Ham Sandwich is Equivalent to Borsuk-Ulam

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Abstract

The Borsuk-Ulam theorem is a fundamental result in algebraic topology, with applications to various areas of Mathematics. A classical application of the Borsuk-Ulam theorem is the Ham Sandwich theorem: The volumes of any n compact sets in $\mathbb{R}^n$ can always be simultaneously bisected by an $(n-1)$-dimensional hyperplane.

In this paper, we demonstrate the equivalence between the Borsuk-Ulam theorem and the Ham Sandwich theorem. The main technical result we show towards establishing the equivalence is the following: For every odd polynomial restricted to the hypersphere $f : S^n \to \mathbb{R}$, there exists a compact set $A \subseteq \mathbb{R}^{n+1}$, such that for every $x \in S^n$ we have $f(x) = \mathrm{vol}(A \cap H^+) - \mathrm{vol}(A \cap H^-)$, where H is the oriented hyperplane containing the origin with $\vec{x}$ as the normal. A noteworthy aspect of the proof of the above result is the use of hyperspherical harmonics.

Finally, using the above result we prove that there exist constants $n_0, \varepsilon_0 > 0$ such that for every $n \geq n_0$ and $\varepsilon \leq \varepsilon_0/\sqrt{48n}$, any query algorithm to find an ε-bisecting $(n-1)$-dimensional hyperplane of n compact sets in $[-n^{4.51}, n^{4.51}]^n$, even with success probability $2^{-\Omega(n)}$, requires $2^{\Omega(n)}$ queries.

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1 Introduction

The Borsuk-Ulam theorem states that every continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point [6]. This result has countless applications in Mathematics [21]. In particular it implies the Brouwer's Fixed Point Theorem [7, 16] which is the basis of several important results in Economics [5], for example Nash's theorem [23]. Soon after the Borsuk-Ulam theorem was established, the Ham Sandwich theorem was proven using it [28, 29]. The Ham Sandwich theorem states that the volumes of any n compact sets in $\mathbb{R}^n$ can always be simultaneously bisected by an $(n-1)$-dimensional hyperplane. However, as far as we know, there is no result in previous literature establishing the equivalence of the Borsuk-Ulam theorem and the Ham Sandwich theorem.

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From a computational perspective, the computation of Brouwer fixed points has been studied extensively in various models of computations such as the Time/Computational complexity model [24, 12, 10, 9, 26, 27], the Query complexity model [18, 8, 11, 3, 27], and the Communication model [25, 4]. From these results one may obtain a reasonable understanding of the problem of computing equally valued antipodal points in a Borsuk-Ulam function by utilizing the constructive reduction from the Brouwer fixed point computation problem [30, 32]. However, computational aspects of the Ham Sandwich theorem have been poorly understood. In particular, no hardness result or non-trivial lower bounds in *any* model of computation are known in literature for the Ham Sandwich problem.

In this paper, we prove that the Borsuk-Ulam theorem and the Ham Sandwich theorem are indeed equivalent! Moreover, we use this equivalence to prove a query complexity lower bound on the Ham Sandwich problem.

1.1 Our Results

Our main result is a reduction from the Borsuk-Ulam theorem to the Ham Sandwich theorem. A key result in establishing the above reduction is that of establishing the equivalence between the two theorems for polynomial functions:

► **Proposition 1.** *For every polynomial $f : S^n \to \mathbb{R}^n$ restricted to the hypersphere, there exist n compact sets $A_1, \ldots, A_n \subseteq \mathbb{R}^{n+1}$, such that for every $x \in S^n$ and $i \in [n]$, we have the following:*

$$f_i(x) - f_i(-x) = vol(A_i \cap H^+) - vol(A_i \cap H^-),$$

where $f_i(x)$ is the projection of $f(x)$ to the i^{th} coordinate, and H is the oriented hyperplane containing the origin with $\vec{x}$ as the normal.

After establishing the above result, we use the Stone-Weierstrass theorem to note that any continuous function can be arbitrarily well approximated by polynomial functions, and prove the Borsuk-Ulam theorem for all continuous functions.

Next, we consider the Ham Sandwich problem in the query model: the input to the problem is n compact sets $A_1, \ldots, A_n \subseteq \left[-n^k, n^k\right]^n$, for some constant $k > 0$, and each query is an oriented hyperplane H and the answer is $\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)$, for all $i \in [n]$. The goal is to find a $(n-1)-$dimensional hyperplane H such that each set is ε-bisected by H, i.e., for all $i \in [n]$, we have $|\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)| \leq \varepsilon$. We show the following lower bound for the Ham Sandwich problem:

► **Theorem 2.** *There exist constants $n_0, \varepsilon_0 > 0$ such that for any $n \geq n_0$, $\varepsilon \leq \varepsilon_0/\sqrt{48n}$, $p = 2^{-\Omega(n)}$, and $k \geq 4.51$ the following holds: any query algorithm to find an ε-bisecting $(n-1)$-dimensional hyperplane of n compact sets in $\left[-n^k, n^k\right]^n$, even with success probability p, requires $2^{\Omega(n)}$ queries.*

By assuming a notion of Lipschitz continuity, we can show that the number of queries needed to compute an $\varepsilon-$bisecting hyperplane is $2^{O(n \log n)}$ by querying translations of hyperplanes over $\left[-n^k, n^k\right]^n$ whose normals form an $O(\varepsilon)$-net over S^n. Thus, the above lower bound is tight up to logarithmic multiplicative factor in the exponent. Furthermore, we remark here that Theorem 2 is the first nontrivial lower bound for the Ham Sandwich problem in *any* model of computation.

