Opinion Dynamics in Networks: Convergence, Stability and Lack of Explosion*†

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Abstract

Inspired by the work of Kempe et al. [Kempe, Kleinberg, Oren, Slivkins, EC 2013], we introduce and analyze a model on opinion formation; the update rule of our dynamics is a simplified version of that of [Kempe, Kleinberg, Oren, Slivkins, EC 2013]. We assume that the population is partitioned into types whose interaction pattern is specified by a graph. Interaction leads to population mass moving from types of smaller mass to those of bigger mass. We show that starting uniformly at random over all population vectors on the simplex, our dynamics converges point-wise with probability one to an independent set. This settles an open problem of [Kempe, Kleinberg, Oren, Slivkins, EC 2013], as applicable to our dynamics. We believe that our techniques can be used to settle the open problem for the Kempe et al. dynamics as well.

Next, we extend the model of Kempe et al. by introducing the notion of birth and death of types, with the interaction graph evolving appropriately. Birth of types is determined by a Bernoulli process and types die when their population mass is less than ϵ (a parameter). We show that if the births are infrequent, then there are long periods of "stability" in which there is no population mass that moves. Finally we show that even if births are frequent and "stability" is not attained, the total number of types does not explode: it remains logarithmic in $1/\epsilon$.

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1 Introduction

The birth, growth and death of political parties, organizations, social communities and product adoption groups (e.g., whether to use Windows, Mac OS or Linux) often follows common patterns, leading to the belief that the dynamics underlying these processes has much in common. Understanding this commonality is important for the purposes of predictability and hence has been the subject of study in mathematical social science for many years [4, 7, 8, 14, 26]. In recent years, the growth of social communities on the Internet, and their increasing economic and social value, has provided fresh impetus to this study [1, 2, 5, 17].

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In this paper, we continue along these lines by building on a natural model proposed by Kempe et al. [15]. Their model consists of an influence graph G on n vertices (types, parties) into which the entire population mass is partitioned. Their main tenet is that individuals in smaller parties tend to get influenced by those in bigger parties [1]. Individuals in the two vertices connected by an edge can interact with each other. These interactions result in individuals moving from smaller to bigger in population vertices. Kempe et al. characterize stable equilibria of this dynamics via the notion of Lyapunov stability, and they show that under any stable equilibrium, the entire mass lies in an independent set, i.e., the population breaks into non-interacting islands. The message of this result is clear: a population is (Lyapunov) stable, in the sense that the system does not change by much under small perturbations, only if people of different opinions do not interact. They also showed convergence to a fixed point, not necessarily an independent set, starting from any initial population vector and influence graph. One of their main open problems was to determine whether starting uniformly at random over all population vectors on the unit simplex, their dynamics converge with probability one to an independent set.

We first settle this open problem in the affirmative for a modification of the dynamics, which however is similar to that of Kempe et al. in spirit in that it moves mass from smaller to bigger parties (the dynamics is defined below along with a justification). We believe that the ideas behind our analysis can be used to settle the open problem for the dynamics of Kempe et al. as well, via a more complicated spectral analysis of the Jacobian of the update rule of the dynamics (see Section 3.2).

Whereas the model of Kempe et al. captures and studies the effects of migration of individuals across types in a very satisfactory manner, it is quite limited in that it does not include the birth and death of types. In this paper, we model birth and death of types. In order to arrive at realistic definitions of these notions, we first conducted case studies of political parties in several countries. We present below a case study on Greek politics, but similar phenomena arise in India, Spain, Italy and Holland (see Wikipedia pages).

The Siriza party in Greece provides an excellent example of birth of a party (this information is readily available in Wikipedia pages). This party was essentially in a dormant state until the first 2012 elections in which it got 16.8% of the vote, mostly taken away from the Pasok party, which dropped from 43.9% to 13.2% in the process (Wikipedia). In the second election in 2012, Siriza increased its vote to 26.9% and Pasok dropped to 12.3%. Finally, in 2015, Siriza increased to 36.3% and Pasok dropped further to 4.7%. Another party, Potami, was formed in 2015 and got 6.1% of the vote, again mainly from Pasok. However, in a major 2016 poll, it seems to have collapsed and is likely be absorbed by other parties. In contrast, the KKE party in Greece, which had almost no interactions with the rest of the parties (and was like a disconnected component), has remained between 4.5-8.5% of the vote over the last 26 years.

Motivated by these examples, we have modeled birth and death of types in the following manner. We model population as a continuum, as is standard in population dynamics, and time is discrete. This is the same as arXiv Version 1 of [15], which is what we will refer to

[1] Changes in the sizes of political parties and other organizations can occur for a multitude of possible reasons, such as changes in economic conditions, immigration flows, wars and terrorism, and drastic changes in technology (such as the introduction of the Internet, smart phones and social media). Studying changes due to these multitude of reasons in a systematic quantitative manner is unrealistic. For this reason, many authors in computer science and the social sciences have limited their work to studying the effects of relative sizes of the groups, in itself a key factor, e.g. see [15] and references therein. Following these works, our paper also takes a similar approach.

throughout this paper; the later versions study the continuous time analog. The birth of a new type in our model is determined by a Bernoulli process, with parameter p. The newly born type absorbs mass from all other types via a randomized process given by an arbitrary distribution with finite support (see Section 2.2). After birth, the new type is connected to an arbitrary, though non-empty, set of other types. Our model has a parameter ϵ, and when the size of a type drops below ϵ, it simply dies, moving its mass equally among its neighbors.

