

# Greedy Minimization of Weakly Supermodular Set Functions

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## Abstract

This paper defines *weak- $\alpha$ -supermodularity* for set functions. It shows that minimizing such functions under cardinality constraints is a common task in machine learning and data mining. Moreover, any problem whose objective function exhibits this property benefits from a greedy extension phase. Explicitly, let  $S^*$  be the optimal set of cardinality  $k$  that minimizes  $f$  and let  $S_0$  be an initial solution such that  $f(S_0) \leq \rho f(S^*)$ . Then, a greedy extension  $S \supset S_0$  of size  $|S| \leq |S_0| + \lceil \alpha k \ln(\rho/\varepsilon) \rceil$  yields  $f(S) \leq (1 + \varepsilon)f(S^*)$ .

Example usages of this framework give streamlined proofs and new bi-criteria results for  $k$ -means, sparse regression, column subset selection, and sparse convex function minimization. Sparse regression and column subset selection are special cases of a new, more general, sparse multiple linear regression problem that is of independent interest. This paper also corrects a brittleness of the proof of Natarajan for the properties of the greedy algorithm for sparse regression.

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## 1 Introduction

Many problems in data mining and unsupervised machine learning take the form of minimizing a set function with cardinality constraints. More explicitly, denote by  $[n]$  the set  $\{1, \dots, n\}$  and  $f(S) : 2^{[n]} \rightarrow \mathbb{R}_+$ . Our goal is to minimize  $f(S)$  subject to  $|S| \leq k$ . These problems include clustering and covering problems as well as sparse regression, matrix approximation problems and many others. These combinatorial problems are hard to minimize in general. Finding good (e.g. constant factor) approximate solutions for them requires significant sophistication and highly specialized algorithms.

In this paper we analyze the behavior of the greedy algorithm to all of these problems. We start by claiming that the functions above are special. A trivial observation is that they are non-negative and non-increasing, that is,  $f(S) \geq f(S \cup T) \geq 0$  for any  $S, T \subseteq [n]$ . This immediately shows that expanding solution sets is (at least potentially) beneficial in terms of reducing the function value. But, monotonicity is not sufficient to ensure that any number of greedy extensions of a given solution would significantly reduce the objective function.

To this end we need to somehow quantify the gain of adding a single element (greedily) to a solution set. Let  $f(S) - f(S \cup T)$  be the reduction in  $f$  one gains by adding a set



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of elements  $T$  to the current solution  $S$ . Then, the average gain of adding elements from  $T$  *sequentially* is  $[f(S) - f(S \cup T)]/|T \setminus S|$ . One would hope that there exists an element in  $i \in T \setminus S$  such  $f(S) - f(S \cup \{i\}) \geq [f(S) - f(S \cup T)]/|T \setminus S|$ . However, that would be false in general because different element contributions are not independent of each other. Nevertheless, it is true for supermodular functions (see Fact 3).

Combining this fact with the idea that  $T$  could be any set, including the optimal solution  $S^*$ , already gives some useful results for minimizing supermodular set functions. Specifically those for which  $f(S^*)$  is bounded away from zero. Notice that  $k$ -means clustering (defined below) is exactly this kind of problem. Section 4 gives some new bicriteria results obtainable for  $k$ -means via the greedy extension algorithm of Section 3.

Alas, most problems of interest, such as regression, column subset selection, and feature selection are not supermodular. In Section 2 we define the notion of weak- $\alpha$ -supermodularity. Intuitively, weak- $\alpha$ -supermodular functions are those conducive to greedy type algorithms. The property requires that there exists an element  $i \in T \setminus S$  such that adding  $i$  *first* gains at least  $[f(S) - f(S \cup T)]/\alpha|T \setminus S|$  for some  $\alpha \geq 1$ .

An analogous relaxation of the submodular property for set functions was considered in [5] (see definition 2.3). They define a *submodularity-ratio* for set functions which are not submodular. They show that if the *submodularity-ratio* is bounded, a greedy algorithm can be used to obtain bi-criteria results for the maximization problem. The work of [5] can be viewed as a direct extension of well know fact. Namely that the greedy algorithm provides a  $(1 - 1/e)$ -factor approximation for maximizing set functions  $g(S)$  subject to  $|S| \leq k$  if  $g$  for positive, monotone non-decreasing and submodular set functions [15].

This paper complements both [15] and [5] for the intuitively related process of greedily minimizing supermodular functions. While our setting is not significantly more complex it is quite different. In contrast to maximizing submodular functions, minimizing supermodular functions is, in general, hard [10]. The difficulty arises from the fact that a value of zero of the objective function could force any constant factor approximation algorithm to find an optimal solution. Our work cannot overcome this fundamental (and unresolvable) difficulty.

