text{if} \quad m_1 = m_2 = 0, \\ g(0) & \text{if} \quad m_1 = 0, m_2 = 1, \\ g(1) & \text{if} \quad m_1 = m_2 = 1, \end{cases} \\ &= m_1. \end{aligned}$$

□

Corollary 3.1. *If a, b be are {0,1}-elements such that $a < b$ and $a, b, w(a,b)$ are in $\mathbf{A} \le \mathbf{M}^I$ and if $h : \mathbf{A} \to \mathbf{M}$ is a homomorphism then $h(a) = a_i$ and $h(b) = b_i$ if and only if $h(w(a,b)) = w_i$.*

Proof. Recall that for $w(a,b)$ the definition of w-element says that $w_i = a_i + b_i$. Assuming that $h(a) = a_i$ and $h(b) = b_i$ then by Lemma 3.9 we have $h(w(a,b)) = a_i + b_i = w_i$. Conversely if $h(w(a,b)) = w_i = a_i + b_i$ then $h(a) = a_i$ and $h(b) = b_i$. □

Lemma 3.10. *If a, b are {0,1}-elements such that $a < b$ and $a, b, w(a,b)$ are in $\mathbf{A} \le \mathbf{M}^I$ and if $h : \mathbf{A} \to \mathbf{M}$ is a homomorphism then there exists $i \le n$ such that $h(a) = a_i$, $h(b) = (b_i)$ and $h(w(a,b)) = w_i$.*

Proof. As $a < b$ there exist $j, j', \check{j}$ such that $a_j = b_j = 0$, $a_{j'} = 0$, $b_{j'} = 1$ and $a_{\check{j}} = b_{\check{j}} = 1$. By Lemmas 3.3 and 3.8 there are only three cases. If $h(a) = h(b) = 0$

then let $i = j$. If $h(a) = 0$, and $h(b) = 1$ then let $i = j'$. If $h(a) = h(b) = 1$ then let $i = \check{j}$. Thus $h(a) = a_i$ and $h(b) = b_i$ and by Corollary 3.1 $h(w(a,b)) = w_i$. $\square$

Lemmas 3.11 and 3.12 state that certain ladders can be added to a subalgebra of $\mathbf{M}^I$ to obtain a new subalgebra. Then Lemma 3.13 says that a homomorphism is determined completely by its behaviour on the foot elements.

Lemma 3.11. *If $\mathbf{A} \leq \mathbf{M}^I$ then $\mathbf{A}' = \langle A \cup L_{v^j}, \{f,g\}\rangle$ is a subalgebra of $\mathbf{M}^I$.*

Proof. Pick $a \in A'$. If $a \in A$ then $f(a), g(a) \in A \subset A'$ as $\mathbf{A}$ is a subalgebra. If $a \in L_{v^j}$ and $a \neq v^j$ then $g(a) \in L_{v^j} \subset A'$ and if $a = v^j$ then $g(a) = c_{\mathbf{A}}(0) \in A \subset A'$. Similarly if $a \in L_{v^j}$ and $a \neq u^j$ then $f(a) \in L_{v^j} \subset A'$ and if $a = u^j$ then $f(a) = c_{\mathbf{A}}(\mu) \in A \subset A'$. So for all $a \in A', f(a), g(a) \in A'$ and $\mathbf{A}'$ is a subalgebra of $\mathbf{M}^I$. $\square$

Lemma 3.12. *If $\mathbf{A} \leq \mathbf{M}^I$ and $a, b \in A$ are $\{0,1\}$-elements with $a < b$ then $\mathbf{A}' = \langle A \cup L_{w(a,b)}, \{f,g\}\rangle$ is a subalgebra of $\mathbf{M}^I$.*

Proof. Let $A' = A \cup L_{w(a,b)}$ and pick $a' \in A$. If $a' \in A$ then $f(a'), g(a') \in A \subseteq A'$ as $\mathbf{A}$ is a subalgebra. If $a' \in L_{w(a,b)}$ and $\mu = 2$ then $a' = w(a,b)$ and so $f(a') = f(b) \in A \subseteq A'$ and $g(a') = a \in A \subset A'$. Otherwise if $\mu > 2$ and $a' \in L_{w(a,b)}$ then there are three cases. First if $a' = w(a,b)$ then $g(a') = a \in A \subseteq A'$ and $f(a') \in L_{w(a,b)} \subset A'$. Next if $a' = f^{\mu-2}(w(a,b))$ then $f(a') = f^{\mu-1}(b) \in A \subseteq A'$ and $g(a') \in L_{w(a,b)} \subset A'$. Finally if $a' \neq w(a,b)$ and $a' \neq f^{\mu-2}(w(a,b))$ then $f(a') \in L_{w(a,b)} \subset A'$ and $g(a') \in L_{w(a,b)} \subset A'$. So for all $a' \in A'$, both $f(a'), g(a')$ are in A' and $\mathbf{A}'$ is thus a subalgebra of $\mathbf{M}^I$. $\square$

Lemma 3.13. *For $\mathbf{A} \leq \mathbf{M}^I$ let $S \subseteq M^I$ such that $A = \bigcup\{L_x | x \in S\}$. If the map $h : S \to M$ extends to a homomorphism $h^* : \mathbf{A} \to \mathbf{M}$ then h^* is unique.*

Proof. Assume $h : S \to M$ and $h^* : \mathbf{A} \to \mathbf{M}$ is a homomorphism that agrees with h on S. Pick y in A. Then $y = x$ or $y = f^k(x)$ for some x in S. If $y = x$ then $h^*(x) = h(x)$. If $y = f^k(x)$ then $h^*(y) = h^*(f^k(x)) = f^k(h^*(x))$ as h^* is a homomorphism and so $h^*(y) = f^k(h(x))$. Thus

$$h^* = \begin{cases} h(x) & \text{if} \quad x \in S \\ f^k(h(x)) & \text{if} \quad y = f^k(x) \quad \text{for some } x \in S \end{cases}$$

is the unique extension of h to $\mathbf{A}$. □

Uniqueness follows directly from Theorem 6.2 of [1] but the above explicitly defines h^* for later use.

3.4 The w-Algebras.

Recall the definition of v^j given on page 29. For any $n \geq 3$ define T_n as follows:

$$T_n = C_{M^n} \cup L_{v^{n-1}} \cup L_{v^{n-2}} \cup \ldots \cup L_{v^1}.$$

By Lemma 3.11 $\mathbf{T}_n = \langle T_n, \{f, g\} \rangle$ is a subalgebra of $\mathbf{M}^n$. A ***w*-algebra** is a subalgebra of $\mathbf{M}^n$ that has as its universe the union of the set T_n with some subset, possibly empty, of ladders, $L_{w(a,b)}$, where a, b are $\{0,1\}$-elements in $\mathbf{T}_n$ with $a < b$. Lemma 3.12 shows this definition is well-defined.

Lemma 3.14. *For any w-algebra $\mathbf{W}$ if $h : \mathbf{W} \to \mathbf{M}$ is a responsible homomorphism then h is a projection.*

Proof. Recall that $c(0) < v^{n-1} < \cdots < v^1 < c(1)$. First we need to show that there exists an $i \leq n$ such that $h(v^j) = \pi_i(v^j)$ for all $j < n$. If h is responsible then there are three cases.

Case 1 : Assume $h(v^j) = 0$ for $1 \leq j < n$. Since $\pi_1(v^j) = 0$ we have $h(v^j) = \pi_1(v^j)$.

Case 2 : Assume $h(v^j) = 1$ for $1 \leq j < n$. Again $\pi_n(v^j) = 1$ so $h(v^j) = \pi_n(v^j)$.

Case 3 : There is some $1 < i \leq n$ such that $h(v^i) = 0$ and $h(v^{i-1}) = 1$. If $j \geq i$ then $h(v^j) = 0$ and if $j < i$ then $h(v^j) = 1$, as h is responsible. By the definition of v^j, if $j \geq i$ then $\pi_i(v^j) = 0$ and if $j < i$ then $\pi_i(v^j) = 1$. So $h(v^j) = \pi_i(v^j)$.

Thus there exists i with $h(v^j) = \pi_i(v^j)$. Now we need to show that h is π_i restricted to $\mathbf{W}$. By Lemma 3.6 we know that if $x \in L_{v^j}$ then $h(x) = \pi_i(x)$. By Lemma 3.1 we have that if $w(a,b) \in \mathbf{W}$, then $h(w(a,b)) = \pi_i(w(a,b))$; so, for all $x \in L_{w(a,b)}$ it follows that $h(x) = \pi_i(x)$. Finally note that $h(x) = \pi_i(x)$ for all $x \in C_W$. So for all $x \in \mathbf{W}$ we have $h(x) = \pi_i(x)$. □

Now we consider algebras that have subsets of a specific structure. Later we construct homomorphisms on some of these algebras that have infinite rank. If $\mathbf{A} \leq \mathbf{M}^n$ where $n \geq 7$ then $\mathbf{A}$ has a ***primary section*** at v^j of size k (denoted $\langle v^j, k\rangle$-section) if $6 \leq j < n$, and there exists an even integer $4 \leq k \leq j-2$ such that $v^j, v^{j-1}, \ldots, v^{j-k-1} \in \mathbf{A}$ and the w-elements in $\mathbf{A}$ connecting $v^j, v^{j-1}, \ldots, v^{j-k-1}$ are precisely

$$w(v^j, v^{j-1}), w(v^j, v^{j-3}), \ldots, w(v^j, v^{j-k+1}) \quad \text{and}$$

$$w(v^{j-2}, v^{j-k-1}), w(v^{j-4}, v^{j-k-1}), \ldots, w(v^{j-k}, v^{j-k-1}).$$

The next lemma deals with homomorphisms that are irresponsible on a $\langle v^j, k\rangle$-section.

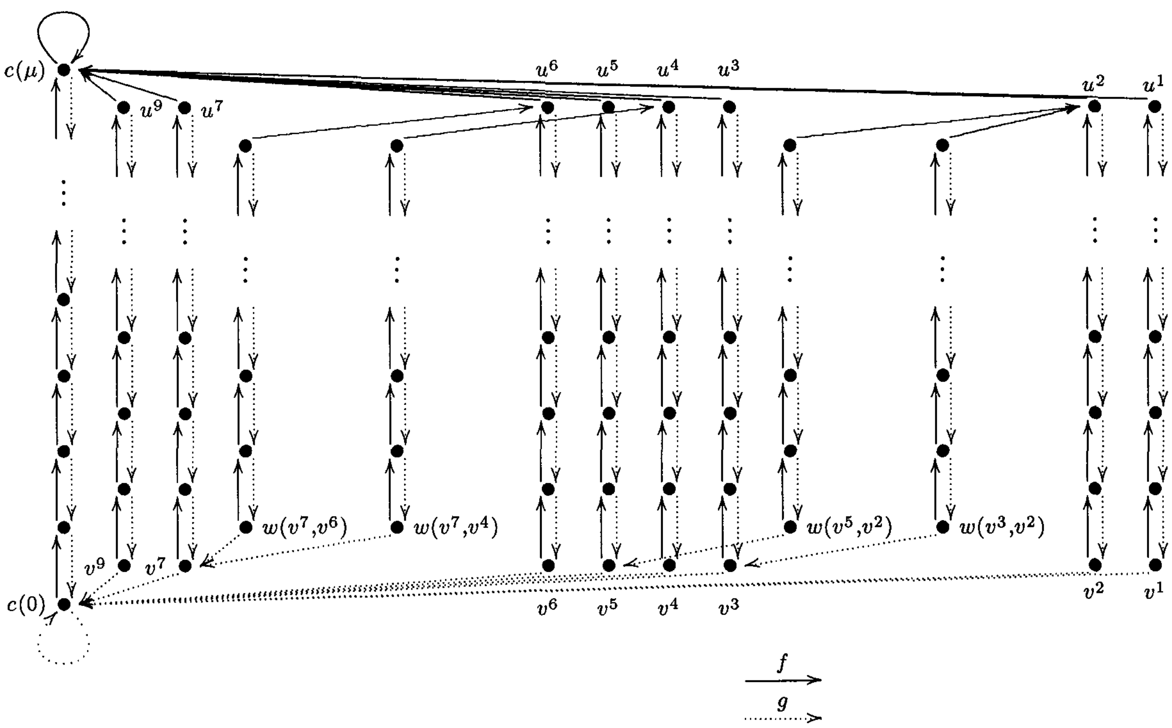

Figure 3.4: A subalgebra of $\mathbf{M}^{10}$ with a $\langle v^7, 4\rangle$-section