Our rule for migration of mass, which is somewhat different from that of Kempe et al. is motivated by the following considerations. For a type u, $\boldsymbol{x}_u$ will denote the fraction of population that is of type u. Assume that types u and v have an edge, i.e., their populations interact. If so, we will assume that some individuals of the smaller type get influenced by the larger one and move to the larger one. The question is what is a reasonable assumption on the population mass that moves.

For arriving at the rule proposed in this paper, consider three situations. If $\boldsymbol{x}_u = .02$ and $\boldsymbol{x}_v = .25$, i.e., the smaller type is very small, then clearly not many people will move. If $\boldsymbol{x}_u = .22$ and $\boldsymbol{x}_v = .25$, i.e., the types are approximately of the same population size, then again we expect not many people to move. Finally, if $\boldsymbol{x}_u = .15$ and $\boldsymbol{x}_v = .25$, i.e., both types are reasonably big and their difference is also reasonably big, then we expect several people to move from the smaller to the bigger type. From these considerations, we propose that the amount of population mass moving from v to u, assuming $\boldsymbol{x}_v < \boldsymbol{x}_u$, is given by the rule

$$f^{(t)}_{v\to u} = \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v \cdot F_{uv}(\boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_v),$$

where $F_{uv}(z) = F_{vu}(z)$ is a function that captures the level of influence between u, v. We assume that $F_{uv} : [-1, 1] \to [-1, 1]$ is continuously differentiable, $F_{uv}(0) = 0$ (there is no population flow between two neighboring types if they have the same fraction of population), is increasing and finally it is odd, i.e., $F_{uv}(-z) = -F_{uv}(z)$ (so that $f^{(t)}_{v\to u} = -f^{(t)}_{u\to v}$).

In this simplified setting we have made the assumption that the system is closed, i.e., that it does not get influence from outside factors (e.g., economical crisis, immigrations flows, terrorism etc).

1.1 Our results and techniques

We first study our migration dynamics without birth and death and settle the open problem of Kempe et al., as it applies to our dynamics.

We show that the dynamics converges set-wise to a fixed point, i.e., there is a set S containing only fixed points such that the distance between the trajectory of the dynamics and S goes to zero for all starting population vectors. To show this convergence result, we use a simple potential function of the population mass namely, the ℓ_2^2 norm of the population vector, and we show that this potential is strictly increasing at each time step (unless the dynamics is at a fixed point). Moreover, the potential is bounded, hence the result follows.

Next, we strengthen this result by showing point-wise convergence as well. The latter result is technically deeper and more difficult, since it means that every trajectory converges to a specific fixed point $\boldsymbol{p}$. We show point-wise convergence by constructing a *local* potential function that is decreasing in a small neighborhood of the limit point $\boldsymbol{p}$. The potential function is always non-zero in that small neighborhood and is zero only at $\boldsymbol{p}$.

Using the latter result and one of the most important theorems in dynamical systems, the Center Stable Manifold Theorem, we prove that with probability one, under an initial population vector picked uniformly at random from the unit simplex, our dynamics converges point-wise to a fixed point $\boldsymbol{p}$, where the *active* types w in $\boldsymbol{p}$, i.e., $w \in V(G)$ so that $\boldsymbol{p}_w > 0$, form an independent set of G. This involves characterization of the linearly stable (see

Section 2.3 for definition) fixed points and proving that the update rule of the dynamics is a local diffeomorphism[2]. This settles the open problem of Kempe et al., mentioned in the Introduction, for our dynamics. This result is important because it allows us to perform a long-term average case analysis of the behavior of our dynamics and make predictions.

Next, we introduce birth and death in our model. Clearly there will be no convergence in this case since new parties are created all the time. Instead we define and study a notion of “stability" which is different from the classical notions that appear in dynamical systems (see Section 2.3 for the definition of the classical notion and Definition 12 for our notion). A dynamics is (T, d)-stable if and only if $\forall t: \ T \leq t \leq T + d$, no population mass moves at step t. We show that despite birth and death, there are arbitrarily long periods of “stability" with high probability, for a sufficiently small p. Finally, we show that in the long run, with high probability, for a sufficiently large p, the number of types in the population will be $O(\log(1/\epsilon))$. This may seem counter-intuitive, since with a large p new types will be created often; however, since new types absorb mass from old types, the old types die frequently. In contrast, in the short term (from the definition of ϵ) we can have up to $\Theta(1/\epsilon)$ types.

Let us give an interpretation of the results of the previous paragraph in terms of political parties of certain countries (information obtained from Wikipedia). Countries do have periods of political stability, e.g., during 1981-85, 2004-07, no new major (with more than 1% of the vote) parties were formed in Greece, moreover there was no substantial change in the percentage of votes won by parties in successive elections. The parameter ϵ can be interpreted as the fraction of people that can form a party that participates in elections. The minimum size of a party arises for organizational and legal reasons, and is $\Theta(1/Q)$, where Q is the population of the country and therefore ϵ is inversely related to population. The message of the latter theorem is that the number of political parties grows at most as the logarithm of the population of the country, i.e. $O(\log Q)$ [3]. The following data supports this fact. The population of Greece, Spain and India in 2015 was $1.1e7, 4.6e7$ and $1.2e9$, respectively, and the number of parties that participated in the general elections was 20, 32 and 50, respectively.

1.2 Related work

As stated above, we build on the work of Kempe et al. [15]. They model their dynamics in a similar way, i.e., there is a flow of population for every interacting pair of types u, v. The flow goes from smaller to bigger types; in our case the mass is just the population of a type. One very interesting common trait between the two dynamics is that the fixed points have similar description: all types with positive mass belonging to the same connected component C have the same mass. Stable fixed points also have the same properties in both dynamics, namely they are independent sets. The update rules of the two dynamics are somewhat different; our simpler dynamics helps us in proving stronger results.