1.2 Our Techniques and Proof Overview

We provide below a proof-sketch of the reduction from the Borsuk-Ulam theorem to the Ham Sandwich theorem. The basic idea is to find for a given continuous odd function on S^n taking values in $\mathbb{R}$, a compact measurable set in $\mathbb{R}^{n+1}$, such that the given function is the difference of volumes of the set on the positive and negative side of an oriented hyperplane through the origin. This makes sense as the oriented hyperplanes through the origin are parametrised by S^n on which its positive unit normal takes values, so we actually get a continuous odd function on S^n. Then an oriented hyperplane bisects the set if and only if the given odd function vanishes at the point on S^n corresponding to the positive unit normal of the hyperplane. In particular, if we have have an odd continuous function from S^n to $\mathbb{R}^n$, we can make the above argument for every component. Then an oriented hyperplane bisects the sets if and only if the given odd function taking values in $\mathbb{R}^n$ vanishes at the point on S^n corresponding to the positive unit normal of the hyperplane.

A compact measurable set may be constructed by starting with a solid $(n+1)$-dimensional ball of unit radius centred at the origin and then radially scaling it by a continuous function on S^n taking values in $\mathbb{R}$ that is positive everywhere. Then the volume contained in a solid angle sector would be given by integrating an expression proportional to the $(n+1)$-th power of the scaling function over the region on S^n corresponding to the solid angle sector. Thus the difference of volumes on either side of a hyperplane is related to the $(n+1)$-th power of the scaling function by a linear integral transform. It turns out that this linear map becomes diagonal in the basis of hyperspherical harmonics and in order to invert this integral transform we work in this basis.

Since the basis is infinite-dimensional, there may be issues with convergence. We tackle this by first constructing the inverse transform for functions that are restrictions of polynomial functions on $\mathbb{R}^{n+1}$ to S^n (see Proposition 1), since in these cases, only finitely many elements in the basis suffice. This means that the reduction of Borsuk-Ulam theorem to the Ham Sandwich theorem holds for all functions that are restrictions of polynomial functions on $\mathbb{R}^{n+1}$ to S^n. And then we use the Stone-Weierstrass theorem, which states that any continuous function on $[-1,1]^{n+1}$ may be uniformly approximated using polynomial functions, to extend the reduction to all continuous functions on S^n.

In order to prove Theorem 2, we start from the randomized query complexity lower bound recently obtained by Rubinstein [27] (building on the works of Hirsch et al. [18] and Babichenko [3]) for the computation of *approximate* fixed points in a Brouwer function in the Euclidean norm. We then show that computing approximately equal-valued antipodal points in the query model is as hard as computing approximate fixed points in a Brouwer function by using Su's constructive proof of the Brouwer fixed point theorem from the Borsuk-Ulam theorem [30]. Finally, we use multivariate Bernstein polynomials to approximate the Borsuk-Ulam function and construct an input of the Ham Sandwich problem from Proposition 1 to obtain the randomized query complexity lower bound for the Ham-Sandwich problem.

1.3 Related Works

Papadimitriou considered the Ham Sandwich problem in the computational complexity model: given $2n^2$ points in general position in $\mathbb{R}^n$, separated into n groups with $2n$ points each, find a hyperplane which divides all groups in half. Papadimitriou showed that this search problem is in the complexity class **PPA** [24, 1]. However, no hardness result is known for this problem. On the other hand, there are many algorithms proposed in literature to solve this problem [13, 20, 33]. In particular, the best known algorithm for finding a hyperplane simultaneously bisecting n point-sets $A_1, \dots, A_n$ in $\mathbb{R}^d$, is $O\left(|A|^{d-1}\right)$ [20], where $A = \bigcup_{i\in[n]} A_i$.

A variant of the Ham-Sandwich problem was considered by Knauer et al.: Given $d+1$ point-sets in $\mathbb{R}^d$, is there a hyperplane which simultaneously bisects all the point-sets? They showed that this decision problem is **NP**-hard and **W**[1]-hard (with respect to d) [19], even when one of the point-sets is just a single point. However, it is not easy to construct a meaningful decision version of the Ham Sandwich problem because of its totality.

1.4 Organization of the Paper

This paper is organized as follows. In Section 2, we introduce the notations used in the rest of the paper, provide some key results about hyperspherical harmonics, and formally describe the query model of computation. In Section 3, we provide the complete reduction from the Borsuk-Ulam theorem to the Ham Sandwich theorem. In Section 4, we prove the randomized query complexity lower bound for the Ham Sandwich problem. Finally, in Section 5, we conclude by highlighting some open directions for future research.

2 Preliminaries

We formally state the two theorems of interest to this paper.

▶ **Theorem 3** (Borsuk-Ulam Theorem, [6]). *Let S^n denote the set of all points on the unit n-dimensional sphere. If $n \geq 0$ then for any continuous mapping $f : S^n \to \mathbb{R}^n$ there is a point $x \in S^n$ for which $f(x) = f(-x)$.*

▶ **Theorem 4** (Ham Sandwich Theorem, [28, 29]). *Given n compact sets in $\mathbb{R}^n$ there is a $(n-1)$-dimensional hyperplane which bisects each set into two sets of equal measure.*

Below, we list some notations and standard definitions that are used through out the paper.

2.1 Notations

The L^p norm of a vector $x \in \mathbb{R}^n$ is defined in the standard way as follows:

$$\|x\|_p = \left(\sum_{i\in[n]} |x_i|^p\right)^{1/p}.$$

Moreover, we define $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$, $S^n_\infty = \{x \in \mathbb{R}^{n+1} \mid \|x\|_\infty = 1\}$, and $B^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$.

A hyperplane in $\mathbb{R}^{n+1}$ is the set of solutions of an equation of the form

$$a_0 + \sum_{i=1}^{n+1} a_i x_i = 0.$$

The unit normals of the hyperplane are the vectors $\pm(a_1, a_2, \dots, a_{n+1})$. A choice of one of the two possible unit normals is said to be an orientation on the hyperplane which is referred to as being oriented and the chosen unit normal as the positive unit normal (the other one is said to be the negative unit normal).

The volume of a compact set, assumed to be measurable, is simply its measure.

2.2 Hyperspherical Harmonics

We gather together some definitions and results we need regarding hyperspherical harmonics.

