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ON THE MINIMUM ORDER OF EXTREMAL GRAPHS TO HAVE A PRESCRIBED GIRTH*

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Abstract. We show that any n-vertex extremal graph G without cycles of length at most k has girth exactly $k+1$ if $k \geq 6$ and $n > (2(k-2)^{k-2}+k-5)/(k-3)$. This result provides an improvement of the asymptotical known result by Lazebnik and Wang [*J. Graph Theory*, 26 (1997), pp. 147–153] who proved thatthe girth is exactly $k+1$ if $k \geq 12$ and $n \geq 2^{a^2+a+1}k^a$, where $a = k-3-\lfloor (k-2)/4 \rfloor$. Moreover, we prove that the girth of G is at most $k+2$ if $n > (2(t-2)^{k-2}+t-5)/(t-3)$, where $t = \lceil (k+1)/2 \rceil \geq 4$. In general, for $k \geq 5$ we show that the girth of G is at most $2k-4$ if $n \geq 2k-2$.

Key words. extremal graphs, girth, forbidden cycles, cages

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1. Introduction. Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [2] for terminology and definitions.

Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of a graph G, respectively. The order of G is denoted by $|V(G)| = n$ and the size by $|E(G)| = e(G)$. The minimum length of a cycle contained in G is the *girth* $g(G)$ of G. A cycle of minimum length is said to be *a girdle* and if G does not contain a cycle, we set $g(G) = \infty$. By C_r we will denote a cycle of length r, $r \geq 3$.

Let $\mathcal{F}$ be a family of graphs. The extremal number $ex(n, \mathcal{F})$ is the maximum number of edges in a graph of order n that does not contain any graph of $\mathcal{F}$ as a subgraph. The graphs of order n and size $ex(n, \mathcal{F})$ not containing any $F \in \mathcal{F}$ as a subgraph are the extremal graphs and are denoted by $EX(n, \mathcal{F})$. We refer to graphs from $EX(n, \mathcal{F})$ as *extremal* $\mathcal{F}$-free graphs of order n, or just *extremal*.

By $ex(n; \{C_3, C_4, \ldots, C_k\})$ we denote the maximum number of edges in a graph of order n and girth at least $k+1$, and by $EX(n; \{C_3, C_4, \ldots, C_k\})$ we denote the set of all graphs of order n, girth at least $k+1$, and with $ex(n; \{C_3, C_4, \ldots, C_k\})$ edges. Erdös and Sachs [3] showed that an r-regular graph of girth at least $k+1$ with the least possible number of vertices has girth equal to $k+1$. (A proof of this result can be found in Lovász [7, pp. 66, 384, 385, and the references therein].) These graphs are called $(r; k+1)$-*cages*.

In this paper we consider a similar question asked by Garnick and Nieuwejaar in [5] on extremal graphs with a relatively large girth. Is there a constant c such that for all $k \geq 5$ and all $n \geq ck$, the girth of any extremal graph with girth $\geq k+1$ is $k+1$? They give an affirmative answer for $k = 4$. Lazebnik and Wang [6] showed that the

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answer is negative for $c = 2$ and affirmative if $k = 5$ or if n is large in comparison with k. More precisely they proved the following result.

Theorem A. *Let* $k \geq 12$, $a = k - 3 - \lfloor (k-2)/4 \rfloor$, $n \geq 2^{a^2+a+1}k^a$, *and* $G \in EX(n; \{C_3, C_4, \ldots, C_k\})$. *Then the girth* $g(G) = k+1$.

In order to prove Theorem A, Lazebnik and Wang used the following result, which they also stated in [6].

Theorem B. *Let* $k \geq 3$, $G \in EX(n; \{C_3, C_4, \ldots, C_k\})$, *and the maximum degree be* $\Delta(G) \geq k$. *Then* $g(G) = k+1$.