One of the most studied models is the following: there is a graph G in which each vertex denotes an individual having two possible opinions. At each time step, an individual is chosen at random who next chooses his opinion according to the majority (best response) opinion among his neighbors. This has been introduced by Galam[10] and appeared in [22, 9], where they address the question: in which classes of graphs do individuals reach consensus. The same dynamics, but with each agent choosing his opinion according to noisy

[2] Continuously differentiable, the inverse exists and is also continuously differentiable (in some small neighborhood of each point).

[3] This is just an upper bound, countries like UK, US satisfy this rule too.

best response (the dynamics is a Markov chain) has been studied in [21, 16] and many other papers referenced therein. They give bounds for the hitting time and expected time of the consensus state (risk dominant) respectively.

Another well-known model for the dynamics of opinion formation in multi-agent systems is Hegselmann-Krause [11]. Individuals are discrete entities and are modeled as points in some opinion space (e.g., real line). At every time step, each individual moves to the mass center of all the individuals within unit distance. Typical questions are related to the rate of convergence (see [6] and references therein). Finally, another classic model is the voter model, where there is a fixed graph G among the individuals, and at every time step, a random individual selects a random neighbor and adopts his opinion [12]. For more information on opinion formation dynamics of an individual using information learned from his neighbors, see [13] for a survey (also see [29] for more information on opinion formation models).

Other works, including dynamical systems that show convergence to fixed points, are [23, 19, 18, 24, 20, 27]. [27] focuses on quadratic dynamics and they show convergence in the limit. On the other hand [3] shows that sampling from the distribution this dynamics induces at a given time step is PSPACE-complete. In [23, 19], it is shown that replicator dynamics in linear congestion and 2-player coordination games converges to pure Nash equilibria, and in [18, 24] it is shown that gradient descent converges to local minima, avoiding saddle points even in the case where the fixed points are uncountably many.

Organization: In Section 2 we describe our dynamics formally and give the necessary definitions about dynamical systems. In Section 3 we show that our dynamics without births/deaths converges with probability one to fixed points $\boldsymbol{p}$ so that the set of types with positive population, i.e., active types, form an independent set of G. Finally, in Section 4 we first show that there is no explosion in the number of types (i.e., the order never becomes $\Theta(1/\epsilon)$) and also we perform stability analysis using our notion.

2 Preliminaries

Notation: We denote the probability simplex on a set of size n as Δ_n. Vectors in $\mathbb{R}^n$ are denoted in boldface and $\boldsymbol{z}_j$ denotes the jth coordinate of a given vector $\boldsymbol{z}$. Time indices are denoted by superscripts. Thus, a time indexed vector $\boldsymbol{z}$ at time t is denoted as $\boldsymbol{z}^{(t)}$. We use the letters $J, \mathbb{J}$ to denote the Jacobian of a function and finally we use f^t to denote the composition of f by itself t times.

2.1 Migration dynamics

Let $G = (V, E)$ be an undirected graph on n vertices (which we also call types), and let N_v denote the set of neighbors of v in G. During the whole dynamical process, each vertex v has a non-negative population mass representing the fraction of the population of type v. We consider a discrete-time process and let $\boldsymbol{x}_v^{(t)}$ denote the mass of v at time step t. It follows that the condition

$$\sum_{v \in V(G)} \boldsymbol{x}_v^{(t)} = 1,$$

must be maintained for all t, i.e., $\boldsymbol{x}^{(t)} \in {\Delta_n}^4$ for all $t \in \mathbb{N}$.

[4] Recall that Δ_n denotes the unit simplex of dimension $n + 1$, where $|V(G)| = n$.

Additionally, we consider a dynamical migration rule where the population can move along edges of G at each step. The movement at step t is determined by $\boldsymbol{x}^{(t)}$. Specifically, for $uv \in E(G)$, the amount of mass moving from v to u at step t is given by

$$f^{(t)}_{v\to u} = \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v F_{uv}(\boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_v).$$

For all $uv \in E(G)$ we assume that $F_{uv} : [-1, 1] \to [-1, 1]$ is a continuously differentiable function such that:

1. $F_{uv}(0) = 0$ (there is no population flow between two neighboring types if they have the same fraction of population),
2. F_{uv} is increasing (the larger $\boldsymbol{x}_u - \boldsymbol{x}_v$, the more population moving from v to u),
3. F_{uv} is odd i.e., $F_{uv}(-z) = -F_{uv}(z)$ (so that $f^{(t)}_{v\to u} = -f^{(t)}_{u\to v}$).

It can be easily derived from the assumptions that $F_{uv}(z) \geq F_{uv}(0) = 0$ for $z \geq 0$ and $F'_{uv}(-z) = F'_{uv}(z)$ for $z \in [-1, 1]$, where F'_{uv} denotes the derivative of F_{uv}. Note that $f^{(t)}_{v\to u} > 0$ implies that population is moving from v to u, and $f^{(t)}_{v\to u} < 0$ implies that population is moving in the other direction. The update rule for the population of type u can be written as

$$\begin{aligned} \boldsymbol{x}^{(t+1)}_u &= \boldsymbol{x}^{(t)}_u + \sum_{v\in N_u} f^{(t)}_{v\to u} && (1)\\ &= \boldsymbol{x}^{(t)}_u + \sum_{v\in N_u} \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v F_{uv}(\boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_v). && (2) \end{aligned}$$

