Inapproximability of Maximum Edge Biclique, Maximum Balanced Biclique and Minimum k-Cut from the Small Set Expansion Hypothesis*†

Pasin Manurangsi

University of California, Berkeley, CA, USA
pasin@berkeley.edu

Abstract

The *Small Set Expansion Hypothesis* (SSEH) is a conjecture which roughly states that it is NP-hard to distinguish between a graph with a small set of vertices whose expansion is almost zero and one in which all small sets of vertices have expansion almost one. In this work, we prove conditional inapproximability results for the following graph problems based on this hypothesis:

- *Maximum Edge Biclique* (MEB): given a bipartite graph G, find a complete bipartite subgraph of G with maximum number of edges. We show that, assuming SSEH and that NP $\not\subseteq$ BPP, no polynomial time algorithm gives $n^{1-\varepsilon}$-approximation for MEB for every constant $\varepsilon > 0$.
- *Maximum Balanced Biclique* (MBB): given a bipartite graph G, find a balanced complete bipartite subgraph of G with maximum number of vertices. Similar to MEB, we prove $n^{1-\varepsilon}$ ratio inapproximability for MBB for every $\varepsilon > 0$, assuming SSEH and that NP $\not\subseteq$ BPP.
- *Minimum k-Cut*: given a weighted graph G, find a set of edges with minimum total weight whose removal splits the graph into k components. We prove that this problem is NP-hard to approximate to within $(2-\varepsilon)$ factor of the optimum for every $\varepsilon > 0$, assuming SSEH.

The ratios in our results are essentially tight since trivial algorithms give n-approximation to both MEB and MBB and 2-approximation algorithms are known for Minimum k-Cut [35].

Our first two results are proved by combining a technique developed by Raghavendra, Steurer and Tulsiani [33] to avoid locality of gadget reductions with a generalization of Bansal and Khot's long code test [4] whereas our last result is shown via an elementary reduction.

1998 ACM Subject Classification F.2 Analysis of Algorithms and Problem Complexity

Keywords and phrases Hardness of Approximation, Small Set Expansion Hypothesis

Digital Object Identifier 10.4230/LIPIcs.ICALP.2017.79

1 Introduction

Since the PCP theorem was proved two decades ago [2, 3], our understanding of approximability of combinatorial optimization problems has grown enormously; tight inapproximability results have been obtained for fundamental problems such as Max-3SAT [15], Max Clique [14] and Set Cover [27, 9]. Yet, for other problems, including Vertex Cover and Max Cut, known NP-hardness of approximation results come short of matching best known algorithms.

The introduction of the Unique Games Conjecture (UGC) by Khot [19] propelled another wave of development in hardness of approximation that saw many of these open problems resolved (see e.g. [23, 21]). Alas, some problems continue to elude even attempts at proving

* A full version of the paper is available at https://arxiv.org/abs/1705.03581.

† This material is based upon work supported by the National Science Foundation under Grants No. CCF 1540685 and CCF 1655215.

44th International Colloquium on Automata, Languages, and Programming (ICALP 2017).
Editors: Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl;
Article No. 79; pp. 79:1–79:14
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

UGC-hardness of approximation. For a class of such problems, the failure stems from the fact that typical reductions are local in nature; many reductions from unique games to graph problems could produce disconnected graphs. If we try to use such reductions for problems that involve some forms of expansion of graphs (e.g. Sparsest Cut), we are out of luck.

One approach to overcome the aforementioned issue is through the *Small Set Expansion Hypothesis* (SSEH) of Raghavendra and Steurer [32]. To describe the hypothesis, recall that, on a d-regular undirected unweighted graph $G = (V, E)$, the edge expansion $\Phi(S)$ of $S \subseteq V$ is defined as

$$\Phi(S) = \frac{|E(S, V \setminus S)|}{d \min\{|S|, |V \setminus S|\}}$$

where $E(S, V \setminus S)$ is the set of edges across the cut $(S, V \setminus S)$. The small set expansion problem SSE(δ, η), where η, δ are two parameters that lie in (0, 1), can be defined as follows.

▶ **Definition 1** (SSE(δ, η))**.** Given a regular graph $G = (V, E)$, distinguish between:
- *(Completeness)* There exists $S \subseteq V$ of size $\delta|V|$ such that $\Phi(S) \leqslant \eta$.
- *(Soundness)* For every $S \subseteq V$ of size $\delta|V|$, $\Phi(S) \geqslant 1 - \eta$.

For simplicity, we always assume that the input graphs of SSE are regular and unweighted and defer the treatment of arbitrary weighted graphs to the full version.

Roughly speaking, SSEH asserts that it is NP-hard to distinguish between a graph that has a small non-expanding subset of vertices and one in which all small subsets of vertices have almost perfect edge expansion. More formally, the hypothesis can be stated as follows.

▶ **Conjecture 1** (SSEH [32])**.** *For every $\eta > 0$, there is $\delta > 0$ such that SSE(δ, η) is NP-hard.*

Interestingly, SSEH not only implies UGC [32], but it is also equivalent to a stregthened version of the latter, in which the graph is required to have almost perfect small set expansion [33].

Since its proposal, SSEH has been used as a starting point for proving inapproximability of many problems whose hardnesses are not known otherwise. Most relevant to us is the work of Raghavendra, Steurer and Tulsiani (henceforth RST) [33] who devised a technique that exploited structures of SSE instances to avoid locality in reductions. In doing so, they obtained hardness of approximation for Min Bisection, Balanced Separator, and Minimum Linear Arrangement; these problems are not known to be hard to approximate under UGC.

1.1 Maximum Edge Biclique and Maximum Balanced Biclique

Our first result is adapting RST technique to prove inapproximability results for Maximum Edge Biclique (MEB) and Maximum Balanced Biclique (MBB). For both problems, the input is a bipartite graph. The goal for the former is to find a complete bipartite subgraph that contains as many edges as possible whereas, for the latter, the goal is to find a balanced complete bipartite subgraph that contains as many vertices as possible.

Both problems are NP-hard. MBB was stated (without proof) to be NP-hard in [12, page 196]; several proofs of this exist such as one provided in [17]. For MEB, it was proved to be NP-hard more recently by Peeters [31]. Unfortunately, much less is known when it comes to approximability of both problems. Similar to Maximum Clique, folklore algorithms give $O(n/\operatorname{polylog} n)$ approximation ratio for both MBB and MEB, and no better algorithm is known. However, not even NP-hardness of approximation of some constant ratio is known for the problems. This is in stark contrast to Maximum Clique for which strong inapproximability results are known [14, 18, 22, 39]. Fortunately, the situation is not completely hopeless as the problems are known to be hard to approximate under stronger complexity assumptions.

