A Counterexample to Thiagarajan's Conjecture on Regular Event Structures*

Jérémie Chalopin[1] and Victor Chepoi[1]

1 **LIF, CNRS and Aix-Marseille Université, Marseille, France**
jeremie.chalopin@lif.univ-mrs.fr
2 **LIF, CNRS and Aix-Marseille Université, Marseille, France**
victor.chepoi@lif.univ-mrs.fr

Abstract

We provide a counterexample to a conjecture by Thiagarajan (1996 and 2002) that regular prime event structures correspond exactly to those obtained as unfoldings of finite 1-safe Petri nets. The same counterexample is used to disprove a closely related conjecture by Badouel, Darondeau, and Raoult (1999) that domains of regular event structures with bounded $\natural$-cliques are recognizable by finite trace automata. Event structures, trace automata, and Petri nets are fundamental models in concurrency theory. There exist nice interpretations of these structures as combinatorial and geometric objects and both conjectures can be reformulated in this framework. Namely, the domains of prime event structures correspond exactly to pointed median graphs; from a geometric point of view, these domains are in bijection with pointed CAT(0) cube complexes.

A necessary condition for both conjectures to be true is that domains of respective regular event structures admit a regular nice labeling. To disprove these conjectures, we describe a regular event domain (with bounded $\natural$-cliques) that does not admit a regular nice labeling. Our counterexample is derived from an example by Wise (1996 and 2007) of a nonpositively curved square complex $\mathbf{X}$ whose universal cover $\widetilde{\mathbf{X}}$ is a CAT(0) square complex containing a particular plane with an aperiodic tiling.

1998 ACM Subject Classification F.1.1 Models of Computation, G.2.2 Graph Theory

Keywords and phrases Discrete event structures, Trace automata, Median graphs and CAT(0) cube Complexes, Unfoldings and universal covers

Digital Object Identifier 10.4230/LIPIcs.ICALP.2017.101

1 Introduction

Event structures, introduced by Nielsen, Plotkin, and Winskel [18, 29, 30], are a widely recognized abstract model of concurrent computation. An event structure is a partially ordered set of events together with a conflict relation. The partial order captures the causal dependency of events. The conflict relation models incompatibility of events so that two events that are in conflict cannot simultaneously occur in any state of the computation. Consequently, two events that are neither ordered nor in conflict may occur concurrently. The domain of an event structure consists of all computation states, called configurations. Each computation state is a subset of events subject to the constraints that no two conflicting events can occur together in the same computation and if an event occurred in a computation then all events on which it causally depends have occurred too. Therefore, the domain of an event structure $\mathcal{E}$ is the set $\mathcal{D}(\mathcal{E})$ of all finite configurations ordered by inclusion. An event e

* A full version of the paper is available at https://arxiv.org/abs/1605.08288.

44th International Colloquium on Automata, Languages, and Programming (ICALP 2017).
Editors: Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl;
Article No. 101; pp. 101:1–101:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

is said to be enabled by a configuration c if $e \notin c$ and $c \cup \{e\}$ is a configuration. The degree of an event structure $\mathcal{E}$ is the maximum number of events enabled by a configuration of $\mathcal{E}$. The future of a configuration c is the set of all finite configurations c' containing c.

Among other things, the importance of event structures stems from the fact that several fundamental models of concurrent computation lead to event structures. Nielsen, Plotkin, and Winskel [18] proved that every 1-safe Petri net N unfolds into an event structure $\mathcal{E}_N$. Later results of [19] and [30] show in fact that 1-safe Petri nets and event structures represent each other in a strong sense. In the same vein, Stark [25] established that the domains of configurations of trace automata are exactly the conflict event domains; a presentation of domains of event structures as trace monoids (Mazurkiewicz traces) or as asynchronous transition systems was given in [22] and [6], respectively. In both cases, the events of the resulting event structure are labeled in a such a way that any two events enabled by the same configuration are labeled differently (such a labeling is usually called a nice labeling). To deal with *finite* 1-safe Petri nets, Thiagarajan [26, 27] introduced the notions of regular event structure and regular trace event structure. A regular event structure $\mathcal{E}$ is an event structure with a finite number of isomorphism types of futures of configurations and finite degree. A regular trace event structure is an event structure $\mathcal{E}$ whose events can be nicely labeled by the letters of a finite trace alphabet $M = (\Sigma, I)$ in a such a way that any two concurrent events define a pair of I and there exists only a finite number of isomorphism types of labeled futures of configurations. These definitions were motivated by the fact that the event structures $\mathcal{E}_N$ arising from *finite* 1-safe Petri nets N are regular: Thiagarajan [26] proved that event structures of *finite* 1-safe Petri nets correspond to regular trace event structures. This lead Thiagarajan to formulate the following conjecture:

► **Conjecture 1** ([26, 27]). *An event structure $\mathcal{E}$ is isomorphic to the event structure $\mathcal{E}_N$ arising from a finite 1-safe Petri net N if and only if $\mathcal{E}$ is regular.*

Badouel, Darondeau, and Raoult [2] formulated two similar conjectures about conflict event domain that are recognizable by finite trace automata. The first one is equivalent to Conjecture 1, while the second one is formulated in a more general setting with an extra condition. We formulate their second conjecture in the particular case of event structures:

► **Conjecture 2** ([2]). *The domain of an event structure $\mathcal{E}$ is recognizable if and only if $\mathcal{E}$ is regular and has bounded $\natural$-cliques.*

In view of previous results, to establish Conjecture 1, it is necessary for a regular event structure $\mathcal{E}$ to define a regular nice labeling with letters from some trace alphabet (Σ, I). Nielsen and Thiagarajan [20] proved in a technically involved but very nice combinatorial way that all regular conflict-free event structures satisfy Conjecture 1. In a equally difficult and technical proof, Badouel et al. [2] proved that their conjectures hold for context-free event domains, i.e., for domains whose underlying graph is a context-free graph sensu Müller and Schupp [17]. In this paper, we present a counterexample to Thiagarajan's Conjecture based on a more geometric and combinatorial view on event structures. We show that our example also provides a counterexample to Conjecture 2 of Badouel et al.

We use the striking bijections between the domains of event structures, median graphs, and CAT(0) cube complexes. Median graphs have many nice properties and admit numerous characterizations. They have been investigated in several contexts for more than half a century, and play a central role in metric graph theory; for more detailed information, the interested reader can consult the surveys [3, 4]. On the other hand, CAT(0) cube complexes are central objects in geometric group theory [23, 24, 33]. They have been characterized in a nice combinatorial way by Gromov [12] as simply connected cube complexes in which the links of 0-cubes are simplicial flag complexes. It was proven in [9, 21] that 1-skeleta of CAT(0) cube complexes are exactly the median graphs. Barthélemy and Constantin [5] proved that the Hasse diagrams of domains of event structures are median graphs and every pointed median graph is the domain of an event structure. The bijection between pointed median graphs and event domains established in [5] can be viewed as the classical characterization of prime event domains as prime algebraic coherent partial orders provided by Nielsen, Plotkin, and Winskel [18]. Via these bijections, the events of an event structure $\mathcal{E}$ correspond to the parallelism classes of edges of the domain $D(\mathcal{E})$ viewed as a median graph.