We denote the update rule of the dynamics as $g : \Delta_n \to \Delta_n$, i.e., we have that

$$\boldsymbol{x}^{(t+1)} = g(\boldsymbol{x}^{(t)}).$$

Therefore it holds that $\boldsymbol{x}^{(t)} = g^t(\boldsymbol{x}^{(0)})$, where g^t denotes the composition of g by itself t times. It is easy to see g is well-defined for $\sup_{z\in[-1,1]} |F_{uv}(z)| \leq 1$ for all $uv \in E(G)$, in the sense that if $\boldsymbol{x}^{(t)} \in \Delta_n$ then $\boldsymbol{x}^{(t+1)} \in \Delta_n$. This is true since for all u we get (using induction, i.e., $\boldsymbol{x}^{(t)} \in \Delta_n$)

$$\begin{aligned} \boldsymbol{x}^{(t+1)}_u &= \boldsymbol{x}^{(t)}_u + \sum_{v\in N_u} \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v F_{uv}(\boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_v)\\ &\geq \boldsymbol{x}^{(t)}_u - \sum_{v\in N_u} \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v \geq \boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_u(1 - \boldsymbol{x}^{(t)}_u) \geq 0, \end{aligned}$$

moreover it holds

$$\begin{aligned} \boldsymbol{x}^{(t+1)}_u &= \boldsymbol{x}^{(t)}_u + \sum_{v\in N_u} \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v F_{uv}(\boldsymbol{x}^{(t)}_u - \boldsymbol{x}^{(t)}_v)\\ &\leq \boldsymbol{x}^{(t)}_u + \sum_{v\in N_u} \boldsymbol{x}^{(t)}_u \boldsymbol{x}^{(t)}_v \leq \boldsymbol{x}^{(t)}_u + \boldsymbol{x}^{(t)}_u(1 - \boldsymbol{x}^{(t)}_u) \leq \boldsymbol{x}^{(t)}_u + 1 - \boldsymbol{x}^{(t)}_u = 1, \end{aligned}$$

and also $\sum_u \boldsymbol{x}^{(t+1)}_u = \sum_u \boldsymbol{x}^{(t)}_u = 1$ (the other terms cancel out).

2.2 Birth and death of types

Political parties or social communities don't tend to survive once their size becomes "small" and hence there is a need to incorporate death of parties in our model. We will define a global parameter ϵ in our model. When the population mass of a type v becomes smaller than some fixed value ϵ, we consider it to be dead and move its mass arbitrarily to existing types. Formally, if $\boldsymbol{x}_v^{(t)} \leq \epsilon$ then $\boldsymbol{x}_v^{(t)} \leftarrow 0$ and $\boldsymbol{x}_u^{(t)} \leftarrow \boldsymbol{x}_u^{(t)} + \boldsymbol{x}_v^{(t)} / |N_v|$ for all $u \in N_v$. Also, vertex v is removed and edges are added arbitrarily on its neighbors to ensure connectivity of the resulting graph.

▶ Remark. It is not hard to see that the maximum number of types is $1/\epsilon$ (by definition). We say that we have explosion in the number of types if they are of $\Theta(1/\epsilon)$. In Theorem 16 we show that in the long run, the number of types is much smaller – it is $O(\log(1/\epsilon))$ with high probability.

Every so often, new political opinions emerge and like-minded people move from the existing parties to create a new party, which then follows the normal dynamics to either survive or die out. To model birth of new types, at each time step, with probability p, we create a new type v such that v takes a portion of mass from each existing type independently. The amount of mass going to v from each u follows an arbitrary distribution in the range $[\beta_{\min}, \beta_{\max}]$. Specifically, let $\boldsymbol{Z}_u \sim \mathcal{D}$ where $\mathcal{D}$ is a distribution with support $[\beta_{\min}, \beta_{\max}]$, the amount of mass going from u to v is $\boldsymbol{Z}_u \boldsymbol{x}_u$. We connect v to the existing graph arbitrarily such that it remains connected.

Additionally, we make a small change to the migration dynamics defined in Section 2.1 to make it more realistic. Our tenet is that population mass migrates from smaller to bigger types because of influence. However, if the two types are of approximately the same size, the difference is size is not discernible and hence migration should not happen. To incorporate this, we introduce a new parameter $\delta > 0$ and if $|\boldsymbol{x}_u - \boldsymbol{x}_v| \leq \delta$, we assume that no population moves from u to v.

Finally, each step of the dynamics consists of there phases in the following order:

1. Migration: the dynamics follows the update rule from Section 2.1.
2. Birth: with probability p, a new type v is created and takes mass from the existing types.
3. Death: a type with mass smaller than ϵ dies out and move its mass to the existing types.

▶ Remark. For any different order of phases, all proofs in the paper still go through with minimal changes.