We consider the case where either the objective function is bounded away from zero or one could obtain an approximate initial solution. In that case, supermodularity (or weak- $\alpha$ -supermodularity) is shown to be sufficient for obtaining good bi-criteria results using the greedy algorithm. Section 2 includes notations and concepts that will be used throughout the paper. In section 3 we present two generic greedy algorithms and analyze their guaranties for weak- $\alpha$ -supermodular functions.

Many important problems in data mining and machine learning fall into this regime. As a warm-up, in Section 4 we obtain new bi-criteria results for  $k$ -means clustering, the objective function of which is supermodular. Section 5 presents the sparse multiple linear regression (SMLR) and shows that it is weakly- $\alpha$ -supermodular. We then streamline and slightly improve the result of [14] for sparse regression, also known as feature selection. Column Subset Selection (CSS) for matrix approximation is an instance of SMLR. Section 7 gives new bi-criteria results for CSS with little additional effort. Finally, we recreate the result of [16] for minimizing smooth and strongly convex functions with sparse solutions. The result is equivalent but the proof is simpler and shorter.

## 2 Preliminaries and definitions

Throughout the manuscript we denote by  $[n]$  the set  $\{1, \dots, n\}$ . We concern ourselves with non-negative set function  $f(S) : 2^{[n]} \rightarrow \mathbb{R}_+$ . More specifically monotone non-increasing set function such that  $f(S) \geq f(S \cup T)$  for any two sets  $S \subseteq [n]$  and  $T \subseteq [n]$ .

**Algorithm 1** Greedy Extension Algorithm

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**Input:** Weakly- $\alpha$ -supermodular function  $f(S)$ , initial set  $S_0$ , parameters  $k \in \mathbb{Z}_+$  and the sequence  $\Lambda_1, \Lambda_2, \dots$

**while**  $t \leq \lceil \alpha k \ln \Lambda_t \rceil$  **do**

$S_t \leftarrow S_{t-1} \cup \arg \min_{i \in [n]} f(S_{t-1} \cup \{i\})$

**Output:**  $S_t$

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► **Definition 1.** A set function  $f(S) : 2^{[n]} \rightarrow \mathbb{R}_+$  is said to be *supermodular* if for any two sets  $S, T \subseteq [n]$

$$f(S \cap T) + f(S \cup T) \geq f(S) + f(T). \quad (1)$$

► **Definition 2.** A non-negative non-increasing set function  $f(S) : 2^{[n]} \rightarrow \mathbb{R}_+$  is said to be *weakly- $\alpha$ -supermodular* if there exists  $\alpha \geq 1$  such that for any two sets  $S, T \subseteq [n]$

$$f(S) - f(S \cup T) \leq \alpha \sum_{i \in T \setminus S} (f(S) - f(S \cup \{i\})). \quad (2)$$

This property is useful because we will later try to minimize  $f$ . It asserts that if adding  $T \setminus S$  is beneficial then there is an element  $i \in T \setminus S$  that contributes at least a fraction of that. The reason for the name of this property might also be explained by the following definition and lemma.

► **Fact 3.** A non-increasing non-negative supermodular function  $f$  is weakly- $\alpha$ -supermodular with parameter  $\alpha = 1$ .

**Proof.** For  $S, T \subseteq [n]$  order the set  $T \setminus S$  in an arbitrary order, i.e.  $T \setminus S = \{i_1, \dots, i_{|T \setminus S|}\}$ . Define  $R_0 = \emptyset$  and  $R_t = \{i_1, \dots, i_t\}$  for  $t > 0$ . By supermodularity we have for any  $t$

$$f(S) - f(S \cup \{i_t\}) \geq f(S \cup R_{t-1}) - f(S \cup R_{t-1} \cup \{i_t\}) \quad (3)$$

We note that  $R_{t-1} \cup \{i_t\} = R_t$  and sum up Equation (3).

$$\begin{aligned} \sum_{t=1}^{|T \setminus S|} [f(S) - f(S \cup \{i_t\})] &\geq \sum_{t=1}^{|T \setminus S|} f(S \cup R_{t-1}) - f(S \cup R_{t-1} \cup \{i_t\}) \\ &= f(S) - f(S \cup T). \end{aligned}$$

Since  $|T \setminus S| \cdot \max_{i \in T \setminus S} [f(S) - f(S \cup \{i\})] \geq \sum_{t=1}^{|T \setminus S|} [f(S) - f(S \cup \{i_t\})]$  this implies weak-1-supermodularity. ◀