Our counter-example is based on Wise's [31, 32] nonpositively curved square complex $\mathbf{X}$ with one vertex and six squares, whose edges are colored in five colors, and whose colored universal cover $\widetilde{\mathbf{X}}$ contains a particular plane with an aperiodic tiling. As a result, $\widetilde{\mathbf{X}}$ is a CAT(0) square complex whose edges are colored by the colors of their images in $\mathbf{X}$ and are directed in such a way that all edges in the same parallelism class are oriented in the same way. With respect to this orientation, all vertices of $\widetilde{\mathbf{X}}$ are equivalent up to automorphism. We modify the complex $\mathbf{X}$ by taking its barycentric subdivision and by adding to the middles of the edges of $\mathbf{X}$ directed paths of five different lengths in order to encode the colors of the edges of $\mathbf{X}$ (and $\widetilde{\mathbf{X}}$) and to obtain a nonpositively curved square complex W. The universal cover $\widetilde{W}$ of W is a directed (but no longer colored) CAT(0) square complex. Since $\widetilde{W}$ is the universal cover of a finite complex W, $\widetilde{W}$ has a finite number of equivalence classes of vertices up to automorphism. From $\widetilde{W}$ we derive a domain of a regular event structure $\widetilde{W}_{\tilde{v}}$ by considering the future of an arbitrary vertex $\tilde{v}$ of $\widetilde{\mathbf{X}}$. Using the fact that $\widetilde{\mathbf{X}}$ contains a particular plane with an aperiodic tiling, we prove that $\widetilde{W}_{\tilde{v}}$ does not admit a regular nice labeling, thus $\widetilde{W}_{\tilde{v}}$ does not have a regular trace labeling.

Due to space limitations, some proofs are omitted; a full version of the paper is available on arXiv [8].

2 Event structures

2.1 Event structures and domains

An *event structure* is a triple $\mathcal{E} = (E, \leq, \#)$, where

- E is a set of *events*,
- $\leq \subseteq E \times E$ is a partial order of *causal dependency*,
- $\# \subseteq E \times E$ is a binary, irreflexive, symmetric relation of *conflict*,
- $\downarrow e := \{e' \in E : e' \leq e\}$ is finite for any $e \in E$,
- $e\#e'$ and $e' \leq e''$ imply $e\#e''$.

What we call here an event structure is usually called a *prime event structure*. Two events e', e'' are *concurrent* (notation $e' \| e''$) if they are order-incomparable and they are not in conflict. The conflict $e'\#e''$ between two elements e' and e'' is said to be *minimal* (notation, $e'\#_\mu e''$) if there is no event $e \neq e', e''$ such that either $e \leq e'$ and $e\#e''$ or $e \leq e''$ and $e\#e'$. Also define the binary relation $\lessdot \subseteq E \times E$ as follows: set $e \lessdot e'$ if and only if $e \leq e', e \neq e'$, and for every e'' if $e \leq e'' \leq e'$, then $e'' = e$ or $e'' = e'$. Given two event structures $\mathcal{E}_1 = (E_1, \leq_1, \#_1)$ and $\mathcal{E}_2 = (E_2, \leq_2, \#_2)$, a map $f : E_1 \to E_2$ is an isomorphism if f is a bijection such that $e \leq_1 e'$ iff $f(e) \leq_2 f(e')$ and $e\#_1 e'$ iff $f(e)\#_2 f(e')$ for every $e, e' \in E_1$. If such an isomorphism exists, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be isomorphic; notation $\mathcal{E}_1 \equiv \mathcal{E}_2$.

A *labeled event structure* $\mathcal{E}^\lambda = (\mathcal{E}, \lambda)$ is defined by an *underlying event structure* $\mathcal{E} = (E, \leq, \#)$ and a *labeling* λ that is a map from E to some alphabet Σ. Two labeled event structures $\mathcal{E}_1^{\lambda_1} = (\mathcal{E}_1, \lambda_1)$ and $\mathcal{E}_2^{\lambda_1} = (\mathcal{E}_2, \lambda_2)$ are isomorphic (notation $\mathcal{E}_1^{\lambda_1} \equiv \mathcal{E}_2^{\lambda_2}$) if there exists an isomorphism f between the underlying event structures $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\lambda_2(f(e_1)) = \lambda_1(e_1)$ for every $e_1 \in E_1$.

A *configuration* of an event structure $\mathcal{E} = (E, \leq, \#)$ is any finite subset $c \subset E$ of events which is *conflict-free* ($e, e' \in c$ implies that e, e' are not in conflict) and *downward-closed* ($e \in c$ and $e' \leq e$ implies that $e' \in c$) [30]. Notice that $\varnothing$ is always a configuration and that $\downarrow e$ and $\downarrow e \setminus \{e\}$ are configurations for any $e \in E$. The *domain* of an event structure is the set $\mathcal{D} := \mathcal{D}(\mathcal{E})$ of all configurations of $\mathcal{E}$ ordered by inclusion; (c', c) is a (directed) edge of the Hasse diagram of the poset $(\mathcal{D}(\mathcal{E}), \subseteq)$ if and only if $c = c' \cup \{e\}$ for an event $e \in E \setminus c$. An event e is said to be *enabled* by a configuration c if $e \notin c$ and $c \cup \{e\}$ is a configuration. Denote by $en(c)$ the set of all events enabled at the configuration c. Two events are called *co-initial* if they are both enabled at some configuration c. Note that if e and e' are co-initial, then either $e \#_\mu e'$ or $e \| e'$. It is easy to see that two events e and e' are in minimal conflict $e \#_\mu e'$ if and only if $e \# e'$ and e and e' are co-initial. The *degree* $\deg(\mathcal{E})$ of an event structure $\mathcal{E}$ is the least positive integer d such that $|en(c)| \leq d$ for any configuration c of $\mathcal{E}$. We say that $\mathcal{E}$ has *finite degree* if $\deg(\mathcal{E})$ is finite. The *future* (or the *filter*) $\mathcal{F}(c)$ of a configuration c is the set of all configurations c' containing c: $\mathcal{F}(c) = \uparrow c := \{c' \in \mathcal{D}(\mathcal{E}) : c \subseteq c'\}$, i.e., $\mathcal{F}(c)$ is the principal filter of c in the ordered set $(\mathcal{D}(\mathcal{E}), \subseteq)$.

For an event structure $\mathcal{E} = (E, \leq, \natural)$, let $\natural$ be the least irreflexive and symmetric relation on the set of events E such that $e_1 \natural e_2$ if (1) $e_1 \| e_2$, or (2) $e_1 \#_\mu e_2$, or (3) there exists an event e_3 that is co-initial with e_1 and e_2 at two different configurations such that $e_1 \| e_3$ and $e_2 \#_\mu e_3$. If $e_1 \natural e_2$ and this comes from condition (3), then we write $e_1 \natural_{(3)} e_2$. A $\natural$-*clique* is a subset S of events such that $e_1 \natural e_2$ for any $e_1, e_2 \in S$.

A labeling $\lambda : E \to \Sigma$ of an event structure $\mathcal{E}$ (or of its domain $\mathcal{D}(\mathcal{E})$) is called a *nice labeling* if any two events that are co-initial have different labels [22]. A nice labeling of $\mathcal{E}$ can be reformulated as a coloring of the directed edges of the Hasse diagram of its domain $\mathcal{D}(\mathcal{E})$ subject to the following local conditions:

- **Determinism:** The edges outgoing from the same vertex of $\mathcal{D}(\mathcal{E})$ have different colors.
- **Concurrency:** the opposite edges of each square of $\mathcal{D}(\mathcal{E})$ are colored with the same color.

