Continuity and Rational Functions*

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Abstract

A word-to-word function is continuous for a class of languages $\mathcal{V}$ if its inverse maps $\mathcal{V}$_languages to $\mathcal{V}$. This notion provides a basis for an algebraic study of transducers, and was integral to the characterization of the sequential transducers computable in some circuit complexity classes.

Here, we report on the decidability of continuity for functional transducers and some standard classes of regular languages. Previous algebraic studies of transducers have focused on the structure of the underlying input automaton, disregarding the output. We propose a comparison of the two algebraic approaches through two questions: When are the automaton structure and the continuity properties related, and when does continuity propagate to superclasses?

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1 Introduction

The algebraic theory of regular languages is tightly interwoven with fundamental questions about the computing power of Boolean circuits and logics. The most famous of these braids revolves around $\mathcal{A}$, the class of *aperiodic* or *counter-free* languages. Not only is it expressed using the logic FO[<], but it can be seen as the basic building block of AC^0, the class of languages recognized by circuit families of polynomial size and constant depth, this class being in turn expressed by the logic FO[arb] (see [18] for a lovely account). This pervasive interaction naturally prompts to lift this study to the functional level, hence to *rational functions*. This was started in [4], where it was shown that a subsequential (i.e., input-deterministic) transducer computes an AC^0 function iff it preserves the regular languages of AC^0 by inverse image. Buoyed by this clean, semantic characterization, we wish to further investigate this latter property for different classes: say that a function $f\colon A^* \to B^*$ is $\mathcal{V}$_continuous, for a class of languages $\mathcal{V}$, if for every language $L \subseteq B^*$ of $\mathcal{V}$, the language $f^{-1}(L)$ is also a language of $\mathcal{V}$. Our main focus will be on deciding $\mathcal{V}$_continuity for rational functions; before listing our main results, we emphasize two additional motivations.

First, there has been some historical progression towards this goal. Noting, in [9], that inverse rational functions provide a uniform and compelling view of a wealth of natural operations on regular languages, Pin and Sakarovitch initiated in [10] a study of regular-continuous functions. It was already known at the time, by a result of Choffrut (see [3,

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Theorem 2.7]), that regular-continuity together with some uniform continuity property *characterize* functions computed by subsequential transducers. This characterization was instrumental in the study of Reutenauer and Schützenberger [15], who already noticed the peculiar link between uniform continuity for some distances on words and continuity for certain classes of languages. This link was tightened by Pin and Silva [11] who formalized this topological approach and generalized it to rational relations. More recently [12], the same authors made precise the link unveiled by Reutenauer and Schützenberger, and developed a fascinating and robust framework in which language continuity has a topological interpretation (see the beginning of Section 3, as we build upon this theory). Pin and Silva [13] notably proposed thereafter a study of functions that propagate continuity for a class to subclasses.

Second, the interweaving between languages, circuits, and logic that was alluded to previously can in fact be formally stated (see again [18, 19]). As a central property towards this formalization is the correspondence between "cascade products" of automata, stacking of circuits, and nesting of formulas, respectively. Strikingly, these operations can all be seen as inverse rational functions [19]. These operations being intrinsic in the construction of complex objects, decompositions are often naturally used to specify languages, circuits, and formulas (see, e.g., [17, Section 5.5]). We remark that a sufficient condition for the result of the composition to be in some given class (of languages, circuits, or logic formulas), is that each rational function be continuous for that class. Hence deciding continuity allows to give a sufficient condition for this membership question *without* computing the result of the composition, which is subject to combinatorial blowup.

Here, we report on three questions, the first two relating continuity to the main other algebraic approach to transducers, while allowing a more gentle introduction to the evaluation of *profinite words* by transducers:

- When is the transducer *structure* (i.e., its so-called *transition monoid*) impacting its continuity? The results of Reutenauer and Schützenberger [15] can indeed be seen as the starting point of two distinct algebraic theories for rational functions; on the one hand, the study of continuity, and on the other the study of the transition monoid of the transducer (by disregarding the output). This latter endeavor was carried by [5].
- What is the impact of *variety inclusion* on the inclusion of the related classes of continuous rational functions? When the focus is solely on the structure of the transducer, there is a natural propagation to superclasses; when is it the case for continuity?
- When is $\mathcal{V}$_continuity decidable for rational functions? We show decidability for the varieties $\mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{DA}, \mathcal{A}, \mathcal{C}om, \mathcal{A}b, \mathcal{G}_{\mathrm{sol}}$, and $\mathcal{G}$; these constitute our main results.

2 Preliminaries

We assume some familiarity with the theory of automata and transducers, and concepts related to metric spaces (see, e.g., [3, 8] for presentations pertaining to our topic). Apart from these prerequisites, for which the notation is first settled, the presentation is self-contained.

We will use A and B for alphabets, and A^* for words over A, with 1 the empty word. For each word u, there is a smallest v, called the *primitive root* of u, such that $u = v^c$ for some c; if $c = 1$, then u is itself *primitive*. We write $|u|$ for the length of a word $u \in A^*$ and $\mathsf{alph}(u)$ for the set of letters that appear in u. For a word $u \in A^*$ and a language $L \subseteq A^*$, we write $u^{-1}L$ for $\{v \mid u \cdot v \in L\}$, and symmetrically for Lu^{-1}, these two operations being called the left and right quotients of L by u, respectively. We naturally extend concatenation and quotients to relations, in a component-wise fashion, e.g., for $R \subseteq A^* \times A^*$ and a pair $\rho \in A^* \times A^*$, we may use $\rho^{-1}R$ and $R\rho^{-1}$. We write L^c for the complement of L. A *variety* is a mapping $\mathcal{V}$ which associates with each alphabet A a set $\mathcal{V}(A^*)$ of regular languages closed under the Boolean operations and quotient, and such that for any morphism $h\colon A^* \to B^*$ and any $L \in \mathcal{V}(B^*)$, it holds that $h^{-1}(L) \in \mathcal{V}(A^*)$. Reg is the variety that maps every alphabet A to the set $\mathrm{Reg}(A^*)$ of regular languages over A. Given two languages $K, L \subseteq A^*$, we say that they are $\mathcal{V}$-separable if there is a $S \in \mathcal{V}(A^*)$ such that $K \subseteq S$ and $L \cap S = \emptyset$.

Transducers. A transducer τ is a 9-tuple $(Q, A, B, \delta, I, F, \lambda, \mu, \rho)$ where (Q, A, δ, I, F) forms an automaton (i.e., Q is a state set, A an input alphabet, $\delta \subseteq Q \times A \times Q$ a transition set, $I \subseteq Q$ a set of initial states, and $F \subseteq Q$ a set of final states), and additionally, B is an output alphabet and $\lambda\colon I \to B^*, \mu\colon \delta \to B^*, \rho\colon F \to B^*$ are the output functions. We write $\tau_{q,q'}$ for τ with $I := \{q\}$ and $F := \{q'\}$, adjusting λ and ρ to output 1 if they were undefined on these states. Similarly, $\tau_{q,\bullet}$ is τ with $I := \{q\}$ and F unchanged, and symmetrically for $\tau_{\bullet,q}$. For $q \in Q$ and $u \in A^*$, we write $q.u$ for the set of states reached from q by reading u. We assume that all the transducers and automata under study have no useless state, that is, that all states appear in some accepting path.