2.3 Definitions and basics

A recurrence relation of the form $\boldsymbol{z}^{(t+1)} = f(\boldsymbol{z}^{(t)})$ is a discrete time dynamical system, with update rule $f : \mathcal{S} \to \mathcal{S}$ (for our purposes, the set $\mathcal{S}$ is Δ_n). The point $\boldsymbol{p}$ is called a *fixed point* or *equilibrium* of f if $f(\boldsymbol{p}) = \boldsymbol{p}$. A fixed point $\boldsymbol{p}$ is called *Lyapunov stable* (or just stable) if for every $\varepsilon > 0$, there exists a $\zeta = \zeta(\varepsilon) > 0$ such that for all $\boldsymbol{z}$ with $\|\boldsymbol{z} - \boldsymbol{p}\| < \zeta$ we have that $\left\|f^k(\boldsymbol{z}) - \boldsymbol{p}\right\| < \varepsilon$ for every $k \geq 0$. We call a fixed point $\boldsymbol{p}$ *linearly stable* if, for the Jacobian $J(\boldsymbol{p})$ of f, it holds that its spectral radius is at most one. It is true that if a fixed point $\boldsymbol{p}$ is stable then it is linearly stable but the converse does not hold in general [25]. A sequence $(f^t(\boldsymbol{z}^{(0)}))_{t \in \mathbb{N}}$ is called a *trajectory* of the dynamics with $\boldsymbol{z}^{(0)}$ as starting point. A common technique to show that a dynamical system converges to a fixed point is to construct a function $P : \Delta_m \to \mathbb{R}$ such that $P(f(\boldsymbol{z})) > P(\boldsymbol{z})$ unless $\boldsymbol{z}$ is a fixed point. We call P a *potential* or *Lyapunov* function.

3 Convergence to independent sets almost surely

In this section we prove that the deterministic dynamics (assuming no death/birth of types, namely the graph G remains fixed) converges point-wise to fixed points $\boldsymbol{p}$ where $\{v : \boldsymbol{p}_v > 0\}$ (set of active types) is an independent set of the graph G, with probability one assuming that the starting point $\boldsymbol{x}^{(0)}$ follows an atomless distribution with support in Δ_n. To do that, we show that for all starting points $\boldsymbol{x}^{(0)}$, the dynamics converges point-wise to fixed points. Moreover we prove that the update rule of the dynamics is a diffeomorphism and that the linearly stable fixed points $\boldsymbol{p}$ of the dynamics satisfy the fact that the set of active types in $\boldsymbol{p}$ is an independent set of G. Finally, our main claim of the section follows by using Center-Stable Manifold theorem.

Structure of fixed points. The fixed points of the dynamics (1) are vectors $\boldsymbol{p}$ such that for each $uv \in E(G)$, at least one of the following conditions must hold:

1. $\boldsymbol{p}_v = \boldsymbol{p}_u$, 2. $\boldsymbol{p}_v = 0$, 3. $\boldsymbol{p}_u = 0$.

Therefore, for each fixed point $\boldsymbol{p}$, the set of active types (types with non-zero population mass) with respect to $\boldsymbol{p}$ must form a set of connected components such that all types in each component have the same population mass. We first prove that the dynamics converges point-wise to fixed points.

3.1 Point-wise convergence

Initially, we consider the function $\Phi(\boldsymbol{x}) = \sum_v \boldsymbol{x}_v^2$ and state the following lemma on Φ.

▶ **Lemma 1 (Lyapunov (potential) function).** *Let $\boldsymbol{x}$ be a point with $\boldsymbol{x}_u > \boldsymbol{x}_v$. Let $\boldsymbol{y}$ be another point such that $\boldsymbol{y}_v = \boldsymbol{x}_v - d$, $\boldsymbol{y}_u = \boldsymbol{x}_u + d$ for some $0 < d \leq \boldsymbol{x}_v$ and $\boldsymbol{y}_z = \boldsymbol{x}_z$ for all $z \neq u, v$. Then $\Phi(\boldsymbol{x}) < \Phi(\boldsymbol{y})$.*

If we think of $\boldsymbol{x}$ as a population vector, Lemma 1 implies that $\Phi(\boldsymbol{x})$ increases if population is moving from a smaller type to a bigger type.

▶ **Theorem 2 (Set-wise convergence).** *$\Phi(\boldsymbol{x}^{(t)})$ is strictly increasing along every nontrivial trajectory, i.e., $\Phi(\boldsymbol{x}^{t+1}) = \Phi(g(\boldsymbol{x}^{(t)})) \geq \Phi(\boldsymbol{x}^{(t)})$ with equality only when $\boldsymbol{x}^{(t)}$ is a fixed point. As a corollary, the dynamics converges to fixed points (set-wise convergence).*

Using the above theorem (Theorem 2) along with the construction of a local Lyapunov function, we can show the following theorem:

▶ **Theorem 3 (Point-wise convergence).** *The dynamics converges point-wise to fixed points.*

Proof Sketch of Theorems 2 and 3. We show Theorem 2 by first breaking the migration step from $\boldsymbol{x}^{(t)}$ to $\boldsymbol{x}^{(t+1)}$ into multiple steps which involve migration between two types only and using Lemma 1. Moreover, given a limit point $\boldsymbol{p}$ of a trajectory with initial population vector $\boldsymbol{x}^{(0)}$, we create a local Lyapunov function Ψ that depends on $\boldsymbol{p}$, i.e., $\Psi(\boldsymbol{x}, \boldsymbol{p}) = \sum_{v:\boldsymbol{p}_v>0}(\boldsymbol{p}_v - \boldsymbol{x}_v)$. Ψ is decreasing and nonnegative in a neighborhood of $\boldsymbol{p}$, and zero only at $\boldsymbol{p}$. Since $\boldsymbol{p}$ is a limit point, there is a subsequence of times $t_k \to \infty$ so that the dynamics for these times converges to $\boldsymbol{p}$, therefore the dynamics converges to $\boldsymbol{p}$ as $t \to \infty$, with initial condition $\boldsymbol{x}^{(0)}$. ◀