Feige [10] showed that, assuming that random 3SAT formulae cannot be refuted in polynomial time, both problems[1] cannot be approximated to within n^ε of the optimum in polynomial time for some $\varepsilon > 0$. Later, Feige and Kogan [11] proved $2^{(\log n)^\varepsilon}$ ratio inapproximability for both problems for some $\varepsilon > 0$, assuming that 3SAT $\notin$ DTIME($2^{n^{3/4+\delta}}$) for some $\delta > 0$. Moreover, Khot [20] showed, assuming 3SAT $\notin$ BPTIME(2^{n^δ}) for some $\delta > 0$, that no polynomial time algorithm achieves n^ε-approximation for MBB for some $\varepsilon > 0$. Ambühl et al. [1] subsequently built on Khot's result and showed a similar hardness for MEB. Recently, Bhangale et al. [6] proved that both problems are hard to approximate to within[2] $n^{1-\varepsilon}$ factor for every $\varepsilon > 0$, assuming a certain strengthened version of UGC and NP $\neq$ BPP. In addition, while not stated explicitly, the author's recent reduction for Densest k-Subgraph [25] yields $n^{1/\operatorname{polyloglog} n}$ ratio inapproximability for both problems under the Exponential Time Hypothesis [16] (3SAT $\notin$ DTIME($2^{o(n)}$)) and this ratio can be improved to $n^{f(n)}$ for any $f \in o(1)$ under the stronger Gap Exponential Time Hypothesis [8, 26] (no $2^{o(n)}$ time algorithm can distinguish a fully satisfiable 3SAT formula from one which is only $(1-\varepsilon)$-satisfiable for some $\varepsilon > 0$); these ratios are better than those in [11] but worse than those in [20, 1, 6].

In this work, we prove strong inapproximability results for both problems, assuming SSEH:

▶ **Theorem 2.** *Assuming SSEH, there is no polynomial time algorithm that approximates MEB or MBB to within $n^{1-\varepsilon}$ factor of the optimum for every $\varepsilon > 0$, unless NP $\subseteq$ BPP.*

We note that the only part of the reduction that is randomized is the gap amplification via randomized graph product [5, 7]. If one is willing to assume only that NP $\neq$ P (and SSEH), our reduction still implies that both are hard to approximate to within any constant factor.

Only Bhangale et al.'s result [6] and our result achieve the inapproximability ratio of $n^{1-\varepsilon}$ for every $\varepsilon > 0$; all other results achieve at most n^ε ratio for some $\varepsilon > 0$. Moreover, only Bhangale et al.'s reduction and ours are candidate NP-hardness reductions, whereas each of the other reductions either uses superpolynomial time [11, 20, 1, 25] or relies on an average-case assumption [10]. It is also worth noting here that, while both Bhangale et al.'s result and our result are based on assumptions which can be viewed as stronger variants of UGC, the two assumptions are incomparable and, to the best of our knowledge, Bhangale et al.'s technique does not apply to SSEH. Due to space constraint, we defer a more in-depth discussion on the differences between the two assumptions to a longer version of this work.

Along the way, we prove inapproximability of the following hypergraph bisection problem, which may be of independent interest: given a hypergraph $H = (V_H, E_H)$ find a bisection[3] (T_0, T_1) of V_H such that the number of uncut hyperedges is maximized. We refer to this problem as *Max UnCut Hypergraph Bisection* (MUCHB). Roughly speaking, we show that, assuming SSEH, it is hard to distinguish a hypergraph whose optimal bisection cuts only ε fraction of hyperedges from one in which every bisection cuts all but ε fraction of hyperedges:

---

1 While Feige only stated this for MBB, the reduction clearly works for MEB too.

2 In [6], the inapproximability ratio is only claimed to be n^ε for *some* $\varepsilon > 0$. However, it is not hard to see that their result in fact implies $n^{1-\varepsilon}$ factor hardness of approximation as well.

3 (T_0, T_1) is a bisection of V_H if $|T_0| = |T_1| = |V_H|/2, T_0 \cap T_1 = \emptyset$ and $V_H = T_0 \cup T_1$.

▶ **Lemma 3.** *Assuming SSEH, for every $\varepsilon > 0$, it is NP-hard to, given a hypergraph $H = (V_H, E_H)$, distinguish between the following two cases:*

- (Completeness) *There is a bisection (T_0, T_1) of V_H s.t.*

$$|E_H(T_0)|, |E_H(T_1)| \geqslant (1/2 - \varepsilon)|E_H|\,.$$

- (Soundness) *For every set $T \subseteq V_H$ of size at most $|V_H|/2$, $|E_H(T)| \leqslant \varepsilon |E_H|$.*

Here $E_H(T) \triangleq \{e \in E_H \mid e \subseteq T\}$ denotes the set of hyperedges inside of the set $T \subseteq V_H$.

Our result above is similar to Khot's quasi-random PCP [20]; roughly speaking, Khot's result states that it is hard (if 3SAT $\notin \bigcap_{\delta>0}$ BPTIME(2^{n^δ})) to distinguish between a d-uniform hypergraph where $1/2^{d-2}$ fraction of hyperedges are uncut in the optimal bisection from one where roughly $1/2^{d-1}$ fraction of hyperedges are uncut in any bisection. In this sense, [20] provides better soundness at the expense of worse completeness compared ours.

1.2 Minimum k-Cut

In addition to the above biclique problems, we prove an inapproximability result for the Minimum k-Cut problem, in which a weighted graph is given and the goal is to find a set of edges with minimum total weight whose removal paritions the graph into (at least) k connected components. For any fixed k, the problem was proved to be in P by Goldschmidt and Hochbaum [13], who also showed that, when k is part of the input, the problem is NP-hard. To circumvent this, Saran and Vazirani [35] devised two simple polynomial time $(2 - 2/k)$-approximation algorithms for the problem. In the ensuing years, different approximation algorithms [29, 38, 34, 37] have been proposed for the problem, none of which are able achieve an approximation ratio of $(2-\varepsilon)$ for some $\varepsilon > 0$. In fact, Saran and Vazirani themselves conjectured that $(2 - \varepsilon)$-approximation is intractible for the problem [35]. In this work, we show that their conjecture is indeed true, if the SSEH holds:

▶ **Theorem 4.** *Assuming SSEH, it is NP-hard to approximate Minimum k-Cut to within $(2 - \varepsilon)$ factor of the optimum for every constant $\varepsilon > 0$.*

Note that the problem was claimed to be APX-hard in [35]. However, to the best of our knowledge, the proof has never been published and no other inapproximability is known.

2 Inapproximability of Minimum k-Cut

We now proceed to prove our main results. Let us start with the simplest: Minimum k-Cut.

Proof of Theorem 4. The reduction from SSE(δ, η) to Minimum k-Cut is simple; the graph G remains the input graph for Minimum k-Cut and we let $k = \delta n + 1$ where $n = |V|$.

Completeness. If there is $S \subseteq V$ of size δn such that $\Phi(S) \leqslant \eta$, then we partition the graph into k groups where the first group is $V \setminus S$ and each of the other groups contains one vertex from S. The edges cut are the edges in $E(S, V \setminus S)$ and the edges within the set S itself. There are $d|S|\Phi(S) \leqslant \eta d|S|$ edges of the former type and only at most $d|S|/2$ of the latter. Hence, the number of edges cut in this partition is at most $(1/2 + \eta)d|S| = (1/2 + \eta)\delta dn$.

Soundness. Suppose that, for every $S \subseteq V$ of size δn, $\Phi(S) \geqslant 1 - \eta$. Let $T_1, \dots, T_k \subseteq V$ be any k-partition of the graph. Assume w.l.o.g. that $|T_1| \leqslant \cdots \leqslant |T_k|$. Let $A = T_1 \cup \cdots \cup T_i$ where i is the maximum index such that $|T_1 \cup \cdots \cup T_i| \leqslant \delta n$.