2.2 Regular event structures

In this subsection, we recall the definitions of regular event structures, regular trace event structures, and regular nice labelings of event structures. We closely follow the definitions and notations of [26, 27, 20]. Let $\mathcal{E} = (E, \leq, \#)$ be an event structure. Let c be a configuration of $\mathcal{E}$. Set $\#(c) = \{e' : \exists e \in c, e \# e'\}$. The *event structure rooted at* c is defined to be the triple $\mathcal{E} \backslash c = (E', \leq', \#')$, where $E' = E \setminus (c \cup \#(c))$, $\leq'$ is $\leq$ restricted to $E' \times E'$, and $\#'$ is $\#$ restricted to $E' \times E'$. It can be easily seen that the domain $\mathcal{D}(\mathcal{E} \backslash c)$ of the event structure $\mathcal{E} \backslash c$ is isomorphic to the filter $\mathcal{F}(c)$ of c in $\mathcal{D}(\mathcal{E})$ such that any configuration c' of $\mathcal{D}(\mathcal{E})$ corresponds to the configuration $c' \setminus c$ of $\mathcal{D}(\mathcal{E} \backslash c)$.

For an event structure $\mathcal{E} = (E, \leq, \#)$, define the equivalence relation $R_{\mathcal{E}}$ on its configurations in the following way: for two configurations c and c' set $c R_{\mathcal{E}} c'$ if and only if $\mathcal{E} \backslash c \equiv \mathcal{E} \backslash c'$. The *index* of an event structure $\mathcal{E}$ is the number of equivalence classes of $R_{\mathcal{E}}$, i.e., the number of isomorphism types of futures of configurations of $\mathcal{E}$. The event structure $\mathcal{E}$ is *regular* [26, 27, 20] if $\mathcal{E}$ has finite index and finite degree.

Now, let $\mathcal{E}^\lambda = (\mathcal{E}, \lambda)$ be a labeled event structure. For any configuration c of $\mathcal{E}$, if we restrict λ to $\mathcal{E} \backslash c$, then we obtain a labeled event structure $(\mathcal{E} \backslash c, \lambda)$ denoted by $\mathcal{E}^\lambda \backslash c$. Analogously, define the equivalence relation $R_{\mathcal{E}^\lambda}$ on its configurations by setting $c R_{\mathcal{E}^\lambda} c'$ if and only if $\mathcal{E}^\lambda \backslash c \equiv \mathcal{E}^\lambda \backslash c'$. The index of $\mathcal{E}^\lambda$ is the number of equivalence classes of $R_{\mathcal{E}^\lambda}$. We say that an event structure $\mathcal{E}$ admits a *regular nice labeling* if there exists a nice labeling λ of $\mathcal{E}$ with a finite alphabet Σ such that $\mathcal{E}^\lambda$ has finite index.

We continue by recalling the definition of regular trace event structures from [26, 27]. A *(Mazurkiewicz) trace alphabet* is a pair $M = (\Sigma, I)$, where Σ is a finite non-empty alphabet set and $I \subset \Sigma \times \Sigma$ is an irreflexive and symmetric relation called the *independence relation*. As usual, Σ^* is the set of finite words with letters in Σ. The independence relation I induces the equivalence relation $\sim_I$, which is the reflexive and transitive closure of the binary relation $\leftrightarrow_I$: *if* $\sigma, \sigma' \in \Sigma^*$ *and* $(a, b) \in I$, *then* $\sigma ab\sigma' \leftrightarrow_I \sigma ba\sigma'$. The relation $D := (\Sigma \times \Sigma) \setminus I$ is called the *dependence relation*. An *M-labeled event structure* is a labeled event structure $\mathcal{E}^\lambda = (\mathcal{E}, \lambda)$, where $\mathcal{E} = (E, \leq, \#)$ is an event structure and $\lambda : E \to \Sigma$ is a labeling function which satisfies the following conditions:

- (LES1) $e \#_\mu e'$ implies $\lambda(e) \neq \lambda(e')$,
- (LES2) $e \lessdot e'$ or $e \#_\mu e'$, then $(\lambda(e), \lambda(e')) \in D$,
- (LES3) if $(\lambda(e), \lambda(e')) \in D$, then $e \leq e'$ or $e' \leq e$ or $e \# e'$.

We call λ a *trace labeling* of $\mathcal{E}$. The conditions (LES2) and (LES3) on the labeling function ensures that the concurrency relation $\|$ of $\mathcal{E}$ respects the independence relation I of M. In particular, since I is irreflexive, from (LES3) it follows that any two concurrent events are labeled differently. Since by (LES1) two events in minimal conflict are also labeled differently, this implies that λ is a finite nice labeling of $\mathcal{E}$.

An M-labeled event structure $\mathcal{E}^\lambda = (\mathcal{E}, \lambda)$ is *regular* if $\mathcal{E}^\lambda$ has finite index. Finally, an event structure $\mathcal{E}$ is called a *regular trace event structure* [26, 27] iff there exists a trace alphabet $M = (\Sigma, I)$ and a regular M-labeled event structure $\mathcal{E}^\lambda$ such that $\mathcal{E}$ is isomorphic to the underlying event structure of $\mathcal{E}^\lambda$. From the definition immediately follows that every regular trace event structure is also a regular event structure. It turns out that the converse is equivalent to Conjecture 1. Namely, [27] establishes the following equivalence (this result dispenses us from giving a formal definition of 1-safe Petri nets; the interested readers can find it in the papers [27, 20]):

▶ **Theorem 3** ([27, Theorem 1]). *$\mathcal{E}$ is a regular trace event structure if and only if there exists a finite 1-safe Petri net N such that $\mathcal{E}$ and $\mathcal{E}_N$ are isomorphic.*

In view of this theorem, Conjecture 1 is equivalent to the following conjecture:

▶ **Conjecture 4.** *$\mathcal{E}$ is a regular event structure iff $\mathcal{E}$ is a regular trace event structure.*

Badouel et al. [2] considered recognizable conflict event domains that are more general than the domains of event structures we consider in this paper. Since the domain of an event structure $\mathcal{E}$ is recognizable if and only if $\mathcal{E}$ is a regular trace event structure (see [16, Section 5]), Conjecture 2 can be reformulated as follows:

▶ **Conjecture 5.** *$\mathcal{E}$ is a regular event structure iff $\mathcal{E}$ is a regular trace event structure and $\mathcal{E}$ has bounded $\natural$-cliques.*

Since any regular trace labeling is a regular nice labeling, any regular event structure $\mathcal{E}$ not admitting a regular nice labeling is a counter-example to Conjecture 4 (and thus to Conjecture 1). If, additionally, $\mathcal{E}$ has bounded $\natural$-cliques, $\mathcal{E}$ is also a counter-example to Conjecture 5 (and thus to Conjecture 2).

3 Domains, median graphs, and CAT(0) cube complexes

In this section, we recall the bijections between domains of event structures and median graphs/CAT(0) cube complexes established in [1] and [5], and between median graphs and 1-skeleta of CAT(0) cube complexes established in [9] and [21].

Let $G = (V, E)$ be a simple, connected, not necessarily finite graph. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths. A graph G is *median* if for any three vertices x, y, z of G, there exists a unique vertex $m = m(x, y, z)$, called the *median* of x, y, z, simultaneously lying on the intervals $I(x, y), I(x, z)$, and $I(y, z)$. Basic examples of median graphs are trees, hypercubes, rectangular grids, and Hasse diagrams of distributive lattices and of median semilattices [3]. With any vertex v of a median graph $G = (V, E)$ is associated a *canonical partial order* $\leq_v$ defined by setting $x \leq_v y$ if and only if $x \in I(v, y)$; v is called the *basepoint* of $\leq_v$. Since G is bipartite, the Hasse diagram G_v of the partial order $(V, \leq_v)$ is the graph G in which any edge xy is directed from x to y if and only if the inequality $d_G(x, v) < d_G(y, v)$ holds. We call G_v a *pointed median graph*.