▶ **Definition 5.** A polynomial $H_\ell(x_1, x_2, \ldots, x_{n+1})$ is homogeneous of degree ℓ in the $n+1$ variables $x_1, x_2, ..., x_{n+1}$ provided $H_\ell(tx_1, tx_2, ..., tx_{n+1}) = t^\ell H_\ell(x_1, x_2, ..., x_{n+1})$. The Laplace operator in $\mathbb{R}^{n+1}$ is given by $\Delta_{n+1} := \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}$. $H_\ell(x_1, x_2, \ldots, x_{n+1})$ is called harmonic if $\Delta_{n+1} H_\ell(x_1, x_2, \ldots, x_{n+1}) = 0$. A hyperspherical harmonic of degree ℓ, denoted $Y_\ell^{(n+1)}(\xi)$, is a harmonic homogeneous polynomial of degree ℓ in $n+1$ variables restricted to S^n.

▶ **Claim 6.** *The dimension of the vector space of hyperspherical harmonics of degree ℓ on S^n is $M(n, \ell)$ where,*

$$M(n,\ell) = \begin{cases} 1 & \text{if } \ell = 0, \\ \frac{2\ell+n-1}{\ell}\binom{\ell+n-2}{\ell-1} & \text{if } \ell > 0. \end{cases}$$

Proof. See Theorem 4.4 in [14]. ◀

▶ **Definition 7.** Let V_{n+1} be the set of all $n+1$-variate polynomials over $\mathbb{R}$ restricted to S^n.

▶ **Claim 8.** *V_{n+1} is an inner product space, with addition and scaling defined for any two polynomials $f, g : S^n \to \mathbb{R}$ which are restricted to the n-sphere as follows:*

$$(f+g)(x) = f(x) + g(x),$$
$$\forall \alpha \in \mathbb{R},\ (\alpha \cdot f)(x) = \alpha \cdot f(x),$$
$$\langle f, g\rangle = \int_{x \in S^n} f(x) \cdot g(x)\ dx.$$

Proof. A linear combination of two polynomials is another polynomial. Furthermore, from the definition of the inner product, it is clear that $\langle f, g\rangle$ is symmetric under interchange of f and g and bilinear. To prove nondegenerateness, we shall show that $\langle f, f\rangle > 0$ whenever f is not identically zero. Assume that f is not identically zero. Then there must be point $x' \in S^n$ such that $f(x) \neq 0$. Because f is continuous there must be an open neighbourhood around this point such that f^2 is positive at all points in the neighbourhood. The integral of f^2 over this neighbourhood is therefore positive and since the integral of $f^2 \geq 0$ over the rest of the sphere has to be at least zero, it follows $\langle f, f\rangle > 0$. This completes the proof. ◀

▶ **Definition 9.** For $n \geq 2$ and each degree ℓ, the set $\left\{ Y_{\ell,m}^{(n+1)} \middle| m \in [M(n,\ell)] \right\}$ is a fixed orthonormal basis for the vector space of hyperspherical harmonics of degree ℓ on S^n.

▶ **Claim 10.** *For every $n \geq 2$, and $d \in \mathbb{Z}_{\geq 0}$, the set $\left\{ Y_{\ell,m}^{(n+1)} \middle| \ell \in \mathbb{Z}_{\geq 0}, \ell \leq d, m \in [M(n,\ell)] \right\}$ is an orthonormal set spanning all $f \in V_{n+1}$ of total degree d.*

Proof. Since every polynomial can be written as a finite sum of homogeneous polynomials of various degrees, it suffices to prove the above for the case where f is the restriction of a homogeneous polynomial $\tilde{f}$ of degree d to S^n. By Theorem 2.18 in [2], we note that there is a unique decomposition as follows,

$$\tilde{f}(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{\lfloor d/2 \rfloor} H_{d-2i}(x_1, \ldots, x_{n+1}) \left(\sum_{j=1}^{n+1} x_j^2 \right)^i,$$

where H_ℓ is a harmonic homogeneous polynomial of degree ℓ. Restricting to S^n gives the following:

$$f = \sum_{i=0}^{\lfloor d/2 \rfloor} H_{d-2i}|_{S^n}.$$

The restriction $H_\ell|_{S^n}$ is a hyperspherical harmonic of degree ℓ and so is a (finite) linear combination of the functions $Y_{\ell,m}^{(n+1)}$ where m varies over $[M(n,\ell)]$. It follows that f is a (finite) linear combination of hyperspherical harmonics of degree at most d. Finally, orthonormality follows from Definition 9 above and Theorem 4.6 in [14]. ◀

▶ **Lemma 11.** *For every $n \geq 2$, and for any odd function f in V_{n+1}, let it be written as follows:*

$$f = \sum_{\ell \in \mathbb{Z}_{\geq 0}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot Y_{\ell,m}^{(n+1)}.$$

Then, for every even integer ℓ, we have that $\alpha_{\ell,m} = 0$.

Proof. Since f is assumed to be odd, we have that $f(x) + f(-x) = 0$ for all $x \in S^n$. Since the sum in the hyperspherical harmonic decomposition of f is finite, we may rearrange the terms to have

$$\begin{aligned} 0 &= \sum_{\ell \in \mathbb{Z}_{\geq 0}^{\text{even}}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot (Y_{\ell,m}^{(n+1)}(x) + Y_{\ell,m}^{(n+1)}(-x)) \\ &\quad + \sum_{\ell \in \mathbb{Z}_{\geq 0}^{\text{odd}}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot (Y_{\ell,m}^{(n+1)}(x) + Y_{\ell,m}^{(n+1)}(-x)) \\ &= 2 \sum_{\ell \in \mathbb{Z}_{\geq 0}^{\text{even}}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot Y_{\ell,m}^{(n+1)}(x). \end{aligned}$$