Our main contribution to this problem is to provide an improvement of Theorem A. More precisely we prove that the girth of $G \in EX(n; \{C_3, C_4, \ldots, C_k\})$ is $k+1$ if either $k = 3$ and $n \geq 5$; or $k = 4$ and $n \geq 9$; or $k = 5$ and $n \geq 8$; or $k = 6$ and $n \geq 171$; or $k \geq 7$ and

$$n \geq \frac{2(k-2)^{k-2} + k - 5}{k-3} + 1.$$

This contribution contains the known results for $k = 3, 4, 5$; see [4, 5, 6]. Furthermore, it gives an answer to the problem for $k = 6$ posed by Lazebnik and Wang [6], who asked to prove the girth of an extremal $\{C_3, C_4, C_5, C_6\}$-free graph is 7.

Moreover, we show that the girth of $G \in EX(n; \{C_3, C_4, \ldots, C_k\})$ is at most $2k-4$ provided that $k \geq 5$ and $n \geq 2k-2$. This clearly implies that for $k = 6$ the girth of an extremal graph is at most 8 for $10 \leq n \leq 170$.

Let $t = \lceil (k+1)/2 \rceil$. We also prove that the girth of $G \in EX(n; \{C_3, C_4, \ldots, C_k\})$ is at most $k+2$ if $k \geq 7$ and

$$n \geq \frac{2(t-2)^{k-2} + t - 5}{t-3} + 1.$$

From this result it follows for $k = 7$ that if $n \geq 64$, then $g(G) \leq 9$.

2. Main results. The set of neighbors of $u \in V(G)$ is denoted by $N_G(u)$. The number of neighbors of u is the degree $d_G(u)$ of u in G, or briefly $d(u)$ when it is clear which graph is meant. The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x-y$ path in G. The greatest distance between any two vertices in G is the diameter $D(G)$ of G. Diameter and girth are related by $g(G) \leq 2D(G) + 1$. Let $e = xy$ be an edge of G. As usual we will denote by $G/\{e\} = G/e$ the graph obtained from G by contracting the edge e into a new vertex v_e, which becomes adjacent to all the former neighbors of x and y. Taking into account that we dealt with simple graphs of girth at least 4 the resultant graph by any edge contraction remains simple.

Throughout the paper $k \geq 3$ is an integer. We begin by proving a technical and useful lemma.

Lemma 2.1. *Let* $G \in EX(n; \{C_3, \ldots, C_k\})$ *have two distinct edges* e_1 *and* e_2 *such that every cycle of* G *containing both of them has a length of at least* $k+3$. *Then the girth is* $g(G) = k+1$ *if the diameter is* $D(G/\{e_1, e_2\}) \geq k-2$.

Proof. Let $G \in EX(n; \{C_3, \ldots, C_k\})$ satisfy the hypothesis of the lemma and suppose that the girth is $g(G) \geq k+2$. The graph $G' = G/\{e_1, e_2\}$ has $g(G') \geq k+1$ because by hypothesis any cycle passing through both edges e_1 and e_2 has a length of at least $k+3$. Let u', v' be two vertices of G' such that $d_{G'}(u', v') = D(G')$; then by hypothesis $d_{G'}(u', v') = D(G') \geq k-2$. Let us consider the graph G^* obtained from G' by adding two new vertices x_1, x_2 and the three edges $u'x_1, x_1x_2$, and x_2v'. We have $g(G^*) = \min\{g(G'), D(G') + 3\} \geq k+1$, $|V(G^*)| = |V(G')|$

$+ 2 = n$, and $e(G^*) = e(G) + 1$, which contradict the maximality of G. Therefore $g(G) = k + 1$. □

As a first consequence of the above lemma, we obtain in the next theorem an upper bound for the girth of any extremal graph which contains the known result $g = k + 1$ for $k = 5$; see [6].