3.2 Diffeomorphism and stability analysis via Jacobian

In this section we compute the Jacobian J of g and then perform spectral analysis on J. The Jacobian of g is the following:

$$\frac{\partial g_u}{\partial \boldsymbol{x}_u} = J_{u,u} = 1 + \sum_{v \in N_u} \boldsymbol{x}_v \left[F_{uv}(\boldsymbol{x}_u - \boldsymbol{x}_v) + \boldsymbol{x}_u F'_{uv}(\boldsymbol{x}_u - \boldsymbol{x}_v)\right],$$

$$\frac{\partial g_u}{\partial \boldsymbol{x}_v} = J_{u,v} = \boldsymbol{x}_u \left[F_{uv}(\boldsymbol{x}_u - \boldsymbol{x}_v) - \boldsymbol{x}_v F'_{uv}(\boldsymbol{x}_u - \boldsymbol{x}_v)\right] \text{ if } uv \in E(G) \text{ else } 0.$$

▶ **Lemma 4** (Local Diffeomorphism). *The Jacobian is invertible on the subspace* $\sum_v \boldsymbol{x}_v = 1$, *for* $\sup_{z\in[-1,1]} |F_{uv}(z)| < \frac{1}{2}$ *for each* $uv \in E(G)$. *Moreover, g is a local diffeomorphism in a neighborhood of* Δ_n.

▶ **Lemma 5** (Linearly stable fixed point ⇒ independent set). *Let* $\boldsymbol{p}$ *be a fixed point such that there exists a connected component C of size greater than 1, and all types* $v \in C$ *have the same positive mass* $\boldsymbol{p}_v > 0$. *Then the Jacobian at* $\boldsymbol{p}$ *has an eigenvalue with absolute value greater than one.*

Proof Sketch of Lemmas 4 and 5. To prove Lemma 4, it suffices to show that the Jacobian $J(\boldsymbol{x})$ is invertible and then use the Inverse Function theorem. Invertibility comes from the fact that $J(\boldsymbol{x})$ is shown to be strictly diagonally dominant for $|F_{uv}(z)| < \frac{1}{2}$ for each $uv \in E(G)$ and $z \in [-1, 1]$. Moreover, to show Lemma 5, we can show that if a fixed point $\boldsymbol{p}$ does not induce an independent set, then the trace of the Jacobian of size $l \times l$ (after removing columns and rows of non-active types) at $\boldsymbol{p}$ is greater than l. Since the trace of a matrix is equal to the sum of its eigenvalues, the maximum eigenvalue in absolute value is greater than one and the claim follows. ◀

▶ Remark. Lemmas 4 and 5 are the key Lemmas for the next subsection in which we prove our first main result (Theorem 6 and Corollary 10). If one wants to prove such a result for the model of Kempe et al., these are the two lemmas that need to be adapted to their setting. Analyzing the Jacobian of the update rule of that model is very challenging since the update rule is a rational function, compared to our model which is generic but the derivatives are simpler to compute and analyze.

3.3 Center-stable manifold and average case analysis

In this section we prove our first main result, Corollary 10, which is a consequence of the following theorem:

▶ **Theorem 6.** *Assume that* $\max_{z\in[-1,1]} |F_{uv}(z)| < 1/2$ *for all* $uv \in E(G)$. *The set of points* $\boldsymbol{x} \in \Delta_n$ *such that dynamics 1 starting at* $\boldsymbol{x}$ *converges to a fixed point* $\boldsymbol{p}$ *whose active types do not form an independent set of G has measure zero.*

To prove Theorem 6, we are going to use arguably one of the most important theorems in dynamical systems, called *Center Stable Manifold Theorem*:

▶ **Theorem 7** (Center-stable Manifold Theorem [28]). *Let* $\boldsymbol{p}$ *be a fixed point for the* C^r *local diffeomorphism* $f : U \to \mathbb{R}^m$ *where* $U \subset \mathbb{R}^m$ *is an open neighborhood of* $\boldsymbol{p}$ *in* $\mathbb{R}^m$ *and* $r \geq 1$. *Let* $E^s \oplus E^c \oplus E^u$ *be the invariant splitting of* $\mathbb{R}^m$ *into generalized eigenspaces of the Jacobian* $J(\boldsymbol{p})$ *that correspond to eigenvalues of absolute value less than one, equal to one, and greater than one. To the* $J(\boldsymbol{p})$ *invariant subspace* $E^s \oplus E^c$ *there is an associated local f invariant* C^r *embedded disc* W^{sc}_{loc} *tangent to the linear subspace at* $\boldsymbol{p}$ *and a ball B around* $\boldsymbol{p}$ *such that:*

$$f(W^{sc}_{loc}) \cap B \subset W^{sc}_{loc}. \textit{ If } f^m(\boldsymbol{x}) \in B \textit{ for all } m \geq 0, \textit{ then } \boldsymbol{x} \in W^{sc}_{loc}. \tag{3}$$

Since an n-dimensional simplex Δ_n in $\mathbb{R}^n$ has dimension $n-1$, we need to take a projection of the domain space ($\sum_v \boldsymbol{x}_v = 1$) and accordingly redefine the map g. Let $\boldsymbol{x}$ be a point mass in Δ_n. Let u be a fixed type and define $h : \mathbb{R}^n \to \mathbb{R}^{n-1}$ so that we exclude the variable $\boldsymbol{x}_u$ from $\boldsymbol{x}$, i.e., $h(\boldsymbol{x}) = \boldsymbol{x}_{-u}$. We substitute the variable $\boldsymbol{x}_u$ with $1 - \sum_{v\neq u} \boldsymbol{x}_v$ and let g' be the resulting update rule of the dynamics $g'(\boldsymbol{x}_{-u}) = g(\boldsymbol{x})$. The following lemma gives a relation between the eigenvalues of the Jacobians of functions g and g'.