With $w \in A^*$, let $t_1 t_2 \cdots t_{|w|} \in \delta^*$ be an accepting path for w, starting in a state $q \in I$ and ending in some $q' \in F$. The output of this path is $\lambda(q)\mu(t_1)\mu(t_2)\cdots\mu(t_n)\rho(q')$, and we write $\tau(w)$ for the set of outputs of such paths. We use τ for both the transducer and its associated partial function from A^* to subsets of B^*. Relations of the form $\{(u, v) \mid v \in \tau(u)\}$ are called *rational relations*.

The transducer τ is *unambiguous* if there is at most one accepting path for each word. In that case $\tau_{q,q'}$ is also an unambiguous transducer for any states q, q'. When τ is unambiguous, it realizes a word-to-word function: the set of functions computed by unambiguous transducers is the set of *rational functions*. Further restricting, if the underlying automaton is deterministic, we say that τ is *subsequential*. If τ is a finite union of subsequential rational functions of disjoint domains, we say that τ is *plurisubsequential*.

Word distances, profinite words. For a variety $\mathcal{V}$ of regular languages, we define a distance between words for which, intuitively, two words are close if it is hard to separate them with $\mathcal{V}$ languages. Define $d_{\mathcal{V}}(u, v)$, for words $u, v \in A^*$, to be 2^{-r} where r is the size of the smallest automaton that recognizes a language of $\mathcal{V}(A^*)$ that separates $\{u\}$ from $\{v\}$; if no such language exists, then $d_{\mathcal{V}}(u, v) = 0$. It can be shown that this distance is a *pseudo-ultrametric* [8, Section VII.2]; we make only implicit and innocuous use of this fact.

We simply write d for d_{Reg}. The complete metric space that is the completion of (A^*, d) is denoted $\widehat{A^*}$ and is called the *free profinite monoid*, its elements being the *profinite words*, and the concatenation being naturally extended. By definition, if $(u_n)_{n>0}$ is a Cauchy sequence, it should hold that for any regular language L, there is a N such that either all u_n with $n > N$ belong to L, or none does. For any $x \in A^*$, define the profinite word $x^\omega = \lim x^{n!}$, and more generally, $x^{\omega - c} = \lim x^{n! - c}$. That $(x^{n!})_{n>0}$ is a Cauchy sequence is a starting point of the profinite theory [8, Proposition VI.2.10]; it is also easily checked that $x^{c\times\omega} = \lim x^{c\times n!}$ is equal to x^ω for any integer $c \geq 1$. Given a language $L \subseteq A^*$, we write $\overline{L} \subseteq \widehat{A^*}$ for its closure, and we note that if L is regular, $\overline{L}^{\mathrm{c}} = \overline{L^{\mathrm{c}}}$ and for L' regular, $\overline{L \cup L'} = \overline{L} \cup \overline{L'}$, and similarly for intersection (see [8, Theorem VI.3.15]).

Equations. For $u, v \in \widehat{A^*}$, a language $L \subseteq A^*$ *satisfies the (profinite) equation* $u = v$ if for any words $s, t \in A^*$, $[s \cdot u \cdot t \in \overline{L} \Leftrightarrow s \cdot v \cdot t \in \overline{L}]$. Similarly, a class of languages satisfies an equation if all the languages of the class satisfy it. For a variety $\mathcal{V}$, we write $u =_{\mathcal{V}} v$, and say that u is equal to v in $\mathcal{V}$, if $\mathcal{V}(A^*)$ satisfies $u = v$. For a partial function f, $f(u) = \_\mathcal{V} f(v)$ means that either both $f(u)$ and $f(v)$ are undefined, or they are both defined and equal in $\mathcal{V}$.

Given a set E of equations over $\widehat{A^*}$, the class of languages *defined* by E is the class of languages over A^* that satisfy all the equations of E. Reiterman's theorem shows in particular that for any variety $\mathcal{V}$ and any alphabet A, $\mathcal{V}(A^*)$ is defined by a set of equations (the precise form of which being studied in [6]).

More on varieties. Borrowing from Almeida and Costa [2], we say that a variety $\mathcal{V}$ is *supercancellative* when for any alphabet A, any $u, v \in \widehat{A^*}$ and $x, y \in A$, if $u \cdot x = \_\mathcal{V} v \cdot y$ or $x \cdot u = \_\mathcal{V} y \cdot v$, then $u = \_\mathcal{V} v$ and $x = y$. This implies in particular that for any word $w \in A^*$, both $w \cdot A^*$ and $A^* \cdot w$ are in $\mathcal{V}(A^*)$. We further say that a variety $\mathcal{V}$ *separates words* if for any $s, t \in A^*$, $\{s\}$ and $\{t\}$ are $\mathcal{V}$_separable.

Our main applications revolve around some classical varieties, that we define over any possible alphabet A as follows, where x, y range over all of A^*, and a, b over A:

- $\mathcal{J}$, def. by $(xy)^\omega \cdot x = y \cdot (xy)^\omega = (xy)^\omega$
- $\mathcal{R}$, def. by $(xy)^\omega \cdot x = (xy)^\omega$
- $\mathcal{L}$, def. by $y \cdot (xy)^\omega = (xy)^\omega$
- $\mathcal{DA}$, def. by $x^\omega \cdot z \cdot x^\omega = x^\omega$ for all $z \in \mathsf{alph}(x)^*$
- $\mathcal{A}$, def. by $x^{\omega+1} = x^\omega$
- $\mathcal{C}om$, def. by $ab = ba$
- $\mathcal{A}b$, def. by $ab = ba$ and $a^\omega = 1$
- $\mathcal{G}_{\text{nil}}$, the languages rec. by nilpotent groups
- $\mathcal{G}_{\text{sol}}$, the languages rec. by solvable groups
- $\mathcal{G}$, the languages rec. by groups

The varieties included in $\mathcal{A}$ are called *aperiodic varieties* and those in $\mathcal{G}$ are called *group varieties*. Precise definitions, in particular for the group varieties, can be found in [18, 14]; we simply note that in group varieties, x^ω equals 1 for all $x \in A^*$. All these varieties except for $\mathcal{A}b$ and $\mathcal{C}om$ separate words, and only $\mathcal{DA}$ and $\mathcal{A}$ are supercancellative. They verify:

$$\mathcal{J} = \mathcal{R} \cap \mathcal{L} \subsetneq \mathcal{R}, \mathcal{L} \subsetneq \mathcal{DA} \subsetneq \mathcal{A} \qquad \mathcal{A}b = \mathcal{G} \cap \mathcal{C}om \subsetneq \mathcal{G}_{\text{nil}} \subsetneq \mathcal{G}_{\text{sol}} \subsetneq \mathcal{G}, \quad \mathcal{A}b \subsetneq \mathcal{C}om$$

On transducers and profinite words. For a profinite word u and a state q of an unambiguous transducer τ, the set $q.u$ is well-defined; indeed, with $u = \lim u\_n$, the set $q.u\_n$ is eventually constant, as otherwise for some state q', the domain of $\tau_{q,q'}$ would be a regular language that separates infinitely many u_n's.