THEOREM 2.2. *Let $G \in EX\,(n; \{C_3, \ldots, C_k\})$ be for $k \geq 5$ and $n \geq 2k-2$. Then G has a girth of $g(G) \leq 2k-4$.*

Proof. Let $G \in EX\,(n; \{C_3, \ldots, C_k\})$ satisfy the hypothesis of the theorem, and assume the girth of G is $g \geq 2k-2$. Let $C : u_0u_1 \cdots u_{g-1}u_0$ be a girdle in G, and notice that $g \geq k+3$ because $k \geq 5$. The graph $G' = G/\{u_0u_1, u_1u_2\}$ clearly has girth $g(G') \geq 2k-4$; hence the diameter is $D(G') \geq \lfloor g(G')/2 \rfloor \geq \lfloor (2k-4)/2 \rfloor = k-2$. By Lemma 2.1 we have $g = g(G) = k+1$, yielding $2k-2 \leq k+1$, which is a contradiction because $k \geq 5$. Therefore the girth of G is $g \leq 2k-3$. Assume the girth of G is exactly $g = 2k-3$. As $n \geq 2k-2$ the graph G must contain a vertex y not belonging to C. Without loss of generality, suppose that u_0y is an edge of G. Notice that u_{k-2} and u_{k-1}, both belonging to C, satisfy that $d_C(u_0, u_{k-2}) = d_C(u_0, u_{k-1}) = k-2$. Then both $u_0 - u_{k-2}$ and $u_0 - u_{k-1}$ paths contained in C must be the unique shortest $u_0 - u_{k-2}$ and $u_0 - u_{k-1}$ paths in G, because $k-2 = (g-1)/2$. This implies that $d_G(y, u_{k-2}) \geq k-2$ and $d_G(y, u_{k-1}) \geq k-2$ so that every cycle, if any, containing both edges u_0y and $u_{k-2}u_{k-1}$ must have a length of at least $g+1 = 2k-2$, which is at least $k+3$ because $k \geq 5$. Now let $G'' = G/\{u_0y, u_{k-2}u_{k-1}\}$. Clearly, $D(G'') \geq d_{G''}(u_1, u_k) = d_G(u_1, u_k) = k-2$. By Lemma 2.1 we obtain $g(G) = g = k+1$, i.e., $2k-3 \leq k+1$, which is impossible for $k \geq 5$. Hence the girth of G is at most $2k-4$ and the theorem is valid. □

Next, we obtain the following result which is an improvement of Theorem A and also contains the known results for $k = 3, 4, 5$; see [4, 5, 6].

THEOREM 2.3. *Let $G \in EX\,(n; \{C_3, \ldots, C_k\})$. Then $g(G) = k+1$ if either $k = 3$ and $n \geq 5$; or $k = 4$ and $n \geq 9$; or $k = 5$ and $n \geq 8$; or $k = 6$ and $n \geq 171$; or $k \geq 7$ and*

$$n \geq \frac{2(k-2)^{k-2} + k - 5}{k-3} + 1.$$

Proof. From Theorem 2.2 it follows that any graph $G \in EX\,(n; \{C_3, C_4, C_5\})$ for $n \geq 8$ has girth of 6. Therefore we can assume $k = 3, 4$ or $k \geq 6$. Let $G \in EX\,(n; \{C_3, \ldots, C_k\})$ and suppose that its girth is $g(G) \geq k+2$. Then, by Theorem B we have $\Delta \leq k-1$, where Δ denotes the maximum degree of G. Let D be the diameter of G and let us take two vertices x, y at distance $d_G(x, y) = D$. Then $D \leq k-1$ because otherwise by adding the edge xy to G we would obtain a graph G' of order n having girth $g(G') \geq k+1$ and more edges than G, which contradicts the maximality of G. Let us consider the two cases $D = k-1$ and $D \leq k-2$ separately.

Case 1. $D = k-1$. Define the set $N_G^{k-1}(x) = \{y \in V(G) : d_G(x, y) = k-1\}$. Clearly, $|N_G^{k-1}(x)| \geq 1$, because $y \in N_G^{k-1}(x)$. Let us see that $|N_G^{k-1}(x)| = 1$.

