# Covering Lattice Points by Subspaces and Counting Point-Hyperplane Incidences*

Martin Balko[1], Josef Cibulka[2], and Pavel Valtr[3]

1 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic; and
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary
balko@kam.mff.cuni.cz

2 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
cibulka@kam.mff.cuni.cz

3 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic; and
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

## Abstract

Let $d$ and $k$ be integers with $1 \leq k \leq d-1$. Let $\Lambda$ be a $d$-dimensional lattice and let $K$ be a $d$-dimensional compact convex body symmetric about the origin. We provide estimates for the minimum number of $k$-dimensional linear subspaces needed to cover all points in $\Lambda \cap K$. In particular, our results imply that the minimum number of $k$-dimensional linear subspaces needed to cover the $d$-dimensional $n \times \cdots \times n$ grid is at least $\Omega(n^{d(d-k)/(d-1)-\varepsilon})$ and at most $O(n^{d(d-k)/(d-1)})$, where $\varepsilon > 0$ is an arbitrarily small constant. This nearly settles a problem mentioned in the book of Brass, Moser, and Pach [7]. We also find tight bounds for the minimum number of $k$-dimensional affine subspaces needed to cover $\Lambda \cap K$.

We use these new results to improve the best known lower bound for the maximum number of point-hyperplane incidences by Brass and Knauer [6]. For $d \geq 3$ and $\varepsilon \in (0,1)$, we show that there is an integer $r = r(d,\varepsilon)$ such that for all positive integers $n, m$ the following statement is true. There is a set of $n$ points in $\mathbb{R}^d$ and an arrangement of $m$ hyperplanes in $\mathbb{R}^d$ with no $K_{r,r}$ in their incidence graph and with at least $\Omega\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right)$ incidences if $d$ is odd and $\Omega\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right)$ incidences if $d$ is even.

**1998 ACM Subject Classification** G.2.1 Combinatorics

**Keywords and phrases** lattice point, covering, linear subspace, point-hyperplane incidence

**Digital Object Identifier** 10.4230/LIPIcs.SoCG.2017.12

## 1 Introduction

In this paper, we study the minimum number of linear or affine subspaces needed to cover points that are contained in the intersection of a given lattice with a given 0-symmetric convex body. We also present an application of our results to the problem of estimating the

* The first and the third author acknowledge the support of the grants GAČR 14-14179S of Czech Science Foundation, ERC Advanced Research Grant no 267165 (DISCONV), and GAUK 690214 of the Grant Agency of the Charles University. The first author is also supported by the grant SVV–2016–260332.


33rd International Symposium on Computational Geometry (SoCG 2017).
Editors: Boris Aronov and Matthew J. Katz; Article No. 12; pp. 12:1–12:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

maximum number of incidences between a set of points and an arrangement of hyperplanes. Consequently, this establishes a new lower bound for the time complexity of so-called partitioning algorithms for Hopcroft's problem. Before describing our results in more detail, we first give some preliminaries and introduce necessary definitions.

## 1.1 Preliminaries

For linearly independent vectors $b_1,\dots,b_d \in \mathbb{R}^d$, the $d$-dimensional *lattice* $\Lambda = \Lambda(b_1,\dots,b_d)$ with *basis* $\{b_1,\dots,b_d\}$ is the set of all linear combinations of the vectors $b_1,\dots,b_d$ with integer coefficients. We define the *determinant of* $\Lambda$ as $\det(\Lambda) := |\det(B)|$, where $B$ is the $d \times d$ matrix with the vectors $b_1,\dots,b_d$ as columns. For a positive integer $d$, we use $\mathcal{L}^d$ to denote the set of $d$-dimensional lattices $\Lambda$, that is, lattices with $\det(\Lambda) \neq 0$.

A convex body $K$ is *symmetric about the origin* 0 if $K = -K$. We let $\mathcal{K}^d$ be the set of $d$-dimensional compact convex bodies in $\mathbb{R}^d$ that are symmetric about the origin.

For a positive integer $n$, we use the abbreviation $[n]$ to denote the set $\{1,2,\dots,n\}$. A point $x$ of a lattice is called *primitive* if whenever its multiple $\lambda \cdot x$ is a lattice point, then $\lambda$ is an integer. For $K \in \mathcal{K}_d$, let $\mathrm{vol}(K)$ be the $d$-dimensional Lebesgue measure of $K$. We say that $\mathrm{vol}(K)$ is the *volume of* $K$. The closed $d$-dimensional ball with the radius $r \in \mathbb{R}$, $r \geq 0$, centered in the origin is denoted by $B^d(r)$. If $r = 1$, we simply write $B^d$ instead of $B^d(1)$. For $x \in \mathbb{R}^d$, we use $\|x\|$ to denote the Euclidean norm of $x$.

Let $X$ be a subset of $\mathbb{R}^d$. We use $\mathrm{aff}(X)$ and $\mathrm{lin}(X)$ to denote the *affine hull of* $X$ and the *linear hull of* $X$, respectively. The dimension of the affine hull of $X$ is denoted by $\dim(X)$.