Lemma 3.15. *Let* $\mathbf{A}$ *have a* $\langle v^j, k\rangle$*-section and* $h : \mathbf{A} \to \mathbf{M}$ *be a homomorphism such that* $h(v^j) = 1$ *and* $h(v^{j-k-1}) = 0$*. Then for all elements* x *of the primary section,* $h(x)$ *is uniquely determined, and in particular*

$$h(v^{j-1}) = h(v^{j-3}) = \cdots = h(v^{j-k+1}) = 1 \quad \textit{and}$$

$$h(v^{j-2}) = h(v^{j-4}) = \cdots = h(v^{j-k}) = 0.$$

Additionally

$$h(w(v^j, v^{j-1})) = h(w(v^j, v^{j-3})) = \cdots = h(w(v^j, v^{j-k+1})) = 2 \quad \textit{and}$$

$$h(w(v^{j-2}, v^{j-k-1})) = h(w(v^{j-4}, v^{j-k-1})) = \cdots = h(w(v^{j-k}, v^{j-k-1})) = 0.$$

Proof. Let $i \in \{j-1, j-3, \ldots, j-k+1\}$ then, by Lemma 3.7, $g(w(v^j, v^i)) = v^j$ implying that $h(w(v^j, v^i)) = 2$ as $h(v^j) = 1$ is assumed. Lemma 3.7 also says that $f^{\mu-1}(w(v^j, v^i)) = f^{\mu-1}(v^i)$. Hence

$$\begin{aligned} f^{\mu-1}(h(v^i)) &= h(f^{\mu-1}(v^i)) \\ &= h(f^{\mu-1}(w(v^j, v^i))) \\ &= f^{\mu-1}(h(w(v^j, v^i))) \\ &= f^{\mu-1}(2) \\ &= \mu \end{aligned}$$

and so $h(v^i) \geq 1$. By Lemma 3.3 it follows that $h(v^i) = 1$.

Let $i \in \{j-2, j-4, \ldots, j-k\}$ then by using the two parts of Lemma 3.7 again we get

$$\begin{aligned}
f^{\mu-1}(h(w(v^i, v^{j-k-1}))) &= h(f^{\mu-1}(w(v^i, v^{j-k-1}))) \\
&= h(f^{\mu-1}(v^{j-k-1})) \\
&= f^{\mu-1}(h(v^{j-k-1})) \\
&= f^{\mu-1}(0) \\
&= \mu - 1
\end{aligned}$$

which implies that $h(w(v^i, v^{j-k-1})) = 0$. As $g(w(v^i, v^{j-k-1})) = v^i$ we also get $h(v^i) = 0$. $\square$

For $\mathbf{A} \leq \mathbf{M}^n, 1 < i < n$, and $k > 0$ define the repetition of coordinates mapping $\alpha_i^{n,k} : \mathbf{A} \to \mathbf{M}^{n+k}$ by adding k copies of the ith coordinate between the ith coordinate and the $(i+1)$th coordinate. So for $x \in \mathbf{A}$

$$\alpha_i^{n,k}(x) = (x_1, x_2 \ldots, x_i, \underbrace{x_i, \ldots, x_i}_{k \text{ copies}}, x_{i+1}, \ldots, x_n).$$

We now start to construct a commuting diagram as in Figure 2.8. We start with a w-algebra $\mathbf{A}$ in place of $\mathbf{B}$ in the diagram and using the map $\alpha_i^{n,k}$ as σ.

Let $\mathbf{A} \leq \mathbf{M}^n$ be a w-algebra, k an even integer greater than 3 and let $h : \mathbf{A} \to \mathbf{M}$ be a homomorphism. Let $\mathbf{A}' = \alpha_i^{n,k}(\mathbf{A}) \leq \mathbf{M}^{n+k}$. As $\mathbf{A}$ is a w-algebra the {0,1}-elements are precisely

$$v^{1,n}, v^{2,n}, \ldots, v^{i-1,n}, v^{i,n}, v^{i+1,n} \ldots, v^{n-1,n}.$$

The corresponding {0,1}-elements in $\mathbf{A}'$ are

$$v^{1,n+k}, v^{2,n+k}, \ldots, v^{i-1,n+k}, v^{i+k,n+k}, v^{i+k+1,n+k}, \ldots, v^{n+k-1,n+k}.$$

Assume $w(v^{i,n}, v^{i-1,n})$ is not in $\mathbf{A}$ then $w(v^{i+k,n+k}, v^{i-1,n+k})$ is not in $\mathbf{A}'$ and we can define $\mathcal{S}(\mathbf{A}')$ to be the w-algebra created by adding all the necessary elements to form a $\langle v^{i+k}, k\rangle$-section to $\mathbf{A}'$. Specifically the universe of $\mathcal{S}(\mathbf{A}')$ is

$$\begin{aligned} S' = A' &\cup L_{v^{i+k-1}} \cup L_{v^{i+k-2}} \cup \cdots \cup L_{v^i} \\ &\cup L_{w(v^{i+k}, v^{i+k-1})} \cup L_{w(v^{i+k}, v^{i+k-3})} \cdots \cup L_{w(v^{i+k}, v^{i+1})} \\ &\cup L_{w(v^{i+k-2}, v^{i-1})} \cup L_{w(v^{i+k-4}, v^{i-1})} \cdots \cup L_{w(v^{i}, v^{i-1})} \end{aligned}$$

This is possible as none of $v^{i+k-1}, v^{i+k-2}, \ldots, v^i$ are in $\mathbf{A}'$ and so there are no w-elements connected to these {0,1}-elements in $\mathbf{A}'$.

Lemma 3.16. *Let $\mathbf{A}' \leq \mathbf{M}^{n+k}$. Given $h' : \mathbf{A}' \to \mathbf{M}$ an irresponsible homomorphism with $h'(v^{i+k}) = 1$ and $h'(v^{i-1}) = 0$, then h' extends to $h^+ : \mathcal{S}(\mathbf{A}') \to \mathbf{M}$.*

Proof. Define $I_1 = \{i+k-1, i+k-3, \ldots, i+1\}$ and $I_2 = \{i+k-2, i+k-4, \ldots, i\}$ and

$$h^+(x) = \begin{cases} h'(x) & \text{if } \ x \in A' \\ f^s(1) & \text{if } \ x = f^s(v^t) \text{ for } t \in I_1, \quad 0 \le s \le \mu - 1, \\ f^s(0) & \text{if } \ x = f^s(v^t) \text{ for } t \in I_2, \quad 0 \le s \le \mu - 1, \\ f^s(2) & \text{if } \ x = f^s(w(v^{i+k}, v^t)) \text{ for } t \in I_1, \quad 0 \le s \le \mu - 2, \\ f^s(0) & \text{if } \ x = f^s(w(v^t, v^{i-1})) \text{ for } t \in I_2, \quad 0 \le s \le \mu - 2. \end{cases}$$

To show that h^+ is a homomorphism we need to show that $f(h^+(x)) = h^+(f(x))$ and $g(h^+(x)) = h^+(g(x))$ for all five cases.

Case 1 : $x \in A'$.

If x is in A' then $f(x)$ and $g(x)$ are in A'. Thus $f(h^+(x)) = f(h'(x)) = h'(f(x)) = h^+(f(x))$ and $g(h^+(x)) = g(h'(x)) = h'(g(x)) = h^+(g(x))$.

Case 2 : $x = f^s(v^t)$ for $t \in \{i+k-1, i+k-3, \ldots, i+1\}$ and $0 \le s \le \mu - 1$.

If $s = \mu - 1$ then $h^+(f(x)) = h^+(f(f^{\mu-1}(v^t))) = h^+(c(\mu)) = \mu = f^\mu(1) = f(f^{\mu-1}(1)) = f(h^+(x))$. Otherwise if $s < \mu - 1$ then $h^+(f(x)) = h^+(f^{s+1}(x)) = f^{s+1}(1) = f(f^s(1)) = f(h^+(x))$. For $s > 0$ we have $h^+(g(x)) = h^+(f^{s-1}(v^t)) = f^{s-1}(1) = g(f^s(1)) = g(h^+(x))$. If $s = 0$ then $h^+(g(x)) = h^+(c(0)) = 0 = g(1) = g(h^+(x))$.

Case 3 : $x = f^s(v^t)$ for $t \in \{i+k-2, i+k-4, \ldots, i\}$ and $0 \le s \le \mu - 1$.

If $s = \mu - 1$ then $h^+(f(x)) = h^+(f(f^{\mu-1}(v^t))) = h^+(c(\mu)) = \mu = f^\mu(0) = f(f^{\mu-1}(0)) = f(h^+(x))$. If, on the other hand, $s < \mu - 1$ then $h^+(f(x)) = h^+(f^{s+1}(x)) = f^{s+1}(0) = f(f^s(0)) = f(h^+(x))$. For $s > 0$ we have $h^+(g(x)) = h^+(f^{s-1}(v^t)) = f^{s-1}(0) = g(f^s(0)) = g(h^+(x))$. If $s = 0$ then $h^+(g(x)) = h^+(c(0)) =$

$0 = g(0) = g(h^+(x))$.

CASE 4 : $x = f^s(w(v^j, v^t))$ for $t \in \{i+k-1, i+k-3, \ldots, i+1\}$ and $0 \leq s \leq \mu - 2$.

If $s = \mu - 2$ then by Lemma 3.7 we have $f^{\mu-1}(w(v^j, v^t)) = f^{\mu-1}(v^t)$. So $h^+(f(x)) = h^+(f(f^s(w(v^j, v^t)))) = h^+(f^{\mu-1}(w(v^j, v^t))) = h^+(f^{\mu-1}(v^t)) = f^{\mu-1}(1) = \mu = f^{\mu-2}(2) = f(f^s(2)) = f(h^+(x))$. Otherwise, if $s < \mu - 2$ then $h^+(f(x)) = f^{s+1}(2) = f(f^s(2)) = f(h^+(x))$. For $s = 0$ we have $h^+(g(x)) = h^+(v^j) = 1 = g(2) = g(h^+(x))$. If $s > 0$ then $h^+(g(x)) = h^+(f^{s-1}(w(v^j, v^t))) = f^{s-1}(2) = g(f^s(2)) = g(h^+(x))$.

CASE 5 : $x = f^s(w(v^t, v^{i-1}))$ for $t \in \{i+k-2, i+k-4, \ldots, i\}$ and $0 \leq s \leq \mu - 2$.

Finally if $s = \mu - 2$ then by Lemma 3.7 we have $f^{\mu-1}(w(v^t, v^{i-1})) = f^{\mu-1}(v^{i-1})$. So $h^+(f(x)) = h^+(f(f^s(w(v^t, v^{i-1})))) = h^+(f^{\mu-1}(w(v^t, v^{i-1}))) = h^+(f^{\mu-1}(v^{i-1})) = f^{\mu-1}(0) = f(f^{\mu-2}(0)) = f(h^+(x))$. If $s < \mu - 2$ then $h^+(f(x)) = f^{s+1}(0) = f(f^s(0)) = f(h^+(x))$. For $s = 0$ we have $h^+(g(x)) = h^+(v^t) = 0 = g(0) = g(h^+(x))$. If $s > 0$ then $h^+(g(x)) = h^+(f^{s-1}(w(v^t, v^{i-1}))) = f^{s-1}(0) = g(f^s(0)) = g(h^+(x))$.