A transducer $\tau \colon A^* \to B^*$ is a $\mathcal{V}$_*transducer*,[1] for a variety $\mathcal{V}$, if for some set of equations E defining $\mathcal{V}(A^*)$, for all $(u = v) \in E$ and all states q of τ, it holds that $q.u = q.v$. A rational function is $\mathcal{V}$_*realizable* if it is realizable by a $\mathcal{V}$_transducer.

Continuity. For a variety $\mathcal{V}$, a function $f \colon A^* \to B^*$ is $\mathcal{V}$_continuous[2] iff for any $L \in \mathcal{V}(B^*)$, $f^{-1}(L) \in \mathcal{V}(A^*)$. We mostly restrict our attention to rational functions, and their being

[1] The usual definition of $\mathcal{V}$_transducer is based on the so-called transition monoid of τ, see, e.g., [15]; the definition here is easily seen to be equivalent by [1, Lemma 3.2] and [4, Lemma 1].

[2] A note on terminology: There has been some fluctuation on the use of the term "continuous" in the literature, mostly when a possible incompatibility arises with topology. In [13], the authors use the term "preserving" in the more general context of functions from monoids to monoids. In our study, we focus on word to word functions, in which the natural topological context provides a solid basis for the use of "continuous," as used in [11, 4].

computed by transducers implies that they are countably many. We note that much more Reg_continuous functions exist, in particular uncomputable ones:

▶ **Proposition 1.** *There are uncountably many Reg_continuous functions.*

3 Continuity: The profinite approach

We build upon the work of Pin and Silva [11] and develop tools specialized to rational functions. In Section 3.1, we present a lemma asserting the equivalence between $\mathcal{V}$_continuity and the “preservation” of the defining equations for $\mathcal{V}$. In the sections thereafter, we specialize this approach to rational functions. As noted in [11], it often occurs that results about rational functions can be readily applied to the larger class of Reg_continuous functions; here, this is in particular the case for the Preservation Lemma of Section 3.1.

Our main appeal to a classical notion of continuity is given by the:

▶ **Theorem 2** ([12, Theorem 4.1])**.** *Let $f\colon A^* \to B^*$. It holds that f is $\mathcal{V}$_continuous iff f is uniformly continuous for the distance $d\_\mathcal{V}$.*

Consequently, if f is Reg_continuous then it has a unique extension to the free profinite monoids, written $\widehat{f}\colon \widehat{A^*} \to \widehat{B^*}$. The salient property of this mapping is that it is continuous in the *topological sense* (see, e.g., [8]). For our specific needs, we simply mention that it implies that for any regular language L, we have that $\widehat{f}^{-1}(\overline{L})$ is closed (that is, it is the closure of some set).

3.1 The Preservation Lemma: Continuity is preserving equations

The Preservation Lemma gives us a key characterization in our study: it ties together continuity and some notion of preservation of equations. This can be seen as a generalization to functions of equation satisfaction for languages. We will need the following technical lemma that extends [8, Proposition VI.3.17] from morphisms to arbitrary Reg_continuous functions; interestingly, this relies on a quite different proof.

▶ **Lemma 3.** *Let $f\colon A^* \to B^*$ be a Reg_continuous function and L a regular language. It holds that $\widehat{f}^{-1}(\overline{L}) = \overline{f^{-1}(L)}$.*

▶ **Lemma 4** (Preservation Lemma)**.** *Let $f\colon A^* \to B^*$ be a Reg_continuous function and E a set of equations that defines $\mathcal{V}(A^*)$. The function f is $\mathcal{V}$_continuous iff for all $(u = v) \in E$ and words $s, t \in A^*$, $\widehat{f}(s \cdot u \cdot t) =\_\mathcal{V} \widehat{f}(s \cdot v \cdot t)$.*

Proof. *(Only if)* Suppose f is $\mathcal{V}$_continuous. Let $u, v \in \widehat{A^*}$ such that $u =\_\mathcal{V} v$, and $s, t \in A^*$. Since by $\mathcal{V}$_continuity $f^{-1}(B^*) \in \mathcal{V}(A^*)$, either both $s \cdot u \cdot t$ and $s \cdot v \cdot t$ belong to the closure of this language, or they both do not. The latter case readily yields the result, hence suppose we are in the former case.

By definition, $u = \lim u\_n$ and $v = \lim v\_n$ for some Cauchy sequences of words $(u\_n)\_n > 0$ and $(v\_n)\_n > 0$. Since $s \cdot u \cdot t =\_\mathcal{V} s \cdot v \cdot t$, the hypothesis yields that $d\_\mathcal{V}(s \cdot u\_n \cdot t, s \cdot v\_n \cdot t)$ tends to 0. By Theorem 2, f is uniformly continuous for $d\_\mathcal{V}$, hence $d\_\mathcal{V}(f(s \cdot u\_n \cdot t), f(s \cdot v\_n \cdot t))$ also tends to 0 (note that both $f(s \cdot u\_n \cdot t)$ and $f(s \cdot v\_n \cdot t)$ are defined for all n big enough). This shows that $\widehat{f}(s \cdot u \cdot t) =\_\mathcal{V} \widehat{f}(s \cdot v \cdot t)$.

(If) Suppose that f preserves the equations of E as in the statement. Let $L \in \mathcal{V}(B^*)$, we wish to verify that $L' = f^{-1}(L) \in \mathcal{V}(A^*)$, or equivalently by definition, that L' satisfies all the equations of E. Let $(u = v) \in E$ be one such equation, and $s, t \in A^*$; we must show that $s \cdot u \cdot t \in \overline{L'} \Leftrightarrow s \cdot v \cdot t \in \overline{L'}$.