### 3 General Greedy Extension Algorithms

We are given a weakly- $\alpha$ -supermodular set function  $f(S)$  and would like to solve the following optimization problem

$$\min\{f(S) : |S| \leq k\}. \quad (4)$$

Let  $0 < \Lambda_1 \leq \Lambda_2 \leq \dots$  be a non-decreasing bounded sequence of reals, i.e.  $\max_t \Lambda_t < +\infty$ . Our algorithm works in phases and we may assume that  $\Lambda_t$  is computed on step  $t$  of the algorithm. Consider a simple greedy algorithm that starts with some initial solution  $S_0$  of value  $f(S_0)$  (maybe  $S_0 = \emptyset$ ) and sequentially and greedily adds elements to it to minimize  $f$ .

Note that since the sequence  $\Lambda_t$  is bounded the algorithm terminates after at most  $\lceil \alpha k \ln(\max_t \Lambda_t) \rceil$  iterations.

**Algorithm 2** Greedy Extension Algorithm

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**Input:** Weakly- $\alpha$ -supermodular function  $f(S)$ , initial set  $S_0$ ,  $k \in \mathbb{Z}_+$

**while**  $t \leq \lceil \alpha k \ln(f(S_0)/\varepsilon f(S_{t-1})) \rceil$  **do**

$S_t \leftarrow S_{t-1} \cup \arg \min_{i \in [n]} f(S_{t-1} \cup \{i\})$

**Output:**  $S_t$

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► **Lemma 4.** *Let  $S_\tau$  be the output of Algorithm 1. Then  $|S_\tau| \leq |S_0| + \lceil \alpha k \ln \Lambda_\tau \rceil$  and  $f(S_\tau) \leq f(S^*) + \frac{f(S_0) - f(S^*)}{\Lambda_{\tau+1}}$  where  $S^*$  is an optimal solution of the optimization problem (4).*

**Proof.** The fact that  $|S_\tau| \leq |S_0| + \lceil \alpha k \ln \Lambda_\tau \rceil$  is a trivial observation. For the second claim consider an arbitrary iteration  $t \in [\tau]$  and consider the set  $S^* \setminus S_{t-1}$ . By monotonicity and weak  $\alpha$ -supermodularity

$$\begin{aligned} f(S_{t-1}) - f(S^*) &\leq f(S_{t-1}) - f(S_{t-1} \cup S^*) \\ &\leq \alpha \cdot \sum_{i \in S^* \setminus S_{t-1}} f(S_{t-1}) - f(S_{t-1} \cup \{i\}) \\ &\leq \alpha k \cdot \max_{i \in [n]} f(S_{t-1}) - f(S_{t-1} \cup \{i\}) \\ &= \alpha k \cdot (f(S_{t-1}) - f(S_t)). \end{aligned}$$

By rearranging the above equation and recursing over  $t$  we get

$$f(S_t) - f(S^*) \leq (f(S_{t-1}) - f(S^*)) (1 - 1/\alpha k) \leq (f(S_0) - f(S^*)) (1 - 1/\alpha k)^t$$

Substituting  $\tau + 1 > \lceil \alpha k \ln \Lambda_{\tau+1} \rceil$  for the last step of the algorithm completes the proof.

$$\begin{aligned} f(S_\tau) - f(S^*) &\leq (f(S_0) - f(S^*)) (1 - 1/\alpha k)^{\alpha k \ln \Lambda_{\tau+1}} \\ &\leq (f(S_0) - f(S^*)) e^{-\ln \Lambda_{\tau+1}} \leq \frac{f(S_0) - f(S^*)}{\Lambda_{\tau+1}}. \end{aligned} \quad \blacktriangleleft$$

► **Theorem 5.** *Let  $S_\tau$  be the output of Algorithm 2 which is an instantiation of Algorithm 1 with parameters  $\Lambda_t = f(S_0)/\varepsilon f(S_{t-1})$  for some error  $\varepsilon \geq 0$ . Then  $|S_\tau| \leq |S_0| + \lceil \alpha k \ln(f(S_0)/\varepsilon f(S^*)) \rceil$  and  $f(S_\tau) \leq f(S^*)/(1 - \varepsilon)$  where  $S^*$  is an optimal solution of the optimization problem (4).*