For functions $f,g\colon \mathbb{N} \to \mathbb{N}$, we write $f(n) \leq O(g(n))$ if there is a fixed constant $c_1$ such that $f(n) \leq c_1 \cdot g(n)$ for all $n \in \mathbb{N}$. We write $f(n) \geq \Omega(g(n))$ if there is a fixed constant $c_2 > 0$ such that $f(n) \geq c_2 \cdot g(n)$ for all $n \in \mathbb{N}$. If the constants $c_1$ and $c_2$ depend on some parameters $a_1,\dots,a_t$, then we emphasize this by writing $f(n) \leq O_{a_1,\dots,a_t}(g(n))$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(g(n))$, respectively. If $f(n) \leq O_{a_1,\dots,a_t}(n)$ and $f(n) \geq \Omega_{a_1,\dots,a_t}(n)$, then we write $f(n) = \Theta_{a_1,\dots,a_t}(n)$.

## 1.2 Covering lattice points by subspaces

We say that a collection $\mathcal{S}$ of subsets in $\mathbb{R}^d$ *covers* a set of points $P$ from $\mathbb{R}^d$ if every point from $P$ lies in some set from $\mathcal{S}$.

Let $d$, $k$, $n$, and $r$ be positive integers that satisfy $1 \leq k \leq d-1$. We let $a(d,k,n,r)$ be the maximum size of a set $S \subseteq \mathbb{Z}^d \cap B^d(n)$ such that every $k$-dimensional *affine* subspace of $\mathbb{R}^d$ contains at most $r-1$ points of $S$. Similarly, we let $l(d,k,n,r)$ be the maximum size of a set $S \subseteq \mathbb{Z}^d \cap B^d(n)$ such that every $k$-dimensional *linear* subspace of $\mathbb{R}^d$ contains at most $r-1$ points of $S$. We also let $g(d,k,n)$ be the minimum number of $k$-dimensional linear subspaces of $\mathbb{R}^d$ necessary to cover $\mathbb{Z}^d \cap B^d(n)$.

In this paper, we study the functions $a(d,k,n,r)$, $l(d,k,n,r)$, and $g(d,k,n)$ and their generalizations to arbitrary lattices from $\mathcal{L}^d$ and bodies from $\mathcal{K}^d$. We mostly deal with the last two functions, that is, with covering lattice points by linear subspaces. In particular, we obtain new upper bounds on $g(d,k,n)$ (Theorem 4), lower bounds on $l(d,k,n,r)$ (Theorem 5), and we use the estimates for $a(d,k,n,r)$ and $l(d,k,n,r)$ to obtain improved lower bounds for the maximum number of point-hyperplane incidences (Theorem 9). Before doing so, we first give a summary of known results, since many of them are used later in the paper.

The problem of determining $a(d,k,n,r)$ is essentially solved. In general, the set $\mathbb{Z}^d \cap B^d(n)$ can be covered by $(2n+1)^{d-k}$ affine $k$-dimensional subspaces and thus we have an upper bound $a(d,k,n,r) \leq (r-1)(2n+1)^{d-k}$. This trivial upper bound is asymptotically almost tight for all fixed $d$, $k$, and some $r$, as Brass and Knauer [6] showed with a probabilistic argument that for every $\varepsilon > 0$ there is an $r = r(d, \varepsilon, k) \in \mathbb{N}$ such that for each positive integer $n$ we have

$$a(d,k,n,r) \geq \Omega_{d,\varepsilon,k}\left(n^{d-k-\varepsilon}\right). \tag{1}$$

For fixed $d$ and $r$, the upper bound is known to be asymptotically tight in the cases $k = 1$ and $k = d-1$. This is showed by considering points on the modular moment surface for $k = 1$ and the modular moment curve for $k = d-1$; see [6].

Covering lattice points by linear subspaces seems to be more difficult than covering by affine subspaces. From the definitions we immediately get $l(d,k,n,r) \leq (r-1)g(d,k,n)$. In the case $k = d-1$ and $d$ fixed, Bárány, Harcos, Pach, and Tardos [5] obtained the following asymptotically tight estimates for the functions $l(d,d-1,n,d)$ and $g(d,d-1,n)$:

$$l(d,d-1,n,d) = \Theta_d(n^{d/(d-1)}) \qquad \text{and} \qquad g(d,d-1,n) = \Theta_d(n^{d/(d-1)}).$$

In fact, Bárány et al. [5] proved stronger results that estimate the minimum number of $(d-1)$-dimensional linear subspaces necessary to cover the set $\Lambda \cap K$ in terms of so-called successive minima of a given lattice $\Lambda \in \mathcal{L}^d$ and a body $K \in \mathcal{K}^d$.

For a lattice $\Lambda \in \mathcal{L}^d$, a body $K \in \mathcal{K}^d$, and $i \in [d]$, we let $\lambda_i(\Lambda, K)$ be the *ith successive minimum of* $\Lambda$ *and* $K$. That is, $\lambda_i(\Lambda,K) := \inf\{\lambda \in \mathbb{R} \colon \dim(\Lambda \cap (\lambda \cdot K)) \geq i\}$. Since $K$ is compact, it is easy to see that the successive minima are achieved. That is, there are linearly independent vectors $v_1, \dots, v_d$ from $\Lambda$ such that $v_i \in \lambda_i(\Lambda,K) \cdot K$ for every $i \in [d]$. Also note that we have $\lambda_1(\Lambda,K) \leq \cdots \leq \lambda_d(\Lambda,K)$ and $\lambda_1(\mathbb{Z}^d, B^d(n)) = \cdots = \lambda_d(\mathbb{Z}^d, B^d(n)) = 1/n$.