Now, for any $\ell' \in \mathbb{Z}_{\geq 0}^{\text{even}}$, we may multiply the above equation by $Y_{\ell',m}^{(n+1)}(x)$ on both sides and integrate over S^n so that, by virtue of Claim 10 we have $0 = 2\alpha_{\ell',m}$. Since $\ell' \in \mathbb{Z}_{\geq 0}^{\text{even}}$ was arbitrary , the result to be proved follows. ◀

▶ **Definition 12.** Let the sign function sgn on the interval $[-1,1]$ be defined as follows.

$$\forall \xi \in [-1,1],\ \mathsf{sgn}(\xi) = \begin{cases} -1 & \text{if } \xi < 0, \\ 0 & \text{if } \xi = 0, \\ 1 & \text{if } \xi > 0. \end{cases}$$

▶ **Definition 13.** For every $\ell \in \mathbb{Z}_{\geq 0}$, $n \geq 2$, and $\xi \in [-1,1]$, $P_\ell^{(n+1)}(\xi)$ is the ℓ^{th}-Gegenbauer polynomial in $n+1$ dimensions defined as follows:

$$P_\ell^{(n+1)}(\xi) = \frac{(-1)^\ell}{2^\ell \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} (1-\xi^2)^{(2-n)/2} \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^\ell (1-\xi^2)^{\ell+(n-2)/2}.$$

▶ **Theorem 14** (Funk-Hecke theorem, [15, 17]). *Let $x \in S^n$, $f : [-1,1] \to \mathbb{R}$ a bounded measurable function, and $Y_\ell^{(n+1)}$ a hyperspherical harmonic polynomial of degree ℓ. Then,*

$$\int_{y \in S^n} f(\langle x, y\rangle) Y_\ell^{(n+1)}(y)\ \mathrm{d}y = s_{n-1} Y_\ell^{(n+1)}(x) \cdot \int_{-1}^{1} f(t) P_\ell^{(n+1)}(t)(1-t)^{n/2-1}\ \mathrm{d}t,$$

where s_{n-1} is the volume of the $(n-1)$-sphere, i.e., S^{n-1}.

Proof. See Theorem 4.24 in [14]. ◀

▶ **Lemma 15.** *Let $x \in S^n$, $f : [-1,1] \to \mathbb{R}$ a bounded measurable function, and $Y_\ell^{(n+1)}$ a hyperspherical harmonic polynomial of odd degree ℓ. Then,*

$$Y_\ell^{(n+1)}(x) = \frac{n}{2 s_{n-1}} \cdot \frac{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i + n + 1)}{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i)} \int_{y \in S^n} \mathsf{sgn}(\langle x, y\rangle) \cdot Y_\ell^{(n+1)}(y)\ \mathrm{d}y.$$

Proof. Plugging in the sign function in Theorem 14, gives us:

$$\int_{y \in S^n} \mathsf{sgn}(\langle x, y\rangle) Y_\ell^{(n+1)}(y)\ \mathrm{d}y = s_{n-1} Y_\ell^{(n+1)}(x) \cdot \int_{-1}^{1} \mathsf{sgn}(t) P_\ell^{(n+1)}(t)(1-t)^{n/2-1}\ \mathrm{d}t.$$

So it remains to evaluate the below when ℓ is odd:

$$\int_{-1}^{1} \mathsf{sgn}(t) P_\ell^{(n+1)}(t)(1-t)^{n/2-1}\ \mathrm{d}t.$$

We plug in Definition 13 into the above

$$\int_{-1}^{1} \frac{\mathsf{sgn}(t)(-1)^\ell}{2^\ell \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell (1-t^2)^{\ell + (n-2)/2} \mathrm{d}t.$$

When ℓ is odd, the function under the integral is even, so we have:

$$\begin{aligned}
&\int_{-1}^{1} \frac{\mathsf{sgn}(t)(-1)^\ell}{2^\ell \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell (1-t^2)^{\ell + (n-2)/2} \mathrm{d}t \\
&= 2 \int_{0}^{1} \frac{\mathsf{sgn}(t)(-1)^\ell}{2^\ell \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell (1-t^2)^{\ell + (n-2)/2} \mathrm{d}t \\
&= \int_{0}^{1} \frac{(-1)^\ell}{2^{\ell-1} \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell (1-t^2)^{\ell + (n-2)/2} \mathrm{d}t.
\end{aligned}$$

The term under the integral is a total derivative, so the integral may be simplified to

$$\left[\frac{(-1)^\ell}{2^{\ell-1} \cdot \prod_{i=0}^{\ell-1} (\ell + (n-2)/2 - i)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell-1} (1-t^2)^{\ell + (n-2)/2} \right]_{t=0}^{t=1}.$$

Note that $\ell + (n-2)/2 = (\ell-1) + (n+2-2)/2$, so we may again use Definition 13 to write the above as

$$\left[-\frac{1}{n/2} \cdot (1-t^2)^{n/2} \cdot P_{\ell-1}^{(n+3)}(t)\right]_{t=0}^{t=1}.$$

When $t = 1$, the expression inside the square brackets vanishes. So all we are left with is $(2/n) \cdot P_{\ell-1}^{(n+3)}(0)$. The recurrence relation from Proposition 4.21 in [14] tells us that for all $\ell \geq 1$ we have

$$(\ell - 1 + n) \cdot P_{\ell-1}^{(n+3)}(0) + (\ell - 2) \cdot P_{\ell-3}^{(n+3)}(0) = 0.$$

This, along with the observation that $P_0^{(n+3)}(0) = 1$ may be used to determine $P_{\ell-1}^{(n+3)}(0)$ to be

$$P_{\ell-1}^{(n+3)}(0) = \frac{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i)}{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i + n + 1)}.$$

This completes the proof. ◀

2.3 Query Model

In this paper, we refer to the query model as described in [3]: every problem is specified by the allowed possible inputs, the desired outputs, and the queries which are specified types of questions that can be asked and by the answers that are provided. A query algorithm, is a procedure that asks queries in an adaptive manner and generates an output for every input. For this paper, a highly relevant remark is that there is no computational constraints on the way the query algorithm generates the next query or the output, given the previous answers.