**Proof.** By Lemma 4 we have

$$|S_\tau| \leq |S_0| + \lceil \alpha k \ln \Lambda_\tau \rceil \leq |S_0| + \lceil \alpha k \ln(f(S_0)/\varepsilon f(S^*)) \rceil$$

and

$$\begin{aligned} f(S_\tau) &\leq f(S^*) + \frac{f(S_0) - f(S^*)}{\Lambda_{\tau+1}} \\ &= f(S^*) + \frac{f(S_0) - f(S^*)}{f(S_0)} \varepsilon f(S_\tau) \leq f(S^*) + \varepsilon f(S_\tau). \end{aligned} \quad \blacktriangleleft$$

► **Theorem 6.** *Assume there exist a  $\rho$ -approximation algorithm creating  $S_0$  such that  $f(S_0) \leq \rho f(S^*)$ . There exists an algorithm for generating  $S$  such that  $|S| \leq |S_0| + \lceil \alpha k (\ln \frac{\rho}{\varepsilon}) \rceil$  and  $f(S) \leq f(S^*)/(1 - \varepsilon)$ .*

**Proof.** Use the  $\rho$ -approximation algorithm to create  $S_0$  for Algorithm 1 and apply Theorem 5. ◀

**Algorithm 3** Greedy Extension Algorithm; an alternative stopping criterion**Input:** Weakly- $\alpha$ -supermodular function  $f$ ,  $S_0$ ,  $f_{\text{stop}}$ **repeat** $S_t \leftarrow S_{t-1} \cup \arg \min_i f(S_{t-1} \cup \{i\})$ **until**  $f(S_t) \leq f_{\text{stop}}$ **Output:**  $S = S_t$ 

► **Theorem 7.** Let  $k'$  be the minimal cardinality of a set  $S'$  such that  $f(S') \leq f'$ . For any  $f_{\text{stop}}$  and an initial set  $S_0$  such that  $f' < f_{\text{stop}} < f(S_0)$  Algorithm 3 outputs  $S$  such that

$$|S| \leq |S_0| + \left\lceil \alpha k' \left( \ln \frac{f(S_0) - f'}{f_{\text{stop}} - f'} \right) \right\rceil$$

**Proof.** Let  $f' = f(S')$ . The proof follows from Lemma 4 by setting  $k = k_f$ ,  $\Lambda_t = \frac{f(S_0) - f'}{f_{\text{stop}} - f'}$  and noticing that  $\frac{f(S_0) - f'}{f_{\text{stop}} - f'} \leq \frac{f(S_0) - f}{f_{\text{stop}} - f}$ . ◀

This alternative algorithm will be used in Section 6

**4**  **$k$ -means Clustering**

As a gentle introduction we begin with deriving new bi-criteria results for the  $k$ -means clustering problem. We begin by defining the constrained  $k$  means problem.

► **Definition 8 (Constrained  $k$ -means).** Given a set of  $n$  points  $X \subset \mathbb{R}^d$ , find a set  $S \subset X$  minimizing  $f(S) = \sum_{x \in X} \min_{x' \in S} \|x - x'\|^2$  subject to  $|S| \leq k$ .

► **Lemma 9.** For the constrained  $k$ -means problem, one can find in  $O(n^2 dk \log(1/\varepsilon))$  time a set  $S$  of size  $|S| = O(k) + k \log(1/\varepsilon)$  such that  $f(S) \leq (1 + \varepsilon)f(S^*)$  where  $f(S^*)$  is the optimal solution.

**Proof.** The constrained  $k$ -means objective function  $f$  is weakly-1-supermodular because it is supermodular (Fact 3). This is both well known and not hard to verify. Using the algorithm of [1] one obtains a set  $S_0$  of size  $|S_0| = O(k)$  points from  $X$  for which  $f(S_0) = O(f(S^*))$ . Their technique improves on the analysis of well known  $k$ -means++ adaptive sampling scheme of [2]. Greedily extending  $S_0$  and applying the analysis of Theorem 5 completes the proof. The quadratic dependency of the running time on the number of data points can be alleviated using the corset construction of [8, 9] ◀

► **Definition 10 (Unconstrained  $k$ -means).** Given a set of  $n$  points  $X \subset \mathbb{R}^d$ , find a set  $S \subset \mathbb{R}^d$  minimizing  $f(S) = \sum_{x \in X} \min_{c \in S} \|x - c\|^2$  subject to  $|S| \leq k$ .

► **Lemma 11.** Let  $f(S^*)$  be the optimal solution to the unconstrained  $k$ -means problem. One can find in time  $O(n^2 dk \log(1/\varepsilon))$  a set  $S \in \mathbb{R}^d$  of size  $|S| = O(k) + k \log(1/\varepsilon)$  such that  $f(S) \leq (2 + \varepsilon)f(S^*)$ .