Suppose $s \cdot u \cdot t \in \overline{L'}$. Since f is Reg_continuous, it holds that $\widehat{f}(s \cdot u \cdot t) \in \overline{L}$ (observe that $\widehat{f}(s \cdot u \cdot t)$ is indeed defined). By hypothesis, $\widehat{f}(s \cdot u \cdot t) =_\mathcal{V} \widehat{f}(s \cdot v \cdot t)$; now since $L \in \mathcal{V}(B^*)$, it must hold that $\widehat{f}(s \cdot v \cdot t) \in \overline{L}$. Taking the inverse image of $\widehat{f}$ on both sides, it thus holds that $s \cdot v \cdot t \in \widehat{f}^{-1}(\overline{L})$, and Lemma 3 then shows that $s \cdot v \cdot t \in \overline{L'}$. As the argument works both ways, this shows that $s \cdot u \cdot t \in \overline{L'} \Leftrightarrow s \cdot v \cdot t \in \overline{L'}$, concluding the proof. ◀

Continuity can be seen as preserving *membership* to $\mathcal{V}$ (by inverse image); this is where the nomenclature "$\mathcal{V}$_preserving function" of [13] stems from. Strikingly, this could also be worded as preserving *nonmembership* to $\mathcal{V}$:

▶ **Proposition 5.** *A Reg_continuous total*[3] *function $f : A^* \to B^*$ is $\mathcal{V}$_continuous iff for all $L \subseteq A^*$ that do* not *belong to $\mathcal{V}(A^*)$, $f(L)$ and $f(L^c)$ are not $\mathcal{V}$_separable.*

3.2 The profinite extension of rational functions

The Preservation Lemma already hints at our intention to see transducers as computing functions from and to the free profinite monoids. Naturally, if τ is a rational function, its being Reg_continuous allows us to do so (by Theorem 2). For $u = \lim u_n$ a profinite word, we will write $\tau(u)$ for $\widehat{\tau}(u)$, i.e., the limit $\lim \tau(u_n)$, which exists by continuity. In this section, we develop a slightly more combinatorial approach to this evaluation, and address two classes of profinite words: those expressed as $s \cdot u \cdot t$ for s, t words and u a profinite word, and those expressed as x^ω for x a word.

Recall that for a transducer state q and a profinite word u, $q.u$ is well-defined. As a consequence, if s and t are words and τ is unambiguous, then there is at most one initial state q_0, one $q \in q_0.s$ and one $q' \in q.u$ such that $q'.t$ is final, and these states exist iff $\tau(s \cdot u \cdot f)$ is defined. Thus:

▶ **Lemma 6.** *Let τ be an unambiguous transducer from A^* to B^*, $s, t \in A^*$ and $u \in \widehat{A^*}$. Suppose $\tau(s \cdot u \cdot f)$ is defined, and let q_0, q, q' be the unique states such that q_0 is initial, $q \in q_0.s$, $q' \in q.u$, and $q'.t$ is final. The following holds: $\tau(s \cdot u \cdot t) = \tau_{\bullet,q}(s) \cdot \tau_{q,q'}(u) \cdot \tau_{q',\bullet}(t)$.*

▶ **Lemma 7.** *Let τ be an unambiguous transducer from A^* to B^* and $x \in A^*$. If $\tau(x^\omega)$ is defined, then there are words $s, y, t \in B^*$ such that: $\tau(x^\omega) = s \cdot y^{\omega-1} \cdot t$.*

These constitute our main ways to effectively evaluate the image of profinite words through transducers. Their use being quite ubiquitous in our study, we will rarely refer to these lemmata nominally.

3.3 The Syncing Lemma: Preservation Lemma applied to transducers

We apply the Preservation Lemma on transducers and deduce a slightly more combinatorial characterization of transducers describing continuous functions. This does not provide an immediate decidable criterion, but our decidability results will often rely on it. The goal of the forthcoming lemma is to decouple, when evaluating $s \cdot u \cdot t$ (with the notations of the Preservation Lemma), the behavior of the u part and that of the s, t part. This latter part will be tested against an *equalizer* set:

[3] In all the varieties we are interested in, one can easily modify any partial function into a total function while preserving its continuity properties.

▶ **Definition 8** (Equalizer set). Let $u, v \in \widehat{A^*}$. The *equalizer set* of u and v in $\mathcal{V}$ is:

$$\mathsf{Equ}\_\mathcal{V}(u, v) = \{(s, s', t, t') \in (A^*)^4 \mid s \cdot u \cdot t =\_\mathcal{V} s' \cdot v \cdot t'\} .$$

▶ Remark. The complexity of equalizer sets can be surprisingly high. For instance, letting $\mathcal{V}$ be the class of languages defined by $\{x^2 = x^3 \mid x \in A^*\}$, there is a profinite word u for which $\mathsf{Equ}\_\mathcal{V}(u, u)$ is undecidable. On the other hand, equalizer sets quickly become less complex for common varieties; for instance, Lemma 12 will provide a simple form for the equalizer sets of aperiodic supercancellative varieties.

▶ **Definition 9** (Input synchronization). Let $R, S \subseteq A^* \times B^*$. The *input synchronization* of R and S is defined as the relation over $B^* \times B^*$ obtained by synchronizing the first component of R and S: $R \bowtie S = \{(u, v) \mid (\exists s)[(s, u) \in R \wedge (s, v) \in S]\}$ $\big(= S \circ R^{-1}\big)$.

Naturally, the input synchronization of two rational functions is a rational relation.

▶ **Lemma 10** (Syncing Lemma). *Let τ be an unambiguous transducer from A^* to B^* and E a set of equations that defines $\mathcal{V}(A^*)$. The function τ is $\mathcal{V}_$continuous iff:*

1. *$\tau^{-1}(B^*) \in \mathcal{V}(A^*)$, and*
2. *For any $(u = v) \in E$, any states p, q, any $p' \in p.u$, and any $q' \in q.v$, and letting $u' = \tau_{p,p'}(u)$ and $v' = \tau_{q,q'}(v)$: $(\tau_{\bullet,p} \bowtie \tau_{\bullet,q}) \times (\tau_{p',\bullet} \bowtie \tau_{q',\bullet}) \subseteq \mathsf{Equ}\_\mathcal{V}(u', v')$.*

3.4 A profinite toolbox for the aperiodic setting

In this section, we provide a few lemmata pertaining to our study of aperiodic continuity. We show that the equalizer sets of aperiodic supercancellative varieties are well-behaved. Intuitively, the larger the varieties are, the more their nonempty equalizer sets will be similar to the identity. For instance, if $s \cdot x^\omega =\_\mathcal{A} x^\omega$, for words s and x, it should hold that s and x have the same primitive root. We first note the following easy fact that will only be used in this section; it is reminiscent of the notion of *equidivisibility*, studied in the profinite context by Almeida and Costa [2].

▶ **Lemma 11.** *Let u, v be profinite words over an alphabet A and $\mathcal{V}$ be a supercancellative variety. Suppose that there are $s, t \in A^*$ such that $u \cdot t =\_\mathcal{V} s \cdot v$, then there is a $w \in \widehat{A^*}$ such that $u =\_\mathcal{V} s \cdot w$ and $v =\_\mathcal{V} w \cdot t$. If moreover $u = v$ and $\mathcal{V}$ is aperiodic, then $u =\_\mathcal{V} s \cdot u \cdot t$.*

▶ **Lemma 12.** *Let u, v be profinite words over an alphabet A and $\mathcal{V}$ be an aperiodic supercancellative variety. Suppose $\mathsf{Equ}\_\mathcal{V}(u, v)$ is nonempty. There are words $x, y \in A^*$ and two pairs $\rho\_1, \rho\_2 \in (A^*)^2$ such that: $\mathsf{Equ}\_\mathcal{V}(u, v) = \Big(Id \cdot \big((x^*, x^*)\rho\_1^{-1}\big)\Big) \times \Big(\big(\rho\_2^{-1}(y^*, y^*)\big) \cdot Id\Big)$.*

▶ **Lemma 13.** *Let x, y be words. For every aperiodic supercancellative variety $\mathcal{V}$, it holds that $\mathsf{Equ}\_\mathcal{V}(x^\omega, y^\omega) = \mathsf{Equ}\_\mathcal{A}(x^\omega, y^\omega)$.*

▶ Remark. For two aperiodic supercancellative varieties $\mathcal{V}$ and $\mathcal{W}$, we could further show that if both $\mathsf{Equ}\_\mathcal{V}(u, v)$ and $\mathsf{Equ}\_\mathcal{W}(u, v)$ are nonempty, then they are equal, for any profinite words u, v. It may however happen that one equalizer set is empty while the other is not; for instance, with $u = (ab)^\omega$ and $v = (ab)^\omega \cdot a \cdot (ab)^\omega$, the equalizer set of u and v in $\mathcal{DA}$ is nonempty, while it is empty in $\mathcal{A}$.