We claim that $|A| \geqslant \delta n - \sqrt{n}$. To see that this is the case, suppose for the sake of contradiction that $|A| < \delta n - \sqrt{n}$. Since $|A \cup T_{i+1}| > \delta n$, we have $T_{i+1} > \sqrt{n}$. Moreover, since $A = T_1 \cup \cdots T_i$, we have $i \leqslant |A| < \delta n - \sqrt{n}$. As a result, we have $n = |T_1 \cup \cdots \cup T_k| \geqslant |T_{i+1} \cup \cdots \cup T_k| \geqslant (k-i)|T_i| > \sqrt{n} \cdot \sqrt{n} = n$, which is a contradiction. Hence, $|A| \geqslant \delta n - \sqrt{n}$.

Now, note that, for every $S \subseteq V$ of size δn, $\Phi(S) \geqslant 1 - \eta$ implies that $|E(S)| \leqslant \eta d \delta n/2$ where $E(S)$ denote the set of all edges within S. Since $|A| \leqslant \delta n$, we also have $|E(A)| \leqslant \eta d \delta n/2$. As a result, the number of edges in the cut $(A, V \setminus A)$ is at least

$$d|A| - \eta d \delta n \geqslant (1-\eta) d \delta n - d\sqrt{n} = \left(1 - \eta - \frac{1}{\delta \sqrt{n}}\right) \delta d n.$$

For every constant $\varepsilon > 0$, by setting $\eta = \varepsilon/20$ and $n > 100/(\varepsilon^2 \delta^2)$, the ratio between the two cases is at least $(2 - \varepsilon)$, which concludes the proof of Theorem 4. ◀

3 Inapproximability of MEB and MBB

Let us now turn our attention to MEB and MBB. First, note that we can reduce MUCHB to MEB/MBB by just letting the two sides of the bipartite graph be E_H and creating an edge (e_1, e_2) iff $e_1 \cap e_2 = \emptyset$. This immediately shows that Lemma 3 implies the following:

▶ **Lemma 5.** *Assuming SSEH, for every $\delta > 0$, it is NP-hard to, given a bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$, distinguish between the following two cases:*
- (Completeness) *G contains $K_{(1/2-\delta)n,(1/2-\delta)n}$ as a subgraph.*
- (Soundness) *G does not contain $K_{\delta n, \delta n}$ as a subgraph.*

Here $K_{t,t}$ denotes the complete bipartite graph in which each side contains t vertices.

Note that Theorem 2 follows from Lemma 5 by gap amplification via randomized graph product [5, 7]. Since this has been analyzed before even for biclique [20, Appendix D], we do not repeat the argument here.

We are now only left to prove Lemma 3; we devote the rest of this section to this task.

3.1 Preliminaries

Before we continue, we need additional notations and preliminaries. For every graph G and every vertex v, we will write $G(v)$ to denote the uniform distribution on its neighbors.

It will be convenient for us to use a different (but equivalent) formulation of SSEH. To state it, we will define a variant of SSE(δ, η) called SSE(δ, η, M); the completeness remains the same whereas the soundness is strengtened to include all S of size between $\frac{\delta|V|}{M}$ and $\delta|V|M$.

▶ **Definition 6** (SSE(δ, η, M)). Given a regular graph $G = (V, E)$, distinguish between:
- *(Completeness)* There exists $S \subseteq V$ of size $\delta|V|$ such that $\Phi(S) \leqslant \eta$.
- *(Soundness)* For every $S \subseteq V$ with $|S| \in \left[\frac{\delta|V|}{M}, \delta|V|M\right]$, $\Phi(S) \geqslant 1 - \eta$.

The new formulation of the hypothesis can now be stated as follows.

▶ **Conjecture 2.** *For every $\eta, M > 0$, there is $\delta > 0$ such that SSE(δ, η, M) is NP-hard.*

Raghavendra et al. [33] showed that this formulation is equivalent to the earlier formulation (Conjecture 1); please refer to Appendix A.2 of [33] for more details of the proof.

While our reduction can be understood without notation of unique games, it is best described in a context of unique games reductions. We provide a definition of unique games below.

▶ **Definition 7** (Unique Game (UG)). A unique game instance $(\mathcal{G}, [R], \{\pi_e\}_{e \in E})$ consists of a bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a label set $[R] = \{1, \dots, R\}$, and, for each $e \in \mathcal{E}$, a permutation $\pi_e : [R] \to [R]$. The goal is to find an assignment $F : V \to [R]$ such that, for as many edges $(u, v) \in \mathcal{E}$ as possible, we have $\pi_{(u,v)}(F(u)) = F(v)$; these edges are said to be *satisfied*.

Khot's UGC [19] states that, for every $\varepsilon > 0$, it is NP-hard to distinguish between a unique game in which there exists an assignment satisfying at least $(1 - \varepsilon)$ fraction of edges from one in which every assignment satisfies at most ε fraction of edges.

Finally, we need some preliminaries in discrete Fourier analysis. We state here only few facts that we need. We refer interested readers to [30] for more details about the topic.

For any discrete probability space Ω, $f : \Omega^R \to [0, 1]$ can be written as $\sum_{\sigma \in [|\Omega|]^R} \hat{f}(\sigma)\phi_\sigma$ where $\{\phi_\sigma\}_{\sigma \in [|\Omega|]^R}$ is the product Fourier basis of $L^2(\Omega^R)$ (see [30, Chapter 8.1]). The degree-d influence on the j-th coordinate of f is $\mathrm{infl}_j^d(f) \triangleq \sum_{\sigma \in [|\Omega|]^R, \sigma_j \neq 1, \#\sigma \leqslant d} \hat{f}^2(\sigma)$ where $\#\sigma \triangleq |\{i \in [R] \mid \sigma_i \neq 1\}|$. It is well known that $\sum_{j=1}^R \mathrm{infl}_j^d(f) \leqslant d$ (see [28, Proposition 3.8]).

We also need the following theorem. It follows easily[4] from the so-called "It Ain't Over Till It's Over" conjecture, which is by now a theorem [28, Theorem 4.9].

▶ **Theorem 8** ([28]). *For any $\beta, \varepsilon_T, \gamma > 0$, there exists $\kappa > 0$ and $t, d \in \mathbb{N}$ such that, if any functions $f_1, \dots, f_t : \Omega^R \to \{0, 1\}$ where Ω is a probability space whose probability of each atom is at least β satisfy*[5]

$$\forall i \in [t], \mathop{\mathbb{E}}_{x \in \Omega^R}[f_i(x)] \leqslant 0.99 \quad \textit{and} \quad \forall j \in [R], \forall 1 \leqslant i_1 \neq i_2 \leqslant t, \min\{\mathrm{infl}_j^d(f_{i_1}), \mathrm{infl}_j^d(f_{i_2})\} \leqslant \kappa,$$

then

$$\Pr_{x \in \Omega^R, D \sim S_{\varepsilon_T}(R)} \left[\bigwedge_{i=1}^{t} f_i(C_D(x)) \equiv 1 \right] < \gamma$$

where $D \sim S_{\varepsilon_T}(R)$ is a random subset of $[R]$ where each $i \in [R]$ is included independently w.p. ε_T, $C_D(x) \triangleq \{x' \mid x'_{[R] \setminus D} = x_{[R] \setminus D}\}$ and $f_i(C_D(x)) \equiv 1$ is short for $\forall x' \in C_D(x), f_i(x') = 1$.