▶ **Theorem 1** ([5]). *For an integer $d \geq 2$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda,K)$ for every $i \in [d]$. If $\lambda_d \leq 1$, then the set $\Lambda \cap K$ can be covered with at most*

$$c2^d d^2 \log_2 d \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

*$(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, where $c$ is some absolute constant.*

*On the other hand, if $\lambda_d \leq 1$, then there is a subset $S$ of $\Lambda \cap K$ of size*

$$\frac{1-\lambda_d}{16d^2} \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}$$

*such that no $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ contains $d$ points from $S$.*

We note that the assumption $\lambda_d \leq 1$ is necessary; see the discussion in [5]. Not much is known for linear subspaces of lower dimension. We trivially have $l(d,k,n,r) \geq a(d,k,n,r)$ for all $d,k,n,r$ with $1 \leq k \leq d-1$. Thus $l(d,k,n,r) \geq \Omega_{d,\varepsilon,k}(n^{d-k-\varepsilon})$ for some $r = r(d,\varepsilon,k)$ by (1). Brass and Knauer [6] conjectured that $l(d,k,n,k+1) = \Theta_{d,k}(n^{d(d-k)/(d-1)})$ for $d$ fixed. This conjecture was refuted by Lefmann [15] who showed that, for all $d$ and $k$ with $1 \leq k \leq d-1$, there is an absolute constant $c$ such that we have $l(d,k,n,k+1) \leq c \cdot n^{d/\lceil k/2 \rceil}$ for every positive integer $n$. This bound is asymptotically smaller in $n$ than the growth rate conjectured by Brass and Knauer for sufficiently large $d$ and almost all values of $k$ with $1 \leq k \leq d-1$.

Covering lattice points by linear subspaces is also mentioned in the book by Brass, Moser, and Pach [7], where the authors pose the following problem.

▶ **Problem 2** ([7, Problem 6 in Chapter 10.2]). *What is the minimum number of $k$-dimensional linear subspaces necessary to cover the $d$-dimensional $n \times \cdots \times n$ lattice cube?*

## 1.3 Point-hyperplane incidences

As we will see later, the problem of determining $a(d,k,n,r)$ and $l(d,n,k,r)$ is related to a problem of bounding the maximum number of point-hyperplane incidences. For an integer $d \geq 2$, let $P$ be a set of $n$ points in $\mathbb{R}^d$ and let $\mathcal{H}$ be an arrangement of $m$ hyperplanes in $\mathbb{R}^d$. An *incidence between $P$ and $\mathcal{H}$* is a pair $(p,H)$ such that $p \in P$, $H \in \mathcal{H}$, and $p \in H$. The number of incidences between $P$ and $\mathcal{H}$ is denoted by $\mathrm{I}(P,\mathcal{H})$.

We are interested in the maximum number of incidences between $P$ and $\mathcal{H}$. In the plane, the famous *Szemerédi–Trotter theorem* [23] says that the maximum number of incidences between a set of $n$ points in $\mathbb{R}^2$ and an arrangement of $m$ lines in $\mathbb{R}^2$ is at most $O((mn)^{2/3} + m + n)$. This is known to be asymptotically tight, as a matching lower bound was found earlier by Erdős [9]. The current best known bounds are $\approx 1.27(mn)^{2/3} + m + n$ [19][1] and $\approx 2.44(mn)^{2/3} + m + n$ [1].

For $d \geq 3$, it is easy to see that there is a set $P$ of $n$ points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of $m$ hyperplanes in $\mathbb{R}^d$ for which the number of incidences is maximum possible, that is $\mathrm{I}(P,\mathcal{H}) = mn$. It suffices to consider the case where all points from $P$ lie in an affine subspace that is contained in every hyperplane from $\mathcal{H}$. In order to avoid this degenerate case, we forbid large complete bipartite graphs in the *incidence graph of $P$ and $\mathcal{H}$*, which is denoted by $G(P,\mathcal{H})$. This is the bipartite graph on the vertex set $P \cup \mathcal{H}$ and with edges $\{p,H\}$ where $(p,H)$ is an incidence between $P$ and $\mathcal{H}$.

With this restriction, bounding $\mathrm{I}(P,\mathcal{H})$ becomes more difficult and no tight bounds are known for $d \geq 3$. It follows from the works of Chazelle [8], Brass and Knauer [6], and Apfelbaum and Sharir [2] that the number of incidences between any set $P$ of $n$ points in $\mathbb{R}^d$ and any arrangement $\mathcal{H}$ of $m$ hyperplanes in $\mathbb{R}^d$ with $K_{r,r} \not\subseteq G(P,\mathcal{H})$ satisfies

$$\mathrm{I}(P,\mathcal{H}) \leq O_{d,r}\left((mn)^{1-1/(d+1)} + m + n\right). \tag{2}$$

We note that an upper bound similar to (2) holds in a much more general setting; see the remark in the proof of Theorem 9. The best general lower bound for $\mathrm{I}(P,\mathcal{H})$ is due to a construction of Brass and Knauer [6], which gives the following estimate.