Median graphs can be obtained from hypercubes by amalgams and median graphs are themselves isometric subgraphs of hypercubes. The canonical isometric embedding of a median graph G into a (smallest) hypercube can be determined by the so called *Djoković-Winkler ("parallelism")* relation Θ on the edges of G [11, 28]. For median graphs, the equivalence relation Θ can be defined as follows. First say that two edges uv and xy are in relation Θ_0 if they are either equal or opposite edges of a 4-cycle $uvxy$ in G. Then let Θ be the transitive closure of Θ_0. We denote by $\{\Theta_i : i \in I\}$ the equivalence classes of the relation Θ (in [5], they were called parallelism classes). Each equivalence class $\Theta_i, i \in I$, is a cutset of G: namely, it splits $V(G)$ in two convex subgraphs A_i, B_i of G (A subgraph H of G is *convex* if for all $u, v \in V(H)$, $I(u, v) \subseteq V(H)$). The equivalence relation Θ is fundamental in the bijection between event structures and median graphs:

► **Theorem 6** ([5])**.** *The Hasse diagram of the domain $(\mathcal{D}(\mathcal{E}), \subseteq)$ of any event structure $\mathcal{E} = (E, \leq, \#)$ is a median graph. Conversely, for any median graph G and any basepoint v of G, the pointed median graph G_v is isomorphic to the Hasse diagram of the domain of an event structure.*

In the construction of an event structure $\mathcal{E}_v$ from a median graph G pointed at a vertex v, the events $e_i, i \in I$ of $\mathcal{E}_v$ correspond to the equivalence classes $\Theta_i, i \in I$ of Θ. Two classes Θ_i and Θ_j define concurrent events if and only if they cross, i.e., there exists a square $uvxy$ where $uv, xy \in \Theta_i$ and $uy, vx \in \Theta_j$. For two events e_i, e_j, we have $e_i \prec e_j$ if and only if the cutset Θ_i separates v from the edges of Θ_j. Finally, two events e_i, e_j are in conflict if and only if Θ_i and Θ_j do not cross and neither separates the other from v.

A *cube complex* is a cell complex X whose cells are unit Euclidean cubes of various dimensions such that any two intersecting cubes of X intersect in a common face. The 0-cubes and the 1-cubes of X are called *vertices* and *edges* of X and define the graph $X^{(1)}$, the 1-*skeleton* of X. The *star* $\mathrm{St}(v, X)$ of a vertex v of X is the subcomplex spanned by all cubes containing v. A cube complex X is *simply connected* if every cycle C of its 1-skeleton is null-homotopic, i.e., it can be contracted to a single point by elementary homotopies. Given two cube complexes X and Y, a *covering (map)* is a surjection $p\colon Y \to X$ mapping cubes to cubes and such that $p|_{\mathrm{St}(v,Y)}\colon \mathrm{St}(v, Y) \to \mathrm{St}(p(v), X)$ is an isomorphism for every vertex v in Y. The space Y is then called a *covering space* of X. A *universal cover* of X is a simply connected covering space; it always exists and it is unique up to isomorphism [13, Sections 1.3 and 4.1]. The universal cover of a complex X will be denoted by $\widetilde{X}$. In particular, if X is simply connected, then its universal cover $\widetilde{X}$ is X itself.

An important class of cube complexes studied in geometric group theory and combinatorics is the class of CAT(0) cube complexes. In this case, being CAT(0) is equivalent to the unicity of geodesics in the ℓ_2 metric; see [7] for this and other properties of CAT(0) spaces. Gromov [12] gave a beautiful combinatorial characterization of CAT(0) cube complexes as simply connected cube complexes satisfying the following condition: *if three* $(k+2)$*-cubes pairwise intersect in a* $(k+1)$*-cube and all three intersect in a* k*-cube, then they are included in a* $(k+3)$*-cube.* A cube complex X satisfying this combinatorial condition is called a *nonpositively curved (NPC)* complex. As a corollary of Gromov's result, for any NPC complex X, its universal cover $\widetilde{X}$ is CAT(0).

There is a well-known bijection between median graphs and CAT(0) cube complexes [9, 21]. Each median graph G gives rise to a cube complex $X(G)$ obtained by replacing all hypercubes of G by Euclidean unit cubes. Endowed with the intrisic ℓ_2-metric, $X(G)$ is a CAT(0) space. Conversely, the 1-skeleton of any CAT(0) cube complex is a median graph. In fact, a graph G is median if and only if its cube complex is simply connected and G satisfies the 3-cube condition [9]: *if three squares of* G *pairwise intersect in an edge and all three intersect in a vertex, then they belong to a 3-cube.*

This link between event domains, median graphs, and CAT(0) cube complexes allows a more geometric and combinatorial approach to several questions on event structures (and to work only with CAT(0) cube complexes viewed as event domains). For example, this allowed [10] to disprove the so-called *nice labeling conjecture* of Rozoy and Thiagarajan [22] asserting that any event structure of finite degree admits a finite nice labeling.

4 Directed NPC Complexes

Since we can define event structures from their domains, universal covers of NPC complexes represent a rich source of event structures. To obtain regular event structures, it is natural to consider universal covers of finite NPC complexes. Moreover, since domains of event structures are directed, it is natural to consider universal covers of NPC complexes whose edges are directed. However, the resulting directed universal covers are not in general domains of event structures. In particular, the domains corresponding to pointed median graphs given by Theorem 6 cannot be obtained in this way. In order to overcome this difficulty, we introduce directed median graphs and directed NPC complexes. Using these notions, one can naturally define regular event structures starting from finite directed NPC complexes.

A *directed median graph* is a pair (G, o), where G is a median graph and o is an orientation of the edges of G in a such a way that opposite edges of squares of G have the same direction. By transitivity of Θ, all edges from the same parallelism class Θ_i of G have the same direction. Since each Θ_i partitions G into two parts, o defines a partial order $\prec_o$ on the vertex-set of G. For a vertex v of G, let $\mathcal{F}_o(v, G) = \{x \in V : v \prec_o x\}$ be the principal filter of v in the partial order $(V(G), \prec_o)$.

The following lemma shows that choosing an arbitrary vertex in a directed median graph as a basepoint, one can define the domain of an event structure.

▶ **Lemma 7.** *For any vertex* v *of a directed median graph* (G, o)*, the following holds:*
1. $\mathcal{F}_o(v, G)$ *induces a convex subgraph of* G*;*
2. *the restriction of the partial order* $\prec_o$ *on* $\mathcal{F}_o(v, G)$ *coincides with the restriction of the canonical basepoint order* $\leq_v$ *on* $\mathcal{F}_o(v, G)$*;*
3. $\mathcal{F}_o(v, G)$ *with* $\prec_o$ *is the domain of an event structure;*
4. *for any vertex* $u \in \mathcal{F}_o(v, G)$*, the principal filter* $\mathcal{F}_o(u, G)$ *is included in* $\mathcal{F}_o(v, G)$ *and* $\mathcal{F}_o(u, G)$ *coincides with the principal filter of* u *with respect to the canonical basepoint order* $\leq_v$ *on* $\mathcal{F}_o(v, G)$*.*

A *directed NPC complex* is a pair (Y, o), where Y is a NPC complex and o is an orientation of the edges of Y in a such a way that the opposite edges of the same square of Y have the same direction. The orientation o of the edges of Y induces in a natural way an orientation $\tilde{o}$ of the edges of its universal cover $\widetilde{Y}$, so that $(\widetilde{Y}, \tilde{o})$ is a directed NPC complex and $(\widetilde{Y}^{(1)}, \tilde{o})$ is a directed median graph. We now formulate the crucial regularity property of directed median graphs $(\widetilde{Y}^{(1)}, \tilde{o})$ when (Y, o) is finite.