3.2 Bansal-Khot Long Code Test and A Candidate Reduction

Theorem 8 leads us nicely to the Bansal-Khot long code test [4]. For UGC hardness reductions, one typically needs a *long code test* (aka dictatorship gadget) which, on input $f_1, \dots, f_t : \{0, 1\}^R \to \{0, 1\}$, has the following properties:

- *(Completeness)* If $f_1 = \dots = f_t$ is a long code[6], the test accepts with large probability.
- *(Soundness)* If $f_1, \dots, f_t$ are balanced (i.e. $\mathbb{E} f_1 = \dots = \mathbb{E} f_t = 1/2$) and are "far from being a long code", then the test accepts with low probability.

  A widely-used notion of "far from being a long code", and one we will use here, is that the functions do not share a coordinate with large low degree influence (i.e. for every $j \in [R]$ and every $i_1 \neq i_2 \in [t]$, at least one of $\mathrm{infl}_j^d(f_{i_1})$, $\mathrm{infl}_j^d(f_{i_2})$ is small),

[4] For more details on how this version follows from there, please refer to [36, page 769].
[5] 0.99 could be replaced by any constant less than one; we use it to avoid introducing more parameters.
[6] A long code is simply j-junta (i.e. a function that depends only on the x_j) for some $j \in [R]$.

Input: A unique game $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), [R], \{\pi_e\}_{e \in E})$ and parameters $\ell \in \mathbb{N}$ and $\varepsilon_T \in (0, 1)$.
Output: A hypergraph $H = (V_H, E_H)$.
The vertex set V_H is $\mathcal{V} \times \{0,1\}^R$ and the hyperedges are distributed as follows:

- Sample $u \sim \mathcal{V}$ and sample $v_1 \sim \mathcal{G}(u), \dots, v_\ell \sim \mathcal{G}(u)$.
- Sample $x \sim \{0,1\}^R$ and a subset $D \sim S_{\varepsilon_T}(R)$.
- Output a hyperedge $e = \{(v_p, x') \mid p \in [\ell], x' \in C_D(x)\}$.

Figure 1 A Candidate Reduction from UG to MUCHB.

Bansal and Khot's long code test works as follows: pick $x \sim \{0,1\}^R$ and $D \sim S_{\varepsilon_T}(R)$. Then, test whether f_i evaluates to 1 on the whole $C_D(x)$. Note that this can be viewed as an "algorithmic" version of Theorem 8; more specifically, the theorem (with $\Omega = \{0, 1\}$) immediately implies the soundness property of this test. On the other hand, it is obvious that, if $f_1 = \cdots = f_t$ is a long code, then the test accepts with probability $1/2 - \varepsilon_T$.

Bansal and Khot used this test to prove tight hardness of approximation of Vertex Cover. The reduction is via a natural composition of the test with unique games. Their reduction also gives a cadidate reduction from UG to MUCHB, which is stated below in Figure 1.

As is typical for gadget reductions, for $T \subseteq V_H$, we view the indicator function $f_u(x) \triangleq \mathbb{1}[(u, x) \in T]$ for each $u \in \mathcal{V}$ as the intended long code. If there exists an assignment ϕ to the unique game instance that satisfies nearly all the constraints, then the bisection corresponding to $f_u(x) = x_{\phi(u)}$ cuts only small fraction of edges, which yields the completeness of MUCHB.

As for the soundness, we would like to decode an UG assignment from $T \subseteq V_H$ of size at most $|V_H|/2$ which contains at least ε fraction of hyperedges. In terms of the tests, this corresponds to a collection of functions $\{f_u\}_{u \in \mathcal{V}}$ such that $\mathbb{E}_{u \sim \mathcal{V}} \mathbb{E}_{x \sim \{0,1\}^R} f_u(x) = 1/2$ and the Bansal-Khot test on $f_{v_1}, \dots, f_{v_t}$ passes with probability at least ε where $v_1, \dots, v_t$ are sampled as in Figure 1. Now, if we assume that $\mathbb{E}_x f_u(x) \leqslant 0.99$ for all $u \in \mathcal{V}$, then such decoding is possible via a similar method as in [4] since Theorem 8 can be applied here.

Unfortunately, the assumption $\mathbb{E}_x f_u(x) \leqslant 0.99$ does not hold for an arbitrary $T \subseteq V_H$ and the soundness property indeed fails. For instance, imagine the constraint graph $\mathcal{G}$ of the starting unique game instance consisting of two disconnected components of equal size; let $\mathcal{V}_0$ and $\mathcal{V}_1$ be the set of vertices in the two components. In this case, if we set $T_0 = \mathcal{V}_0$ and $T_1 = \mathcal{V}_1$, then the bisection (T_0, T_1) does not even cut a single edge! This is regardless of whether there exists an assignment to the UG that satisfies a large fraction of edges.

3.3 RST Technique and The Reduction from SSE to MUCHB

The issue described above is common for graph problems that involves some form of expansion of the graph. The RST technique [33] was indeed invented to specifically circumvent this issue. It works by first reducing SSE to UG and then exploiting the structure of the constructed UG instance when composing it with a long code test; this allows them to avoid extreme cases such as one above. There are four parameters in the reduction: $R, k \in \mathbb{N}$ and ε_V, β. Before we describe the reduction, let us define additional notations here:

- Let $G^{\otimes R}$ denote the R-tensor graph of G; the vertex set of $G^{\otimes R}$ is V^R and there is an edge between A, B if and only if there is an edge between A_i, B_i in G for every $i \in [R]$.
- For each $A \in V^R$, $T_V(A)$ denote the distribution on V^R where the i-th coordinate is set to A_i with probability $1 - \varepsilon_V$ and is randomly sampled from V otherwise.

- Let $\Pi_{R,k}$ denote the set of all permutations π's of $[R]$ such that, for each $j \in [k]$, $\pi(\{R(j-1)/k+1,\ldots,Rj/k\}) = \{R(j-1)/k+1,\ldots,Rj/k\}$.
- Let $\{0,1,\perp\}_\beta$ denote the probability space such that the probability for $0,1$ are both $\beta/2$ and the probability for $\perp$ is $1-\beta$.

The first step of reduction takes an $\mathrm{SSE}(\delta,\eta,M)$ instance $G=(V,E)$ and produces a unique game $\mathcal{U}=(\mathcal{G}=(\mathcal{V},\mathcal{E}),[R],\{\pi_e\}_{e\in E})$ where $\mathcal{V}=V^R$ and the edges are distributed as follows:

1. Sample $A \sim V^R$ and $\tilde{A} \sim T_V(A)$.
2. Sample $B \sim G^{\otimes R}(\tilde{A})$ and $\tilde{B} \sim T_V(B)$.
3. Sample two random $\pi_A,\pi_B \sim \Pi_{R,k}$.
4. Output an edge $e=(\pi_A(\tilde{A}),\pi_B(\tilde{B}))$ with $\pi_{(A,B)}=\pi_B\circ\pi_A^{-1}$.