So h^+ is a homomorphism and by Lemmas 3.13 and 3.15 it is the only such homomorphism. □

If $h : \mathbf{A} \to \mathbf{M}$ is an irresponsible homomorphism such that $h(v^i) = 1$ and $h(v^{i-1}) = 0$ then $h' = h \circ (\alpha_i^{n,k})^{-1}$ is the natural extension of h from $\mathbf{A}$ to $\mathbf{A}'$ and $h'(v^{i+k,n+k}) = 1$ and $h'(v^{i-1,n+k}) = 0$. Lemma 3.16 shows that h' extends uniquely to a homomorphism $h^+ : \mathcal{S}(\mathbf{A}') \to \mathbf{M}$. This construction sets up the commuting diagram for rank starting with algebra $\mathbf{A}$, as in Figure 2.8, and hence will be important in the proof of Theorem 4.1 that escalator algebras have infinite rank.

Example 3.3. Let $n = 3$, $i = 2$, $k = 4$ and $\mathbf{A}$ be as in Figure 3.5. Then $\alpha_2^{3,4}(001) = 0000001 = v^{6,7}$ and $\alpha_2^{3,4}(011) = 0111111 = v^{1,7}$. The algebra $\mathbf{A}' = \alpha_2^{3,4}(\mathbf{A})$ is shown in Figure 3.6. To form an algebra with a $\langle v^6, 4\rangle$-section we add L_{v^5}, L_{v^4}, L_{v^3}, and L_{v^2} to $\mathbf{A}'$ and the appropriate w-elements and their ladders. The resulting algebra $\mathcal{S}(\mathbf{A}')$ is shown in Figure 3.7.

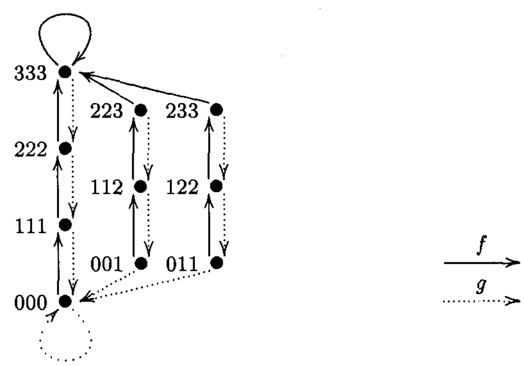

Figure 3.5: An algebra $\mathbf{A} \leq \mathbf{M}^3$ for Example 3.3

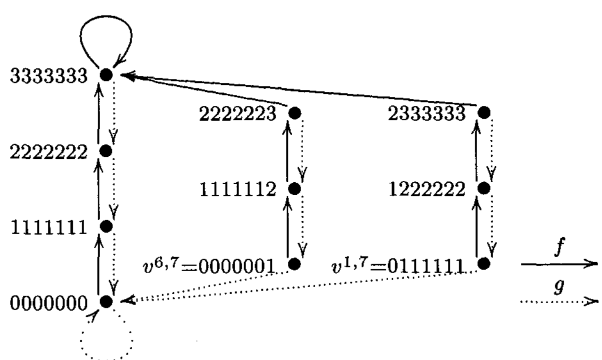

Figure 3.6: An algebra $\alpha_2^{3,4}(\mathbf{A}) = \mathbf{A}' \leq \mathbf{M}^7$ for Example 3.3

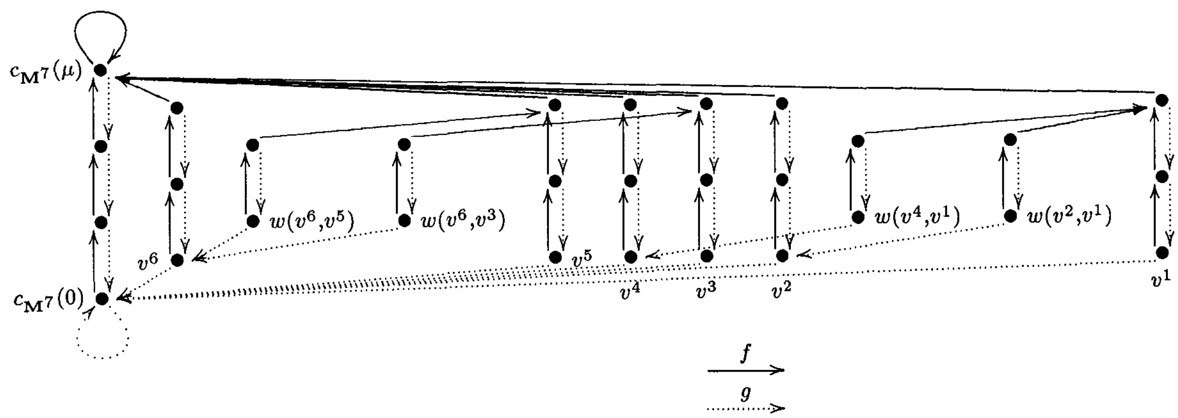

Figure 3.7: An algebra $\mathcal{S}(\mathbf{A}') \leq \mathbf{M}^7$ showing the addition of a $\langle v^6, 4\rangle$-section

Chapter 4

Rank and Duality

4.1 Rank of an Escalator Algebra

In this section we consider the rank of an escalator algebra. Hyndman and Willard's algebra that is dualizable but not fully dualizable for any set of operations is the escalator algebra of length 2. Little's algebra $\mathbf{P_{12}}$ [10] is an escalator algebra of length 3. See Figure 4.1. His work suggests that the rank of this algebra is infinite. We show that this is true for all escalator algebras, not just $\mathbf{P_{12}}$. Recall the definition of the {0,1}-element v^j from page 29.

We will use the notation $\Pi Y(\mathbf{D})$ in place of $\mathbf{D}/Y$. See page 9 for these equivalent concepts.

Lemma 4.1. *Let* $\mathbf{B}' \leq \mathbf{C} \leq \mathbf{M}^n$ *where* $0 \leq k < j < n$ *and* $v^j, v^{j-1}, \ldots, v^{j-k} \in \mathbf{C}$. *Let* $Y \subseteq \operatorname{Hom}(\mathbf{C}, \mathbf{M})$ *be a set of projections. If* $v^j, v^{j-1}, \ldots, v^{j-k}$ *determine distinct elements in* $\Pi Y(\mathbf{C})$ *then* $|Y| \geq k$.

Proof. Assume $v^j, v^{j-1}, \ldots, v^{j-k}$ determine distinct elements in $\Pi Y(\mathbf{C})$. Then for

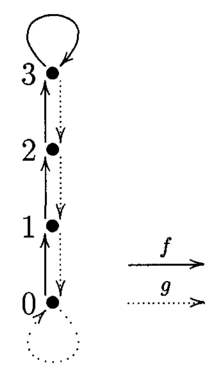

Figure 4.1: The diagram showing Little's $\mathbf{P}_{12}$

all $b_1 \neq b_2 \in \{v^j, v^{j-1}, \ldots, v^{j-k}\}$ there exists a projection $\pi_i \in Y$ such that $\pi_i(b_1) \neq \pi_i(b_2)$. For $\check{\jmath} \in \{j, j-1, \ldots, j-k+1\}$ we have $v^{\check{\jmath}}, v^{\check{\jmath}-1} \in C$ and if $i \neq \check{\jmath}$ then $\pi_i(v^{\check{\jmath}}) = \pi_i(v^{\check{\jmath}-1})$ so $\pi_{\check{\jmath}} \in Y$. Hence we need at least k projections in Y to separate $v^j, v^{j-1}, \ldots, v^{j-k}$. $\square$

Lemma 4.2. *Let $\mathbf{A} \leq \mathbf{M}^n$ be a w-algebra, let $h : \mathbf{A} \to \mathbf{M}$ be a homomorphism such that for some $1 < i < n$ both $h(v^i) = 1$ and $h(v^{i-1}) = 0$, and let k be an even integer greater than 3. Define $\mathbf{A}' = \alpha_i^{n,k}(\mathbf{A})$ and $h' = h \circ ((\alpha_i^{n,k})^{-1})$. For $Y \subseteq \mathrm{Hom}(\mathcal{S}(\mathbf{A}'), \mathbf{M})$, if h' lifts to $\Pi Y(\mathcal{S}(\mathbf{A}'))$ then for $i \leq \check{\jmath} < i+k-1$ we have $\Pi Y(v^{\check{\jmath}}) \neq \Pi Y(v^{\check{\jmath}+1})$.*

Proof. Assume h' lifts to $\Pi Y(\mathcal{S}(\mathbf{A}'))$ then there exists a homomorphism γ such that for all $a \in \mathbf{A}'$ we have $h'(a) = \gamma(\Pi Y(a))$. As $h(v^{i,n}) = 1$ both $h'(v^{i+k,n+k}) = 1$ and $\gamma(\Pi Y(v^{i+k,n+k})) = 1$. Similarly $h(v^{i-1,n}) = 0$ implies that $h'(v^{i-1,n+k}) = 0$ and thus $\gamma(\Pi Y(v^{i-1,n+k})) = 0$. Let $j = i + k$ then $j - k \leq \check{\jmath} < j - 1$ and there are two cases. If $\check{\jmath} \in \{j-k, j-k+2, \ldots, j-2\}$ then by Lemma 3.15, and using $\gamma \circ \Pi Y$ it follows that $\gamma(\Pi Y(v^{\check{\jmath},n+k})) = 0$ and $\gamma(\Pi Y(v^{\check{\jmath}+1})) = 1$. Otherwise if

$\check{\jmath} \in \{j-k+1, j-k+3, \ldots, j-3\}$ then once again by Lemma 3.15 it follows that $\gamma(\Pi Y(v^{\check{\jmath},n+k})) = 1$ and $\gamma(\Pi Y(v^{\check{\jmath}+1})) = 0$. In both cases we have $\Pi Y(v^{\check{\jmath}}) \neq \Pi Y(v^{\check{\jmath}+1})$ as required. □

Corollary 4.1. *For* $\mathbf{W}$ *a w-algebra if* $h : \mathbf{W} \to \mathbf{M}$ *is an irresponsible homomorphism then* $\operatorname{rank}(h) > 1$.