▶ **Lemma 8.** *If (Y, o) is a finite directed NPC complex, then $(\widetilde{Y}^{(1)}, \tilde{o})$ is a directed median graph with at most $|V(Y)|$ isomorphism types of principal filters.*

Combining Lemmas 7 and 8, we obtain the following result.

▶ **Proposition 9.** *Let (Y, o) be a finite directed NPC complex. Then for any vertex $\tilde{v}$ of the universal cover $\widetilde{Y}$ of Y, the principal filter $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{Y}^{(1)})$ with the partial order $\prec_{\tilde{o}}$ is the domain of a regular event structure with at most $|V(Y)|$ different isomorphism types of futures.*

A *square complex* X is a combinatorial 2-complex whose 2-cells are attached by closed combinatorial paths of length 4. Thus, one can consider each 2-cell as a square attached to the 1-skeleton $X^{(1)}$ of X. A square complex X is a VH-*complex* (*vertical-horizontal complex*) if the 1-cells (edges) of X are partitioned into two sets V and H called *vertical* and *horizontal* edges respectively, and the edges in each square alternate between edges in V and H. Notice that if X is a VH-complex, then X satisfies the Gromov's nonpositive curvature condition since no three squares may pairwise intersect on three edges with a common vertex. A VH-complex X is a *complete square complex* (CSC) [32] if any vertical edge and any horizontal edge incident to a common vertex belong to a common square of X. By [32, Theorem 3.8], if X is a complete square complex, then the universal cover $\widetilde{X}$ of X is isomorphic to the Cartesian product of two trees. By a *plane* Π in $\widetilde{X}$ we will mean a convex subcomplex of $\widetilde{X}$ isometric to $\mathbb{R}^2$ tiled by the grid $\mathbb{Z}^2$ into unit squares.

5 Wise's event domain $\widetilde{W}_{\tilde{v}}$

In this section, we construct the domain $\widetilde{W}_{\tilde{v}}$ of a regular event structure (with bounded $\natural$-cliques) that does not admit a regular nice labelling. To do so, we start with a directed colored CSC $\mathbf{X}$ introduced by Wise [32]. In the following, we consider directed colored VH-complexes, in which each edge has an orientation and a color. Such complexes will be denoted by bold letters, like $\mathbf{X}$. Sometimes, we need to forget the colors and the orientations of the edges of these complexes. For a complex $\mathbf{X}$, we denote by X the complex obtained by forgetting the colors and the orientations of the edges of $\mathbf{X}$ (X is called the *support* of $\mathbf{X}$), and we denote by (X, o) the directed complex obtained by forgetting the colors of $\mathbf{X}$.

5.1 Wise's square complex $\mathbf{X}$ and its universal cover $\widetilde{\mathbf{X}}$

The complex $\mathbf{X}$ consists of six squares as indicated in Figure 1 (reproducing Figure 3 of [32]). Each square has two vertical and two horizontal edges. The horizontal edges are oriented from left to right and vertical edges from bottom to top. Denote this orientation of edges by o. The vertical edges of squares are colored white, grey, and black and denoted a, b, and c, respectively. The horizontal edges of squares are colored by single or double arrow, and denoted x and y, respectively. The six squares are glued together by identifying edges of the same color and respecting the directions to obtain the square complex $\mathbf{X}$. Note that $\mathbf{X}$ has a unique vertex, five edges, and six squares. It can be directly checked that $\mathbf{X}$ is a complete square complex, and consequently (X, o) is a directed NPC complex. Let H_X denote the subcomplex of X consisting of the 2 horizontal edges and let V_X denote the subcomplex of X consisting of the 3 vertical edges.

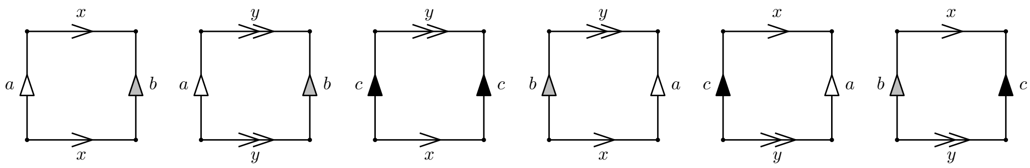

Figure 1 The 6 squares defining the complex $\mathbf{X}$.

The universal cover $\widetilde{H_X}$ of H_X is the 4-regular infinite tree F_4. Its edges inherit the orientations from their images in H_X: each vertex of $\widetilde{H_X}$ has two incoming and two outgoing arcs. Analogously, the universal cover $\widetilde{H_V}$ of H_V is the 6-regular infinite tree F_6 where each vertex has three incoming and three outgoing arcs. Let $\tilde{v}_1$ be any vertex of $\widetilde{H_X}$. Then the principal filter of $\tilde{v}_1$ is the infinite binary tree T_2 rooted at $\tilde{v}_1$: all its vertices except $\tilde{v}_1$ have one incoming and two outgoing arcs, while $\tilde{v}_1$ has two outgoing arcs and no incoming arc. Analogously, the principal filter of any vertex $\tilde{v}_2$ in the ordered set $\widetilde{H_V}$ is the infinite ternary tree T_3 rooted at $\tilde{v}_2$.

Let $\widetilde{\mathbf{X}}$ be the universal cover of $\mathbf{X}$ and let $p : \widetilde{\mathbf{X}} \to \mathbf{X}$ be a covering map. Let $\widetilde{X}$ denote the support of $\widetilde{\mathbf{X}}$. Since $\mathbf{X}$ is a CSC, by [32, Theorem 3.8], $\widetilde{X}$ is the Cartesian product $F_4 \times F_6$ of the trees F_4 and F_6. The edges of $\widetilde{\mathbf{X}}$ are colored and oriented as their images in $\mathbf{X}$, and are also classified as horizontal or vertical edges. The squares of $\widetilde{\mathbf{X}}$ are oriented as their images in $\mathbf{X}$, thus two opposite edges of the same square of $\widetilde{\mathbf{X}}$ have the same direction. This implies that all classes of parallel edges of $\widetilde{\mathbf{X}}$ are oriented in the same direction. Denote this orientation of the edges of $\widetilde{\mathbf{X}}$ by $\tilde{o}$. The 1-skeleton $\widetilde{X}^{(1)}$ of $\widetilde{X}$ together with $\tilde{o}$ is a directed median graph. Let $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ be any vertex of $\widetilde{X}$, where $\tilde{v}_1$ and $\tilde{v}_2$ are the coordinates of $\tilde{v}$ in the trees F_4 and F_6. Then the principal filter $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ of $\tilde{v}$ is the Cartesian product of the principal filters of $\tilde{v}_1$ in F_4 and of $\tilde{v}_2$ in F_6, i.e., is isomorphic to $T_2 \times T_3$.

By Lemma 7, the orientation of the edges of $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ corresponds to the canonical basepoint orientation of $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ with $\tilde{v}$ as the basepoint. Moreover, by Proposition 9, $\mathcal{F}_{\tilde{o}}(\tilde{v}, \widetilde{X}^{(1)})$ is the domain of a regular event structure with one isomorphism type of futures.

5.2 Aperiodicity of $\widetilde{\mathbf{X}}$

We recall here the main properties of $\widetilde{\mathbf{X}}$ established in [32, Section 5]. Let $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ be an arbitrary vertex of $\widetilde{\mathbf{X}}$, where $\tilde{v}_1$ and $\tilde{v}_2$ are defined as before. From the definition of the covering map, the loop of $\mathbf{X}$ colored y gives rise to a bi-infinite horizontal path P_y of $\widetilde{\mathbf{X}}^{(1)}$ passing via $\tilde{v}$ and whose all edges are colored y and are directed from left to right. Analogously, there exists a bi-infinite vertical path P_c of $\widetilde{\mathbf{X}}^{(1)}$ passing via $\tilde{v}$ and whose all edges are colored c and are directed from bottom to top.