4 Intermezzos

We present a few facts of independent interest on continuous rational functions. Through this, we develop a few examples, showing in particular how the Preservation and Syncing

Lemmata can be used to show (non)continuity. In a first part, we study when the structure of the transducer is relevant to continuity, and in a second, when the (non)inclusion of variety relates to (non)inclusion of the class of continuous rational functions.

4.1 Transducer structure and continuity

As noted by Reutenauer and Schützenberger [15, p. 231], there exist numerous natural varieties $\mathcal{V}$ for which any $\mathcal{V}$_realizable rational function is $\mathcal{V}$_continuous. Indeed:

▶ **Proposition 14.** *Let $\mathcal{V}$ be a variety of languages closed under inverse $\mathcal{V}$_realizable rational function. Any $\mathcal{V}$_realizable rational function is $\mathcal{V}$_continuous. This holds in particular for the varieties $\mathcal{A}, \mathcal{G}_{\mathrm{sol}}$, and $\mathcal{G}$.*

▶ **Proposition 15.** *For $\mathcal{V} \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{DA}, \mathcal{AB}, \mathcal{G}_{\mathrm{nil}}, \mathcal{COM}\}$, there are $\mathcal{V}$_realizable rational functions that are not $\mathcal{V}$_continuous.*

The converse concern, that is, whether all $\mathcal{V}$_continuous rational functions are $\mathcal{V}$_realizable, was mentioned by Reutenauer and Schützenberger [15] for $\mathcal{V} = \mathcal{A}$.

▶ **Proposition 16.** *For $\mathcal{V} \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{DA}, \mathcal{A}, \mathcal{AB}, \mathcal{COM}\}$, there are $\mathcal{V}$_continuous rational functions that are not $\mathcal{V}$_realizable.*

Proof. *(The aperiodic cases)* Let $A = \{a\}$, a unary alphabet. Consider the transducer τ that removes every second a: its minimal transducer not being a $\mathcal{A}$_transducer, it is not $\mathcal{A}$_realizable (this is a property of subsequential transducers [15]). However, all the unary languages of $\mathcal{V}$ are either finite or co-finite, and hence for any $L \in \mathcal{V}(A^*)$, $\tau^{-1}(L)$ is either finite or co-finite, hence belongs to $\mathcal{V}(A^*)$.

(The $\mathcal{AB}$ and $\mathcal{COM}$ cases) Over $A = \{a, b\}$, define τ to map words w in aA^* to $(ab)^{|w|}$, and words w in bA^* to $(ba)^{|w|}$. Clearly, a and b cannot act commutatively on the transducer. Now $\tau(ab) = \_\mathcal{COM}\tau(ba)$, and moreover $\tau(x^\omega) = \_\mathcal{AB}(ab)^\omega = \_\mathcal{AB}1 = \tau(1)$, hence τ is continuous for both $\mathcal{AB}$ and $\mathcal{COM}$ by the Preservation Lemma. ◀

We delay the positive answers to that question, namely for $\mathcal{G}_{\mathrm{nil}}, \mathcal{G}_{\mathrm{sol}}, \mathcal{G}$, to Corollary 27 as they constitute our main lever towards the decidability of continuity for these classes.

4.2 Variety inclusion and inclusion of classes of continuous functions

In this section, we study the consequence of variety (non)inclusion on the inclusion of the related classes of continuous rational functions. This is reminiscent of the notion of *heredity* studied by [12], where a function is $\mathcal{V}$_hereditarily continuous if it is $\mathcal{W}$_continuous for each subvariety $\mathcal{W}$ of $\mathcal{V}$. Variety noninclusion provides the simplest study case here:

▶ **Proposition 17.** *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties. If $\mathcal{V} \nsubseteq \mathcal{W}$ then there are $\mathcal{V}$_continuous rational functions that are not $\mathcal{W}$_continuous.*

The remainder of this section focuses on a dual statement:

If $\mathcal{V} \subsetneq \mathcal{W}$, are all $\mathcal{V}$_continuous rational functions $\mathcal{W}$_continuous?

We first focus on group varieties. Naturally, if 1. $\mathcal{V}$_continuous rational functions are $\mathcal{V}$_realizable and 2. $\mathcal{W}$_realizable rational functions are $\mathcal{W}$_continuous, this holds. Appealing to the forthcoming Corollary 27 for point 1 and Proposition 14 for point 2, we then get:

▶ **Proposition 18.** *For $\mathcal{V}, \mathcal{W} \in \{\mathcal{G}_{\text{nil}}, \mathcal{G}_{\text{sol}}, \mathcal{G}\}$ with $\mathcal{V} \subsetneq \mathcal{W}$, all $\mathcal{V}$_continuous rational functions are $\mathcal{W}$_continuous. This however fails for $\mathcal{V} = \mathcal{A}b$ and for any $\mathcal{W} \in \{\mathcal{G}_{\text{nil}}, \mathcal{G}_{\text{sol}}, \mathcal{G}\}$.*

Proof. It remains to show the case $\mathcal{V} = \mathcal{A}b$. This is in fact the same example as in the proof of Proposition 16, to wit, over $A = \{a, b\}$, the rational function τ that maps $w \in aA^*$ to $(ab)^{|w|}$, and words $w \in bA^*$ to $(ba)^{|w|}$. Indeed, we saw that this function is continuous for $\mathcal{A}b$, but it holds that $\tau(a) = ab$ on the one hand, and $\tau(b^\omega a) = (ba)^\omega ba =_{\mathcal{W}} ba$, but $ab \neq_{\mathcal{W}} ba$. The Preservation Lemma then shows that τ is not continuous for $\mathcal{W}$. ◀

▶ **Proposition 19.** *All $\mathcal{A}b$_continuous rational functions are $\mathcal{C}om$_continuous.*

We now turn to aperiodic varieties. For lesser expressive varieties, the property fails:

▶ **Proposition 20.** *For $\mathcal{V} \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}\}$ and $\mathcal{W} \in \{\mathcal{L}, \mathcal{R}, \mathcal{DA}, \mathcal{A}\}$ with $\mathcal{V} \subsetneq \mathcal{W}$, there are $\mathcal{V}$_continuous rational functions that are not $\mathcal{W}$_continuous.*