**Proof.** The proof and the algorithm are identical to the above. The only point to note is that a  $1 + \varepsilon/2$  approximation to the constrained problem is at most a  $2 + \varepsilon$  approximation to the unconstrained one. See [2], for example, for the argument that the minimum of the constrained objective is at most twice that of the unconstrained one. ◀

Alternatively, we can utilize a more computationally expensive approach which goes through a reduction to the  $k$ -median problem.

► **Definition 12** (*k*-Median). We are given a set  $X$  of data points, the set  $\mathcal{C}$  of potential cluster center locations and the nonnegative costs  $w_{ij} \geq 0$  for all  $i, j \in X \times \mathcal{C}$ . Find a set  $S \subset \mathcal{C}$  minimizing  $f(S) = \sum_{i \in X} \min_{j \in \mathcal{C}} w_{ij}$  subject to  $|S| \leq k$ .

It is known that given an instance  $(X, k)$  of the Unconstrained *k*-means problem one can construct in polynomial time an instance of the *k*-Median problem  $(X, \mathcal{C}, w, k)$  where  $\mathcal{C} \subseteq \mathbb{R}^d$  such that for any solution of value  $\Phi$  for the Unconstrained *k*-means problem there exists a solution of value  $(1+\varepsilon)\Phi$  for the corresponding instance of the *k*-Median problem (see Theorem 7 [13]). Moreover,  $|\mathcal{C}| = n^{O(\log(1/\varepsilon)/\varepsilon^2)}$ . Therefore, after applying this transformation on our instance of the Unconstrained *k*-means and using the same initial solution  $S_0$  as in Lemma 11 we derive.

► **Lemma 13.** *Let  $f(S^*)$  be the optimal solution to the unconstrained *k*-means problem. One can find in time  $O(n^{O(\log(1/\varepsilon)/\varepsilon^2)} dk)$  a set  $S \in \mathbb{R}^d$  of size  $|S| = O(k) + k \log(1/\varepsilon)$  such that  $f(S) \leq (1 + \varepsilon)f(S^*)$ .*

## 5 Sparse Multiple Linear Regression

We begin by defining the Sparse Multiple Linear Regression (SMLR) problem. Given two matrices  $X \in \mathbb{R}^{m \times n}$  and  $Y \in \mathbb{R}^{m \times \ell}$ , and an integer  $k$  find a matrix  $W \in \mathbb{R}^{n \times \ell}$  that minimizes  $\|XW - Y\|_F^2$  subject to  $W$  having at most  $k$  non zero rows. We assume for notational brevity (and w.l.o.g.) that the columns of  $X$  have unit norm. An alternative and equivalent formulation of SMLR is as follows. Let  $X_S$  be a submatrix of the matrix  $X$  defined by the columns of  $X$  indexed by the set  $S \subseteq [n]$ . Let  $X_S^+$  be the Moore-Penrose pseudo-inverse of  $X_S$ . It is well-known that the minimizer of  $\|XW - Y\|_F^2$  subject to  $W$  whose non zero rows are indexed by  $S$  is equal to  $\|Y - X_S X_S^+ Y\|_F^2$ . SMLR can therefore be reformulated as

$$\min_{S \subseteq [n]} \{f(S) = \|Y - X_S X_S^+ Y\|_F^2 : |S| \leq k\}.$$

We can consequently apply our methodology from Section 3 to SMLR if we show that  $f(S)$  is  $\alpha$ -weakly-supermodular.

► **Lemma 14.** *For  $X \in \mathbb{R}^{m \times n}$  and  $Y \in \mathbb{R}^{m \times \ell}$  the SMLR minimization function  $f(S) = \|Y - X_S X_S^+ Y\|_F^2$  is  $\alpha$ -weakly-supermodular with  $\alpha = \max_{S'} \|X_{S'}^+\|_2^2$ .*

**Proof.** We first estimate  $f(S) - f(S \cup T)$ . Denote by  $Z_{T \setminus S}$  the matrix whose columns are those of  $X_{T \setminus S}$  projected away from the span of  $X_S$  and normalized. More formally,  $\zeta_i = \|(I - X_S X_S^+) x_i\|$  and  $z_i = (I - X_S X_S^+) x_i / \zeta_i$  for all  $i \in T \setminus S$ . Note that the column span of  $Z_{T \setminus S}$  is orthogonal to that of  $X_S$  and that together they are equal to the column span of  $X_{T \cup S}$ . Using the Pythagorean theorem and the fact that  $X_S X_S^+$  is a projection we obtain  $f(S) = \|Y\|_F^2 - \|X_S X_S^+ Y\|_F^2$  and  $f(S \cup T) = \|Y\|_F^2 - \|X_S X_S^+ Y\|_F^2 - \|Z_{S \setminus T} Z_{S \setminus T}^+ Y\|_F^2$ . Substituting  $T = \{i\}$  also gives  $f(S) - f(S \cup \{i\}) = \|z_i z_i^T Y\|_F^2 = \|z_i^T Y\|_2^2$ .