▶ **Theorem 3** ([6]). *Let $d \geq 3$ be an integer. Then for every $\varepsilon > 0$ there is a positive integer $r = r(d,\varepsilon)$ such that for all positive integers $n$ and $m$ there is a set $P$ of $n$ points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of $m$ hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and*

$$\mathrm{I}(P,\mathcal{H}) \geq \begin{cases} \Omega_{d,\varepsilon}\left((mn)^{1-2/(d+3)-\varepsilon}\right) & \text{if } d \text{ is odd and } d > 3, \\ \Omega_{d,\varepsilon}\left((mn)^{1-2(d+1)/(d+2)^2-\varepsilon}\right) & \text{if } d \text{ is even}, \\ \Omega_{d,\varepsilon}\left((mn)^{7/10}\right) & \text{if } d = 3. \end{cases}$$

For $d \geq 4$, this lower bound has been recently improved by Sheffer [21] in a certain non-diagonal case. Sheffer constructed a set $P$ of $n$ points in $\mathbb{R}^d$, $d \geq 4$, and an arrangement $\mathcal{H}$ of $m = \Theta(n^{(3-3\varepsilon)/(d+1)})$ hyperplanes in $\mathbb{R}^d$ such that $K_{(d-1)/\varepsilon,2} \not\subseteq G(P,\mathcal{H})$ and $\mathrm{I}(P,\mathcal{H}) \geq \Omega\left((mn)^{1-2/(d+4)-\varepsilon}\right)$.

[1] The lower bound claimed by Pach and Tóth [19, Remark 4.2] contains the multiplicative constant $\approx 0.42$. This is due to a miscalculation in the last equation in the calculation of the number of incidences. The correct calculation is $I \approx \cdots = 4n\sum_{r=1}^{1/\varepsilon}\phi(r) - 2n\varepsilon^2\sum_{r=1}^{1/\varepsilon} r^2\phi(r) \approx 4n \cdot 3(1/\varepsilon)^2/\pi^2 - 2n\varepsilon^2(3/2)(1/\varepsilon)^4/\pi^2 = 9n/(\varepsilon^2\pi^2)$. This leads to $c \approx 3\sqrt[3]{3/(4\pi^2)} \approx 1.27$.

## 2 Our results

In this paper, we nearly settle Problem 2 by proving almost tight bounds for the function $g(d,k,n)$ for a fixed $d$ and an arbitrary $k$ from $[d-1]$. For a fixed $d$, an arbitrary $k \in [d-1]$, and some fixed $r$, we also provide bounds on the function $l(d,k,n,r)$ that are very close to the bound conjectured by Brass and Knauer [6]. Thus it seems that the conjectured growth rate of $l(d,k,n,r)$ is true if we allow $r$ to be (significantly) larger than $k+1$.

We study these problems in a more general setting where we are given an arbitrary lattice $\Lambda$ from $\mathcal{L}^d$ and a body $K$ from $\mathcal{K}^d$. Similarly to Theorem 1 by Bárány et al. [5], our bounds are expressed in terms of the successive minima $\lambda_i(\Lambda, K)$, $i \in [d]$.

### 2.1 Covering lattice points by linear subspaces

First, we prove a new upper bound on the minimum number of $k$-dimensional linear subspaces that are necessary to cover points in the intersection of a given lattice with a body from $\mathcal{K}^d$.

▶ **Theorem 4.** *For integers $d$ and $k$ with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \dots, d$. If $\lambda_d \leq 1$, then we can cover $\Lambda \cap K$ with $O_{d,k}(\alpha^{d-k})$ $k$-dimensional linear subspaces of $\mathbb{R}^d$, where*

$$\alpha := \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

We also prove the following lower bound.

▶ **Theorem 5.** *For integers $d$ and $k$ with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \dots, d$. If $\lambda_d \leq 1$, then, for every $\varepsilon \in (0,1)$, there is a positive integer $r = r(d,\varepsilon,k)$ and a set $S \subseteq \Lambda \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, where*

$$\beta := \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)},$$

*such that every $k$-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from $S$.*

We remark that we can get rid of the $\varepsilon$ in the exponent if $k=1$ or $k=d-1$; for details, see Theorem 1 for the case $k=d-1$ and the proof in Section 4 for the case $k=1$. Also note that in the definition of $\alpha$ in Theorem 4 the minimum is taken over the set $\{1,\dots,k\}$, while in the definition of $\beta$ in Theorem 5 the minimum is taken over $\{1,\dots,d-1\}$. There are examples, which show that $\alpha$ cannot be replaced by $\beta$ in Theorem 4. It suffices to consider $d=3$, $k=1$, and let $\Lambda$ be the lattice $\{(x_1/n, x_2/2, x_3/2) \in \mathbb{R}^3 \colon x_1,x_2,x_3 \in \mathbb{Z}\}$ for some large positive integer $n$. Then $\lambda_1(\Lambda, B^3) = 1/n$, $\lambda_2(\Lambda, B^3) = 1/2$, $\lambda_3(\Lambda, B^3) = 1/2$, and thus $\beta = (\lambda_2\lambda_3)^{-1} = 4$. However, it is not difficult to see that we need at least $\Omega(n)$ 1-dimensional linear subspaces to cover $\Lambda \cap B^3$, which is asymptotically larger than $\beta^2 = O(1)$. On the other hand, $\alpha = (\lambda_1\lambda_2\lambda_3)^{-1/2}$ and $O(\alpha^2) = O(n)$ 1-dimensional linear subspaces suffice to cover $\Lambda \cap B^3$. We thus suspect that the lower bound can be improved.