▶ **Proposition 21.** *Any $\mathcal{DA}$_continuous rational function is $\mathcal{A}$_continuous.*

Proof. First note that both $\mathcal{DA}$ and $\mathcal{A}$ satisfy the hypotheses of Lemma 12. Consider a $\mathcal{DA}$_continuous rational function $\tau \colon A^* \to B^*$. By the Syncing Lemma, to show that it is $\mathcal{A}$_continuous, it is enough to show that 1. $\tau^{-1}(B^*) \in \mathcal{A}(A^*)$, and 2. That some input synchronizations of τ, based on equations of the form $x^\omega =_{\mathcal{A}} x^{\omega+1}$, belong to an equalizer set of the form (by Lemma 7):

$$\mathsf{Equ}_{\mathcal{A}}(\alpha \cdot y^\omega \cdot \beta,\ \alpha' \cdot z^\omega \cdot \beta') = \{(s, s', t, t') \mid (s \cdot \alpha,\ s' \cdot \alpha',\ \beta \cdot t,\ \beta' \cdot t') \in \mathsf{Equ}_{\mathcal{A}}(y^\omega,\ z^\omega)\}\ .$$

Applying the Syncing Lemma on τ for the variety $\mathcal{DA}$, we get that point 1 is true, since $\tau^{-1}(B^*) \in \mathcal{DA}(A^*)$. Similarly, point 2 is true since $x^\omega = x^{\omega+1}$ is an equation of $\mathcal{DA}$, and Lemma 13 implies that the equalizer set of the equation above is the same in $\mathcal{DA}$ and $\mathcal{A}$. ◀

▶ **Proposition 22.** *There are nonrational functions that are continuous for both $\mathcal{DA}$ and Reg but are not $\mathcal{A}$_continuous.*

5 Deciding continuity for transducers

5.1 Deciding continuity for group varieties

Reutenauer and Schützenberger showed in [15] that a rational function is $\mathcal{G}$_continuous iff it is $\mathcal{G}$_realizable. Since this is proven effectively, it leads to the decidability of $\mathcal{G}$_continuity. In Proposition 14, we saw that the right-to-left statement also holds for $\mathcal{G}_{\text{sol}}$; we now show that the left-to-right statement holds for all group varieties $\mathcal{V}$ that contain $\mathcal{G}_{\text{nil}}$. As in [15], but with sensibly different techniques, we show that $\mathcal{V}$_continuous transducers are plurisubsequential. The Syncing Lemma will then imply that such transducers are $\mathcal{V}$_transducers. Both properties rely on the following normal form:

▶ **Lemma 23.** *Let τ be a transducer. An equivalent transducer τ' can be constructed by adjoining some codeterministic automaton to τ so that for any states p, q of τ':*

$$\Big[(\exists x, y)\big[\emptyset \neq (\tau'_{p,\bullet} \bowtie \tau'_{q,\bullet}) \subseteq (x, y) \cdot Id\big]\Big] \Rightarrow p = q\ .$$

Alternatively, the "dual" property can be ensured, adjoining a deterministic automaton to τ, so that for any states p, q of τ':

$$\Big[(\exists x, y)\big[\emptyset \neq (\tau'_{\bullet,p} \bowtie \tau'_{\bullet,q}) \subseteq Id \cdot (x, y)\big]\Big] \Rightarrow p = q\ .$$

▶ **Lemma 24.** *Let $\mathcal{V}$ be a variety of group languages that contains $\mathcal{G}_{\text{nil}}$. For any $\mathcal{V}$_continuous unambiguous transducer τ, the transducer obtained by applying the dual of Lemma 23, then applying its first part, is a plurisubsequential $\mathcal{V}$_transducer.*

Proof. Write τ' for the result of the dual part of Lemma 23 on τ, and τ'' for the result of the first part of Lemma 23 on τ'. For these transducers, call a triple a states (p, q, q') a *fork* on a if from p, the transducer can go to q and q' reading one a, and there is a path from q to p reading only a's. Dually, a triple (q, q', p) is a *reverse fork* on a if the transducer can go from q and q' to p reading one a, and there is a path from p to q that reads only a. In both cases, the fork is *proper* if $q \neq q'$. We rely on two facts:

▶ **Fact 25.** *There are no proper forks or reverse forks in τ''.*

▶ **Fact 26.** *For any state p of τ'' and any letter a, it holds that $p \in p.a^\omega$.*

Consider a state p in τ'' and a letter a. As $p \in p.a^\omega$ by Fact 26, there is a cycle of a's on p. Call q the first state of that cycle. Next, let q' be such that (p, a, q') is a transition of τ''. Clearly, (p, q, q') forms a fork, hence by Fact 25, $q = q'$. Thus τ'' is plurisubsequential.

It remains to show that τ'' is a $\mathcal{V}$_transducer. To do so, consider an equation $u = \_\mathcal{V} v$, a state q of τ'', and let $p = q.u$ and $p' = q.v$. We show that $p = p'$, concluding the proof. We rely on the Syncing Lemma, since τ'' is $\mathcal{V}$_continuous; it ensures in particular that:

$$(\tau''_{\bullet,q} \bowtie \tau''_{\bullet,q}) \times (\tau''_{p,\bullet} \bowtie \tau''_{p',\bullet}) \subseteq \mathsf{Equ}\_\mathcal{V}(u', v') \quad \text{with } u' = \tau''_{q,p}(u), v' = \tau''_{q,p'}(v) \ . \qquad (1)$$

Let (s, s, t_1, t_2) be in the left-hand side. It holds that $s \cdot u' \cdot t\_1 = \_\mathcal{V} s \cdot v' \cdot t\_2$, thus $u' \cdot t\_1 = \_\mathcal{V} v' \cdot t\_2$ (here and in the following, we derive equivalent equations by appealing to the fact that the *free group* is embedded, in a precise sense, in $\mathcal{V}$ [16, § 6.1.9]). Now consider another tuple (s', s', t_1', t_2') again in the left-hand side of Equation (1). It also holds that $u' \cdot t\_1' = \_\mathcal{V} v' \cdot t\_2'$, hence we obtain that $t\_1 \cdot t\_2^{-1} = \_\mathcal{V} t\_1' \cdot t\_2'^{-1}$. This is in turn equal in $\mathcal{V}$ to some $\alpha \cdot \beta^{-1}$ such that α and β are words that do not share the same last letter. This shows that $t\_1 = \alpha \cdot t$ and $t\_2 = \beta \cdot t$ for some word t, and similarly for t_1' and t_2'. More generally: $(\tau''_{p,\bullet} \bowtie \tau''_{p',\bullet}) \subseteq (\alpha, \beta) \cdot Id$, and the normal form of Lemma 23 thus shows that $p = p'$. ◀