The projection of P_y on the horizontal factor F_4 is a bi-infinite path P^h of F_4 passing via $\tilde{v}_1$. Analogously, the projection of P_c on the vertical factor F_6 is a bi-infinite path P^v of F_6 passing via $\tilde{v}_2$. Consequently, the convex hull $\mathrm{conv}(P_y \cup P_c)$ of $P_y \cup P_c$ in the graph $\widetilde{\mathbf{X}}^{(1)}$ is isomorphic to the Cartesian product of $P^h \times P^v$ of the paths P^h and P^v. Therefore the subcomplex of $\widetilde{\mathbf{X}}$ spanned by $\mathrm{conv}(P_y \cup P_c)$ is a plane Π_{yc} tiled into squares (recall that each square is of one of 6 types and its sides are colored by the letters a, b, c, x, y), see Figure 2.

In our counterexample we will use the following result of [32] that was used to show that the plane Π_{yc} is not tiled periodically by the preimages of the squares of $\mathbf{X}$. Denote by P_y^+ the (directed) subpath of P_y having $\tilde{v}$ as the origin (this is a one-infinite horizontal path). Analogously, let P_c^+ be the (vertical) subpath of P_c having $\tilde{v}$ as the origin. The convex hull of $P_y^+ \cup P_c^+$ is a quarter of the plane Π_{yc}, which we denote by Π_{yc}^{++}. Any shortest path in

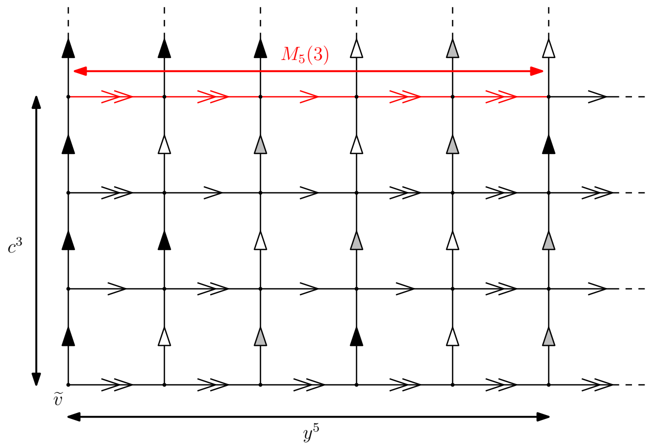

Figure 2 Part of the plane Π_{yc}^{++} appearing in $\widetilde{\mathbf{X}}$.

$\widetilde{\mathbf{X}}^{(1)}$ from $\tilde{v}$ to a vertex $\tilde{u} \in \Pi_{yc}^{++}$ can be viewed as a word in the alphabet $A = \{a, b, c, x, y\}$. For an integer $n \geq 0$, denote by y^n the horizontal subpath of P_y^+ beginning at $\tilde{v}$ and having length n. Analogously, for an integer $m \geq 0$, denote by c^m the vertical subpath of P_c^+ beginning at $\tilde{v}$ and having length m. Let $M_n(m)$ denote the horizontal path of Π_{yc}^{++} of length n beginning at the endpoint of the vertical path c^m. $M_n(m)$ determines a word which is the label of the side opposite to y^n in the rectangle which is the convex hull of y^n and c^m (see Figure 2). Let $M_n(m)$ also denote this corresponding word.

▶ **Proposition 10** ([32, Proposition 5.9]). *For each n, the words $\{M_n(m) : 0 \leq m \leq 2^n - 1\}$ are all distinct, and thus, every positive word in x and y of length n is $M_n(m)$ for some m.*

5.3 The square complex W and its universal cover $\widetilde{W}$

Let $\beta\mathbf{X}$ denote the first barycentric subdivision of $\mathbf{X}$: each square C of $\mathbf{X}$ is subdivided into four squares C_1, C_2, C_3, C_4 by adding a middle vertex to each edge of C and connecting it to the center of C by an edge. This way each edge e of C is subdivided into two edges e_1, e_2, which inherit the orientation and the color of e. The four edges connecting the middle vertices of the edges of C to the center of C are oriented from left to right and from bottom to top (see the middle figure of Figure 3). Denote the resulting orientation by o'. This way, $(\beta\mathbf{X}, o')$ is a directed and colored square complex. Again, denote by βX the support of $\beta\mathbf{X}$. The universal cover $\widetilde{\beta X}$ of βX is the Cartesian product $\beta F_4 \times \beta F_6$ of the trees βF_4 and βF_6, where βF_4 is the first barycentric subdivision of F_4 and βF_6 is the first barycentric subdivision of F_6. Additionally, $(\widetilde{\beta X}, \tilde{o}')$ is a directed CAT(0) square complex. We assign a *type* to each vertex of $\widetilde{\beta\mathbf{X}}$: the preimage of the unique vertex of $\mathbf{X}$ is of type 0 and is called a 0-*vertex*, the preimages of the middles of edges of $\mathbf{X}$ are of type 1 and are called 1-*vertices*, and the preimages of centers of squares of $\mathbf{X}$ are of type 2 and are called 2-*vertices*.

To encode the colors of the edges of $\mathbf{X}$, we introduce our central object, the square complex W (whose edges are no longer colored). Let $A = \{a, b, c, x, y\}$ and let $r : A \to \{1, 2, 3, 4, 5\}$ be a bijective map. The complex W is obtained from βX by adding to each 1-vertex z of βX a path R_z of length $r(\alpha)$ if z is the middle of an edge colored $\alpha \in A$ in $\mathbf{X}$. The path R_z has one end at z (called the *root* of R_z) and z is the unique common vertex of R_z and βX (we call such added paths R_z *tips*). Denote by o^* the orientation of the edges of W defined as follows: the edges of βX are oriented as in $(\beta X, o)$ and the edges of tips are oriented away

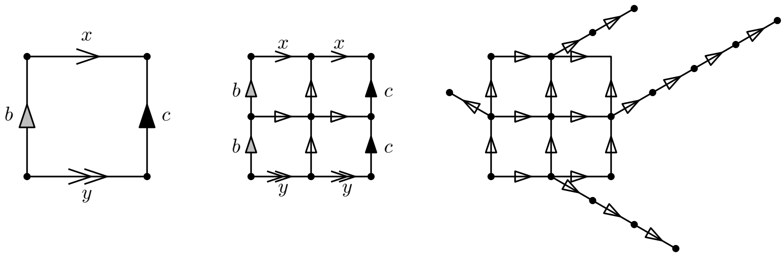

Figure 3 A square of $\mathbf{X}$ and the corresponding subcomplexes in $(\beta\mathbf{X}, o')$ and (W, o^*).

from their roots (see the rightmost figure of Figure 3 for the encoding of the last square of Figure 1). As a result, we obtain a finite directed NPC square complex (W, o^*).

Consider the universal cover $\widetilde{W}$ of W. It can be viewed as the complex $\widetilde{\beta X}$ with a path of length $r(\alpha)$ added to each 1-vertex which encodes an edge of $\widetilde{\mathbf{X}}$ of color $\alpha \in A$. We say that the vertices of $\widetilde{W}$ lying only on tips are of type 3 and they are called 3-*vertices*. Let $\tilde{o}^*$ denote the orientation of the edges of $\widetilde{W}$ induced by the orientation o^* of W. Then $(\widetilde{W}, \tilde{o}^*)$ is a directed CAT(0) square complex. Since W is finite, the directed median graph $(\widetilde{W}^{(1)}, \tilde{o}^*)$ has a finite number of isomorphisms types of principal filters $\mathcal{F}_{\tilde{o}^*}(\tilde{z}, \widetilde{W}^{(1)})$.