Since $\lambda_i(\mathbb{Z}^d, B^d(n)) = 1/n$ for every $i \in [d]$, we can apply Theorem 5 with $\Lambda = \mathbb{Z}^d$ and $K = B^d(n)$ and obtain the following lower bound on $l(d,k,n,r)$.

▶ **Corollary 6.** *Let $d$ and $k$ be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon,k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have*

$$l(d,k,n,r) \geq \Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}).$$

The existence of the set $S$ from Theorem 5 is showed by a probabilistic argument. It would be interesting to find, at least for some value $1 < k < d-1$, some fixed $r \in \mathbb{N}$, and arbitrarily large $n \in \mathbb{N}$, a construction of a subset $R$ of $\mathbb{Z}^d \cap B^d(n)$ of size $\Omega_{d,k}(n^{d(d-k)/(d-1)})$ such that every $k$-dimensional linear subspace contains at most $r-1$ points from $R$. Such constructions are known for $k=1$ and $k=d-1$; see [6, 20].

Since we have $l(d,k,n,r) \leq (r-1)g(d,k,n)$ for every $r \in \mathbb{N}$, Theorem 4 and Corollary 6 give the following almost tight estimates on $g(d,k,n)$. This nearly settles Problem 2.

▶ **Corollary 7.** *Let $d$, $k$, and $n$ be integers with $1 \leq k \leq d-1$. Then, for every $\varepsilon \in (0,1)$, we have*

$$\Omega_{d,\varepsilon,k}(n^{d(d-k)/(d-1)-\varepsilon}) \leq g(d,k,n) \leq O_{d,k}(n^{d(d-k)/(d-1)}).$$

## 2.2 Covering lattice points by affine subspaces

For *affine* subspaces, Brass and Knauer [6] considered only the case of covering the $d$-dimensional $n \times \cdots \times n$ lattice cube by $k$-dimensional affine subspaces. To our knowledge, the case for general $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$ was not considered in the literature. We extend the results of Brass and Knauer to covering $\Lambda \cap K$.

▶ **Theorem 8.** *For integers $d$ and $k$ with $1 \leq k \leq d-1$, a lattice $\Lambda \in \mathcal{L}^d$, and a body $K \in \mathcal{K}^d$, we let $\lambda_i := \lambda_i(\Lambda, K)$ for $i = 1, \ldots, d$. If $\lambda_d \leq 1$, then the set $\Lambda \cap K$ can be covered with $O_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ $k$-dimensional affine subspaces of $\mathbb{R}^d$.*

*On the other hand, at least $\Omega_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1})$ $k$-dimensional affine subspaces of $\mathbb{R}^d$ are necessary to cover $\Lambda \cap K$.*

## 2.3 Point-hyperplane incidences

As an application of Corollary 6, we improve the best known lower bounds on the maximum number of point-hyperplane incidences in $\mathbb{R}^d$ for $d \geq 4$. That is, we improve the bounds from Theorem 3. To our knowledge, this is the first improvement on the estimates for $\mathrm{I}(P, \mathcal{H})$ in the general case during the last 13 years.

▶ **Theorem 9.** *For every integer $d \geq 2$ and $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon) \in \mathbb{N}$ such that for all positive integers $n$ and $m$ the following statement is true. There is a set $P$ of $n$ points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of $m$ hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and*

$$\mathrm{I}(P,\mathcal{H}) \geq \begin{cases} \Omega_{d,\varepsilon}\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right) & \text{if } d \text{ is odd,} \\ \Omega_{d,\varepsilon}\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) & \text{if } d \text{ is even.} \end{cases}$$

We can get rid of the $\varepsilon$ in the exponent for $d \leq 3$. That is, we have the bounds $\Omega((mn)^{2/3})$ for $d=2$ and $\Omega((mn)^{7/10})$ for $d=3$. For $d=3$, our bound is the same as the bound from Theorem 3. For larger $d$, our bounds become stronger. In particular, the exponents in the lower bounds from Theorem 9 exceed the exponents from Theorem 3 by $1/((d+2)(d+3))$ for $d > 3$ odd and by $d^2/((d+2)^2(d^2+2d-2))$ for $d$ even. However, the bounds are not tight.

In the non-diagonal case, when one of $n$ and $m$ is significantly larger that the other, the proof of Theorem 9 yields the following stronger bound.