For randomized query algorithms, errors are allowed in the output. To be precise, we require that for all inputs, the answer is correct only with probability $p < 1$. The randomized query complexity of a problem is the minimal number t such that given an input there exists a randomized query algorithm that makes at most t queries and outputs the correct answer with probability p. We denote the randomized query complexity of a problem Π by $\mathbf{QC}_p(\Pi)$. As noted by Babichenko [3] this measure of randomized query complexity is closely related to another measure: the expected number of queries for outputting the correct answer with probability p. Therefore, any lower bounds on $\mathbf{QC}_p(\Pi)$ can be easily translated to lower bounds on the expected number of queries.

3 Equivalence of Ham Sandwich and Borsuk-Ulam Theorems

In this section, we give the reduction from the Borsuk-Ulam theorem to the Ham Sandwich theorem. First, we show the reduction for polynomials restricted to the hypersphere.

▶ **Proposition 1.** *For every polynomial $f : S^n \to \mathbb{R}^n$ restricted to the hypersphere, there exist n compact sets $A_1, \ldots, A_n \subseteq \mathbb{R}^{n+1}$, such that for every $x \in S^n$ and $i \in [n]$, we have the following:*

$$f_i(x) - f_i(-x) = vol(A_i \cap H^+) - vol(A_i \cap H^-),$$

where $f_i(x)$ is the projection of $f(x)$ to the i^{th} coordinate, and H is the oriented hyperplane containing the origin with $\vec{x}$ as the normal.

Proof. We consider n projection functions of f: $f_1, \ldots, f_n : S^n \to \mathbb{R}$. Let d_i be the total degree of f_i. For every $i \in [n]$ we define $g_i(x) = f_i(x) - f_i(-x)$. Note that the g_is are odd functions. We define below n new functions $h_1, \ldots, h_n : S^n \to \mathbb{R}$ from the g_is. For every $i \in [n]$, from h_i, we construct A_i as follows:

$$A_i = \left\{ k \cdot (h_i(x_1, ..., x_{n+1}) \cdot x_1, \ldots, h_i(x_1, ..., x_{n+1}) \cdot x_{n+1}) \,\middle|\, (x_1, ..., x_{n+1}) \in S^n, 0 \le k \le 1 \right\}.$$

Note that we can define the volume of A_i as follows:

$$\mathrm{vol}(A_i) = \int_{y \in S^n} (h_i(y))^{n+1}/(n+1) \; \mathrm{d}y. \tag{1}$$

We will now define the h_is. We fix $i \in [n]$. From Claim 10, we have that g_i can be written as a linear combination of the hyperspherical harmonics.

$$g_i = \sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot Y^{(n+1)}_{\ell,m}. \tag{2}$$

Note that in the above decomposition of g_i into hyperspherical harmonics, we have that only the odd spherical harmonics appear in the support (from Lemma 11). Next, we define a couple of constants (depending on ℓ and m). For every $\ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}$, where $\ell \le d_i$, we have,

$$\gamma_\ell = \frac{n}{2 s_{n-1}} \cdot \frac{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i + n + 1)}{\prod_{i=1}^{(\ell-1)/2} (\ell - 2i)},$$

$$\beta_{\ell,m} = \alpha_{\ell,m} \cdot \gamma_\ell \cdot (n+1), \tag{3}$$

where s_n is the volume of the n-sphere S^n. Let $\Gamma_i : S^n \to \mathbb{R}$ be a function defined as follows:

$$\Gamma_i = \sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \beta_{\ell,m} \cdot Y^{(n+1)}_{\ell,m}. \tag{4}$$

Note that Γ_i is well defined because f is a polynomial function, which implies g_i is a polynomial function. We have the following bound on Γ_i:

▶ **Claim 16.** *Let* $\psi_i = \max_{x \in S^n} |\Gamma_i(x)|$. *Then,*

$$\psi_i < (n+1)^{(n+7)/2} \cdot (d_i + 1)^{3/2} \cdot \left(1 + \frac{d_i}{n}\right)^n \cdot \max_{x \in S^n} |g_i(x)|.$$

Finally, we define h_i as follows:

$$h_i = \sqrt[n+1]{(\Gamma_i + \psi_i + 1)}, \tag{5}$$

This completes the construction of the n compact sets $A_1, \ldots, A_n$. Fix $i \in [n]$. Let H be some n-dimensional (oriented) hyperplane containing the origin and let x_H be the unit normal of H.

$$\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-) = \frac{1}{(n+1)} \cdot \int_{y \in S^n} \mathsf{sgn}(\langle x_H, y \rangle) \cdot (h_i(y))^{n+1} \; \mathrm{d}y \qquad \text{(From (1))}$$

$$= \frac{1}{(n+1)} \cdot \int_{y \in S^n} \mathsf{sgn}(\langle x_H, y\rangle) \cdot (\Gamma_i(y) + \psi_i + 1)\ \mathrm{d}y \qquad \text{(From (5))}$$

$$= \frac{1}{(n+1)} \cdot \int_{y \in S^n} \mathsf{sgn}(\langle x_H, y\rangle) \cdot \Gamma_i(y)\ \mathrm{d}y$$

$$= \frac{1}{(n+1)} \cdot \int_{y \in S^n} \mathsf{sgn}(\langle x_H, y\rangle) \cdot \left(\sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \beta_{\ell,m} \cdot Y^{(n+1)}_{\ell,m}(y) \right) \mathrm{d}y \qquad \text{(From (4))}$$

$$= \sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \beta_{\ell,m} \cdot \frac{1}{(n+1)} \cdot \int_{y \in S^n} \mathsf{sgn}(\langle x_H, y\rangle) \cdot Y^{(n+1)}_{\ell,m}(y)\ \mathrm{d}y$$

$$= \sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \beta_{\ell,m}/\gamma_\ell \cdot \frac{1}{(n+1)} \cdot Y^{(n+1)}_{\ell,m}(x_H) \qquad \text{(From Lemma 15)}$$

$$= \sum_{\substack{\ell \le d_i, \\ \ell \in \mathbb{Z}^{\mathrm{odd}}_{\ge 0}.}} \sum_{m=1}^{M(n,\ell)} \alpha_{\ell,m} \cdot Y^{(n+1)}_{\ell,m}(x_H) \qquad \text{(From (3))}$$

$$= g_i(x_H) = f_i(x_H) - f_i(-x_H) \qquad \text{(From (2))}$$

This completes the proof. ◀

Below we provide the complete reduction from the Borsuk-Ulam theorem to the Ham Sandwich theorem.