Let $\tilde{v}$ be any 0-vertex of $\widetilde{W}$. Denote by $\widetilde{W}_{\tilde{v}}$ the principal filter $\mathcal{F}_{\tilde{o}^*}(\tilde{v}, \widetilde{W}^{(1)})$ of vertex $\tilde{v}$ in $(\widetilde{W}^{(1)}, \prec_{\tilde{o}^*})$. By Proposition 9, $\widetilde{W}_{\tilde{v}}$ together with the partial order $\prec_{\tilde{o}^*}$ is the domain of a regular event structure, which we call *Wise's event domain.* Since vertices of different types of $\widetilde{W}$ are incident to a different number of outgoing squares, any isomorphism between two filters of $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{o}^*})$ preserves the types of vertices. We summarize all this in the following:

▶ **Proposition 11.** *$(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{o}^*})$ is the domain of a regular event structure. Any isomorphism between any two filters of $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{o}^*})$ preserves the types of vertices.*

5.4 $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{o}^*})$ does not have a regular nice labeling

In this subsection we prove that the event structure associated to Wise's event domain is a counterexample to Thiagarajan's conjecture (Theorem 12) and to the conjecture of Badouel et al. [2] (Theorem 12 and Proposition 13).

▶ **Theorem 12.** *$(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{o}^*})$ does not admit a regular nice labeling.*

Proof. Since $\widetilde{W}_{\tilde{v}}$ is the principal filter of a 0-vertex $\tilde{v}$, $\widetilde{W}_{\tilde{v}}$ contains all vertices of $\widetilde{\mathbf{X}}$ located in the quarter of plane Π^{++}_{yc} of $\widetilde{\mathbf{X}}$, in particular it contains the vertices of the paths P^+_c and P^+_y. Notice also that $\widetilde{W}_{\tilde{v}}$ contains the barycenters and the tips corresponding to the edges of Π^{++}_{yc}. Suppose by way of contradiction that $\widetilde{W}_{\tilde{v}}$ has a regular nice labeling λ. Since $\widetilde{W}_{\tilde{v}}$ has only a finite number of isomorphism types of labeled filters, the vertical path P^+_c contains two 0-vertices, $\tilde{z}'$ and $\tilde{z}''$, which have isomorphic labeled principal filters. Let $\tilde{z}'$ be the end of the vertical subpath c^k of P^+_c and $\tilde{z}''$ be the end of the vertical subpath c^m of P^+_c, and suppose without loss of generality that $k < m$. Let $n > 0$ be a positive integer such that $m \leq 2^n - 1$. Consider the horizontal convex paths $M_n(k)$ and $M_n(m)$ of Π^{++}_{yc} of length n beginning at the vertices $\tilde{z}'$ and $\tilde{z}''$, respectively. For any $0 \leq i \leq n$, denote by $\tilde{z}_{k,i}$ the ith vertex of $M_n(k)$ (in particular, $\tilde{z}_{k,0} = \tilde{z}'$). Analogously, denote by $\tilde{z}_{m,i}$ the ith vertex of $M_n(m)$ (in particular, $\tilde{z}_{m,0} = \tilde{z}''$). In $\widetilde{W}_{\tilde{v}}$, the paths $M_n(k)$ and $M_n(m)$ give rise to two convex horizontal paths $M^*_n(k)$ and $M^*_n(m)$ obtained from $M_n(k)$ and $M_n(m)$ by subdividing their edges. Denote by $\tilde{u}_{k,i}$ the unique common neighbor of $\tilde{z}_{k,i}$ and $\tilde{z}_{k,i+1}$, $0 \leq i < n$, in

$M_n^*(k)$ (and in $\widetilde{W}^{(1)}$). Analogously, denote by $\tilde{u}_{m,i}$ the unique common neighbor of $\tilde{z}_{m,i}$ and $\tilde{z}_{m,i+1}$, $0 \leq i < n$. The paths $M_n^*(k)$ and $M_n^*(m)$ belong to the principal filters $\mathcal{F}_{\tilde{\delta}^*}(\tilde{z}', \widetilde{W}^{(1)})$ and $\mathcal{F}_{\tilde{\delta}^*}(\tilde{z}'', \widetilde{W}^{(1)})$, respectively.

By Proposition 10, the words $M_n(k)$ and $M_n(m)$ are different. Let f be an isomorphism between the filters $\mathcal{F}_{\tilde{\delta}^*}(\tilde{z}_{k,0}, \widetilde{W}^{(1)})$ and $\mathcal{F}_{\tilde{\delta}^*}(\tilde{z}_{m,0}, \widetilde{W}^{(1)})$. Since the words $M_n(k)$ and $M_n(m)$ are different, from the choice of the lengths of tips in the complexes W and $\widetilde{W}$ it follows that f cannot map the path $M_n^*(k)$ to the path $M_n^*(m)$ by a vertical translation, i.e., there exists an index $0 \leq j < n$ such that $f(\tilde{z}_{k,j+1}) \neq \tilde{z}_{m,j+1}$; let i be the smallest such index. Set $\tilde{z} := f(\tilde{z}_{k,i+1})$ and $\tilde{u} := f(\tilde{u}_{k,i})$. Since f preserves the types of vertices, $\tilde{z}$ is a 0-vertex and $\tilde{u}$ is a 1-vertex. Since f maps a convex path $M_n^*(k)$ to a convex path, $\tilde{u}$ is the unique common neighbor of $\tilde{z}_{m,i}$ and $\tilde{z}$. Since each 1-vertex is the barycenter of a unique edge of $\widetilde{\mathbf{X}}$ and $\tilde{z} \neq \tilde{z}_{m,i+1}$, we deduce that $\tilde{u} \neq \tilde{u}_{m,i}$. The edge $\tilde{z}_{k,i}\tilde{u}_{k,i}$ is directed from $\tilde{z}_{k,i}$ to $\tilde{u}_{k,i}$. Analogously the edges $\tilde{z}_{m,i}\tilde{u}_{m,i}$ and $\tilde{z}_{m,i}\tilde{u}$ are directed from $\tilde{z}_{m,i}$ to $\tilde{u}_{m,i}$ and $\tilde{u}$, respectively. Since $\tilde{z}_{k,i}\tilde{u}_{k,i}$ and $\tilde{z}_{m,i}\tilde{u}_{m,i}$ are parallel edges, they define the same event and therefore $\lambda(\tilde{z}_{k,i}\tilde{u}_{k,i}) = \lambda(\tilde{z}_{m,i}\tilde{u}_{m,i})$. On the other hand, since f maps the edge $\tilde{z}_{k,i}\tilde{u}_{k,i}$ to the edge $\tilde{z}_{m,i}\tilde{u}$ and the map f preserves the labels, we have $\lambda(\tilde{z}_{k,i}\tilde{u}_{k,i}) = \lambda(\tilde{z}_{m,i}\tilde{u})$. As a result, $\tilde{z}_{m,i}$ has two outgoing edges, $\tilde{z}_{m,i}\tilde{u}_{m,i}$ and $\tilde{z}_{m,i}\tilde{u}$, having the same label, contrary to the assumption that λ is a nice labeling. This contradiction shows that $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\delta}^*})$ does not admit a regular nice labeling. This concludes the proof of the theorem. ◀

▶ **Proposition 13.** *Wise's event domain $(\widetilde{W}_{\tilde{v}}, \prec_{\tilde{\delta}^*})$ has bounded $\natural$-cliques.*

6 Conclusions and open questions

In this paper, we presented an example of a regular event domain $\widetilde{W}_{\tilde{v}}$ with bounded degree and bounded $\natural$-cliques which does not admit a regular nice labeling. Consequently, the domain $\widetilde{W}_{\tilde{v}}$ is not recognizable and the prime event structure whose domain is $\widetilde{W}_{\tilde{v}}$ is not a regular trace event structure. This provides a counterexample to Conjecture 1 of Thiagarajan and Conjecture 2 of Badouel, Darondeau, and Raoult.