Let $W = \{w \in V(G) : d_G(x, w) + d_G(w, y) = k-1\}$ and suppose that there exists a vertex $u \in V(G) \setminus W$. Then $d_G(x, u) + d_G(u, y) \geq k$ or, in other words, all the possible paths passing through u that connect x with y have a length of at least k. Take any vertex $v \in N_G(u)$ and consider the graph G' resulting by contracting the edge uv in G. The girth of this new graph is $g(G') \geq k+1$ and the diameter $D(G') = D = k-1$. So let $x', y' \in V(G')$ be such that $d_{G'}(x', y') = k-1$, and denote by G^* the graph obtained from G' by adding a new vertex x^* and the edges

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$x'x^*$ and x^*y'. Clearly, $|V(G^*)| = |V(G')| + 1 = n$, and girth $g(G^*) = k+1$, but $e(G^*) = e(G') + 2 = e(G) + 1$, which contradicts the maximality of G. Hence, $V(G) = W$, which readily implies that y is the only vertex at distance $D = k-1$ from x and the number of vertices at distance $D-1 = k-2$ from x is at most Δ, since these vertices must be neighbors of y.

Therefore, if $k=3$, then $n \le 1+\Delta+1 \le 1+k = 4$, contradicting the hypothesis for this case. If $k=4$, then $n \le 1+\Delta+\Delta+1 \le 2k = 8$, contradicting again the hypothesis for this case. So assume that $k \ge 6$. As for $1 \le i \le D-2 = k-3$, the maximum number of vertices at distance i from x is $\Delta(\Delta-1)^{i-1}$, we obtain

$$\begin{aligned} n &\le 1 + \Delta\sum_{i=0}^{k-4}(\Delta-1)^i + \Delta + 1 \le 1 + (k-1)\sum_{i=0}^{k-4}(k-2)^i + k \\ &= \frac{(k-1)(k-2)^{k-3}-2}{k-3} + k \\ &< \frac{(k-1)(k-2)^{k-3}-2}{k-3} + (k-2)^{k-3} = \frac{2(k-2)^{k-2}-2}{k-3}. \end{aligned}$$

This contradicts the hypothesis of the theorem, so $g(G) = k+1$ in the case $D = k-1$.

Case 2. $D \le k-2$. Notice that $k = 3, 4$ are impossible for this case because $D \ge \lfloor g/2 \rfloor \ge \lfloor (k+2)/2 \rfloor$. So we have $k \ge 6$.

Let x^* be a vertex of G with degree $d_G(x^*) = \delta$, where δ is the minimum degree of G, and let us denote by $\epsilon(x^*) = \max\{d_G(x^*, y) : y \in V(G)\}$ the eccentricity of x^*. As the diameter is the maximum of the eccentricities we have $\epsilon(x^*) \le D \le k-2$. Suppose first that $\epsilon(x^*) \le k-3$. As for $1 \le i \le k-3$, the maximum number of vertices at distance i from x^* is $\delta(\Delta-1)^{i-1}$, it is immediate that

$$n \le 1 + \delta\sum_{i=0}^{k-4}(\Delta-1)^i \le 1 + (k-1)\sum_{i=0}^{k-4}(k-2)^i \le \frac{(k-1)(k-2)^{k-3}-2}{k-3},$$

which is a contradiction. Therefore $\epsilon(x^*) = k-2$, which means $D = k-2$. Let us consider the set $N_G^{k-2}(x^*) = \{y \in V(G) : d_G(x^*, y) = k-2\}$. Let us prove the following claim.

Claim. Given any vertex $y \in N_G^{k-2}(x^*)$, every neighbor of vertex y is at a distance of $k-3$ from x^*.

Otherwise suppose that there exists a vertex $y_1 \in N_G^{k-2}(x^*) \cap N_G(y)$. Let us denote by $x^* = x_0x_1x_2\cdots x_{k-2} = y$ any shortest $x^* - y$ path. Clearly, every cycle containing both edges x^*x_1 and yy_1, if any, has a length of at least $k+3$ because $k \ge 6$. Then we consider the new graph G' obtained from G by contracting the edges x^*x_1 and yy_1. If the diameter of G' is $D(G') = k-2$, then by Lemma 2.1 we would have $g(G) = k+1$, which is a contradiction with our assumption $g(G) \ge k+2$. Therefore $D(G') = k-3$, which implies that for all $z \in N(x^*)$, $d_G(z, y') = k-3$ for all $y' \in N_G^{k-2}(x^*)$. Consequently, the edge yy_1 and any vertex $z \in N_G(x^*)$ lies on a cycle in G of length at most $2k-5$, which is impossible for $k=6$ because $g \ge k+2$. Hence every neighbor of vertex y is at a distance of $k-3$ from x^* when $k=6$ and the claim is true for this case.