Proof. The proof is by contradiction. Assume $\operatorname{rank}(h) \leq 1$. We have $\mathbf{W} \leq \mathbf{M}^n$ where $n \geq 3$. Since h is irresponsible there exists $1 < i < n$ such that $h(v^{i,n}) = 1$, and $h(v^{i-1,n}) = 0$. We can construct the commuting diagram for rank (Figure 2.8) as follows. Let k be any even integer greater than 4. Let $\mathbf{B} = \mathbf{W}, \sigma = \alpha_i^{n,k}, \mathbf{B}' = \sigma(\mathbf{B}) = \mathbf{W}', \mathbf{C} = \mathbf{D} = \mathcal{S}(\mathbf{W}')$ and $h' = h \circ (\alpha_i^{n,k})^{-1}$. Then h^+ exists and is the natural extension of h' from $\mathbf{W}'$ to $\mathcal{S}(\mathbf{W}')$ as discussed on page 42. By the definition of rank there exists an N_0 and a $Y \subseteq \operatorname{Hom}(\mathcal{S}(\mathbf{W}'), \mathbf{M})$ such that $|Y| \leq N_0$, the map h' lifts to $\Pi Y(\mathcal{S}(\mathbf{W}'))$, and for all $g \in Y$ we have $\operatorname{rank}(g) < 1$. So we know that Y is a set of projections and by Lemma 4.2 that $\Pi Y(v^i) \neq \Pi Y(v^{i+1})$ which implies that $\pi_{i+1} \in Y$. For $i < t \leq i+k$ we have $\pi_{i+1}(v^t) = 0 \neq 1 = \pi_{i+1}(v^i)$ and so $\Pi Y(v^i) \neq \Pi Y(v^t)$ and thus h' separates $v^{i+k}, v^{i+k-1}, \ldots, v^i$. With $j = i+k$ by Lemma 4.1 we get $|Y| \geq k$. The only restriction on our choice of k was that it be an even integer greater than 4 so let $k = 2N_0 + 2$. Then $|Y| \geq k = 2N_0 + 2 > N_0$. This is a contradiction as $|Y| \leq N_0$. Therefore $\operatorname{rank}(h) > 1$. □

Theorem 4.1. *The rank of the escalator algebra of length* μ *is infinity.*

Proof. Assume for contradiction that there is an ordinal β_1 such that $\operatorname{rank}(\mathbf{M}) = \beta_1$. Let β be the least ordinal such that there exists a w-algebra, $\mathbf{W} \leq \mathbf{M}^n$, and an irresponsible homomorphism $h : \mathbf{W} \to \mathbf{M}$ with $\operatorname{rank}(h) = \beta \leq \beta_1$. Let N witness this. By Corollary 4.1, $\beta \geq 2$. As h is irresponsible there exists $i < n$ with $h(v^i) = 1$ and $h(v^{i-1}) = 0$. Let $k = 2(N+2)$ and $\sigma = \alpha_i^{n,k}$. As shown at the end of the last chapter we can construct the commuting diagram in Figure 4.2.

$$
\begin{tikzcd}
\mathbf{W} \arrow[r, phantom, "\Rrightarrow_{\alpha_i^{n,k}}"] \arrow[dr, "h"] & \mathbf{W}' \arrow[d, "h'"] \arrow[r, phantom, "\leq"] & \mathcal{S}(\mathbf{W}') \arrow[r, phantom, "\leq"] & \mathcal{S}(\mathbf{W}') \arrow[dll, dashed, "h^+"'] \arrow[r, phantom, "\leq"] & \mathbf{M}^{n+k} \\
& \mathbf{M} & & &
\end{tikzcd}
$$

Figure 4.2: A commuting diagram for a w-algebra

In the definition of rank we found an integer $N \geq 1$, a set $Y \subseteq \mathrm{Hom}(\mathcal{S}(\mathbf{W}'), \mathbf{M})$ with $|Y| \leq N$, where h' lifts to $\Pi Y(\mathbf{C})$ such that for all $g \in Y$ we have $\mathrm{rank}(g|_{\mathcal{S}(\mathbf{W}')}) < \beta$. As $k = 2(N+2)$ then $|Y| \leq N = \frac{k-4}{2} < k$ so by Lemma 4.1 there exists at least one homomorphism, g_1, in Y that is not a projection. By Lemma 3.14, the homomorphism g_1 is irresponsible. Since $\mathcal{S}(\mathbf{W}')$ is a w-algebra our initial assumption says that $\mathrm{rank}(g_1) \geq \beta$ and we have our contradiction. So $\mathrm{rank}(\mathbf{M})$ is infinite. $\square$

4.2 M is Dualizable

The next goal is to show that every escalator algebra is dualizable. In [9], Hyndman and Willard showed that $\mathbf{M}_2$ is dualizable. The general proof follows the structure of their proof quite closely but requires additional algebraic relations in the alter ego.

Given $\mathbf{M}$ we need to find $\mathbb{M}$ such that $\mathbb{M}$ is a finitary alter ego for $\mathbf{M}$. First we define two families of relations. In the case where $\mu = 2$ these relations are precisely the relations E and R of Hyndman and Willard's proof in [9]. Let

$$\Sigma = \{\tau \mid \tau \text{ is a unary term operation of } \mathbf{M}\}.$$

Recall that an element $y \in \mathbf{M}^k$ is a k-tuple $(y_1, y_2, \dots, y_k)$. For $2 < k \leq \mu + 1$ let

$$S_k = \{y \in M^k : y_1 < y_2 \cdots < y_k\}.$$

Recall for a unary term operation τ that $\tau(x_1, x_2, \dots, x_n) = \tau^{\mathbf{M}^n}(x_1, x_2, \dots, x_n) = (\tau^{\mathbf{M}}(x_1), \tau^{\mathbf{M}}(x_2), \dots, \tau^{\mathbf{M}}(x_n))$. We are omitting the superscripts for convenience. For all $y \in S_k$ define

$$P_y = \{p \in M^{2k-2} : \exists \tau \in \Sigma,\ \tau(p) = (y_1, y_2, y_2, \dots, y_{k-1}, y_{k-1}, y_k)\}$$
$$Q_y = \{q \in M^{2k-2} : q_1 \leq q_2 \leq \cdots \leq q_{2k-2}\} \setminus P_y.$$

For each $2 < k \leq \mu + 1$ we define

$$Q_k = \bigcap \{Q_y \mid y \in S_k\}.$$

Then $Q_k \subseteq M^{2k-2}$ and for all $y \in S_k$ we have $Q_k \cap P_y = \emptyset$ as $Q_y \cap P_y = \emptyset$. Note that $01\dots1$ is in $\mathbf{M}^{2k-2}$ and $01\dots1$ is not in P_y. Otherwise if $01\dots1 \in P_y$ then there exists $\tau \in \Sigma$ with $\tau(01\dots1) = (y_1, y_2, y_2, \dots, y_{k-1}, y_{k-1}, y_k)$ which would result in $y_2 = y_3 = \cdots = y_k$. Thus for all $y \in S_k$ we have $01\dots1 \in Q_y$ and so $Q_k \neq \emptyset$. For

$1 \leq j < \mu$ define

$$E_j = \{(x, x), (x, f^1(x)), \ldots, (x, f^j(x)) \mid x \in M\} \subseteq M^2.$$

Lemma 4.3. *Both* $\mathbf{E_j} = \langle E_j; f, g\rangle$ *and* $\mathbf{Q_k} = \langle Q_k; f, g\rangle$ *are algebras.*

Proof. To show this we need only show that the E_j and Q_k are closed with respect to f and g.

Let $p \in E_j$. Then for some $0 \leq i \leq j$ we have $p = (x, f^i(x))$. Thus $f(p) = f(x, f^i(x)) = (f(x), f(f^i(x))) = (f(x), f^i(f(x)))$ which is an element of E_j as $f(x) \in M$. Similarly $g(p) = g(x, f^i(x)) = (g(x), g(f^i(x)))$. Let

$$\begin{aligned} t &= g(f^i(x)) - g(x) \\ &= \begin{cases} 0 & \text{if} \quad x = 0, i = 0 \\ i - 1 & \text{if} \quad x = 0 \\ f^i(x) - x & \text{if} \quad x > 0 \end{cases} \\ &\leq x + i - x = i. \end{aligned}$$

So $0 \leq t \leq i \leq j < \mu$. As $g(x) \in \mathbf{M}$ and $f^t(g(x)) = g(x) + t = g(x) + (g(f^i(x)) - g(x)) = g(f^i(x))$ we have $g(p) = (g(x), f^t(g(x)))$ is in E_j as desired.

Assume Q_k is not closed with respect to f and g. Then there exists some $\tau \in \Sigma$ and some $q \in Q_k$ such that $\tau(q) \notin Q_k$. As q is ordered we know that $\tau(q)$ is also ordered and hence $\tau(q) \in P_y$ for some y. This implies that $q \in P_y$ but this is a contradiction as $q \in Q_k$. Both $\mathbf{E_j}$ and $\mathbf{Q_k}$ are algebras. $\square$

The alter ego that dualizes $\mathbf{M}$ is

$$\mathbb{M} = \langle M, \wedge, \vee, E_1, E_2, \ldots, E_{\mu-1}, Q_3, \ldots, Q_{\mu+1}, \mathcal{T}_d \rangle$$

where $\mathcal{T}_d$ is the discrete topology.

Hyndman and Willard define two algebraic relations as follows:

$$E = \{(x, y) : x \leq y \text{ and } (x, y) \neq (0, 2)\}$$
$$R = \{(x, y, z, w) : x \leq y \leq z \leq w \text{ and } x = y \text{ or } z = w\}$$

They then show that the topological structure $\mathbb{M} = \langle M, \wedge, \vee, E, R, \mathcal{T}_d \rangle$ dualizes $\mathbf{M_2}$. As $\mu = 2$ the only possible value for k is 3. Then our alter ego for $\mathbf{M}_2$ is $\mathbb{M} = \langle M, \wedge, \vee, E_1, E_2, \ldots, E_{\mu-1}, Q_3, \ldots, Q_{\mu+1}, \mathcal{T}_d \rangle = \langle M, \wedge, \vee, E_1, Q_3, \mathcal{T}_d \rangle$. Thus to show that our construction matches Hyndman and Willard's we need only show that $E_1 = E$ and $Q_3 = R$.

$$S_3 = \{y \in \mathbf{M}^3 : y_1 < y_2 < y_3\} = \{(0, 1, 2)\} \quad \text{and,}$$
$$P_{012} = \{p \in \mathbf{M}^4 : \exists \tau \in \Sigma, \tau(p) = (0, 1, 1, 2)\}$$
$$= \{(0, 1, 1, 2)\}.$$

From this we determine that

$$
\begin{aligned}
Q_{012} =& \{q \in \mathbf{M}^4 : q_1 \leq q_2 \leq q_3 \leq q_4\} \setminus P_{012} \\
=& \{(0,0,0,0),(0,0,0,1),(0,0,1,1),(0,0,1,2),(0,0,2,2), \\
& (0,1,1,1),(0,1,2,2),(0,2,2,2),(1,1,1,1),(1,1,1,2), \\
& (1,1,2,2),(1,2,2,2),(2,2,2,2)\}.
\end{aligned}
$$

As there is only one element in S_3 we have $Q_3 = Q_{012} = R$. We also have

$$
\begin{aligned}
E_1 &= \{(x,x),(x,f^1(x)) \mid x \in M\} \\
&= \{(0,0),(1,1),(2,2),(0,1),(1,2)\} \\
&= E
\end{aligned}
$$

So we have $E = E_1$ and $Q_3 = R$ as desired.

Now we present three lemmas. The first two (Lemma 4.4 and Lemma 4.5) are required for the third and have very similar proofs. Then we have Lemma 4.6 which is required for the proof that $\mathbf{M}$ is dualizable.

Lemma 4.4. *If $\tau \in \Sigma$, $p, q \in \mathbf{M}$ and $\tau(p) < \tau(q)$ then $p < q$.*

Proof. First assume that length of τ is 1. Thus $\tau = f$ or $\tau = g$. If $\tau = f$ then $f(p) < f(q) \leq \mu$ which implies that $p < \mu$ and $f(p) = p+1$. If $q = \mu$ then $p < q$. If $q < \mu$ then $f(q) = q+1$ and $p+1 < q+1$ which also implies that $p < q$. Similarly, if $\tau = g$ then $0 \leq g(p) < g(q)$ which implies that $q > 0$ and $g(q) = q-1$. If $p = 0$ then $p < q$. If $p > 0$ then $p-1 < q-1$, which again gives $p < q$.