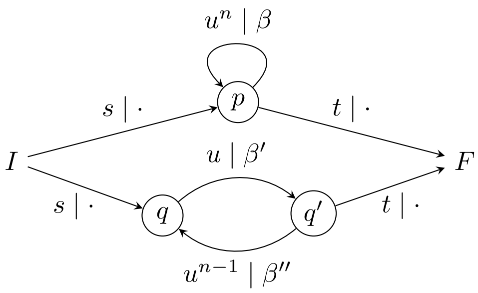
▶ **Corollary 27.** *For $\mathcal{V} \in \{\mathcal{G}_{\text{nil}}, \mathcal{G}_{\text{sol}}, \mathcal{G}\}$, any $\mathcal{V}$_continuous rational function is $\mathcal{V}$_realizable.*

▶ **Theorem 28.** *Let $\mathcal{V}$ be a variety of group languages that includes $\mathcal{G}_{\text{nil}}$ and that is closed under inverse $\mathcal{V}$_realizable rational functions. It is decidable, given an unambiguous transducer, whether it realizes a $\mathcal{V}$_continuous function. This holds in particular for $\mathcal{G}_{\text{sol}}$ and $\mathcal{G}$.*

5.2 Deciding continuity for aperiodic varieties

We saw in Section 4.1 that the approach of the previous section cannot work: there is no correspondence between continuity and realizability for aperiodic varieties. Herein, we use the Syncing Lemma to decide continuity in two main steps. First, note that all of our aperiodic varieties are defined by an infinite number of equations for each alphabet. The Syncing Lemma would thus have us check an infinite number of conditions; our first step is to reduce this to a finite number, which we stress through the forthcoming notion of "pertaining triplet" of states. Second, we have to show that the inclusion of the second point of the Syncing Lemma can effectively be checked. This will be done by simplifying this condition, and showing a decidability property on rational relations.

▶ **Definition 29.** A triplet of states (p, q, q') is *pertaining* if there are words s, u, t and an integer n such that:



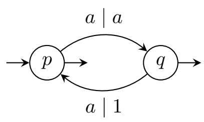
where $\cdot$ means "any word." Further, a pertaining triplet is *empty* if, in the above picture, $\beta = \beta'\beta'' = 1$ and *full* if both words are nonempty; it is *degenerate* if only one of β or $\beta'\beta''$ is empty.

It is called "pertaining" as the second point of the Syncing Lemma elaborates on properties of such a triplet, in particular, since $u^\omega = u^{\omega+1}$ is an equation of $\mathcal{A}$. The following characterization of $\mathcal{A}$_continuity is then made *without appeal* to equations or profinite words:

▶ **Lemma 30.** *A transducer $\tau\colon A^* \to B^*$ is $\mathcal{A}$_continuous iff all of the following hold:*

1. $\tau^{-1}(B^*) \in \mathcal{A}(A^*)$*;*
2. *For all full pertaining triplets (p, q, q'), there exist $x, y \in B^*$ and $\rho\_1, \rho\_2 \in (B^*)^2$ such that $\tau_{\bullet,p} \bowtie \tau_{\bullet,q} \subseteq Id \cdot \big((x^*, x^*)\rho\_1^{-1}\big)$ and $\tau_{p,\bullet} \bowtie \tau_{q',\bullet} \subseteq \big(\rho\_2^{-1}(y^*, y^*)\big) \cdot Id$;*
3. *For all empty pertaining triplets (p, q, q') it holds that $(\tau_{\bullet,p} \bowtie \tau_{\bullet,q}) \cdot (\tau_{p,\bullet} \bowtie \tau_{q',\bullet}) \subseteq Id$;*
4. *No pertaining triplet is degenerate.*

▶ **Example 31.** We show that the transducer of Proposition 16 is $\mathcal{A}$_continuous. Let τ be:



First, the function is total, hence the first point of Lemma 30 is verified. Second, there are no empty nor degenerate pertaining triplets, hence the third and fourth points are verified. Now the full pertaining triplets are $(p, p, p), (p, p, q), (q, q, q)$, and (q, q, p). We check that the pertaining triplet (p, p, q) verifies the second condition of Lemma 30, the other cases being similar or clear. The first half of the condition is immediate. Now $\tau_{p,\bullet} \bowtie \tau_{q,\bullet} = \{(a^{\lfloor n+1/2 \rfloor}, a^{\lfloor n/2 \rfloor}) \mid n \geq 0\}$ which verifies the condition.

We now show that the property of Lemma 30 is indeed decidable:

▶ **Proposition 32.** *It is decidable, given a rational relation $R \subseteq A^* \times A^*$, whether there is a word $x \in A^*$ and a pair $\rho \in (A^*)^2$, such that $R \subseteq Id \cdot \big((x^*, x^*)\rho^{-1}\big)$.*

▶ Remark. In general, the problem of deciding, given a rational relation R and a *recognizable* relation K, whether $R \subseteq Id \cdot K$, is undecidable. Indeed, testing $R \cap Id = \emptyset$ is undecidable [3], and equivalent to testing:

$$R \subseteq Id \cdot \big((A^+ \times \{1\}) \cup (\{1\} \times A^+) \cup \bigcup\_a \neq b \in A (a \cdot A^* \times b \cdot A^*)\big) \ ,$$

the right-hand side being of the form $Id \cdot K$.

▶ **Theorem 33.** *It is decidable, given an unambiguous transducer, whether it realizes an* $\mathcal{A}$*_continuous function.*

The same approach, with carefully tweaked conditions, yields:

▶ **Theorem 34.** *For* $\mathcal{V} = \mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{DA}$*, it is decidable, given an unambiguous transducer, whether it realizes a* $\mathcal{V}$*_continuous function.*

5.3 Deciding $\mathcal{C}om$- and $\mathcal{A}b$-continuity

The case of $\mathcal{C}om$ and $\mathcal{A}b$ is comparatively much simpler, in particular because these varieties are defined using a finite number of equations for each alphabet. However, the argument relies on different ideas:

▶ **Theorem 35.** *For* $\mathcal{V} = \mathcal{C}om, \mathcal{A}b$*, it is decidable, given an unambiguous transducer, whether it realizes a* $\mathcal{V}$*_continuous function.*

Proof. We apply the Syncing Lemma. Its first point is clearly decidable. We reduce its second point to decidable properties about semilinear sets (see, e.g., [7]). We also rely on the notion of Parikh image, that is, the mapping $\mathsf{Pkh}\colon A^* \to \mathbb{N}^A$ such that $\mathsf{Pkh}(w)$ maps $a \in A$ to the number of a's in the word w.

Since every $\mathcal{A}b$_continuous function is $\mathcal{C}om$_continuous (Proposition 19), the conditions to test for $\mathcal{A}b$_continuity are included in those for $\mathcal{C}om$_continuity—this can also be seen as a consequence of the fact that if u, v are words, $\mathsf{Equ}\_\mathcal{A}b(u, v) = \mathsf{Equ}\_\mathcal{C}om(u, v)$.