▶ **Theorem 17** (Theorem 3 restated for $n \ge 3$). *For every $n \ge 3$, if $f : S^n \to \mathbb{R}^n$ is continuous then there exists an $x \in S^n$ such that, $f(-x) = f(x)$.*

Proof. Given a continuous function $f_i : S^n \to \mathbb{R}$ we may use the Tietze Extension Theorem to extend it to a continuous function $\tilde{f}_i$ on $[-1,1]^{n+1}$ and then use the Stone-Weierstrass theorem to note that for any real $\varepsilon > 0$ we may find an $(n+1)$-variate polynomial function p_i such that $|\tilde{f}_i(x) - p_i(x)| < \varepsilon$ for all $x := (x_1, \ldots, x_{n+1}) \in [-1,1]^{n+1}$. In particular $|f(x) - p(x)| < \varepsilon$ for all $x \in S^n$.

By Proposition 1, we know that there exist n compact sets $A_1, \ldots, A_n \subseteq \mathbb{R}^{n+1}$, such that for every $x \in S^n$ and $i \in [n]$, we have $p_i(x) - p_i(-x) = \mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)$ where x is the unit normal of the oriented hyperplane H. We introduce another compact set A_{n+1} which is a closed ball centred at the origin, so that any hyperplane bisecting its volume has to necessarily pass through the origin.

By the Ham Sandwich Theorem, we know that there is an oriented hyperplane H' such that $\mathrm{vol}(A_i \cap H'^+) = \mathrm{vol}(A_i \cap H'^-)$ for all $i \in [n+1]$, which is to say, there is an oriented hyperplane H' through the origin such that $\mathrm{vol}(A_i \cap H'^+) = \mathrm{vol}(A_i \cap H'^-)$ for all $i \in [n]$. But this means that $p_i(x') - p_i(-x') = 0$ for all $i \in [n]$ (where x' is the unit normal of H'), and so $|f_i(x') - f_i(-x')| < 2\varepsilon$ for all $i \in [n]$. The map $x \mapsto |f_i(x) - f_i(-x)|$ where $x \in S^n$ is continuous and defined on a compact domain. So it must attain a minimum value somewhere, which is nonnegative. But we have already shown that $|f_i(x) - f_i(-x)| < 2\varepsilon$ for all $\varepsilon > 0$ and $i \in [n]$. It follows that the minimum value attained by this map is 0 (simultaneously for all $i \in [n]$). Let it be attained at $x'' \in S^n$. Then $f(x'') = f(-x'')$. This completes the proof. ◀

4 Query Complexity Lower Bounds

In this section, we show query complexity lower bounds on the Ham Sandwich problem, by using the connection established through Proposition 1.

4.1 Borsuk-Ulam problem in Query Model

The query complexity of computing an approximate fixed-point of a Brouwer function in the *max norm* was studied by Hirsch et al. [18] in the deterministic setting. Recently, Babichenko [3] extended their lower bounds to the randomized setting. Rubinstein [27], furthered this direction to the case of fixed point computation in the *Euclidean norm*. Before stating the result of Rubinstein, we formally define the approximate fixed point problem in the query model as follows:

$\mathsf{AFP}^{\mathbf{Q}}(n,\lambda,\varepsilon)$ Problem:
Input: λ-Lipschitz function $f:[-1,1]^n\to[-1,1]^n$.
Output: $x\in[-1,1]^n$ such that $\|f(x)-x\|_2^2\le\varepsilon\cdot n$.
Queries: Each query is a point $x\in[-1,1]^n$ and the answer is $f(x)$.

We have the following lower bound on $\mathbf{QC}_p(\mathsf{AFP}^{\mathbf{Q}}(n,\lambda,\varepsilon))$.

▶ **Theorem 18** (Rubinstein [27]). *There exist constants $\varepsilon_0,\lambda_0,n_0>0$ such that for any $n\ge n_0$, $\varepsilon\le\varepsilon_0$, and $\lambda\ge\lambda_0$, and for $p=2^{-\Omega(n)}$ the following holds:*

$$\mathbf{QC}_p(\mathsf{AFP}^{\mathbf{Q}}(n,\lambda,\varepsilon))=2^{\Omega(n)}.$$

Next, we define the approximate equally valued antipodal point problem in the query model as follows:

$\mathsf{AAP}^{\mathbf{Q}}(n,\lambda,\varepsilon)$ Problem:
Input: λ-Lipschitz function $f:\sqrt{n+1}\cdot S^n\to\sqrt{n}\cdot B^n$.
Output: $x\in B^n$ such that $\|f(x)-f(-x)\|_2^2\le\varepsilon\cdot n$.
Queries: Each query is a point $x\in\sqrt{n+1}\cdot S^n$ and the answer is $f(x)$.

We have the following lower bound on $\mathbf{QC}_p(\mathsf{AAP}^{\mathbf{Q}}(n,\lambda,\varepsilon))$.