Now assume that if the length of $\tau_0 = s$ then $\tau_0(p) < \tau_0(q)$ implies that $p < q$, also assume length of τ is $s+1$. Then $\tau = f \circ \tau_0$ or $\tau = g \circ \tau_0$ for some τ_0 with length s. If $\tau = f \circ \tau_0$ then $\tau(p) < \tau(q) \leq \mu$ implies that $(f \circ \tau_0)(p) < (f \circ \tau_0)(q) \leq \mu$ and so $\tau_0(p) < \mu$ and $f(\tau_0(p)) = \tau_0(p) + 1$. If $\tau_0(q) = \mu$ then $\tau_0(p) < \tau_0(q)$ and by assumption $p < q$. If $\tau_0(q) < \mu$ then $\tau_0(p) + 1 < \tau_0(q) + 1$ which implies that $\tau_0(p) < \tau_0(q)$ and again by assumption $p < q$. Alternately if $\tau = g \circ \tau_0$ then $0 \leq \tau(p) < \tau(q)$ gives $0 \leq (g \circ \tau_0)(p) < (g \circ \tau_0)(q)$ which implies that $\tau_0(q) > 0$ and $g(\tau_0(q)) = \tau_0(q) - 1$. If $\tau_0(p) = 0$ then $\tau_0(p) < \tau_0(q)$ and by assumption $p < q$. If $\tau_0(p) > 0$ then $\tau_0(p) - 1 < \tau_0(q) - 1$ and so $\tau_0(p) < \tau_0(q)$ and again by assumption $p < q$. So by induction $p < q$ for all terms $\tau \in \Sigma$. $\square$

Lemma 4.5. *If $p_1, p_2, p_3, p_4 \in \mathbf{M}$ and $\tau \in \Sigma$ such that $p_1 \leq p_2 < p_3 \leq p_4$ and $\tau(p_2) = \tau(p_3)$, then either $\tau(p_1) = \tau(p_2)$ or $\tau(p_3) = \tau(p_4)$.*

Proof. First note that for any $x \leq y \in \mathbf{M}$ if $f(x) = f(y)$ then either $x = y$ or $x = \mu - 1$ and $y = \mu$. Similarly, if $g(x) = g(y)$ then either $x = y$ or $x = 0$ and $y = 1$.

The proof is by induction on the length of terms. Assume that length of τ is 1. Thus $\tau = f$ or $\tau = g$. If $\tau = f$ then $f(p_2) = f(p_3)$ which implies that $p_2 = \mu - 1$ and $p_3 = \mu = p_4$. If $\tau = g$ then $g(p_2) = g(p_3)$ which implies that $p_2 = 0 = p_1$ and $p_3 = 1$. Thus either $\tau(p_1) = \tau(p_2)$ or $\tau(p_3) = \tau(p_4)$.

Now assume that if the length of $\tau_0 = s$ then $\tau_0(p_2) = \tau_0(p_3)$ implies that either $\tau_0(p_1) = \tau_0(p_2)$ or $\tau_0(p_3) = \tau_0(p_4)$. Assume the length of τ is $s+1$, then $\tau = f \circ \tau_0$ or $\tau = g \circ \tau_0$ for some τ_0 of length s.

If $\tau = f \circ \tau_0$ then $\tau(p_2) = \tau(p_3)$ implies that $(f \circ \tau_0)(p_2) = (f \circ \tau_0)(p_3)$ and thus we have two cases. If $\tau_0(p_2) = \tau_0(p_3)$ then by assumption $\tau_0(p_1) = \tau_0(p_2)$ or $\tau_0(p_3) =$

$\tau_0(p_4)$. Otherwise $\tau_0(p_2) = \mu - 1$ and $\tau_0(p_3) = \mu = \tau_0(p_4)$. So $f \circ \tau_0(p_3) = f \circ \tau_0(p_4)$, that is, $\tau(p_3) = \tau(p_4)$.

If $\tau = g \circ \tau_0$ then $\tau(p_2) = \tau(p_3)$ implies that $(g \circ \tau_0)(p_2) = (g \circ \tau_0)(p_3)$ and again we have 2 cases. If $\tau_0(p_2) = \tau_0(p_3)$ then by assumption $\tau_0(p_1) = \tau_0(p_2)$ or $\tau_0(p_3) = \tau_0(p_4)$. Otherwise $\tau_0(p_3) = 1$ and $\tau_0(p_2) = 0 = \tau_0(p_1)$ and $\tau(p_2) = \tau(p_1)$. $\square$

Lemma 4.6. *If $p \in \mathbf{M}^{2k-2}$ with $k > 2$ such that there exists a $\tau \in \Sigma$ and $\tau(p) = \langle y_1, y_2, y_2, \ldots, y_{k-1}, y_{k-1}, y_k \rangle$ with $y_1 < y_2 < \cdots < y_k$, then $p = \langle b_1, b_2, b_2, \ldots b_{k-1}, b_{k-1}, b_k \rangle$ for some elements $b_1, b_2, \ldots, b_k$ in $\mathbf{M}$ with $b_1 < b_2 < \cdots < b_k$.*

Proof. Let

$$p = \langle p_1, p_2, p_3, \ldots, p_{2k-2} \rangle \in \mathbf{M}^{2k-2}$$

and $\tau \in \Sigma$ such that

$$\tau(p) = \langle y_1, y_2, y_2, \ldots, y_{k-1}, y_{k-1}, y_k \rangle .$$

Looking at each coordinate we see that

$$\tau(p_i) = \begin{cases} y_{\frac{i+1}{2}} & \text{if } \; i \text{ is odd,} \\ y_{\frac{i}{2}+1} & \text{if } \; i \text{ is even.} \end{cases}$$

For all $j < k - 1$ we know that $y_j < y_{j+1}$ so for all $i < 2k - 2$ we have $\tau(p_i) \leq \tau(p_{i+1})$ First we need to show that if i is odd then $p_i < p_{i+1}$ and that if i is even then $p_i = p_{i+1}$.

Assume i is odd then $\tau(p_{i+1}) = y_{\frac{i+1}{2}+1} = y_{\frac{i+3}{2}} > y_{\frac{i+1}{2}} = \tau(p_i)$.

Assume i is even then $\tau(p_{i+1}) = y_{\frac{i+1+1}{2}} = y_{\frac{i+2}{2}} = y_{\frac{i}{2}+1} = \tau(p_i)$.

So $\tau(p_1) < \tau(p_2) = \tau(p_3) < \cdots < \tau(p_{2k-4}) = \tau(p_{2k-3}) < \tau(p_{2k-2})$ and by Lemma 4.5 and Lemma 4.4 we have $p_1 < p_2 = p_3 < \cdots < p_{2k-4} = p_{2k-3} < p_{2k-2}$.

Let $b_i = \begin{cases} p_1 & \text{if} \quad i = 1, \\ p_{2i-2} & \text{if} \quad i > 1. \end{cases}$

Then

$$\begin{aligned} p &= \langle p_1, p_2, p_3, \ldots, p_{2k-2} \rangle \\ &= \langle p_1, p_2, p_2, p_4, \ldots, p_{2k-4}, p_{2k-4}, p_{2k-2} \rangle \\ &= \langle b_1, b_2, b_2, \ldots, b_{k-1}, b_{k-1}, b_k \rangle. \end{aligned}$$

□

Theorem 4.2. $\mathbb{M}$ *dualizes* $\mathbf{M}$.

Proof. Since $\mathbb{M}$ is a total structure whose set of relations is finite, by Theorem 2.1 we need only show that the interpolation condition holds to show that $\mathbb{M}$ dualizes $\mathbf{M}$.

Let $\mathbb{X} \leq \mathbb{M}^n$ and $h : \mathbb{X} \to \mathbb{M}$. To show that the interpolation condition holds we need to show that h is the restriction to X of an n-ary term operation of $\mathbf{M}$. The proof is broken into three cases. Case 2 uses E_j while case 3 uses Q_k

Case 1 : $|\operatorname{range}(h)| = 1$.

In this case h is constant, say $h(x) = \mu - i$ for some fixed i with $0 \leq i \leq \mu$. So h is the restriction to X of $g^i f^\mu(\pi_1(x))$.

Case 2 : $|\operatorname{range}(h)| = 2$.

Assume $\operatorname{range}(h) = \{y_1, y_2\}$ where $y_1, y_2 \in M$ with $y_1 < y_2$. As h can also be viewed as a homomorphism from the finite distributive lattice $\langle X, \wedge, \vee \rangle$ onto a two-element lattice there exists $r, s \in X$ such that

$$h^{-1}(y_1) = \{x \in X : x \leq r\}$$
$$h^{-1}(y_2) = \{x \in X : x \geq s\}.$$

Let $\hat{r} = r \vee s$ and $a = y_2 - y_1$. Since $h(\hat{r}) = y_2$ and $h(r) = y_1$ we now need to choose a coordinate i so that $r_i + a \leq s_i$. This is accomplished by looking at two cases based on the value of a.

If $a = 1$ may choose any i such that $r_i < \hat{r}_i$. This gives us $r_i + a = r_i + 1 \leq \hat{r}_i = r_i \vee s_i = s_i$ as desired.

If $a > 1$ then $\langle h(r), h(\hat{r})\rangle = \langle y_1, y_2\rangle = \langle y_1, y_1 + a\rangle \notin E_j$ for all $j < a$. This implies that for all $j < a$ we have $(r, \hat{r}) \notin E_j$ as h respects all $E_j \in \mathbb{M}$. Then there must exist $i \leq n$ such that $\langle r_i, \hat{r}_i\rangle \notin E_{a-1}$. As $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_{a-1}$ we also get $\langle r_i, \hat{r}_i\rangle \notin E_j$ for all $j < a$. Let $b = \hat{r}_i - r_i$. Then $\langle r_i, \hat{r}_i\rangle = \langle r_i, r_i + b\rangle \in E_b$ so $b \geq a$ and $r_i + a \leq \hat{r}_i = r_i \vee s_i = s_i$ as desired.

For this fixed i define $X_i = \{x_i : x \in X\}$ and $h_i : X_i \to \text{range}(h)$ by $h_i(x_i) = h(x)$ for all $x \in X$. This is possible since if $x, x' \in X$ and $x_i = x'_i$ it is impossible to have $x \leq r$ and $x' \geq s$ or $x' \leq r$ and $x \geq s$, so $h(x) = h(x')$. Thus we have $h_i^{-1}(y_1) = \{x_i \in X_i : x_i \leq r_i\}$ and $h_i^{-1}(y_2) = \{x_i \in X_i : x_i \geq s_i\}$.

Let τ be the term operation $g^{\mu-a-y_1} f^{\mu-a} g^{r_i}$ and recall that $r_i + a \leq s_i$. If $x_i \leq r_i$ then $\tau(x_i) = g^{\mu-a-y_1} f^{\mu-a} g^{r_i}(x_i) = g^{\mu-a-y_1} f^{\mu-a}(0) = g^{\mu-a-y_1}(\mu - a) = y_1$. If $x_i \geq s_i$ then $\tau(x_i) = g^{\mu-a-y_1} f^{\mu-a} g^{r_i}(x_i) = g^{\mu-a-y_1} f^{\mu-a}(x_i - r_i) = g^{\mu-a-y_1}(\mu) = a + y_1 = y_2$. Thus $\tau|_{X_i} = h_i$ and $h = \tau \circ \pi_i|_X$.

Case 3 : $|\ \text{range}(h)\ | = k$ for $2 < k \leq \mu$.

Then $\mathrm{range}(h) = \{y_1, y_2, \ldots y_k\}$. As each y_i is a distinct element of $\mathbf{M}$ we may assume that $y_1 < y_2 < \cdots < y_k$ so $y = \langle y_1, y_2, \ldots y_k \rangle \in S_k$. Thus h is a lattice homomorphism from $(X, \wedge, \vee)$ onto a k element chain and there exists $r^1, r^2, \ldots, r^{k-1}, s^2, \ldots, s^k \in X$ such that:

$$\begin{aligned}
h^{-1}(y_1) &= \{x \in X \mid x \leq r^1\} \\
h^{-1}(y_2) &= \{x \in X \mid s^2 \leq x \leq r^2\} \\
h^{-1}(y_3) &= \{x \in X \mid s^3 \leq x \leq r^3\} \\
\vdots \qquad & \qquad\qquad \vdots \\
h^{-1}(y_{k-1}) &= \{x \in X \mid s^{k-1} \leq x \leq r^{k-1}\} \\
h^{-1}(y_k) &= \{x \in X \mid s^k \leq x\}.
\end{aligned}$$

Note: It is possible to define $s^1 = \bigwedge X$ and $r^k = \bigvee X$ so that $s^1 \leq x \leq r^k$ is always satisfied. This allows us to rewrite the above set of equations as

$$h^{-1}(y_i) = \{x \in X \mid s^i \leq x \leq r^i\} \qquad \text{for all } 1 \leq i \leq k.$$

For $1 \leq j < k$ define $\hat{r}^j = r^j \vee s^{j+1}$. As h preserves $\vee$ we get $h(\hat{r}^j) = h(r^j) \vee h(s^{j+1}) = y_j \vee y_{j+1} = y_{j+1}$ and we have $s^{j+1} \leq \hat{r}^j \leq r^{j+1}$.