Here ε_V is a small constant, k is large and R/k should be think of as $\Theta(1/\delta)$. When there exists a set $S\subseteq V$ of size $\delta|V|$ with small edge expansion, the intended assignment is to, for each $A\in V^R$, find the first block $j\in[k]$ such that $|A(j)\cap S|=1$ where $A(j)$ denotes the multiset $\{A_{R(j-1)/k+1},\ldots,A_{Rj/k}\}$ and let $F(A)$ be the coordinate of the vertex in that intersection. If no such j exists, we assign $F(A)$ arbitrarily. Note that, since $R/k=\Theta(1/\delta)$, $\Pr[|A(j)\cap S|=1]$ is constant, which means that only $2^{-\Omega(k)}$ fraction of vertices are assigned arbitrarily. Moreover, it is not hard to see that, for the other vertices, their assignments rarely violate constraints as ε_V and $\Phi(S)$ are small. This yields the completeness. In addition, the soundness was shown in [32, 33], i.e., if every $S\subseteq V$ of size $\delta|V|$ has near perfect expansion, no assignment satisfies many constraints in $\mathcal{U}$ (see Lemma 13).

The second step is to reduce this UG instance to a hypergraph $H=(V_H,E_H)$. Instead of making the vertex set $V^R\times\{0,1\}^R$ as in the previous candidate reduction, we will instead make $V_H=V^R\times\Omega^R$ where $\Omega=\{0,1,\perp\}_\beta$ and β is a small constant. This does not seem to make much sense from the UG reduction standpoint because we typically want to assign which side of the bisection $(A,x)\in V_H$ is in according to $x_{F(A)}$ but $x_{F(A)}$ could be $\perp$ in this construction. However, it makes sense when we view this as a reduction from SSE directly: let us discard all coordinates i's such that $x_i=\perp$ and define $A(j,x)\triangleq\{A_i\mid i\in\{R(j-1)/k+1,\ldots,Rj/k\}\wedge x_i\neq\perp\}$. Then, let $j^*(A,x)\triangleq\min\{j\mid |A(j,x)\cap S|=1\}$ and let $i^*(A,x)$ be the coordinate in the intersection between $A(j^*(A,x),x)$ and S, and assign (A,x) to $T_{x_{i^*(A,x)}}$. (If $j^*(A,x)$ does not exists, then assign (A,x) arbitrarily.)

Observe that, in the intended solution, the side that (A,x) is assigned to does not change if (1) A_i is modified for some $i\in[R]$ s.t. $x_i=\perp$ or (2) we apply some permutation $\pi\in\Pi_{R,k}$ to both A and x. In other words, we can "merge" two vertices (A,x) and (A',x') that are equivalent through these changes together in the reduction. For notational convenience, instead of merging vertices, we will just modify the reduction so that, if (A,x) is included in some hyperedge, then every (A',x') reachable from (A,x) by these operations is also included in the hyperedge. More specifically, if we define $M_x(A)\triangleq\{A'\in V^R\mid A'_i=A_i \text{ for all } i\in[R] \text{ such that } x_i\neq\perp\}$ corresponding to the first operation, then we add $\pi(A',x)$ to the hyperedge for every $A'\in M_x(A)$ and $\pi\in\Pi_{R,k}$. The full reduction is shown in Figure 2.

Note that the test we apply here is slightly different from Bansal-Khot test as our test is on $\Omega=\{0,1,\perp\}_\beta$ instead of $\{0,1\}$ used in [4]. Another thing to note is that now our vertices and hyperedges are weighted, the vertices according to the product measure of $V^R\times\Omega^R$ and the edges according to the distribution produced from the reduction. We write μ_H to denote the measure on the vertices, i.e., for $T\subseteq V^R\times\{0,1,\perp\}^R$, $\mu_H(T)=\Pr_{A\sim V^R,x\sim\Omega^R}[(A,x)\in T]$, and we abuse the notation $E_H(T)$ and use it to denote the probability that a hyperedge as generated in Figure 2 lies completely in T. We note here that, while the MUCHB as stated

Input: A graph G with vertex set V and parameters $R, k, \ell \in \mathbb{N}$ and $\varepsilon_T, \varepsilon_V, \beta \in (0,1)$.
Output: A hypergraph $H = (V_H, E_H)$.
$V_H \triangleq V^R \times \Omega^R$ where $\Omega \triangleq \{0, 1, \perp\}_\beta$ and the hyperedges are distributed as follows:

- Sample $A \sim V^R$ and $\tilde{A}^1, \dots, \tilde{A}^\ell \sim T_V(A)$.
- Sample $B^1 \sim G^{\otimes R}(\tilde{A}^1), \dots, B^\ell \sim G^{\otimes R}(\tilde{A}^\ell)$ and $\tilde{B}^1 \sim T_V(B^1), \dots, \tilde{B}^\ell \sim T_V(B^\ell)$
- Sample $x \in \Omega^R$ and a subset $D \sim S_{\varepsilon_T}(R)$.
- Output a hyperedge $e = \{\pi(B', x') \mid p \in [\ell], \pi \in \Pi_{R,k}, x' \in C_D(x), B' \in M_{x'}(\tilde{B}^p)\}$.

Figure 2 Reduction from SSE to Max UnCut Hypergraph Bisection.

in Lemma 3 is unweighted, it is not hard to see that we can go from weighted version to unweighted by copying each vertex and each edge proportional[7] to their weights.

The advantage of this reduction is that the vertex "merging" makes gadget reduction non-local; for instance, it is clear that even if the starting graph V has two connected components, the resulting hypergraph is now connected. In fact, Raghavendra et al. [33] show a much stronger quantitative bound. To state this, let us consider any $T \in V_H$ with $\mu_H(T) = 1/2$. From how the hyperedges are defined, we can assume w.l.o.g. that, if $(A, x) \in T$, then $\pi(A', x) \in T$ for every $A' \in M_x(A)$ and every $\pi \in \Pi_{R,k}$. Again, let $f_A(x) \triangleq \mathbb{1}[(A, x) \in T]$. The following bound on the variance of $\mathbb{E}_x f_A(x)$ is implied by the proof of Lemma 6.6 in [33]:

$$\mathop{\mathbb{E}}_{A \sim V^R} \left(\mathop{\mathbb{E}}_{x \sim \Omega^R} f_A(x) - 1/2 \right)^2 \leqslant \beta.$$

The above bound implies that, for most A's, the mean of f_A cannot be too large. This will indeed allow us to ultimately apply Theorem 8 on a certain fraction of the tuples $(\tilde{B}^1, \dots, \tilde{B}^\ell)$ in the reduction, which leads to an UG assignment with non-negligible value.

3.4 Completeness

In the completeness case, we define a bisection similar to that described above. This bisection indeed cuts only a small fraction of hyperedges; quantitatively, this yields the following lemma. Since its proof consists mainly of calculations, we omit it from this extended abstract.