▶ **Theorem 19.** *There exist constants $\varepsilon_0,n_0>0$ such that for any $n\ge n_0$, $\varepsilon\le\varepsilon_0/12n$, and $\lambda\ge5\sqrt{n}$, and for $p=2^{-\Omega(n)}$ the following holds:*

$$\mathbf{QC}_p(\mathsf{AAP}^{\mathbf{Q}}(n,\lambda,\varepsilon))=2^{\Omega(n)}.$$

Proof. We show that $\mathbf{QC}_p(\mathsf{AFP}^{\mathbf{Q}}(n,\lambda,\varepsilon))\le2\cdot\mathbf{QC}_p\left(\mathsf{AAP}^{\mathbf{Q}}\left(n,5\sqrt{n},\varepsilon^2/12n\right)\right)$ by using the construction of Su [30]. We start from a λ-Lipschitz continuous function $f:[-1,1]^n\to[-1,1]^n$ which is the input of $\mathsf{AFP}^{\mathbf{Q}}$ and have the following reduction to $\mathsf{AAP}^{\mathbf{Q}}$.

Adopting Su's Construction. Below, we describe the function $g_{\mathsf{Su}}:S^n_\infty\to[-3,3]^n$, constructed by Su to build an instance of Borsuk-Ulam by starting from an instance of Brouwer. Let P be the projection function on to the first n coordinates. We define g_{Su} as follows:

$$g_{\mathsf{Su}}(x_1,\dots,x_{n+1})=\begin{cases}P(x)-f(P(x)) & \text{if } x_{n+1}=1,\\ P(x)+f(P(-x)) & \text{if } x_{n+1}=-1,\\ P(x)+\frac{g_{\mathsf{Su}}(P(x),1)+g_{\mathsf{Su}}(P(x),-1)}{2} & \text{if } x_{n+1}=0,\\ x_{n+1}\cdot g_{\mathsf{Su}}(P(x),1)+(1-x_{n+1})\cdot g_{\mathsf{Su}}(P(x),0) & \text{if } 0\le x_{n+1}\le1,\\ -x_{n+1}\cdot g_{\mathsf{Su}}(P(x),-1)+(1+x_{n+1})\cdot g_{\mathsf{Su}}(P(x),0) & \text{if -1}\le x_{n+1}\le0.\end{cases}$$

Using the above function, we can construct $g : \sqrt{n+1} \cdot S^n \to [-1,1]^n$ from g_{Su} as follows:

$$\forall x \in \sqrt{n+1} \cdot S^n,\ g(x) = \frac{1}{3} \cdot g_{\mathsf{Su}}\left(\frac{x}{\|x\|_\infty}\right).$$

First, we observe that g is an odd function:

▶ **Claim 20.** *For every $x \in \sqrt{n+1} \cdot S^n$, we have $g(x) = -g(-x)$.*

Next, we compute the Lipschitz constant of g below.

▶ **Claim 21.** *g is $5\sqrt{n}$-Lipschitz continuous.*

Furthermore, we note that we can obtain approximate fixed points of f from approximate equally valued antipodal points of g in a natural way as follows.

▶ **Claim 22.** *Fix $x \in \sqrt{n+1} \cdot S^n$. If $\|g(x) - g(-x)\|_2^2 \le (\varepsilon^2/12n) \cdot n$ then,*

$$\left\| f\left(P\left(\frac{x}{\|x\|_\infty}\right)\right) - P\left(\frac{x}{\|x\|_\infty}\right)\right\|_2^2 \le \varepsilon \cdot n.$$

Finally, the proof follows by noting that in order to compute g at a point, we need to query f in at most two points. ◀

Note that there is an easy deterministic query algorithm for $\mathsf{AAP}^{\mathbf{Q}}(n, \lambda, \varepsilon)$ which solves it with $\left(1 + \frac{4\lambda}{\sqrt{\varepsilon}}\right)^{n+1}$ queries by building an $\frac{\sqrt{\varepsilon}}{2\lambda}$-net (Lemma 5.2 in [31]). In other words we have that $\mathbf{QC}_p\left(\mathsf{AAP}^{\mathbf{Q}}(n, \lambda, \varepsilon)\right) \le 2^{O(n \log n)}$. Thus, the above lower bound is tight up to logarithmic multiplicative factor in the exponent.

Finally, we define the problem of interest for this section below.

4.2 Ham Sandwich Problem in Query Model

The approximate bisecting hyperplane problem in the query model is defined as follows:

> $\mathsf{ABH}^{\mathbf{Q}}(n, k, \varepsilon)$ Problem:
> **Input**: n compact sets $A_1, \ldots, A_n \subseteq \left[-n^k, n^k\right]^n$.
> **Output**: $(n-1)$-dimensional hyperplane H such that $\forall i \in [n], |\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)| \le \varepsilon$.
> **Queries**: Each query is an oriented hyperplane H and the answer is $\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)$, for every $i \in [n]$.

We have the following lower bound on $\mathbf{QC}_p(\mathsf{ABH}^{\mathbf{Q}}(n, k, \varepsilon))$.

▶ **Theorem 2.** *There exist constants $n_0, \varepsilon_0 > 0$ such that for any $n \ge n_0$, $\varepsilon \le \varepsilon_0/\sqrt{48n}$, $p = 2^{-\Omega(n)}$, and $k \ge 4.51$ the following holds: any query algorithm to find an ε-bisecting $(n-1)$-dimensional hyperplane of n compact sets in $\left[-n^k, n^k\right]^n$, even with success probability p, requires $2^{\Omega(n)}$ queries.*

Proof. We show $\mathbf{QC}_p\left(\mathsf{AAP}^{\mathbf{Q}}\left(n, 5\sqrt{n}, \varepsilon^2/12n\right)\right) \le 2 \cdot \mathbf{QC}_p\left(\mathsf{ABH}^{\mathbf{Q}}\left(n+1, 4.51, \varepsilon/\sqrt{48n}\right)\right)$ by using Proposition 1. We start from a $5\sqrt{n}$-Lipschitz continuous function $f : \sqrt{n+1} \cdot S^n \to \sqrt{n} \cdot B^n$ which is the input of $\mathsf{AAP}^{\mathbf{Q}}$ and have the following preprocessing step.