Now define $\bar{a} = \langle r^1, \hat{r}^1, r^2, \hat{r}^2, \ldots, r^{k-1}, \hat{r}^{k-1} \rangle$. Then

$$\begin{aligned}
h(\bar{a}) &= \langle h(r^1), h(\hat{r}^1), h(r^2), h(\hat{r}^2), \ldots, h(r^{k-1}), h(\hat{r}^{k-1}) \rangle \\
&= \langle y_1, y_2, y_2, \ldots, y_{k-1}, y_{k-1}, y_k \rangle .
\end{aligned}$$

and using $\tau(x) = x$ we obtain $h(\bar{a}) \in P_y$ as well as $h(\bar{a}) \notin Q_y$. As h respects Q_y there exists $i \leq n$ such that there is an element p of M^{2k-2}, where

$$p = \langle r_i^1, \hat{r}_i^1, r_i^2, \hat{r}_i^2, \ldots, r_i^{k-1}, \hat{r}_i^{k-1} \rangle \notin Q_y.$$

Because $\bar{a}$ is ordered we know that p is also ordered and thus $p \in P_y$. This means that there exists a term τ with $\tau(p) = \langle y_1, y_2, y_2, \ldots, y_{k-1}, y_{k-1}, y_k \rangle$ and, by Lemma 4.6, $r_i^j < \hat{r}_i^j$ and thus $r_i^{j+1} = \hat{r}_i^j = r_i^j \vee s_i^{j+1} = s_i^{j+1}$ as these elements are in the chain M.

Let σ be the term operation $g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau \pi_i$ and let $x \in X$. If $h(x) = y_1$ then $x \in h^{-1}(y_1)$. It follows that $x \leq r^1$ and $x_i \leq r_i^1 = p_1$. Thus

$$\begin{aligned}
\sigma(x) &= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau \pi_i(x) \\
&= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau(x_i) \\
&= g^{\mu-y_k} f^{\mu-y_k+y_1}(0) \quad \text{as } \tau(x_i) \leq \tau(p_1) = y_1 \\
&= g^{\mu-y_k}(\mu - y_k + y_1) = y_1 = h(x).
\end{aligned}$$

If $h(x) = y_k$ then $x \in h^{-1}(y_k)$ and we have $x \geq s^k$, $x_i \geq s_i^k = p_{2k-2}$, and $\tau(x_i) \geq \tau(p_k) = y_k$, so

$$\begin{aligned}\sigma(x) &= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau\pi_i(x) \\ &= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau(x_i) \\ &= g^{\mu-y_k} f^{\mu-y_k+y_1} (\tau(x_i) - y_1) \\ &= g^{\mu-y_k}(\mu) = y_k = h(x).\end{aligned}$$

Finally if $h(x) = y_j$ for $2 \leq j \leq k-1$ then $x \in h^{-1}(y_j)$ and $s^j \leq x \leq r^j$ and $p_j = \hat{r}^{j-1} = s_i^j \leq x_i \leq r_i^j = p_j$ or, more succinctly, $x_i = p_j$. This gives us

$$\begin{aligned}\sigma(x) &= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau\pi_i(x) \\ &= g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1} \tau(x_i) = g^{\mu-y_k} f^{\mu-y_k+y_1} g^{y_1}(y_j) \quad \text{as } \tau(x_i) = \tau(p_j) = y_j \\ &= g^{\mu-y_k} f^{\mu-y_k+y_1}(y_j - y_1) \\ &= g^{\mu-y_k}(y_j + \mu - y_k) \\ &= y_j \\ &= h(x).\end{aligned}$$

$\square$

So for all x we have $\sigma|_X(x) = h(x)$ as required. For each case h is the restriction to X of a term operation and so $\mathbf{M}$ is dualizable by Theorem 2.1.

4.3 M is Not Strongly Dualizable

In this section we show that $\mathbf{M}$ is not strongly dualizable using two lemmas from [13]. The first of these states the existence of a given structure for any $n \in \mathbb{N}$. Hyndman and Willard give a more general version of the lemma that does not restrict itself to ω but uses any cardinal number [Lemma 4.1 [9]].

Lemma 4.7. (Lemma 4.1 [13]) *There is a chain* $\mathbf{\Gamma} = \langle \Gamma; \leq \rangle$ *and a partially ordered set* $\mathbf{\Gamma}' = \langle \Gamma; \lhd \rangle$ *such that* $\lhd$ *is strictly contained in* $\leq$ *and the following condition holds:*

for all $c, d \in \Gamma$ *with* $c \leq d$ *and* $c \ntriangleleft d$*, there exists* $\{c_n \mid n \in \mathbb{N}\} \cup \{d_n \mid n \in \mathbb{N}\} \subseteq \Gamma$ *such that* $c \lhd c_n$ *and* $d_n \lhd d$ *and* $c_n \leq d_n \leq c_{n+1}$*, for every* $n \in \mathbb{N}$.

Lemma 4.8. (Lemma 4.2 [13]) *Let* $\mathbf{M}$ *be a finite algebra. Given* $\mathbf{\Gamma}$ *and* $\mathbf{\Gamma}'$ *as in Lemma 4.7 let* $\mathbf{B} \leq \mathbf{A}$ *in* $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ *with* $\Gamma \subseteq B$. *Assume there is a chain* $\mathbf{C} = \langle C; \leq \rangle$*, with* $C \subseteq M$*, such that the maps* $|_\Gamma : \mathrm{Hom}(\mathbf{A}, \mathbf{M}) \to \mathrm{Hom}(\mathbf{\Gamma}, \mathbf{C})$ *and* $|_\Gamma : \mathrm{Hom}(\mathbf{B}, \mathbf{M}) \to \mathrm{Hom}(\mathbf{\Gamma}', \mathbf{C})$ *are bijections. Then the algebra* $\mathbf{M}$ *is not strongly dualizable.*

We now show how we can apply Lemma 4.8 to escalator algebras. Fix Γ as in Lemma 4.7. Define the set $\Gamma^+ = \Gamma \cup \{\top, \bot\}$ where $\bot < \gamma < \top$ for all $\gamma \in \Gamma$. Also let $\hat{\mathbf{\Gamma}} = \langle \Gamma; \unlhd \rangle$ where $\unlhd$ is some relation contained in $\leq$ and let $\mathbf{C} = \langle \{0,1\}; \leq \rangle$ be the chain with $\{0,1\} \subseteq M$.

Define the mapping $q : \Gamma^+ \to M^{\Gamma^+}$ by $q(\top) = c_{\mathbf{M}^{\Gamma^+}}(1)$ and $q(\bot) = c_{\mathbf{M}^{\Gamma^+}}(0)$ and for all γ in Γ we have $q(\gamma) = a_\gamma \in M^{\Gamma^+}$ where

$$a_\gamma(i) = \begin{cases} 1 & \text{if} \quad i \leq \gamma, \\ 0 & \text{if} \quad \gamma < i. \end{cases}$$

Clearly a_γ is a $\{0{,}1\}$-element as $\bot < \gamma < \top$. If $\gamma_i \trianglelefteq \gamma_j$ then $\gamma_i \leq \gamma_j$ and $a_{\gamma_i} \leq a_{\gamma_j}$. For $\gamma_i \neq \gamma_j$, $w(a_{\gamma_i}, a_{\gamma_j})$ is defined and is in M^{Γ^+}. For brevity we will denote $w(a_{\gamma_i}, a_{\gamma_j})$ by w_{ij}. Let $\mathbf{D}$ be the subalgebra of $\mathbf{M}^{\Gamma^+}$ generated by $S = \{a_\gamma \mid \gamma \in \Gamma\} \cup \{w_{ij} \mid \gamma_i \trianglelefteq \gamma_j\}$. Define the maps $\zeta : \Gamma \to \{a_\gamma \mid \gamma \in \Gamma\}$ by $\zeta(\gamma) = a_\gamma$ and $|_\Gamma : \mathrm{Hom}\,(\mathbf{D}, \mathbf{M}) \to \mathrm{Hom}\,(\hat{\mathbf{\Gamma}}, \mathbf{C})$ by $|_\Gamma(h) = h \circ \zeta$. Denote $|_\Gamma(h)$ by $h|_\Gamma$. (See Figure 4.3.)

$$\begin{tikzcd}
\mathbf{\Gamma} \arrow[r, "\zeta"] \arrow[dr, dashed, "h|_\Gamma"'] & \{a_\gamma \mid \gamma \in \Gamma\} \subseteq & \mathbf{D} \arrow[dl, "h"] \\
 & \mathbf{M} &
\end{tikzcd}$$

Figure 4.3: A diagram for $|_\Gamma : \mathrm{Hom}\,(\mathbf{D}, \mathbf{M}) \to \mathrm{Hom}\,(\hat{\mathbf{\Gamma}}, \mathbf{C})$

We now show that for h in $\mathrm{Hom}\,(\mathbf{D}, \mathbf{M})$, we have $h|_\Gamma \in \mathrm{Hom}\,(\hat{\mathbf{\Gamma}}, \mathbf{C})$. Start by picking h in $\mathrm{Hom}\,(\mathbf{D}, \mathbf{M})$. Then $h : \mathbf{D} \to \mathbf{M}$ is a homomorphism and each a_γ is a $\{0{,}1\}$-element and so by Lemma 3.3 we have $h|_\Gamma : \Gamma \to \{0, 1\}$. Pick $\gamma_i, \gamma_j \in \Gamma$ such that $\gamma_i \trianglelefteq \gamma_j$ and $\gamma_i \neq \gamma_j$. Then a_{γ_i}, a_{γ_j}, $w(\gamma_i, \gamma_j) \in \mathbf{D}$ and by Lemma 3.10 there exists an i such that $h(a_{\gamma_i}) = a_{\gamma_i}(i)$ and $h(a_{\gamma_j}) = a_{\gamma_j}(i)$. The fact that $\gamma_i \trianglelefteq \gamma_j$ implies that $a_{\gamma_i} \trianglelefteq a_{\gamma_j}$, so $a_{\gamma_i}(i) \trianglelefteq a_{\gamma_j}(i)$ and thus $h(a_{\gamma_i}) \trianglelefteq h(a_{\gamma_j})$. Finally we have that $h|_\Gamma : \Gamma \to C$ and that $h|_\Gamma \in \mathrm{Hom}\,(\hat{\Gamma}, \mathbf{C})$.

Now pick $h \in \mathrm{Hom}\,(\hat{\Gamma}, \mathbf{C})$. Then $h : \Gamma \to \{0,1\}$ and respects the relation $\trianglelefteq$. Extend h to the mapping $h' : S \to \{0,1,2\}$ by

$$h'(a) = \begin{cases} h(\gamma) & \text{if} \quad a = a_\gamma \quad \text{for some } \gamma \in \Gamma \\ h(\gamma_i) + h(\gamma_j) & \text{if} \quad a = w_{ij} \quad \text{for some } \gamma_i, \gamma_j \in \Gamma \text{ with } \gamma_i \trianglelefteq \gamma_j. \end{cases}$$

This mapping respects the functions f and g restricted to S. To see this pick $x \in S$ such that $g(x) \in S$. Then $x = w(a,b)$ and $g(x) = a$ for some $a, b \in \{a_\gamma \mid \gamma \in \Gamma\}$ and $a < b$ so $h(a) \leq h(b)$. If $h(a) = h(b) = 0$ then $h'(x) = 0$ and $h'(g(x)) = h'(a) = 0 = g(0) = g(h'(x))$. If $h(a) = 0$ and $h(b) = 1$ then $h'(x) = 1$ and $h'(g(x)) = h'(a) = 0 = g(1) = g(h'(x))$. If $h(a) = h(b) = 1$ then $h'(x) = 2$ and $h'(g(x)) = h'(a) = 1 = g(2) = g(h'(x))$. Note that there is no way of picking $x \in S$ such that $f(x) \in S$.