▶ **Theorem 10.** *For all integers $d$ and $k$ with $0 \leq k \leq d-2$ and for $\varepsilon \in (0,1)$, there is an $r = r(d,\varepsilon,k) \in \mathbb{N}$ such that for all positive integers $n$ and $m$ the following statement is true.*

*There is a set $P$ of $n$ points in $\mathbb{R}^d$ and an arrangement $\mathcal{H}$ of $m$ hyperplanes in $\mathbb{R}^d$ such that $K_{r,r} \not\subseteq G(P,\mathcal{H})$ and*

$$\mathrm{I}(P,\mathcal{H}) \geq \Omega_{d,\varepsilon,k}\left(n^{1-(k+1)/((k+2-1/d)(d-k))-\varepsilon} m^{1-(d-1)/(dk+2d-1)-\varepsilon}\right).$$

For example, in the case $m = \Theta(n^{(3-3\varepsilon)/(d+1)})$ considered by Sheffer [21], Theorem 10 gives a slightly better bound than $I(P,\mathcal{H}) \geq \Omega((mn)^{1-2/(d+4)-\varepsilon})$ if we set, for example, $k = \lfloor (d-1)/4 \rfloor$. However, the forbidden complete bipartite subgraph in the incidence graph is larger than $K_{(d-1)/\varepsilon,2}$.

The following problem is known as the counting version of *Hopcroft's problem* [6, 10]: given $n$ points in $\mathbb{R}^d$ and $m$ hyperplanes in $\mathbb{R}^d$, how fast can we count the incidences between them? We note that the lower bounds from Theorem 9 also establish the best known lower bounds for the time complexity of so-called *partitioning algorithms* [10] for the counting version of Hopcroft's problem; see [6] for more details.

In the proofs of our results, we make no serious effort to optimize the constants. We also omit floor and ceiling signs whenever they are not crucial.

## 3 Proof of Theorem 4

Here we sketch the proof of the upper bound on the minimum number of $k$-dimensional linear subspaces needed to cover points from a given $d$-dimensional lattice that are contained in a body $K$ from $\mathcal{K}^d$. We first prove Theorem 4 in the special case $K = B^d$ (Theorem 14) and then we extend the result to arbitrary $K \in \mathcal{K}^d$. Since the proof is rather long and complicated, we only prove a weaker bound (Corollary 16) and then we give a high-level overview of the main ideas of the full proof, which can be found in the full version of the paper [3].

### 3.1 Sketch of the proof for balls

We first introduce some auxiliary results that are used later. The following classical result is due to Minkowski [18] and shows a relation between $\mathrm{vol}(K)$, $\det(\Lambda)$, and the successive minima of $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$.

▶ **Theorem 11** (Minkowski's second theorem [18]). *Let $d$ be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, we have*

$$\frac{1}{2^d} \cdot \frac{\mathrm{vol}(K)}{\det(\Lambda)} \leq \frac{1}{\lambda_1(\Lambda,K)\cdots\lambda_d(\Lambda,K)} \leq \frac{d!}{2^d} \cdot \frac{\mathrm{vol}(K)}{\det(\Lambda)}.$$

A result similar to the first bound from Theorem 11 can be obtained if the volume is replaced by the point enumerator; see Henk [13].

▶ **Theorem 12** ([13, Theorem 1.5]). *Let $d$ be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, we have*

$$|\Lambda \cap K| \leq 2^{d-1} \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(\Lambda,K)} + 1 \right\rfloor.$$

For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$, let $v_1,\dots,v_d$ be linearly independent vectors such that $v_i \in \Lambda \cap (\lambda_i(\Lambda,K) \cdot K)$ for every $i \in [d]$. For $d > 2$, the vectors $v_1,\dots,v_d$ do not necessarily form a basis of $\Lambda$ [22, see Section X.5]. However, the following theorem shows that there exists a basis with vectors of lengths not much larger than the lengths of $v_1,\dots,v_d$.

▶ **Theorem 13** (First finiteness theorem [22, see Lemma 2 in Section X.6]). *Let $d$ be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, there is a basis $\{b_1, \ldots, b_d\}$ of $\Lambda$ with $b_i \in (3/2)^{i-1}\lambda_i(\Lambda, K) \cdot K$ for every $i \in [d]$.*

Now, let $\Lambda$ be a $d$-dimensional lattice with $\lambda_d(\Lambda, B^d) \leq 1$. Throughout this section, we use $\lambda_i$ to denote the $i$th successive minimum $\lambda_i(\Lambda, B^d)$ for $i = 1, \ldots, d$. Let $k$ be an integer with $1 \leq k \leq d-1$. We show the following result.

▶ **Theorem 14.** *There is a constant $C = C(d, k)$ such that the set $\Lambda \cap B^d$ can be covered with $C \cdot \alpha^{d-k}$ $k$-dimensional linear subspaces of $\mathbb{R}^d$, where*

$$\alpha := \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

As the first step towards the proof of Theorem 14, we show a weaker bound on the number of $k$-dimensional linear subspaces needed to cover $\Lambda \cap B^d$; see Corollary 16. To do so, we prove the following lemma that is also used later in the proof of Theorem 8.