Preprocessing Step. Fix $i \in [n]$. Let $f_i : \sqrt{n+1} \cdot S^n \to [-\sqrt{n}, \sqrt{n}]$ be the i-th component of f which is $5\sqrt{n}$-Lipschitz continuous. We define f_i' as follows: $f_i'(x) = f_i(\sqrt{n+1} \cdot x)$. Note that f_i' is a function from S^n to $\sqrt{n} \cdot B^n$ and is $(\sqrt{n+1} \cdot 5\sqrt{n})$-Lipschitz continuous. Now by the Tietze extension theorem f_i' may be extended to a continuous function $\tilde{f}_i'$ on the cube $[-1,1]^{n+1} \supset S^n$ without increasing the Lipschitz constant [22]. Then from the Stone-Weierstrass theorem we have that for any $\varepsilon' > 0$ there is a polynomial function $\tilde{p}_i : [-1,1]^{n+1} \to \mathbb{R}$ such that $|\tilde{f}_i'(x) - \tilde{p}_i(x)| \leq \varepsilon'$ for all $x \in [-1,1]^{n+1}$. Let p_i be the restriction of $\tilde{p}_i$ to S^n. So, we have a polynomial function $p_i : S^n \to [-\sqrt{n} - \varepsilon/4\sqrt{12n}, \sqrt{n} + \varepsilon/4\sqrt{12n}]$ such that for all $x \in S^n$, we have $|f_i'(x) - p_i(x)| \leq \varepsilon/4\sqrt{12n}$ by setting $\varepsilon' = \varepsilon/4\sqrt{12n}$. Furthermore, we have that the degree of p_i is $O(n^5)$ (using multivariate Bernstein polynomials).

Adopting Proposition 1. We have from Proposition 1, that there exist n compact sets $A_1', \ldots, A_n' \subseteq \mathbb{R}^{n+1}$, such that for every $x \in S^n$ and $i \in [n]$, $p_i(x) - p_i(-x) = \mathrm{vol}(A_i' \cap H^+) - \mathrm{vol}(A_i' \cap H^-)$, where H is the oriented hyperplane containing the origin with $\vec{x}$ as the normal. Fix $i \in [n]$. From the construction in proof of Proposition 1, we have that $A_i \subseteq [-h_i^\star, h_i^\star]^{n+1}$, where $h_i^\star = \max\limits_{x \in S^n} |h_i(x)|$. We have the following upper bound on $h_i^\star$ from Claim 16:

$$\begin{aligned} h_i^\star &= \max_{x \in S^n} |h_i(x)| = \max_{x \in S^n} \left| \sqrt[n+1]{\Gamma_i(x) + \psi_i + 1} \right| \leq \sqrt[n+1]{2\psi_i + 1} \\ &= O\left(\sqrt[n+1]{\psi_i} \right) = O\left(\sqrt[n+1]{(n)^{(n+1)/2}} \cdot \sqrt[n+1]{n^{4n}} \right) \\ &= O\left(\sqrt{n} \cdot \sqrt[n+1]{n^{4n}} \right) = O\left(n^{4.5}\right). \end{aligned}$$

Next, we know that $|f_i'(x) - f_i'(-x)| \leq |p_i(x) - p_i(-x)| + \varepsilon/2\sqrt{12n}$. Thus, we have:

$$|f_i'(x) - f_i'(-x)| \leq \left|\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)\right| + \varepsilon/2\sqrt{12n}.$$

Therefore, if we are given some hyperplane H such that for every $i \in [n]$, we have $|\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)| \leq \varepsilon/2\sqrt{12n}$ then, we would obtain $x \in S^n$ such that $|f_i'(x) - f_i'(-x)| \leq \varepsilon/\sqrt{12n}$. This implies that $\|f(\sqrt{n+1} \cdot x) - f(-\sqrt{n+1} \cdot x)\|_2^2 \leq \varepsilon^2/12$. Finally, we complete the proof by noting that to answer $\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-)$ for an oriented hyperplane H, we need to query f in at most two points. ◀

We note here that one can construct an easier (to solve) problem than $\mathsf{ABH}^{\mathbf{Q}}$, namely the Euclidean$-\mathsf{ABH}^{\mathbf{Q}}$ (or $\mathsf{ABH}^{\mathbf{Q}}_{\mathsf{E}}$ for short), where we need to find an $(n-1)$-dimensional hyperplane H such that $\left(\mathbb{E}_{i \in [n]}\left[(\mathrm{vol}(A_i \cap H^+) - \mathrm{vol}(A_i \cap H^-))^2\right]\right)^{1/2} \leq \varepsilon$, and still obtain the same lower bounds as in Theorem 2, i.e., $\mathbf{QC}_p\left(\mathsf{ABH}^{\mathbf{Q}}_{\mathsf{E}}\left(n+1, 4.51, \varepsilon/\sqrt{48n}\right)\right) = 2^{\Omega(n)}$ by starting from $\mathbf{QC}_p\left(\mathsf{AAP}^{\mathbf{Q}}\left(n, 5\sqrt{n}, \varepsilon^2/12n\right)\right)$.

Finally, we remark that one could obtain lower bounds for the case of fixed dimension, i.e., when the compacts objects to be bisected are in a fixed dimension, by using the lower bounds of Chen and Teng [11] for the fixed point computation in Brouwer functions of fixed dimension.

5 Discussion and Conclusion

In this paper, we established the equivalence between the Borsuk-Ulam theorem and the Ham Sandwich theorem. Further, we used this equivalence to prove a lower bound on the Ham Sandwich problem in the query model.

It would be interesting to extend our lower bounds for the Ham Sandwich problem in the query model where the queries are to a membership oracle. Finally, showing that the Ham Sandwich problem introduced by Papadimitriou [24] is **PPA**-complete, remains an interesting and challenging open problem.

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