▶ **Lemma 9.** *If there is a set $S \subseteq V$ such that $\Phi(S) \leqslant \eta$ and $|S| = \delta|V|$ where $\delta \in \left[\frac{k}{10\beta R}, \frac{k}{\beta R}\right]$, then there is a bisection (T_0, T_1) of V_H such that $E_H(T_0), E_H(T_1) \geqslant 1/2 - O(\varepsilon_T/\beta) - O(\eta\ell/\beta) - O(\varepsilon_V \ell/\beta) - 2^{-\Omega(k)}$ where $O(\cdot)$ and $\Omega(\cdot)$ hide only absolute constants.*

3.5 Soundness

Let us consider any set T such that $\mu_H(T) \leqslant 1/2$. We would like to give an upper bound on $E_H(T)$. From how we define hyperedges, we can assume w.l.o.g. that $(A, x) \in T$ if and only if $\pi(A', x) \in T$ for every $A' \in M_x(A)$ and $\pi \in \Pi_{R,k}$. We call such T *$\Pi_{R,k}$-invariant.*

Let $f : V^R \times \Omega^R \to \{0, 1\}$ denote the indicator function for T, i.e., $f(A, x) = 1$ if and only if $(A, x) \in T$. Note that $\mathbb{E}_{A \sim V^R, x \sim \Omega^R} f(A, x) = \mu_H(T) \leqslant 1/2$. Following notation from [33], we write $f_A(x)$ as a shorthand for $f(A, x)$. In addition, for each $A \in V^R$, we will write $\tilde{B} \sim \Gamma(A)$ as a shorthand for $\tilde{B}$ generated randomly by sampling $\tilde{A} \sim T_V(A)$, $B \sim G^{\otimes R}(\tilde{A})$

[7] Note that this is doable since we can pick $\beta, \varepsilon_V, \varepsilon_T$ to be rational.

and $\tilde{B} \sim T_V(B)$ respectively. Let us restate Raghavendra et al.'s [33] Lemma regarding the variance of $\mathbb{E}_x f_A(x)$ in a more convenient formulation below.

▶ **Lemma 10** ([33, Lemma 6.6][8]). *For every $A \in V^R$, let $\mu_A \triangleq \mathbb{E}_{x\sim\Omega^R} f_A(x)$. We have*

$$\mathop{\mathbb{E}}_{A\sim V^R}\left(\mathop{\mathbb{E}}_{\tilde{B}\sim\Gamma(A)} \mu_{\tilde{B}} - \mu_H(T)\right)^2 \leqslant \beta.$$

To see how the above lemma helps us decode an UG assignment, observe that, if our test accepts on $f_{\tilde{B}^1}, \dots, f_{\tilde{B}^\ell}, x, D$, then it also accepts on any subset of the functions (with the same x, D); hence, to apply Theorem 8, it suffices that t of the functions have means $\leqslant 0.99$. We will choose ℓ to be large compared to t. Using above lemma and a standard tail bound, we can argue that Theorem 8 is applicable for almost all tuples $\tilde{B}^1, \dots, \tilde{B}^\ell$, as stated below. Due to space constraint, we omit its proof from this extended abstract.

▶ **Lemma 11.** *For any positive integer $t \leqslant 0.01\ell$,*

$$\Pr_{A\sim V^R, \tilde{B}^1,\dots,\tilde{B}^\ell\sim\Gamma(A)}[|\{i \in [\ell] \mid \mu_{\tilde{B}^i} \leqslant 0.99\}| \geqslant t] \geqslant 1 - 10\beta - 2^{-\ell/100}.$$

3.5.1 Decoding an Unique Games Assignment

With Lemma 11 ready, we can now decode an UG assignment via a similar technique from [4].

▶ **Lemma 12.** *For any $\varepsilon_T, \gamma, \beta > 0$, let $t = t(\varepsilon_T, \gamma, \beta), \kappa = \kappa(\varepsilon_T, \gamma, \beta)$ and $d = d(\varepsilon_T, \gamma, \beta)$ be as in Theorem 8. For any integer $\ell \geqslant 100t$, if there exists $T \subseteq V_H$ of such that $\mu_H(T) \leqslant 1/2$ and $E_H(T) \geqslant 2\gamma + 10\beta + 2^{-\ell/100}$, then there exists $F : V^R \to [R]$ such that*

$$\Pr_{A\sim V^R, \tilde{B}\sim\Gamma(A), \pi_A, \pi_B\sim\Pi_{R,k}}[\pi_A^{-1}(F(\pi_A(\tilde{A}))) = \pi_B^{-1}(F(\pi_B(\tilde{B})))] \geqslant \frac{\gamma\kappa^2}{4d^2\ell^2}.$$

Proof. The decoding procedure is as follows. For each $A \in V^R$, we construct a set of candidate labels $\mathrm{Cand}[A] \triangleq \{j \in [R] \mid \mathrm{infl}_j^d(f_A) \geqslant \kappa\}$. We generate F randomly by, with probability 1/2, setting $F(A)$ to be a random element of Cand$[A]$ and, with probability 1/2, sampling $\tilde{B} \sim \Gamma(A)$ and setting $F(A)$ to be a random element from Cand$[B]$. Note that, if the candidate set is empty, then we simply pick an arbitrary assignment.

From our assumption that T is $\Pi_{R,k}$-invariant, it follows that, for every $A \in V^R, \pi \in \Pi_{R,k}$ and $j \in [R]$, $\Pr_F[\pi^{-1}(F(\pi(A))) = j] = \Pr_F[F(A) = j]$. In other words, we have

$$\begin{aligned}&\Pr_{F, A\sim V^R, \tilde{B}\sim\Gamma(A), \pi_A, \pi_B\sim\Pi_{R,k}}[\pi_A^{-1}(F(\pi_A(\tilde{A}))) = \pi_B^{-1}(F(\pi_B(\tilde{B})))] \\ &\qquad = \Pr_{F, A\sim V^R, \tilde{B}\sim\Gamma(A)}[F(\tilde{A}) = F(\tilde{B})]. \qquad (1)\end{aligned}$$

Next, note that, from how our reduction is defined, $E_H(T)$ can be written as

$$E_H(T) = \Pr_{A\sim V^R, \tilde{B}^1,\dots,\tilde{B}^\ell\sim\Gamma(A), x\sim\Omega^R, D\sim S_{\varepsilon_T}(R)}\left[\bigwedge_{i=1}^{\ell} f_{\tilde{B}^i}(C_D(x)) \equiv 1\right].$$

[8] Lemma 6.6 in [33] involves symmetrizing f's, but we do not need it here since T is $\Pi_{R,k}$-invariant.