▶ **Lemma 15.** *Let $d$ and $s$ be integers with $0 \leq s \leq d-1$. There is a positive integer $r = r(d, s)$ and a projection $p$ of $\mathbb{R}^d$ along $s$ vectors of $\Lambda$ onto a $(d-s)$-dimensional linear subspace $N$ of $\mathbb{R}^d$ such that $\Lambda \cap B^d$ is mapped to $\Lambda \cap N \cap B^d(r)$ and such that $\lambda_i(\Lambda \cap N, B^d(r) \cap N) = \Theta_{d,s}(\lambda_{i+s})$ for every $i \in [d-s]$.*

**Proof.** If $s = 0$, then we set $p$ to be the identity on $\mathbb{R}^d$ and $r := 1$. Thus we assume $s \geq 1$.

For $j = 0, \ldots, d-1$, we set $r_j := (2^{d^2}+1)^j$. For $j = 0, \ldots, d-1$ and a lattice $\Lambda_j \in \mathcal{L}^{d-j}$, we show that there is a projection $p_j$ of $\mathbb{R}^{d-j}$ along a vector $v_j \in \Lambda_j$ onto a $(d-j-1)$-dimensional linear subspace $N$ of $\mathbb{R}^{d-j}$ such that $\Lambda_j \cap B^{d-j}(r_j)$ is mapped to $\Lambda_j \cap N \cap B^{d-j}(r_{j+1})$ by $p_j$ and such that

$$\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))/(2^{d^2}+1) \leq \lambda_i(\Lambda_j \cap N, B^{d-j}(r_{j+1}) \cap N) \leq \lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))$$

for every $i \in [d-j-1]$. We let $p_j$ be the projection for $\Lambda_j := p_{j-1}(\Lambda_{j-1})$ for every $j = 1, \ldots, s-1$, where $\Lambda_0 := \Lambda$ and $p_0$ is the projection for $\Lambda_0$. The statement of the lemma is then obtained by setting $p := p_{s-1} \circ \cdots \circ p_0$.

Let $B = \{b_1, \ldots, b_{d-j}\}$ be a basis of $\Lambda_j$ such that $b_i \in (3/2)^{i-1}\lambda_i(\Lambda_j, B^{d-j}(r_j)) \cdot B^{d-j}(r_j)$ for every $i \in [d-j]$. Such basis exists by the First finiteness theorem (Theorem 13). In particular, $b_1$ is the shortest vector from $\Lambda_j \cap B^{d-j}(r_j)$. Let $v_j := b_1$ and let $N$ be the linear subspace generated by $b_2, \ldots, b_{d-j}$. Let $\Lambda_N$ be the set $\Lambda_j \cap N$. Note that $\Lambda_N$ is a $(d-j-1)$-dimensional lattice with the basis $\{b_2, \ldots, b_{d-j}\}$.

We consider the projection $p_j$ onto $N$ along $v_j$. That is, every $x \in \mathbb{R}^{d-j}$ is mapped to $p_j(x) = \sum_{i=2}^{d-j} t_i b_i$, where $x = \sum_{i=1}^{d-j} t_i b_i$, $t_i \in \mathbb{R}$, is the expression of $x$ with respect to the basis $B$.

We show that $p_j(z) \in \Lambda_N \cap B^{d-j}(r_{j+1})$ for every $z \in \Lambda_j \cap B^{d-j}(r_j)$. We have $p_j(z) \in \Lambda_N$, since $B$ is a basis of $\Lambda_j$ and $B \setminus \{b_1\}$ is a basis of $\Lambda_N$. Let $z = \sum_{i=1}^{d-j} t_i b_i$, $t_i \in \mathbb{Z}$, be the expression of $z$ with respect to $B$ and let $v$ be the Euclidean distance between $b_1$ and $N$.

From the definitions of $\Lambda_N$ and $B$, we have

$$\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j)) \leq \lambda_i(\Lambda_N, B^{d-j}(r_j) \cap N) \leq (3/2)^i \lambda_{i+1}(\Lambda_j, B^{d-j}(r_j)) \quad (3)$$

for every $i \in [d-j-1]$. Using Minkowski's second theorem (Theorem 11) twice, the upper bound in (3), and the choice of $b_1$, we obtain

$$\begin{aligned}\frac{\operatorname{vol}(B^{d-j}(r_j))}{2^{d-j}\det(\Lambda_j)} &\leq \frac{1}{\lambda_1(\Lambda_j, B^{d-j}(r_j))\cdots\lambda_{d-j}(\Lambda_j, B^{d-j}(r_j))}\\ &\leq \frac{r_j}{\|b_1\|}\cdot\frac{(3/2)^{(d-j)(d-j-1)/2}}{\lambda_1(\Lambda_N, B^{d-j}(r_j)\cap N)\cdots\lambda_{d-j-1}(\Lambda_N, B^{d-j}(r_j)\cap N)}\\ &\leq \frac{r_j}{\|b_1\|}\cdot\frac{(3/2)^{(d-j)(d-j-1)/2}\cdot(d-j-1)!\cdot\operatorname{vol}(B^{d-j}(r_j)\cap N)}{2^{d-j-1}\cdot\det(\Lambda_N)}.\end{aligned}$$

Since $\det(\Lambda_j) = v\cdot\det(\Lambda_N)$, we can rewrite this expression as

$$\|b_1\| \leq \frac{r_j\cdot(3/2)^{(d-j)(d-j-1)/2}\cdot(d-j-1)!\cdot 2^{d-j}\cdot\operatorname{vol}(B^{d-j}(r_j)\cap N)\cdot\det(\Lambda_j)}{2^{d-j-1}\cdot\operatorname{vol}(B^{d-j}(r_j))\cdot\det(\Lambda_N)} \leq 2^{d^2}\cdot v.$$