The map h' extends to the map $h^* : \mathbf{D} \to \mathbf{M}$ as follows:

$$h^*(x) = \begin{cases} h'(x) & \text{if} \quad x \in S \\ f^k(h'(a)) & \text{if} \quad x = f^k(a) \text{ with } a \in S. \end{cases}$$

The next lemma shows that h^* is a homomorphism and by Lemma 3.13 it is the unique extension of h' to $\mathbf{D}$.

Lemma 4.9. *If h is an element of* $\mathrm{Hom}\,(\hat{\Gamma}, \mathbf{C})$ *then the unique extension of h' to* $\mathbf{D}$, *$h^* : \mathbf{D} \to \mathbf{M}$ is a homomorphism.*

Proof. We need to show that $h^*(f(x)) = f(h^*(x))$ and $h^*(g(x)) = g(h^*(x))$. Pick

$x \in D = \{L_{a_\gamma} \mid \gamma \in \Gamma\} \cup \{L_{w_{ij}} \mid \gamma_i \trianglelefteq \gamma_j\}$. The construction of $\mathbf{D}$ gives us 4 cases. If $x \in L_{a_\gamma}$ then $x = a_\gamma$ or $x = f^k(a_\gamma)$ for some $1 \leq k \leq \mu - 1$. Otherwise if $x \in L_{w_{ij}}$ then $x = w_{ij}$ or $x = f^k(w_{ij})$ for some $1 \leq k \leq \mu - 2$.

Case 1 : Assume $x = a_\gamma$.

It follows that $h^*(f(a_\gamma)) = f(h'(a_\gamma)) = f(h^*(a_\gamma))$. By Lemma 3.3, $h^*(x) = h^*(a_\gamma) \in \{0, 1\}$ and so $g(h^*(x)) = 0 = h^*(c_{\mathbf{D}}(0)) = h^*(g(a_\gamma)) = h^*(g(x))$.

Case 2 : Assume $x = f^k(a_\gamma)$ with $1 < k \leq \mu - 1$.

We have

$$\begin{aligned} h^*(f(x)) &= h^*(f(f^k(a_\gamma))) = h^*(f^{k+1}(a_\gamma)) \\ &= f^{k+1}(h'(a_\gamma)) = f(f^k(h'(a_\gamma))) = f(h^*(f^k(a_\gamma))) \\ &= f(h^*(x)). \end{aligned}$$

As $x \in L_{a_\gamma}$ and $x \neq a_\gamma$ we have $gf^k(a_\gamma) = f^{k-1}(a_\gamma)$ and thus

$$\begin{aligned} h^*(g(x)) &= h^*(g(f^k(a_\gamma))) = h^*(f^{k-1}(a_\gamma)) \\ &= f^{k-1}(h'(a_\gamma)) \\ &= g(f^k(h'(a_\gamma))) = g(h^*(f^k(a_\gamma))) = g(h^*(x)). \end{aligned}$$

Case 3 : Assume $x = w_{ij}$.

Then $h^*(f(x)) = h^*(f(w_{ij})) = f(h'(w_{ij})) = f(h^*(w_{ij})) = f(h^*(x))$.

As $h'|_S$ respects g and w_{ij} is in S we have $g(h'(x)) = h'(g(x))$ and

$$\begin{aligned} h^*(g(x)) &= h^*(g(w_{ij})) = h^*(a_{\gamma_i}) \\ &= h'(g(w_{ij})) \\ &= g(h'(w_{ij})) = g(h^*(w_{ij})) = g(h^*(x)). \end{aligned}$$

CASE 4 : Assume $x = f^k(w_{ij})$ with $1 < k \leq \mu - 1$.

It follows that

$$\begin{aligned} h^*(f(x)) &= h^*(f(f^k(w_{ij}))) = h^*(f^{k+1}(w_{ij})) \\ &= f^{k+1}(h'(w_{ij})) = f(f^k(h'(w_{ij}))) = f(h^*(f^k(w_{ij}))) \\ &= f(h^*(x)), \end{aligned}$$

and

$$\begin{aligned} h^*(g(x)) &= h^*(g(f^k(w_{ij}))) = h^*(f^{k-1}(w_{ij})) \\ &= f^{k-1}(h'(w_{ij})) \\ &= g(f^k(h'(w_{ij}))) = g(h^*(f^k(w_{ij}))) = g(h^*(x)). \end{aligned}$$

So for all cases we have $h^*(f(x)) = f(h^*(x))$ and $h^*(g(x)) = g(h^*(x))$ and so h is a homomorphism. By Lemma 3.13 this homomorphism is the unique extension of h'. $\square$

Lemma 4.10. *If* $\mathbf{D} = Sg_{\mathbf{M}^{\Gamma+}}(\{a_\gamma \mid \gamma \in \Gamma\} \cup \{w(a_{\gamma_i}, a_{\gamma_j}) \mid \gamma_i \trianglelefteq \gamma_j\})$ *then the map* $|_\Gamma : \mathrm{Hom}\,(\mathbf{D}, \mathbf{M}) \to \mathrm{Hom}\,(\hat{\mathbf{\Gamma}}, \mathbf{C})$ *is a bijection.*

Proof. First pick h in $\operatorname{Hom}(\hat{\mathbf{\Gamma}}, \mathbf{C})$. Then $h^* : \mathbf{D} \to \mathbf{M}$ as constructed above is a homomorphism and $h^*|_\Gamma : \Gamma \to \{0,1\}$. For any γ in Γ we have $h^*|_\Gamma(\gamma) = (h^* \circ \zeta)(\gamma) = h^*(\zeta(\gamma)) = h^*(a_\gamma) = h'(a_\gamma) = h(\gamma)$. So $h^*|_\Gamma = h$ and thus $|_\Gamma$ is onto.

Now pick α and β in $\operatorname{Hom}(\mathbf{D}, \mathbf{M})$ such that $\alpha|_\Gamma = \beta|_\Gamma$. Let x be an element of D. If $x = a_\gamma$ for some $\gamma \in \Gamma$ then $\alpha(x) = \alpha(a_\gamma) = (\alpha \circ \zeta)(\gamma) = \alpha|_\Gamma(\gamma) = \beta|_\Gamma(\gamma) = (\beta \circ \zeta)(\gamma) = \beta(a_\gamma) = \beta(x)$. Alternately if $x = w_{ij}$ for some $\gamma_i \trianglelefteq \gamma_j$ then by Lemma 3.9 we have $\alpha(x) = \alpha(w_{ij}) = \alpha(a_{\gamma_i}) + \alpha(a_{\gamma_j}) = \beta(a_{\gamma_i}) + \beta(a_{\gamma_j}) = \beta(w_{ij}) = \beta(x)$. So $\alpha(x) = \beta(x)$ for all x in S and by Lemma 3.13 we have $\alpha(x) = \beta(x)$ for all x in D. Thus $|_\Gamma$ is one-to-one and hence a bijection. $\square$

Theorem 4.3. $\mathbf{M}$ *is not strongly dualizable.*

Proof. Let $\Gamma = \langle \Gamma; \leq \rangle$, $\Gamma' = \langle \Gamma; \lhd \rangle$ as in Lemma 4.7, and let

$$\begin{aligned}
\mathbf{B} &= Sg_{\mathbf{M}^{\Gamma^+}}(\{a_\gamma \mid \gamma \in \Gamma\} \cup \{w(a_{\gamma_i}, a_{\gamma_j}) \mid \gamma_i \lhd \gamma_j\}), \\
\mathbf{A} &= Sg_{\mathbf{M}^{\Gamma^+}}(\{a_\gamma \mid \gamma \in \Gamma\} \cup \{w(a_{\gamma_i}, a_{\gamma_j}) \mid \gamma_i \leq \gamma_j\}), \quad \text{and} \\
C &= \langle \{0,1\}; \leq \rangle .
\end{aligned}$$

By Lemma 4.10, the maps $|_\Gamma : \operatorname{Hom}(\mathbf{A}, \mathbf{M}) \to \operatorname{Hom}(\mathbf{\Gamma}, \mathbf{C})$ and $|_\Gamma : \operatorname{Hom}(\mathbf{B}, \mathbf{M}) \to \operatorname{Hom}(\mathbf{\Gamma}', \mathbf{C})$ are bijections. Hence by Lemma 4.8 $\mathbf{M}$ is not strongly dualizable. $\square$

4.4 Steps Towards M is Not Fully Dualizable

To show that the escalator algebra of length 2 is not fully dualizable Hyndman and Willard showed that the relations of $\mathbb{M}_2$ were balanced. They then constructed two bi-graphs and relations on each of the bi-graphs. Full dualizability of $\mathbf{M}_2$ would require that one of the relations was the reflexive transitive closure of the other. As this is not the case $\mathbf{M}_2$ is not fully dualizable.

In the attempt to parallel this work we show that the relations of $\mathbb{M}_\mu$ are balanced. We tried to build a graph structure and relations on those graphs that would match the construction of [9]. Instead of getting bi-graphs the structures obtained were multigraphs. It was at this point that work was halted.

4.4.1 E_j and Q_k are Balanced

Recall the definitions of E_j and Q_k from page 49. Before we show that the relations E_j and Q_k are balanced we define balanced and present several lemmas about homomorphisms from $\mathbf{Q_k}$ to $\mathbf{M}$. We also need to show that $\mathbf{E_j}$ and $\mathbf{Q_k}$ are both algebras.

For $\mathbf{A}$ a finite algebra let $\mathbf{S} \leq \mathbf{A}^n$ and S be the corresponding n-ary relation on $\mathbf{A}$. The relation S is ***balanced*** if $|\operatorname{Hom}(\mathbf{S}, \mathbf{A})| = n$ and for $i \neq j$ we have $\pi_i|_S \neq \pi_j|_S$.

Lemma 4.11. *For $1 \leq j \leq 2k-1, 1 \leq n < \mu$ we have v^j and $f^n(v^j)$ in $\mathbf{Q_k} \leq \mathrm{M}^{2k-2}$.*

Proof. As v^j and all its ladder elements are ordered, to show they are in $\mathbf{Q_k}$ we only need to show they are not in P_y for any $y \in S_k$. First assume $v^j \in P_y$ for some $y \in S_k$. Then there is a term operation, τ, such that $\tau(v^j) = (y_1, y_2, y_2, \ldots, y_{k-1}, y_{k-1}, y_k)$. Since $\mathbf{Q_k}$ is only defined for $k > 2$ then $2k-2 \geq 4$ and we can look at the first four coordinates of $\tau(v^j)$. If $j = 1$ or $j = 2$ then $\tau(v^j_3) = \tau(1) = y_2$ and $\tau(v^j_4) = \tau(1) = y_3$ thus $y_2 = y_3$ which is false. If $j > 2$ then $\tau(v^j_1) = \tau(0) = y_1$ and $\tau(v^j_2) = \tau(0) = y_2$

thus $y_1 = y_2$ which also is false. We have $v^j \notin P_y$ and consequently $v^j \notin P_k$. All elements of L_{v^j} being in $\mathbf{Q_k}$ follows directly from the fact that $\mathbf{Q_k}$ is an algebra. □

Recall the definition of u^j from page 29.