▶ **Lemma 8.** *Let J, J' be the Jacobian of g, g' respectively. Let λ be an eigenvalue of J such that λ does not correspond to left eigenvector $(1, \dots, 1)$ (with eigenvalue 1). Then J' has also λ as an eigenvalue.*

Before we proceed with the proof sketch of Theorem 6 and Corollary 10, we state the following which is a corollary of Lemmas 4, 5 and 8 and also uses classic properties for determinants of matrices.

▶ **Corollary 9.** *Let $\boldsymbol{p}$ be a fixed point whose active types do not form an independent set in G. Then J' at $h(\boldsymbol{p})$ has an eigenvalue with absolute value greater than one. Additionally, the Jacobian J' of g' is invertible in $h(\Delta_n)$ and as a result g' is a local diffeomorphism in a neighborhood of $h(\Delta_n)$.*

▶ **Corollary 10** (Convergence to independent sets)**.** *Suppose that $\max_{z\in[-1,1]} |F_{uv}(z)| < 1/2$ for all $uv \in E(G)$. If the initial mass vector $\boldsymbol{x}^{(0)} \in \Delta_n$ is chosen from an atomless distribution, then the dynamics converges point-wise with probability 1 to a point $\boldsymbol{p}$ whose active types form an independent set in G.*

Proof Sketch of Theorem 6 and Corollary 10. The proof of Corollary 10 comes from Theorem 3 and Theorem 6.

The main steps for the proof of Theorem 6 are as follows: Due to Center-Stable Manifold theorem (we can use it since the update rule of the dynamics is a local diffeomorphism, by Lemma 4 and Lemma 9) we have that the set of initial population vectors that stay trapped in a small enough neighborhood of an unstable fixed point is a lower dimensional manifold, hence a zero measure set. Any initial condition that converges point-wise to this unstable fixed point must at some time t reach points in this set. All of these initial conditions can thus be covered by a countable union of pre-images of the zero measure neighborhood implied by the Center-Stable Manifold theorem. Because the update rule is a local diffeomorphism, these pre-images must also be of zero measure and the countable union of zero measure sets imply a zero measure region of attraction for each unstable equilibrium. The only remaining hurdle is to cover the set of linearly unstable fixed points with a countable cover of the small neighborhoods. Finally, by Lemma 5 any fixed point $\boldsymbol{p}$ whose active types do not form an independent set of G is linearly unstable and the claim follows. ◀

Corollary 10 is illustrated in Figure 1 for the case of a 3-path and a triangle. As shown in the figure, if the initial condition is chosen uniformly at random from a point in the simplex, the dynamics converges to an independent set with probability one.

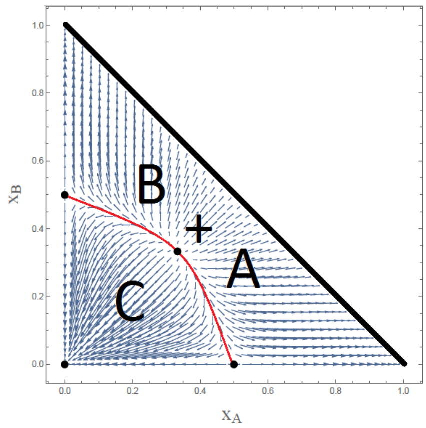
4 Stability and bound on the number of types

In this section we consider dynamical systems with migration, death and birth and prove two probabilistic statements on stability and number of types. The following direct application of Chernoff's bound is used intensively to attain probabilistic guarantees.

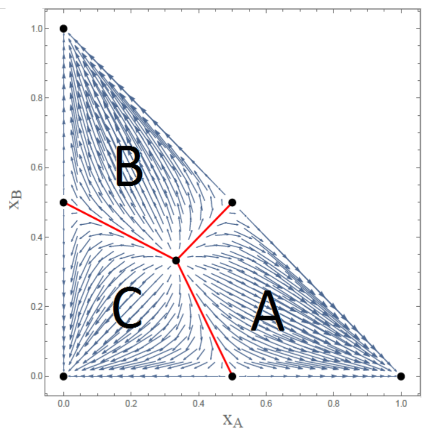
▶ **Lemma 11.** *In a period of t steps, there are at least $tp/2$ births with probability at least $1 - e^{-tp/8}$ and there are at most $3tp/2$ births with probability at least $1 - e^{-tp/6}$.*

4.1 Stability

We define the notion of stability and give a stability result for a dynamical system involving migration, death and birth. For the rest of the paper, we denote by $\alpha_{\min} =$



(a) The region with "C" corresponds to the initial population vectors so that the dynamics converges to the fixed point where all the population is of type C. The region "A+B" corresponds to the initial population masses so that the dynamics converges to a fixed point where part of the population is of type A and the rest of type B.



(b) Each region "A", "B", "C" corresponds to the initial population vectors so that the dynamics converges to all the population being of type A, B, C respectively. It is easy to see that an initial vector $(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C)$ converges to the fixed point where all population is of type $\arg\max_{i\in\{A,B,C\}} \boldsymbol{x}_i$. In case of ties, the limit population is split equally among the tied types (symmetry).

Figure 1 Migration dynamics phase portrait for path and triangle of 3 types A, B, C respectively and for $F_{uv}(z) = 0.25z$ for all $uv \in E(G)$. The black points and the line correspond to the fixed points. $\boldsymbol{x}_A, \boldsymbol{x}_B$ correspond to the fractions of people that are of type A, B. We omit $\boldsymbol{x}_C$ since $\boldsymbol{x}_C = 1 - \boldsymbol{x}_A - \boldsymbol{x}_B$.