The event domain $\widetilde{W}_{\tilde{v}}$ is a 2-dimensional CAT(0) cube complex. The proof that our example $\widetilde{W}_{\tilde{v}}$ does not admit a regular nice labeling strongly uses the fact that the universal cover $\widetilde{\mathbf{X}}$ of Wise's complex $\mathbf{X}$ [32] contains a particular aperiodic tiled plane (that is called *antitorus* by Wise). We think that the relationship between the existence of aperiodic planes and nonexistence of regular labelings is more general. As observed by Kari and Papasoglu [14], any *4-way deterministic* tile-set gives rise to a CAT(0) VH-complex that is the universal cover of a finite NPC complex. In [14], they presented a 4-way deterministic aperiodic tile-set T_{KP}, i.e., all tilings of $\mathbb{R}^2$ using tiles from T_{KP} are aperiodic. Based on this result, Lukkarila [15] proved that for 4-way deterministic tile-sets the tiling problem is undecidable. We conjecture that one can use this result to show that *deciding if a regular event domain admits a regular nice labeling is undecidable.* As a first step in this direction, our proof can be adapted to show that any 4-way deterministic aperiodic tile-set T (in particular, T_{KP}) also provides a counterexample to Conjectures 1 and 2.

Even if Conjecture 1 does not hold in general, it would be interesting to exhibit classes of event structures for which this conjecture is true. Badouel et al. [2] showed that both conjectures hold for context-free domains. Context-free graphs are particular Gromov-hyperbolic graphs. An interesting challenge would be to *establish Conjecture 1 for Gromov-hyperbolic domains.* A positive answer would settle the previous undecidability question. As we noticed already, Conjecture 1 was positively solved by Nielsen and Thiagarajan [20] for conflict-free event structures. A possible way to extend their result is to *consider this* *conjecture for confusion-free domains* introduced by Nielsen et al. [18]. From geometric and combinatorial points of view, context-free and conflict-free domains have quite different structural properties and give rise to different kinds of CAT(0) cube complexes. For instance, in context-free domains (and more generally, hyperbolic domains), isometric square-grids are bounded while conflict-free domains can contain arbitrarily large square-grids.

Acknowledgements. We are grateful to P. S. Thiagarajan for some email exchanges on Conjecture 1 and paper [20] and to our colleague R. Morin for several useful discussions.

References

1. F. Ardila, M. Owen, and S. Sullivant. Geodesics in CAT(0) cubical complexes. *Adv. Appl. Math.*, 48(1):142–163, 2012.
2. E. Badouel, Ph. Darondeau, and J.-C. Raoult. Context-free event domains are recognizable. *Inf. Comput.*, 149(2):134–172, 1999.
3. H.-J. Bandelt and V. Chepoi. Metric graph theory and geometry: a survey. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, volume 453 of *Contemp. Math.*, pages 49–86. AMS, Providence, RI, 2008.
4. H.-J. Bandelt and J. Hedlíková. Median algebras. *Discr. Math.*, 45(1):1–30, 1983.
5. J.-P. Barthélemy and J. Constantin. Median graphs, parallelism and posets. *Discr. Math.*, 111(1-3):49–63, 1993.
6. M. A. Bednarczyk. *Categories of Asynchronous Systems*. PhD thesis, University of Sussex, 1987.
7. M. R. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.
8. J. Chalopin and V. Chepoi. A counterexample to Thiagarajan's conjecture. *arXiv preprint*, 2016. URL: `https://arxiv.org/abs/1605.08288`, `arXiv:1605.08288`.
9. V. Chepoi. Graphs of some CAT(0) complexes. *Adv. Appl. Math.*, 24(2):125–179, 2000.
10. V. Chepoi. Nice labeling problem for event structures: a counterexample. *SIAM J. Comput.*, 41(4):715–727, 2012.
11. D.Ž. Djoković. Distance-preserving subgraphs of hypercubes. *J. Comb. Theory, Ser. B*, 14(3):263–267, 1973.
12. M. Gromov. Hyperbolic groups. In S. M. Gersten, editor, *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
13. A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge,, 2002.
14. J. Kari and P. Papasoglu. Deterministic aperiodic tile sets. *GAFA, Geom. Funct. Anal.*, 9(2):353–369, 1999.
15. V. Lukkarila. The 4-way deterministic tiling problem is undecidable. *Theor. Comput. Sci.*, 410(16):1516–1533, 2009.
16. R. Morin. Concurrent automata vs. asynchronous systems. In *MFCS 2005*, volume 3618 of *LNCS*, pages 686–698. Springer, 2005.
17. D. E. Muller and P. E. Schupp. The theory of ends, pushdown automata, and second-order logic. *Theor. Comput. Sci.*, 37:51–75, 1985.
18. M. Nielsen, G. D. Plotkin, and G. Winskel. Petri nets, event structures and domains, part I. *Theor. Comput. Sci.*, 13:85–108, 1981.
19. M. Nielsen, G. Rozenberg, and P. S. Thiagarajan. Transition systems, event structures and unfoldings. *Inf. Comput.*, 118(2):191–207, 1995.
20. M. Nielsen and P. S. Thiagarajan. Regular event structures and finite Petri nets: the conflict-free case. In *ICATPN 2002*, volume 2360 of *LNCS*, pages 335–351. Springer, 2002.

21 M. Roller. Poc sets, median algebras and group actions. Technical report, Univ. of Southampton, 1998.

22 B. Rozoy and P. S. Thiagarajan. Event structures and trace monoids. *Theor. Comput. Sci.*, 91(2):285–313, 1991.

23 M. Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc.*, s3-71(2):585–617, 1995.

24 M. Sageev. CAT(0) cube complexes and groups. In M. Bestvina, M. Sageev, and K. Vogtmann, editors, *Geometric Group Theory*, volume 21 of *IAS/Park City Mathematics Series*, pages 6–53. AMS, Institute for Advanced Study, 2012.

25 E. W. Stark. Connections between a concrete and an abstract model of concurrent systems. In *Mathematical Foundations of Programming Semantics 1989*, volume 442 of *LNCS*, pages 53–79. Springer, 1989.

26 P. S. Thiagarajan. Regular trace event structures. Technical Report BRICS RS-96-32, Computer Science Department, Aarhus University, Aarhus, Denmark, 1996.

27 P. S. Thiagarajan. Regular event structures and finite Petri nets: A conjecture. In *Formal and Natural Computing*, volume 2300 of *LNCS*, pages 244–256. Springer, 2002.

28 P. M. Winkler. Isometric embedding in products of complete graphs. *Discr. Appl. Math.*, 7(2):221–225, 1984.

29 G. Winskel. *Events in computation.* PhD thesis, Edinburgh Univ., 1980.

30 G. Winskel and M. Nielsen. Models for concurrency. In S. Abramsky, Dov M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science (Vol. 4)*, pages 1–148. Oxford University Press, 1995.

31 D. T. Wise. *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups.* PhD thesis, Princeton University, 1996.

32 D. T. Wise. Complete square complexes. *Comment. Math. Helv*, 82(4):683–724, 2007.

33 D. T. Wise. *From Riches to Raags: 3-manifolds, Right-angled Artin Groups, and Cubical Geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics.* AMS, Providence, RI, 2012.