Lemma 4.12. *If* $h : \mathbf{Q_k} \to \mathbf{M}$ *is a homomorphism then* h *is responsible.*

Proof. Pick $v^i, v^j \in \mathbf{Q_k}$ such that $j < i$ and $h(v^i) = 1$. To show that h is responsible we need to show that $h(v^j) = 1$. By Lemma 3.7 we know that $g(w(v^i, v^j)) = v^i$ and $f^{\mu-1}(w(v^i, v^j)) = f^{\mu-1}(v^j) = u^j$. So $g(h(w(v^i, v^j))) = h(g(w(v^i, v^j))) = h(v^i) = 1$ which implies that $h(w(v^i, v^j)) = 2$. Furthermore we can deduce that

$$h(f^{\mu-1}(w(v^i, v^j))) = f^{\mu-1}(h(w(v^i, v^j))) = f^{\mu-1}(2) = \mu.$$

So we get

$$h(v^j) = h(g^{\mu-1}(u^j)) = g^{\mu-1}(h(u^j)) = g^{\mu-1}(h(f^{\mu-1}(w(v^i, v^j)))) = g^{\mu-1}(\mu) = 1.$$

So h is responsible. □

Lemma 4.13. *If* $h : \mathbf{Q_k} \to \mathbf{M}$ *is a homomorphism then* h *is a projection.*

Proof. Pick $x \in \mathbf{Q_k}$. By Lemma 4.12, h is responsible and thus there are three cases.

CASE 1 : For all $i < 2k - 2$, $h(v^i) = 0$.

Then there exists $v^j \in \mathbf{Q_k}$ such that $f^{s_x-1}(x) = u^j$. So

$$f^{s_x-1}(h(x)) = h(f^{s_x-1}(x)) = h(u^j) = h(f^{\mu-1}(v^j)) = f^{\mu-1}(h(v^j)) = f^{\mu-1}(0) = \mu - 1.$$

This implies that $h(x) + s_x - 1 = \mu - 1$ and because x is ordered we get $h(x) = \mu - s_x = \min(x) = x_1$. So $h(x) = \pi_1(x)$.

CASE 2 : For all $i < 2k-2$, $h(v^i) = 1$.

Then there exists $v^j \in \mathbf{Q_k}$ such that $g^{t_x - 1}(x) = v^j$. So

$$g^{t_x-1}(h(x)) = h(g^{t_x-1}(x)) = h(v^j) = 1.$$

This implies that $h(x) - t_x + 1 = 1$ and we get $h(x) = t_x = \max(x) = x_{2k-2}$ since x is ordered and we have $h(x) = \pi_{2k-2}(x)$.

CASE 3 : There exists $i < 2k-2$ such that for $1 \leq j < 2k-2$ if $j \geq i$ then $h(v^j) = 0$ and if $j < i$ then $h(v^j) = 1$. We claim that $h = \pi_i$.

Then there exists $1 \leq j \leq \check{j} < 2k-2$ such that $g^{t_x-1}(x) = v^{\check{j}}$ and $f^{s_x-1}(x) = u^j$. As $h(v^{\check{j}})$ equals 0 or 1 and $h(u^j)$ equals μ or $\mu - 1$ we have three subcases:

SUBCASE i : $h(v^{\check{j}}) = 1$.

If $h(v^{\check{j}}) = 1$ then $\check{j} < i$ and $v_i^{\check{j}} = 1$. We now have $g^{t_x-1}(x_i) = v_i^{\check{j}} = 1$ which implies that $x_i - t_x + 1 = 1$ and we get $x_i = t_x$. This gives us $g^{t_x-1}(h(x)) = h(g^{t_x-1}(x)) = h(v^{\check{j}}) = 1$ which implies that $h(x) - t_x + 1 = 1$ and $h(x) = t_x = x_i$ and thus $h(x) = \pi_i(x)$.

SUBCASE ii : $h(u^j) = \mu - 1$.

As $f^{s_x-1}(h(x)) = h(f^{s_x-1}(x)) = h(u^j) = \mu - 1$ we get $h(x) + s_x - 1 = \mu - 1$ which implies that $h(x) = \mu - s_x$. To show that $j \geq i$ we only need to note that $h(v^j) = h(g^{\mu-1}(u^j)) = g^{\mu-1}(h(u^j)) = g^{\mu-1}(\mu-1) = 0$. From this we get that $f^{s_x-1}(x_i) = u_i^j = \mu - 1$ which implies that $x_i + s_x - 1 = \mu - 1$ which gives that $x_i = \mu - s_x = h(x)$ and finally $h(x) = \pi_i(x)$.

SUBCASE iii : $h(v^{\jmath}) = 0$ and $h(u^{\jmath}) = \mu$.

To get $j < i \leq \jmath$ we only need to note that $h(v^{\jmath}) = 0$ and that $h(v^{j}) = h(g^{\mu-1}(u^{j})) = g^{\mu-1}(h(u^{j})) = g^{\mu-1}(\mu) = 1$. From this we get $f^{s_x-1}(x_i) = u^j_i = \mu$ which implies $x_i + s_x - 1 \geq \mu$ and $x_i + \mu - \min(x) - 1 \geq \mu$, $x_i - 1 \geq \min(x)$, or $x_i > \min(x) = x_1$. Similarly $g^{t_x-1}(x_i) = 0$ implies $x_i - t_x + 1 \leq 0$, $x_i + 1 \leq t_x$, $x_i < t_x$.

Now assume for contradiction that $h(x) = x_r < x_i$. Let $x' = g^{x_i-1}(x)$. This gives us $x'_i = g^{x_i-1}(x_i) = 1$ and $x'_1 = g^{x_i-1}(x_1) = 0$, as $x_i > x_1$. So $s_{x'} = \mu$ and $f^{s_{x'}-1}(x') = f^{\mu-1}(x')$. Thus $f^{\mu-1}(x'_1) = f^{\mu-1}(0) = \mu-1$ and $f^{\mu-1}(x'_i) = f^{\mu-1}(1) = \mu$ so $f^{\mu-1}(x') = u^{j'}$ for some $j' < i$ which implies that $h(v^{j'}) = 1$ and $h(u^{j'}) = \mu$. Then $h(x') = h(g^{x_i-1}(x)) = g^{x_i-1}(h(x)) = g^{x_i-1}(x_r) = 0$ and $h(f^{\mu-1}(x')) = f^{\mu-1}(h(x')) = f^{\mu-1}(0) = \mu - 1$. But $h(f^{\mu-1}(x')) = h(u^{j'}) = \mu$ which is a contradiction and thus $x_i \leq h(x) < t_x$.

Now assume for contradiction that $h(x) = x_r > x_i$ and without loss of generality assume that $x_{r-1} < x - r$. Let $x'' = f^{\mu-x_r}(x)$. So $x''_r = f^{\mu-x_r}(x_r) = \mu$ and $x''_{r-1} = f^{\mu-x_r}(x_{r-1}) < \mu$, as $x_{r-1} < x - r$. So $t_{x''} = \mu$ and $g^{t_{x''}-1}(x'') = g^{\mu-1}(x'')$. Thus $g^{\mu-1}(x''_{r-1}) = 0$ and $g^{\mu-1}(x''_r) = g^{\mu-1}(\mu) = 1$ so $g^{\mu-1}(x'') = v^{r-1}$, and $h(v^{r-1}) = 0$ as $i \leq r - 1$. Then $h(x'') = h(f^{\mu-x_r}(x)) = f^{\mu-x_r}(h(x)) = f^{\mu-x_r}(x_r) = \mu$ and $h(g^{\mu-1}(x'')) = g^{\mu-1}(h(x'')) = g^{\mu-1}(\mu) = 1$. But $h(g^{\mu-1}(x'')) = h(v^{r-1}) = 0$ which is a contradiction. So $h(x) = \pi_i(x)$. So for all $x \in \mathbf{Q_k}$, $h(x) = \pi_i(x)$. □

Lemma 4.14. *If $h : \mathbf{E_j} \to \mathbf{M}$ is a homomorphism then h is a projection.*

Proof. Assume $h : \mathbf{E_j} \to \mathbf{M}$ is a homomorphism. Given that $E_1 = C_{M^2} \cup L_{01}$ then for $1 < j \leq \mu$ we have $E_j = E_{j-1} \cup L_{0j}$. We will use induction on j to show that h is a projection.

The base case is $j = 1$. Then we have $E_j = E_1$ which consists of the centre and one ladder only. By Lemma 3.2 we know where the centre elements are sent so we only need to determine where the ladder elements are sent. This is completely determined by where the foot element, $(0,1)$, is sent. We see that $g(h(01)) = h(g(01)) = h(00) = 0$ and thus $h(01) = 0 = \pi_1(01)$ or $h(01) = 1 = \pi_2(01)$. So for all $x \in E_1$ we have $h(x) = \pi_i(x)$ for some $i \leq 2$.

Assume that for some j all homomorphisms from $\mathbf{E_j}$ to $\mathbf{M}$ are projections. Then, by an argument similar to the one in the previous paragraph, to determine what all the homomorphisms from $\mathbf{E_{j+1}}$ to $\mathbf{M}$ are we need only look at where $(0, j+1)$ is sent as $E_{j+1} = E_j \cup L_{0(j+1)}$. If $h(0j) = \pi_1(0j) = 0$ then $h(\mu - j, \mu) = h(f^{\mu-j}(0j)) = f^{\mu-j}(h(0j)) = f^{\mu-j}(0) = \mu - j$ and $g(h(0, j+1)) = h(g(0, j+1)) = h(0j) = 0$ which implies that $h(0, j+1) = 0$ or 1. Assume for contradiction that $h(0, j+1) = 1$. Then $h(\mu - j, \mu) = h(f^{\mu-j}(0, j+1)) = f^{\mu-j}(h(0, j+1)) = f^{\mu-j}(1) = \mu - j + 1 \neq \mu - j$ So $h(0, j+1) = 0 = \pi_1(0, j+1)$. If $h(0j) \neq 0$ then by assumption $h(0j) = \pi_2(0j) = j$. Then $g(h(0, j+1)) = h(g(0, j+1)) = h(0j) = j$ and thus $h(0, j+1) = j+1 = \pi_2(0, j+1)$. And so by induction we know that $h : \mathbf{E_j} \to \mathbf{M}$ is a projection for all j.

□

Lemma 4.15. *E_j and Q_k are balanced with respect to* $\mathbf{M}$.

Proof. By Lemma 4.14 we know that all homomorphisms in Hom $(\mathbf{E_j}, \mathbf{M})$ are projections. The element $(0,1) \in E_j$ for all $j \leq \mu$ and $\pi_1((0,1)) = 0 \neq 1 = \pi_1((0,1))$ so π_1 and π_2 are distinct and therefore E_j is balanced.

By Lemma 4.13 we know that all homomorphisms in Hom $(\mathbf{Q_k}, \mathbf{M})$ are projections. Without loss of generality assume $j < i \leq 2k - 2$. Then by Lemma 4.11

$v^j \in Q_k$ and $\pi_j(v^j) = 0 \neq 1 = \pi_i(v^j)$. So there are $2k-2$ distinct projections and therefore Q_k is balanced. $\square$

Chapter 5

Where to go from Here

In this paper we have shown that escalator algebras have infinite rank, and are dualizable but not strongly dualizable. Following the work of Hyndman and Willard in [9] we showed that the relations of $\mathbb{M}$ are balanced which suggest that escalator algebras of length greater than 2 may not be fully dualizable. This work still needs to be completed.

Hyndman and Willard have proven [7] the following:

> Suppose that **M** is some dualizable algebra. Assume $\phi(x, y)$ is a primitive positive formula that defines an acyclic binary relation. If there exists a set $\{0, 1\}$ contained in M such that $\phi(0, 0), \phi(0, 1)$, and $\phi(1, 1)$ hold then **M** is not strongly dualizable.

This result can also be used to show that escalator algebras are not strongly dualizable and thus verifies the results in this thesis.

Open questions that follow directly from this thesis are: whether or not all escalator algebras are not fully dualizable? Do other families of algebras have analogues to w-algebras? If so can they be used to prove infinite rank of those families?

It is possible that answers to the above may provide clues to the following more general questions about unary algebras and their dualizability. For unary algebras with more than three elements, can we find nice conditions for dualizability? Can we find nice conditions for full/not full dualizability, or strong/not strong dualizability? Finally, another question of significant interest is whether there exists an algebra which is not strongly dualizable but is fully dualizable.

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