$\min_{uv\in E(G), z\in[-1,1]} F'_{uv}(z)$ and $\alpha_{\max} = \max_{uv\in E(G), z\in[-1,1]} F'_{uv}(z)$. $\alpha_{\min}, \alpha_{\max}$ are non-negative and finite since F_{uv} is continuously differentiable, increasing and $[-1, 1]$ is a compact set. It can be seen easily that for each $uv \in E(G)$ and $z \in [-1, 1]$,

$$\alpha_{\min}(z - 0) \leq F_{uv}(z) - F_{uv}(0) \leq \alpha_{\max}(z - 0).$$

Since $F_{uv}(0) = 0$, $\alpha_{\min} z \leq F_{uv}(z) \leq \alpha_{\max} z$.

▶ **Definition 12** (((T, d)-Stable dynamics). A dynamics is (T, d)-stable if and only if $\forall T \leq t \leq T + d$, no population mass moves in the migration phase at step t.

We state the following two lemmas whose proofs come from the definition of Φ.

▶ **Lemma 13.** *If the dynamics is not $(t, 0)$-stable, the migration phase at time t increases Φ by at least $2\alpha_{\min}\epsilon\delta^3$.*

▶ **Lemma 14.** *Each birth can decrease Φ by at most $2\beta_{\max}$.*

With the two above lemmas, we can give a theorem on the "stability" of the dynamics:

▶ **Theorem 15** ("Stable" for long enough). *Assume $\alpha_{\min} > 0$. Let $p < \min\left(\frac{\epsilon\delta^3\alpha_{\min}}{3\beta_{\max}}, \frac{2}{3}\right)$ and $t > \frac{1}{\epsilon\delta^3\alpha_{\min} - 3p\beta_{\max}}$. With probability at least $1 - e^{-tp/6}$, the dynamics is $\left(T, \frac{1}{3p}\right)$-stable for some $T \leq t$.*

Proof Sketch. Consider a period of t steps. Lemma 11 guarantees that there are at most $3tp/2$ births in the period with the desired probability. In the migration phases of the period, Φ can either increase if there is a migration or remain unchanged otherwise.

Assume that Φ increases in more than $t/2$ migration phases. Using Lemma 13 and Lemma 14, we can bound the net increase of Φ in the period. Specifically, the net increase is least $t(\alpha_{\min}\epsilon\delta^3 - 3p\beta_{\max})$, which is greater than 1. Therefore, we have reached a contradiction.

It follows that Φ cannot increase in more than $t/2$ migration phases, and must remain unchanged in at least $t/2$ migration phases. Since there are at most $3tp/2$ births, there must be no migration in a period of $1/(3p)$ consecutive steps. ◀

4.2 Bound on the number of types

In this section we investigate a behavior of the dynamics following a long period of time. Specifically, we show that after a large number of steps, the number of types can not be too high. Our goal is to prove the following theorem:

▶ **Theorem 16** (Lack of explosion). *Let $\alpha_{\max} \leq p/512$ and $t \geq (16/p)\log^2(1/\epsilon)$. The dynamics at step t has at most $72\log(1/\epsilon)$ types with probability at least $1 - 3\epsilon$.*

First we give the following lemma, which says that if the number of types is large enough, then after a fixed period of time, it will decrease by a factor of roughly 2.

▶ **Lemma 17.** *Let $\alpha_{\max} \leq p/512$ and k be the number of types at step t_0. If $k \geq 48\log(1/\epsilon)$, with probability at least $1 - 2\epsilon^2$, the number of types at step $t_0 + (16/p)\log(1/\epsilon)$ is at most $k/2 + 24\log(1/\epsilon)$.*

Proof Sketch of Theorem 16. Consider the last $(16/p)\log^2(1/\epsilon)$ steps of the dynamics. We call a period of $(16/p)\log(1/\epsilon)$ steps a *decreasing period* if it satisfies the condition in Lemma 17, i.e, if the number of types k at the beginning of the period is at least $48\log(1/\epsilon)$, and the number of types at the end of the period is at most $k/2 + 24\log(1/\epsilon)$.

Construct a set P of periods of length $(16/p)\log(1/\epsilon)$ as follows. Start with $t' = 0$ and repeat the following step until $t' = t$. If $t' + (16/p)\log(1/\epsilon) \leq t$ and the number of types at t' is at least $48\log(1/\epsilon)$, let i be the period from t' to $t' + (16/p)\log(1/\epsilon)$, and add i to P. Update $t' \leftarrow t' + (16/p)\log(1/\epsilon)$. Else update $t' \leftarrow t' + 1$.

Assume that all periods in P are decreasing periods. By Lemma 17, the probability of such an outcome occurring is at least $1 - 2\epsilon$. With that assumption, if the number of types ever becomes smaller than $48\log(1/\epsilon)$ and reaches $48\log(1/\epsilon)$ again, it will be at least $48\log(1/\epsilon)$ after a period of $(16/p)\log(1/\epsilon)$ steps unless there are less than $(16/p)\log(1/\epsilon)$ subsequent steps. In that case, by Lemma 11, the probability that in the remaining steps, there are at most $24\log(1/\epsilon)$ births is at least $1 - \epsilon$. By union bound, the probability of both outcomes occurring is at least $1 - 3\epsilon$.

Moreover, since the number of types at the beginning is at most $1/\epsilon$, with the assumption, it must become smaller than $48\log(1/\epsilon)$ at some step of the dynamics. The theorem then follows. ◀

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