$$f(S) - f(S \cup T) = \|Z_{T \setminus S} Z_{T \setminus S}^+ Y\|_F^2 \quad (5)$$

$$= \|(Z_{T \setminus S}^T)^+ \cdot Z_{T \setminus S}^T Y\|_F^2 \quad \text{By SVD} \quad (6)$$

$$\leq \|(Z_{T \setminus S}^T)^+\|_2^2 \cdot \|Z_{T \setminus S}^T Y\|_F^2 \quad (7)$$

$$= \|Z_{T \setminus S}^+\|_2^2 \cdot \sum_{i \in T \setminus S} \|z_i^T Y\|_2^2 \quad (8)$$

$$\leq \|X_{T \cup S}^+\|_2^2 \cdot \sum_{i \in T \setminus S} \|z_i^T Y\|_2^2 \quad \text{See below} \quad (9)$$

$$= \alpha \cdot \sum_{i \in T \setminus S} (f(S) - f(S \cup \{i\})) \quad (10)$$

For Equation (9) we use a non trivial transition,  $\|Z_{T \setminus S}^+\|_2 \leq \|X_{T \cup S}^+\|_2$ . By the definition of  $Z_{T \setminus S}$  we can write for  $i \in T \setminus S$  that  $z_i = (x_i - \sum_{j \in S} \alpha_{ij} x_j) / \zeta_i$  and  $\zeta_i = \|(I - X_S X_S^+) x_i\|$ . For any vector  $w$

$$Z_{T \setminus S} w = \sum_{i \in T \setminus S} x_i w_i / \zeta_i + \sum_{j \in S} x_j \sum_{i \in T \setminus S} w_i \alpha_{ij} / \zeta_i = X_{T \cup S} w'$$

where  $w'_i = w_i / \zeta_i$  for  $i \in T \setminus S$  and  $w'_j = \sum_{i \in T \setminus S} w_i \alpha_{ij} / \zeta_i$  for  $j \in S$ . Since,  $\zeta_i = \|(I - X_S X_S^+) x_i\| \leq \|x_i\| = 1$  we have  $\|w'\| \geq \|w\|$ . Finally, consider  $w$  such that  $\|w\| = 1$  and  $\|Z_{T \setminus S} w\| = \|Z_{T \setminus S}^+\|^{-1}$ . This is the right singular vector corresponding to the smallest singular value of  $Z_{T \setminus S}$ . We obtain

$$\|Z_{T \setminus S}^+\|^{-1} = \|Z_{T \setminus S} w\| = \|X_{T \cup S} w'\| \geq \|X_{T \cup S}^+\|^{-1} \|w'\| \geq \|X_{T \cup S}^+\|^{-1}.$$

This completes the proof.  $\blacktriangleleft$

► **Lemma 15.** *Let  $f(S^*)$  be the optimal solution to the Sparse Multiple Linear Regression problem. One can find in time  $O(\alpha k \log(\|Y\|_F^2 / \varepsilon) \cdot n T_f)$  a set  $S \subseteq [n]$  of size  $|S| = \lceil \alpha k \log(\|Y\|_F^2 / \varepsilon) \rceil$  such that  $f(S) \leq f(S^*) / (1 - \varepsilon)$  where  $T_f$  is the time needed to compute  $f(S)$  once.*

## 6 Sparse Regression

The problem of Sparse Regression defined in [14] is an instance of SMLR where the number of columns in  $W$  and  $Y$  is  $\ell = 1$ . Since both  $W$  and  $Y$  are vectors we reduce to the more familiar form of this problem; minimize  $\|Xw - y\|_2^2$  subject to  $\|w\|_0 \leq k$ .