Furthermore, for $k \ge 7$ we have $d_{G'}(v_{x^*x_1}, v_{yy_1}) = k-3$, where $v_{x^*x_1}$ and v_{yy_1} denote the newly arising vertices by the contraction of the edges x^*x_1 and yy_1. Besides, $d_{G'}(v_{x^*x_1}) = d_G(x^*) + d_G(x_1) - 2 \le \delta + \Delta - 2 \le 2(\Delta-1)$ and $d_{G'}(v_{yy_1}) =$

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$d_G(y) + d_G(y_1) \leq 2(\Delta - 1)$. Therefore,

$$V(G') = \{v_{x^*x_1}\} \cup \bigcup_{i=1}^{k-3} N^i_{G'}(v_{x^*x_1}),$$

where $N^i_{G'}(v_{x^*x_1})$ denotes the set of vertices of G' at a distance of i from vertex $v_{x^*x_1}$. Thus $\left|N^i_{G'}(v_{x^*x_1})\right| \leq 2(\Delta-1)(\Delta-1)^{i-1} = 2(\Delta-1)^i$, for $i = 1, \ldots, k-3$, and we get

$$\begin{aligned} n = 2 + |V(G')| &\leq 3 + 2\sum_{i=1}^{k-3}(\Delta-1)^i \\ &\leq 3 + 2\sum_{i=1}^{k-3}(k-2)^i \\ &= 3 + \frac{2(k-2)^{k-2} - 2(k-2)}{k-3} = \frac{2(k-2)^{k-2} + k - 5}{k-3}, \end{aligned}$$

contradicting the hypothesis of the theorem. Thus, every vertex $y \in N^{k-2}_G(x^*)$ has all its neighbors at distance $k-3$ from x^* and the claim holds.

Hence, $|N^i_G(x^*)| \leq \delta(\Delta-1)^{i-1}$, for $i = 1, \ldots, k-3$, and $|N^{k-2}_G(x^*)| \leq (\Delta-1)^{k-3}$. Then, for $k \geq 6$ we have

$$\begin{aligned} n &\leq 1 + \delta\sum_{i=0}^{k-4}(\Delta-1)^i + (\Delta-1)^{k-3} \\ &\leq 1 + \delta\sum_{i=0}^{k-4}(k-2)^i + (k-2)^{k-3} \\ &\leq \frac{(k-1)(k-2)^{k-3} - 2}{k-3} + (k-2)^{k-3} = \frac{2(k-2)^{k-2} - 2}{k-3}. \end{aligned}$$

This contradicts the hypothesis of the theorem, so we conclude that $g(G) = k+1$. ∎

Next, the goal is to provide a lower bound on n in order to guarantee that the girth is at most $k+2$ for $k \geq 7$. To do that first we state that an extremal $\{C_3, \ldots, C_k\}$-free graph with maximum degree $\Delta \geq \lceil (k+1)/2 \rceil$ has necessarily a girth of at most $k+2$.

THEOREM 2.4. *Let $k \geq 7$ be an integer. Let G be a graph belonging to the family $EX(n; \{C_3, \ldots, C_k\})$ with a minimum degree of at least 2 and maximum degree Δ. Then $g(G) \leq k+2$ if $\Delta \geq \lceil (k+1)/2 \rceil$.*