From $E_H(T) \geqslant 2\gamma + 10\beta + 2^{-\ell/100}$ and from Lemma 11, we can conclude that

$$\Pr_{A,\tilde{B}^1,\dots,\tilde{B}^\ell,x,D}\left[\left(\bigwedge_{i=1}^{\ell} f_{\tilde{B}^i}(C_D(x)) \equiv 1\right) \wedge \left(|\{i \in [\ell] \mid \mu_{\tilde{B}^i} \leqslant 0.99\}| \geqslant t\right)\right] \geqslant 2\gamma.$$

From Markov's inequality, we have

$$\begin{aligned}\gamma &\leqslant \Pr_{A,\tilde{B}^1,\dots,\tilde{B}^\ell}\left[\Pr_{x,D}\left[\left(\bigwedge_{i=1}^{\ell} f_{\tilde{B}^i}(C_D(x)) \equiv 1\right) \wedge \left(|\{i \in [\ell] \mid \mu_{\tilde{B}^i} \leqslant 0.99\}| \geqslant t\right)\right] \geqslant \gamma\right] \\ &= \Pr_{A,\tilde{B}^1,\dots,\tilde{B}^\ell}\left[\left(\Pr_{x,D}\left[\bigwedge_{i=1}^{\ell} f_{\tilde{B}^i}(C_D(x)) \equiv 1\right] \geqslant \gamma\right) \wedge \left(|\{i \in [\ell] \mid \mu_{\tilde{B}^i} \leqslant 0.99\}| \geqslant t\right)\right].\end{aligned}$$

A tuple $(A, \tilde{B}^1, \dots, \tilde{B}^\ell)$ is said to be *good* if $\Pr_{x\sim\Omega^R, D\sim S_{\varepsilon_T}(R)}\left[\bigwedge_{i=1}^{\ell} f_{\tilde{B}^i}(C_D(x)) \equiv 1\right] \geqslant \gamma$ and $|\{i \in [\ell] \mid \mu_{\tilde{B}^i} \leqslant 0.99\}| \geqslant t$. For such tuple, Theorem 8 implies that there exist $i_1 \neq i_2 \in [\ell], j \in [R]$ s.t. $\mathrm{infl}_j^d(f_{\tilde{B}^{i_1}}), \mathrm{infl}_j^d(f_{\tilde{B}^{i_2}}) \geqslant \kappa$. This means that $\mathrm{Cand}(\tilde{B}^{i_1}) \cap \mathrm{Cand}(\tilde{B}^{i_2}) \neq \emptyset$.

Hence, if we sample a tuple $(A, \tilde{B}^1, \dots, \tilde{B}^\ell)$ at random, and then sample two different $\tilde{B}, \tilde{B}'$ randomly from $\tilde{B}^1, \dots, \tilde{B}^\ell$, then the tuple is good with probability at least γ and, with probability $1/\ell^2$, we have $\tilde{B} = \tilde{B}^{i_1}, \tilde{B}' = \tilde{B}^{i_2}$. This gives the following bound:

$$\Pr_{A,\tilde{B},\tilde{B}'}\left[\mathrm{Cand}(\tilde{B}) \cap \mathrm{Cand}(\tilde{B}') \neq \emptyset\right] \geqslant \frac{\gamma}{\ell^2}.$$

Now, observe that $\tilde{B}$ and $\tilde{B}'$ above are distributed in the same way as if we pick both of them independently with respect to $\Gamma(A)$. Recall that, with probability $1/2$, $F(A)$ is a random element of $\mathrm{Cand}(\tilde{B})$ where $\tilde{B} \sim \Gamma(A)$ and, with probability $1/2$, $F(\tilde{B}')$ is a random element of $\mathrm{Cand}(\tilde{B}')$. Moreover, since the sum of degree d-influence is at most d [28, Proposition 3.8], the candidate sets are of sizes at most d/κ. As a result, the above bound yields

$$\Pr_{A\sim V^R, \tilde{B}'\sim\Gamma(A)}[F(A) = F(\tilde{B}')] \geqslant \frac{\gamma\kappa^2}{4d^2\ell^2},$$

which, together with (1), concludes the proof of the lemma. ◀

3.5.2 Decoding a Small Non-Expanding Set

To relate our decoded UG assignment back to a small non-expanding set in G, we use the following lemma of [33], which roughly states that, with the right parameters, the soundness case of SSEH implies that only small fraction of constraints in the UG can be satisfied.

▶ **Lemma 13** ([33, Lemma 6.11]). *If there exists $F : V^R \to [R]$ such that*

$$\Pr_{A\sim V^R, \tilde{B}\sim\Gamma(A), \pi_A, \pi_B\sim\Pi_{R,k}}[\pi_A^{-1}(F(\pi_A(\tilde{A}))) = \pi_B^{-1}(F(\pi_B(\tilde{B})))] \geqslant \zeta,$$

then there exists a set $S \subseteq V$ with $\frac{|S|}{|V|} \in \left[\frac{\zeta}{16R}, \frac{3k}{\varepsilon_V R}\right]$ with $\Phi(S) \leqslant 1 - \frac{\zeta}{16k}$.

By combining the above lemma with Lemma 12, we immediately arrive at the following:

▶ **Lemma 14.** *For any $\varepsilon_T, \gamma, \beta > 0$, let $t = t(\varepsilon_T, \gamma, \beta), \kappa = \kappa(\varepsilon_T, \gamma, \beta)$ and $d = d(\varepsilon_T, \gamma, \beta)$ be as in Theorem 8. For any integer $\ell \geqslant 100t$ and any $\varepsilon_V > 0$, if there exists $T \subseteq V_H$ with $\mu_H(T) \leqslant 1/2$ such that $E_H(T) \geqslant 2\gamma + 10\beta + 2^{-\ell/100}$, then there exists a set $S \subseteq V$ with $\frac{|S|}{|V|} \in \left[\frac{\zeta}{16R}, \frac{3k}{\varepsilon_V R}\right]$ with $\Phi(S) \leqslant 1 - \frac{\zeta}{16k}$ where $\zeta = \frac{\gamma\kappa^2}{4d^2\ell^2}$.*

3.6 Putting Things Together

We can now deduce inapproximability of MUCHB by simply picking appropriate parameters.

Proof of Lemma 3. The parameters are chosen as follows:

- Let $\beta = \varepsilon/30, \gamma = \varepsilon/6$, and $k = \Omega(\log(1/\varepsilon))$ so that the term $2^{-\Omega(k)}$ in Lemma 9 is $\leqslant \varepsilon/4$.
- Let $\varepsilon_T = O(\beta\varepsilon)$ so that the error term $O(\varepsilon_T/\beta)$ in Lemma 9 is at most $\varepsilon/4$.
- Let $t = t(\varepsilon_T, \gamma, \beta), \kappa = \kappa(\varepsilon_T, \gamma, \beta)$ and $d = d(\varepsilon_T, \gamma, \beta)$ be as in Theorem 8.
- Let $\zeta = \frac{\gamma\kappa^2}{4d^2\ell^2}$ be as in Lemma 14 and let $\ell = \max\{100t, 1000\log(1/\varepsilon)\}$.
- Let $\varepsilon_V = O(\varepsilon\beta/\ell)$ where so that the error term $O(\varepsilon_V\ell/\beta)$ in Lemma 9 is at most $\varepsilon/4$.
- Let $\eta = \min\{\frac{\zeta}{32k}, O(\varepsilon\beta/\ell)\}$ so that the error term $O(\eta\ell/\beta)$ in Lemma 9 is at most $\varepsilon/4$.
- Let $M = \max\{\frac{16k}{\beta\zeta}, \frac{3\beta}{\varepsilon_V}\}$.
- Finally, let $R = \frac{k}{\beta\delta}$ where $\delta = \delta(\eta, M)$ is the parameter from the SSEH (Conjecture 2).

Let $G = (V, E)$ be an instance of SSE(η, δ, M) and let $H = (V_H, G_H)$ be the hypergraph resulted from our reduction. If there exists $S \subseteq V$ of size $\delta|V|$ of expansion at most η, Lemma 9 implies that there is a bisection (T_0, T_1) of V_H such that $E_H(T_0), E_H(T_1) \geqslant 1/2 - \varepsilon$.