Natarajan [14] analyzes the greedy algorithm for the sparse regression problem. He sets a desired threshold error  $E$  and defines  $k$  to be the minimum cardinality of a solution  $S^*$  that achieves  $f(S^*) \leq E' = E/4$ . He shows that for  $\alpha = \max_{S'} \|X_{S'}^+\|^2$  the greedy algorithm finds a solution  $S$  such that  $f(S) \leq E$  such that

$$|S| \leq \left\lceil 9k\alpha \ln \frac{\|y\|_2^2}{E} \right\rceil.$$

### Natarajan's implicit assumption

In [14] Natarajan uses  $\alpha = \|X^+\|^2$  instead of  $\alpha = \max_{S'} \|X_{S'}^+\|^2$ . This is only correct if the columns of  $X$  are linearly independent which seems to be an implicit assumption. In this setting  $\alpha = \max_{S'} \|X_{S'}^+\|^2 = \|X^+\|^2$  by Cauchy's interlacing theorem. Note that  $\max_{S'} \|X_{S'}^+\| \geq \|X^+\|$  if the columns of  $X$  are linearly dependent. This is the setting in the hardness result of [10] and is inevitable in the under constrained case where the number of columns is larger than their dimension.

Here, we apply Theorem 7 with initial solution  $S_0 = \emptyset$  (which gives  $f(S_0) = \|y\|_2^2$ ) and  $E' = E/4$ . It immediately yields that the greedy algorithm finds a solution of value  $f(S) \leq E$  and

$$|S| \leq \left\lceil k\alpha \ln \frac{\|y\|_2^2 - E/4}{E - E/4} \right\rceil \leq \left\lceil \frac{4}{3} k\alpha \ln \frac{\|y\|_2^2}{E} \right\rceil$$

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using the inequality  $\ln(\frac{4}{3}x - \frac{1}{3}) \leq \frac{4}{3} \ln x$  for  $x \geq 1$ . This improves the result of [14] in three ways

1. the approximation constant is smaller
2. its proof is more streamlined and
3. it extends to viability of the greedy algorithm to the under constrained case where the result of [14] does not hold.

### 7 Column Subset Selection Problem

Given a matrix  $X$ , Column Subset Selection (CSS) is concerned with finding a small set of columns whose span captures as much of the Frobenius norm of  $X$ . It was thoroughly investigated in the context of numerical linear algebra [11, 12, 4]. CSS can be formulated as follows, find a subset  $S \in [n]$ ,  $|S| \leq k$  of matrix columns that minimize  $f(S) = \|X - X_S X_S^+ X\|_F^2$ . This formulation makes it clear that this is a special case of SMLR where  $Y = X$ .

The work of [17] investigates the notion of a curvature  $c \in [0, 1]$  for a nonincreasing set functions. They define it as follows:

$$c = 1 - \min_{j \in [n]} \min_{S, T \subseteq [n] \setminus \{j\}} \frac{f(S) - f(S \cup \{j\})}{f(T) - f(T \cup \{j\})}. \quad (11)$$

They show that there exists a greedy type algorithm that finds a solution of value at most  $1/(1-c)$  times the optimal value of the minimization problem for any objective set function with curvature  $c$  (Corollary 8.5 in [17]).

► **Lemma 16** (Lemma 9.1 from [17]). *Let  $f(S)$  be the objective function for the Column Subset Selection Problem corresponding to the matrix  $X$ . The curvature  $c$  of  $f(S)$  is such that  $\frac{1}{1-c} \leq \kappa^2(X)$  where  $\kappa(X)$  is the condition number of  $X$ .*

Note that for any matrix  $X$  with full column rank if  $\tilde{X}$  is the matrix with normalized columns then  $\|\tilde{X}^+\| \leq \kappa(X)$ . We can find our initial solution  $S_0$  by one of the three known methods:

1. an approximation algorithm from [17] finds a solution  $S_0$  such that  $|S_0| = k$  and performance guarantee  $\rho = \kappa^2(X)$ ;
2. an approximation algorithm from [7, 6] with  $|S_0| = k$  and  $\rho = k + 1$ ;
3. an approximation algorithm from [3] with  $|S_0| = 2k$  and  $\rho = 2$ ;

► **Lemma 17.** *For the column subset selection problem for a column normalized matrix  $X$  and  $\alpha = \max_{S'} \|X_{S'}^+\|_2^2$  one can find a set  $S$  such that*

$$f(S) \leq (1 + \varepsilon)f(S^*) \quad \text{and} \quad |S| = O(k) + \alpha k \left( \ln \frac{\rho}{\varepsilon} \right).$$

**Proof.** Combining one of the above results with the algorithm from Section 3 completes the proof. ◀

### 8 Sparse Convex Function Minimization

One popular extension of the regression problem is to consider

$$f(S) = \min\{R(y) : \text{supp}(y) \subseteq S\} \quad (12)$$

where  $R(y)$  is a convex function and  $\text{supp}(y) = \{i \mid y_i \neq 0\}$ . Following Shalev-Shwartz et al. [16], we consider a special case when the convex function  $R(y)$  satisfies two additional conditions.