Proof. Let $G \in EX(n; \{C_3, \ldots, C_k\})$ satisfy the hypothesis of the theorem, and assume $g(G) \geq k+3$. Let x be a vertex of maximum degree Δ and let $y_1, y_2, \ldots, y_\Delta$ be all the neighbors of x. Since $d_G(y_i) \geq 2$, for each $i = 1, \ldots, \Delta$, there exists $x_i \in V(G) - x$ adjacent to y_i. Notice also that $x_i \neq x_j$ for all $i \neq j$, since $g(G) > 4$. Taking into account that $g(G) \geq k+3$, we deduce that $d_{G-x}(x_i, x_j) \geq g(G) - 4 \geq k-1$, $d_{G-x}(y_i, y_j) \geq g(G) - 2 \geq k+1$, and $d_{G-x}(x_i, y_j) \geq g(G) - 3 \geq k$ for all $i, j = 1, \ldots, \Delta$ with $i \neq j$. Let G^* be the graph obtained from G by first deleting the $\Delta-1$ edges $xy_2, \ldots, xy_\Delta$ and second adding the new Δ edges $y_1x_2, \ldots, y_{\Delta-1}x_\Delta, y_\Delta x_1$. Then G^* has order n and size $e(G^*) = e(G) + 1$. Since G is extremal, G^* must contain a cycle of length at most k. Let us denote by C^* a shortest cycle in G^* (notice that

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$x \notin V(C^*)$, since x has degree 1 in G^*). We denote by C the cycle $x_1y_1x_2y_2\cdots x_\Delta y_\Delta x_1$ which has length $2\Delta \geq k+1$. Observe that C is an induced cycle of G^*, since x_i is nonadjacent to y_j in G, for any $i \neq j$ and the only newly introduced edges are $y_i x_{i+1}$ for $i = 1, \ldots, \Delta - 1$ and $y_\Delta x_1$. Moreover, $C^* \neq C$, since $g(C) \geq k+1$ and $g(C^*) \leq k$. So, we may express $C^* = P_1 \cup P_2$, where P_1 is the longest path whose edges belong to the set $E(C^*)\setminus E(C) \subseteq E(G-x)$, and P_2 is the rest of C^*. Notice that the endvertices of P_1 must belong to $\{x_1, \ldots, x_\Delta\} \cup \{y_1, \ldots, y_\Delta\}$ by the construction of P_1. Observe also that P_2 contains at least one edge of $E(C)$, because otherwise the cycle C^* would be contained in G against the assumption $g(G) \geq k+3$. If the endvertices of P_1 are x_i and x_j for certain $i, j \in \{1, 2\ldots, \Delta\}$, then the edge $y_{i-1}x_i$ or x_iy_i and the edge $y_{j-1}x_j$ or x_jy_j must be contained in P_2 and then $e(P_2) \geq 2$. This implies that $|V(C^*)| = e(C^*) = e(P_1) + e(P_2) \geq d_{G-x}(x_i, x_j) + 2 \geq k - 1 + 2 = k+1$; a contradiction. If the endvertices of P_1 are x_i and y_i, for some $i \in \{1, \ldots, \Delta\}$, then $e(P_1) \geq d_{G-x-\{x_iy_i\}}(x_i, y_i) \geq g(G) - 1 \geq k+2$ and hence $|V(C^*)| = e(C^*) = e(P_1) + e(P_2) \geq k+3$, again a contradiction. Otherwise,

$$e(P_1) \geq \min\{d_{G-x}(y_i, y_j), d_{G-x}(x_i, y_j) : i, j = 1, \ldots, \Delta \text{ and } i \neq j\} \geq k,$$

which implies $|V(C^*)| = e(C^*) = e(P_1) + e(P_2) \geq k+1 > k$, arriving at a contradiction. Hence, $g(G) \leq k+2$. ∎

From Theorem 2.4 we derive the following sufficient condition in terms of the order for an extremal $\{C_3, \ldots, C_k\}$-free graph to have girth at most $k+2$.

THEOREM 2.5. *Let $G \in EX(n; \{C_3, \ldots, C_k\})$ be of a minimum degree of at least 2. Then the girth is $g(G) \leq k+2$ if $k \geq 7$ and*

$$n \geq \frac{2(t-2)^{k-2} + t - 5}{t-3} + 1,$$

where $t = \lceil (k+1)/2 \rceil$.