Let $\tau\colon A^* \to B^*$ be a given transducer. Consider an equation $ab = ba$ and four states p, p', q, q' of τ. Write $u = \tau_{p,p'}(ab)$ and $v = \tau_{q,q'}(ba)$. We ought to check, by the Syncing Lemma, the inclusion in $\mathsf{Equ}\_\mathcal{C}om(u, v) = \{(s, s', t, t') \mid s \cdot u \cdot t =\_\mathcal{C}om s' \cdot v \cdot t'\}$ of some input synchronization. Now this set is the set of (s, s', t, t') such that $\mathsf{Pkh}(s \cdot u \cdot t) = \mathsf{Pkh}(s' \cdot v \cdot t')$, and is thus defined by a simple semilinear property. The input synchronizations themselves, e.g., $\tau_{\bullet,p} \bowtie \tau_{\bullet,q}$, are rational relations, and their component-wise Parikh image is thus a semilinear set. Since the inclusion of semilinear sets is decidable, the inclusion of the second point of the Syncing Lemma is also decidable.

For $\mathcal{A}b$, we should additionally check the equations $a^\omega = 1$. The reasoning is similar. Consider three states (p, p', q), and write $x \cdot u^{\omega-1} \cdot y$ for $\tau_{p,p'}(a^\omega)$. By commutativity and the fact that $u^{\omega-1}$ acts as an inverse of u in the equations holding in $\mathcal{A}b$, we have that $(s, s', t, t') \in \mathsf{Equ}\_\mathcal{A}b(x \cdot u^{\omega-1} \cdot y, 1)$ iff $s \cdot t =\_\mathcal{A}b s' \cdot u \cdot t'$. This again reduces the inclusion of the second point of the Syncing Lemma to a decidable semilinear property. ◀

6 Discussion

We presented a study of continuity in functional transducers, on the one hand focused on general statements (Section 3), on the other hand on continuity for classical varieties. The heart of this contribution resides in decidability properties (Section 5), although we also addressed natural and related questions in a systematic way (Section 4). We single out two main research directions.

First, there is a sharp contrast between the genericity of the Preservation and Syncing Lemma and the technicality of the actual proofs of decidability of continuity. To which extent can these be unified and generalized? We know of two immediate extensions: 1. the generic results of Section 3 readily apply to Boolean algebras of languages closed under quotient, a relaxation of the conditions imposed on varieties, and 2. the varieties $\mathcal{G}\_p$ of languages recognized by p-groups can also be shown to verify Proposition 14 and Lemma 24, hence

$\mathcal{G}_p$_continuity is decidable for transducers. Beyond these two points, we do not know how to show decidability for $\mathcal{G}_{\text{nil}}$ (which is the *join* of the $\mathcal{G}_p$), and the surprising complexity of the equalizer sets for some Burnside varieties (e.g., the one defined by $x^2 = x^3$, see the Remark on page 7) leads us to conjecture that continuity may be undecidable in that case, hence that no unified way to show the decidability of continuity exists.

Second, the notion of continuity may be extended to more general settings. For instance, departing from regular languages, it can be noted that every recursive function is continuous for the class of recursive languages. Another natural generalization consists in studying $(\mathcal{V}, \mathcal{W})$_continuity, that is, the property for a function to map $\mathcal{W}$_languages to $\mathcal{V}$_languages by inverse image. This would provide more flexibility for a sufficient condition for cascades of languages (or stackings of circuits, or nestings of formulas) to be in a given variety.

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References

1 Jorge Almeida. Some algorithmic problems for pseudovarieties. *Publ. Math. Debrecen*, 54(1):531–552, 1999.

2 Jorge Almeida and Alfredo Costa. Equidivisible pseudovarieties of semigroups. *Publicationes Mathematicae Debrecen*, 90(3-4), 2017.

3 Jean Berstel. *Transductions and Context-Free Languages*, volume 38 of *Leitfäden der angewandten Mathematik und Mechanik LAMM*. Teubner, 1979.

4 Michaël Cadilhac, Andreas Krebs, Michael Ludwig, and Charles Paperman. A circuit complexity approach to transductions. In *MFCS*, pages 141–153, 2015. `doi:10.1007/978-3-662-48057-1_11`.

5 Emmanuel Filiot, Olivier Gauwin, and Nathan Lhote. First-order definability of rational transductions: An algebraic approach. In *LICS*, pages 387–396. ACM, 2016. `doi:10.1145/2933575.2934520`.

6 Mai Gehrke, Serge Grigorieff, and Jean-Eric Pin. Duality and equational theory of regular languages. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfsdóttir, and Igor Walukiewicz, editors, *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part II – Track B: Logic, Semantics, and Theory of Programming & Track C: Security and Cryptography Foundations*, volume 5126 of *Lecture Notes in Computer Science*, pages 246–257. Springer, 2008. `doi:10.1007/978-3-540-70583-3_21`.

7 Seymour Ginsburg. *The Mathematical Theory of Context-Free Languages*. McGraw-Hill, Inc., New York, NY, USA, 1966.

8 Jean-Éric Pin. Mathematical foundations of automata theory, 2016.

9 Jean-Éric Pin and Jacques Sakarovitch. Some operations and transductions that preserve rationality. In *Theoretical Computer Science*, pages 277–288. Springer, 1982. `doi:10.1007/BFb0036488`.

10 Jean-Éric Pin and Jacques Sakarovitch. Une application de la representation matricielle des transductions. *Theoretical Computer Science*, 35:271–293, 1985. `doi:10.1016/0304-3975(85)90019-2`.

11 Jean-Éric Pin and Pedro V. Silva. A topological approach to transductions. *Theoretical Computer Science*, 340(2):443–456, 2005. `doi:10.1016/j.tcs.2005.03.029`.

12 Jean-Éric Pin and Pedro V. Silva. On profinite uniform structures defined by varieties of finite monoids. *Int. J. of Algebra and Computation*, 21(01n02):295–314, 2011. doi:10.1142/S0218196711006170.

13 Jean-Éric Pin and Pedro V. Silva. On uniformly continuous functions for some profinite topologies. *Theoretical Computer Science*, 658, Part A:246–262, 2017. Formal Languages and Automata: Models, Methods and Application In honour of the 70th birthday of Antonio Restivo. doi:10.1016/j.tcs.2016.06.013.

14 Jean-Éric Pin and Pascal Weil. Profinite semigroups, Mal'cev products and identities. *Journal of Algebra*, 182(3):604–626, 1996.

15 Christophe Reutenaeur and Marcel-Paul Schützenberger. Variétés et fonctions rationnelles. *Theoretical Computer Science*, 145(1–2):229–240, July 1995. doi:10.1016/0304-3975(94)00180-Q.

16 Derek J. S. Robinson. *A Course in the Theory of Groups*. Springer, 2 edition, 1995.

17 Klaus Schneider. *Verification of Reactive Systems: Formal Methods and Algorithms*. SpringerVerlag, 2004.

18 Howard Straubing. *Finite Automata, Formal Logic, and Circuit Complexity*. Birkhäuser, Boston, 1994. doi:10.1007/978-1-4612-0289-9.

19 Pascal Tesson and Denis Thérien. Logic meets algebra: the case of regular languages. *Logical Methods in Computer Science*, 3(1), 2007.