► **Definition 18.** A function  $R(w)$  is said to be  $\lambda$ -strongly convex for  $\lambda \geq 0$  if for each  $w, u \in \mathbb{R}^d$  we have

$$R(w) \geq R(u) + \langle \nabla R(u), w - u \rangle + \frac{\lambda}{2} \|w - u\|_2^2.$$

► **Definition 19.** A function  $R(w)$  is said to be  $\beta$ -smooth if for each  $w, u \in \mathbb{R}^d$  we have

$$R(w) \leq R(u) + \langle \nabla R(u), w - u \rangle + \frac{\beta}{2} \|w - u\|_1^2.$$

Shalev-Shwartz et al. [16] gave many examples of such convex functions. In particular, they relate our Definition 19 to a class of functions arising in Machine Learning with  $\beta$ -smooth loss functions (see Lemma B1 and Section 3 in [16]).

► **Theorem 20.** *Given the set function  $f(S)$  defined in (12) corresponding to  $\beta$ -smooth  $\lambda$ -strongly convex function  $R(w)$ . The set function  $f(S)$  is  $\alpha$ -weakly-supermodular with  $\alpha = \frac{\beta}{\lambda}$ .*

**Proof.** Let  $y_S \in \mathbb{R}^d$  be a vector minimizing the function  $R(y)$  among vectors with support  $S$  and  $y_{S \cup T} \in \mathbb{R}^d$  be a vector minimizing function  $R(y)$  among vectors with support  $S \cup T$ . For any vector  $x \in \mathbb{R}^d$ , let  $x(j) \in \mathbb{R}$  be its  $j$ -th coordinate.

For each  $j \in T \setminus S$ , we define vector  $\tilde{y}^j \in \mathbb{R}^d$  such that  $\tilde{y}^j(j) = y_{S \cup T}(j)$  and  $\tilde{y}^j(i) = 0$  for all  $i \neq j$ . It is enough to prove the inequality

$$R(y_S) - R(y_{S \cup T}) \leq \frac{\beta}{\lambda} \sum_{j \in T \setminus S} R(y_S) - R\left(y_S + \frac{\lambda}{\beta} \tilde{y}^j\right) \quad (13)$$

to prove the statement of the theorem. By applying Definitions 18 and 19 we derive

$$\begin{aligned} \sum_{j \in T \setminus S} R(y_S) - R\left(y_S + \frac{\lambda}{\beta} \tilde{y}^j\right) &\geq \sum_{j \in T \setminus S} \left( -\left\langle \nabla R(y_S), \frac{\lambda}{\beta} \tilde{y}^j \right\rangle - \frac{\beta}{2} \left\| \frac{\lambda}{\beta} \tilde{y}^j \right\|_1^2 \right) \\ &\geq -\frac{\lambda}{\beta} \left( \sum_{j \in T \setminus S} \langle \nabla R(y_S), \tilde{y}^j \rangle \right) - \frac{\lambda^2}{2\beta} \|y_{S \cup T} - y_S\|_2^2 \\ &= -\frac{\lambda}{\beta} \langle \nabla R(y_S), y_{S \cup T} - y_S \rangle - \frac{\lambda^2}{2\beta} \|y_{S \cup T} - y_S\|_2^2 \\ &\geq \frac{\lambda}{\beta} \left( R(y_S) - R(y_{S \cup T}) + \frac{\lambda}{2} \|y_{S \cup T} - y_S\|_2^2 \right) - \frac{\lambda^2}{2\beta} \|y_{S \cup T} - y_S\|_2^2 \\ &= \frac{\lambda}{\beta} (R(y_S) - R(y_{S \cup T})) \end{aligned}$$

where the first equality follows from the fact that  $\nabla R(y_S)(j) = 0$  for all  $j \in S$ . ◀

Let  $R^*$  be the target value for our convex function  $R(y)$  and  $k_f$  be the minimal cardinality of a set  $S'$  such that  $f(S') \leq R^*$  where  $f(S)$  is defined by (12). Combining Theorem 7 and Theorem 20 we derive

► **Theorem 21.** *For any  $\varepsilon > 0$ , let  $f_{\text{stop}} = R^* + \varepsilon$  then the Algorithm 3 outputs  $S$  such that*

$$|S| \leq \left\lceil \frac{\beta}{\lambda} k_f \left( \ln \frac{R(\emptyset) - R^*}{\varepsilon} \right) \right\rceil.$$

The above theorem is analogous to Theorem 2.8 in [16].

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