Proof. If $\Delta \geq \lceil (k+1)/2 \rceil$, then $g(G) \leq k+2$ for $k \geq 7$ because of Theorem 2.4 and the theorem holds. Hence assume $\Delta \leq \lceil (k+1)/2 \rceil - 1$ and $g(G) \geq k+3$. Let $t = \lceil (k+1)/2 \rceil$. As in the proof of Theorem 2.3 we consider two cases $D = k-1$ and $D \leq k-2$ separately and repeat this proof but taking into account that now $\Delta \leq t-1$ instead of $\Delta \leq k-1$. In this way we arrive at a contradiction, which implies $g(G) \leq k+2$, and the theorem holds. ∎

As an immediate consequence of Theorems 2.3 and 2.5, the following information about the girth of any extremal $\{C_3, \ldots, C_7\}$-free graph is provided.

COROLLARY 2.6. *Let G be a graph belonging to the family $EX(n; \{C_3, \ldots, C_7\})$. Then the girth $g(G) = 8$ if $n \geq 783$, and the girth is $g(G) \leq 9$ if $n \geq 64$.*

3. Conclusions. Theorem 2.3 can be compared with Theorem A. Both results give a sufficient condition on the order of an extremal graph to contain a cycle of minimum length $k+1$. Recall that $a = k - 3 - \lfloor (k-2)/4 \rfloor$; then for $k \geq 12$ we have $2^a > (k-2)^2$. Hence $2^{a^2+a+1} > 2(k-2)^{2a+2} \geq 2(k-2)^{(3k-6)/2}$, and thus $n \geq 2^{a^2+a+1}k^a > 2(k-2)^{(3k-6)/2}k^a$ (which is much larger than the requirement obtained in Theorem 2.3), $n > (2(k-2)^{k-2} + k - 5)/(k-3)$.

Moreover, Theorems 2.2 and 2.3 provide information on the girth of any extremal $\{C_3, C_4, C_5, C_6\}$-free graph G. The girth is $g(G) = 7$ if $n \geq 171$, and the girth is $g(G) \leq 8$ if $n \geq 10$. It is known for $r = 3, 4, 5$ that each $(r; 8)$-cage is the incidence graph of a projective geometry called *generalized quadrangle*; see the survey by Wong [8]. The order of each of these graphs is $30, 80, 170$, respectively. As a referee suggests,

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it appears that a result of Alon, Hoory, and Linial [1] can be used to show these cages do belong to $EX(n;\{C_3,C_4,C_5,C_6,C_7\})$. The question is if these cages are also $\{C_3,C_4,C_5,C_6\}$-free extremal. We would like to suggest the following open problems.

PROBLEM 1. *Prove or disprove that each* $(r;8)$*-cage for* $r=3,4,5$ *is a graph belonging to* $EX\,(n;\{C_3,C_4,C_5,C_6\})$*, for* $n=30,80,170$.

PROBLEM 2. *Is it possible to improve the lower bound on* n *in Theorem* 2.3 *for* $k\geq 7$*?*

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REFERENCES

[1] N. Alon, S. Hoory, and N. Linial, *The Moore bound for irregular graphs*, Graphs Combin., 18 (2002), pp. 53–57.

[2] R. Diestel, *Graph Theory*, 3rd ed., Springer-Verlag, Berlin, 2005.

[3] P. Erdös and H. Sachs, *Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe, 12 (1963), pp. 251–257.

[4] D. K. Garnick, Y. H. H. Kwong, and F. Lazebnik, *Extremal graphs without three-cycles or four-cycles*, J. Graph Theory, 17 (1993), pp. 633–645.

[5] D. K. Garnick and N. A. Nieuwejaar, *Nonisomorphic extremal graphs without three-cycles or four-cycles*, J. Combin. Math. Combin. Comput., 12 (1993), pp. 33–56.

[6] F. Lazebnik and P. Wang, *On the structure of extremal graphs of high girth*, J. Graph Theory, 26 (1997), pp. 147–153.

[7] L. Lovász, *Combinatorial Problems and Exercises*, North–Holland, Amsterdam, 1979.

[8] P. K. Wong, *Cages—A survey*, J. Graph Theory, 6 (1982), pp. 1–22.