As for the soundness, Lemma 14 with our choice of parameters implies that, if there exists a set $T \subseteq V_H$ with $\mu(T) \leqslant 1/2$ and $E_H(T_0) \geqslant \varepsilon$, there exists $S \subseteq V$ with $|S| \in \left[\frac{\delta|V|}{M}, \delta|V|M\right]$ whose expansion is less than $1-\eta$. The contrapositive of this yields the soundness property. ◀

4 Conclusion

In this work, we prove essentially tight inapproximability of MEB, MBB and Minimum k-Cut based on SSEH. Our results, expecially for the biclique problems, demonstrate further the applications of the hypothesis and particularly the RST technique [33] in proving hardness of graph problems that involve some form of expansion. An obvious but intriguing research direction is to try to utilize the technique to other problems. One plausible candidate problem to this end is the 2-Catalog Segmentation Problem [24] since a natural candidate reduction for this problem fails due to a similar counterexample as in Section 3.2.

Acknowledgement. I am grateful to Prasad Raghavendra for providing his insights on the Small Set Expansion problem and techniques developed in [32, 33] and Luca Trevisan for lending his expertise in PCPs with small free bits. I would also like to thank Haris Angelidakis for useful discussions on Minimum k-Cut and Daniel Reichman for inspiring me to work on Maximum Edge Biclique and Maximum Balanced Biclique. Finally, I thank anonymous reviewers for their useful comments and, more specifically, for pointing me to [6].

References

1 Christoph Ambühl, Monaldo Mastrolilli, and Ola Svensson. Inapproximability results for maximum edge biclique, minimum linear arrangement, and sparsest cut. *SIAM J. Comput.*, 40(2):567–596, April 2011.

2 Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *J. ACM*, 45(3):501–555, May 1998.

3 Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of NP. *J. ACM*, 45(1):70–122, January 1998.

4 Nikhil Bansal and Subhash Khot. Optimal long code test with one free bit. In *IEEE FOCS*, pages 453–462, 2009.

5 Piotr Berman and Georg Schnitger. On the complexity of approximating the independent set problem. *Inf. Comput.*, 96(1):77–94, 1992.

6 Amey Bhangale, Rajiv Gandhi, Mohammad Taghi Hajiaghayi, Rohit Khandekar, and Guy Kortsarz. Bicovering: Covering edges with two small subsets of vertices. In *ICALP*, pages 6:1–6:12, 2016.

7 Avrim Blum. Algorithms for approximate graph coloring. Technical report, Massachusetts Institute of Technology, Cambridge, MA, USA, 1991.

8 Irit Dinur. Mildly exponential reduction from gap 3SAT to polynomial-gap label-cover. *ECCC*, 23:128, 2016.

9 Irit Dinur and David Steurer. Analytical approach to parallel repetition. In *ACM STOC*, pages 624–633, 2014.

10 Uriel Feige. Relations between average case complexity and approximation complexity. In *ACM STOC*, pages 534–543, 2002.

11 Uriel Feige and Shimon Kogan. Hardness of approximation of the balanced complete bipartite subgraph problem. Technical report, Weizmann Institute of Science, Israel, 2004.

12 Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.

13 Olivier Goldschmidt and Dorit S Hochbaum. A polynomial algorithm for the k-cut problem for fixed k. *Mathematics of operations research*, 19(1):24–37, 1994.

14 Johan Håstad. Clique is hard to approximate within $n^{1-\varepsilon}$. In *IEEE FOCS*, pages 627–636, 1996.

15 Johan Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, July 2001.

16 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, December 2001.

17 David S. Johnson. The NP-completeness column: An ongoing guide. *J. Algorithms*, 8(5):438–448, September 1987.

18 Subhash Khot. Improved inaproximability results for MaxClique, chromatic number and approximate graph coloring. In *IEEE FOCS*, pages 600–609, 2001.

19 Subhash Khot. On the power of unique 2-prover 1-round games. In *ACM STOC*, pages 767–775, 2002.

20 Subhash Khot. Ruling out ptas for graph min-bisection, dense k-subgraph, and bipartite clique. *SIAM J. Comput.*, 36(4), 2006.

21 Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell. Optimal inapproximability results for MAX-CUT and other 2-variable CSPs? *SIAM J. Comput.*, 37(1):319–357, 2007.

22 Subhash Khot and Ashok Kumar Ponnuswami. Better inapproximability results for Max-Clique, chromatic number and Min-3Lin-Deletion. In *ICALP*, pages 226–237, 2006.

23 Subhash Khot and Oded Regev. Vertex cover might be hard to approximate to within $2-\varepsilon$. *J. Comput. Syst. Sci.*, 74(3):335–349, 2008.

24 Jon Kleinberg, Christos Papadimitriou, and Prabhakar Raghavan. Segmentation problems. *J. ACM*, 51(2):263–280, March 2004.

25 Pasin Manurangsi. Almost-polynomial ratio ETH-hardness of approximating densest k-subgraph. In *ACM STOC*, 2017. To appear.

26 Pasin Manurangsi and Prasad Raghavendra. A birthday repetition theorem and complexity of approximating dense CSPs. In *ICALP*, 2017. To appear.

27 Dana Moshkovitz. The projection games conjecture and the NP-hardness of $\ln n$-approximating set-cover. *Theory of Computing*, 11:221–235, 2015.

28 Elchanan Mossel, Ryan O'Donnell, and Krzysztof Oleszkiewicz. Noise stability of functions with low influences: Invariance and optimality. *Ann. Math.*, pages 295–341, 2010.

29 Joseph (Seffi) Naor and Yuval Rabani. Tree packing and approximating k-cuts. In *ACM-SIAM SODA*, pages 26–27, 2001.
30 Ryan O'Donnell. *Analysis of Boolean Functions.* Cambridge University Press, 2014.
31 René Peeters. The maximum edge biclique problem is NP-complete. *Discrete Appl. Math.*, 131(3):651–654, September 2003.
32 Prasad Raghavendra and David Steurer. Graph expansion and the unique games conjecture. In *ACM STOC*, pages 755–764, 2010.
33 Prasad Raghavendra, David Steurer, and Madhur Tulsiani. Reductions between expansion problems. In *IEEE CCC*, pages 64–73, 2012.
34 R. Ravi and Amitabh Sinha. Approximating k-cuts via network strength. In *ACM-SIAM SODA*, pages 621–622, 2002.
35 Huzur Saran and Vijay V. Vazirani. Finding k cuts within twice the optimal. *SIAM J. Comput.*, 24(1):101–108, February 1995.
36 Ola Svensson. Hardness of vertex deletion and project scheduling. *Theory of Computing*, 9(24):759–781, 2013.
37 Mingyu Xiao, Leizhen Cai, and Andrew Chi-Chih Yao. Tight approximation ratio of a general greedy splitting algorithm for the minimum k-way cut problem. *Algorithmica*, 59(4):510–520, 2011.
38 Liang Zhao, Hiroshi Nagamochi, and Toshihide Ibaraki. Approximating the minimum k-way cut in a graph via minimum 3-way cuts. *J. Comb. Optim.*, 5(4):397–410, 2001.
39 David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, 2007.