To derive the last inequality, we use the well-known formula

$$\operatorname{vol}(B^m(r)) = \begin{cases}\frac{2((m-1)/2)!(4\pi)^{(m-1)/2}}{m!}\cdot r^m & \text{if } m \text{ is odd,}\\ \frac{\pi^{m/2}}{(m/2)!}\cdot r^m & \text{if } m \text{ is even}\end{cases}$$

for the volume of $B^m(r)$, $m, r\in\mathbb{N}$. Since $\operatorname{vol}(B^{d-j}(r_j)\cap N) = \operatorname{vol}(B^{d-j-1}(r_j))$, we have $\operatorname{vol}(B^{d-j}(r_j)\cap N)/\operatorname{vol}(B^{d-j}(r_j)) \leq 2^{d-j}/r_j$. The Euclidean distance between $z$ and $N$ equals $|t_1|\cdot v$, which is at most $r_j$, as $z\in B^{d-j}(r_j)$. Thus, since $|t_1|\leq r_j/v$ and $1/v\leq 2^{d^2}/\|b_1\|$, we obtain $|t_1|\leq 2^{d^2}\cdot r_j/\|b_1\|$. This implies

$$\|p_j(z)\| = \|z - t_1b_1\| \leq \|z\| + |t_1|\cdot\|b_1\| \leq r_j + 2^{d^2}r_j = r_{j+1}$$

and we see that $p_j(z)$ lies in $\Lambda_N\cap B^{d-j}(r_{j+1})$.

Note that $\lambda_i(\Lambda_N, B^{d-j}(r_{j+1})\cap N) = (2^{d^2}+1)^{-1}\cdot\lambda_i(\Lambda_N, B^{d-j}(r_j)\cap N)$ for every $i\in[d-j-1]$. Using this fact together with the bounds in (3), we obtain

$$\frac{\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))}{2^{d^2}+1} \leq \lambda_i(\Lambda_N, B^{d-j}(r_{j+1})\cap N) \leq \frac{(3/2)^{d-j}\lambda_{i+1}(\Lambda_j, B^{d-j}(r_j))}{2^{d^2}+1}$$

for every $i\in[d-j-1]$. ◀

▶ **Corollary 16.** *The set $\Lambda\cap B^d$ can be covered with $O_{d,k}((\lambda_k\cdots\lambda_d)^{-1})$ $k$-dimensional linear subspaces of $\mathbb{R}^d$.*

**Proof.** By Lemma 15, there is a positive integer $r = r(d, k-1)$ and a projection $p$ of $\mathbb{R}^d$ along $k-1$ vectors $b_1,\dots,b_{k-1}\in\Lambda$ onto a $(d-k+1)$-dimensional linear subspace $N$ of $\mathbb{R}^d$ such that $\Lambda\cap B^d$ is mapped to $\Lambda\cap N\cap B^d(r)$ and such that $\lambda'_i := \lambda_i(\Lambda\cap N, B^d(r)\cap N) = \Theta_{d,k}(\lambda_{i+k-1})$ for every $i\in[d-k+1]$. We use $\Lambda_N$ to denote the $(d-k+1)$-dimensional sublattice $\Lambda\cap N$ of $\Lambda$.

We consider the set $\mathcal{S} := \{\operatorname{lin}(\{y, b_1,\dots,b_{k-1}\})\colon y\in(\Lambda_N\setminus\{0\})\cap B^d(r)\}$. Then $\mathcal{S}$ consists of $k$-dimensional linear subspaces and its projection $p(\mathcal{S})$ covers $\Lambda_N\cap B^d(r)$. By Theorem 12, the size of $\mathcal{S}$ is at most

$$|\Lambda_N\cap B^d(r)| \leq 2^{d-k}\prod_{i=1}^{d-k+1}\left\lfloor\frac{2}{\lambda'_i}+1\right\rfloor \leq O_{d,k}\left(\prod_{i=1}^{d-k+1}\frac{1}{\lambda'_i}\right) \leq O_{d,k}((\lambda_k\cdots\lambda_d)^{-1}),$$

where the second inequality follows from the assumption $\lambda_d\leq 1$, as then $\lambda'_{d-k+1}\leq O_{d,k}(\lambda_d)$ implies $\lambda'_1\leq\cdots\leq\lambda'_{d-k+1}\leq O_{d,k}(1)$. The last inequality is obtained from $\lambda'_i\geq\Omega_{d,k}(\lambda_{i+k-1})$ for every $i\in[d-k+1]$. Moreover, $\mathcal{S}$ covers $\Lambda\cap K$, since for every $y\in\Lambda_N\cap B^d(r)$ there is $S\in\mathcal{S}$ with $y\in p(S)$ and $p(z)\in\Lambda_N\cap B^d(r)$ for every $z\in\Lambda\cap B^d$. ◀

Let $q$ be an integer from $\{d-k+1,\dots,d\}$ such that $\alpha = (\lambda_{d-q+1}\cdots\lambda_d)^{-1/(q-1)}$. The bound from Corollary 16 matches the bound from Theorem 14 in the case $k=1$. The case $k=d-1$ was shown by Bárány et al. [5]; see Theorem 1. Thus we may assume $d \geq 4$. Corollary 16 also provides the same bound as Theorem 14 if $q = d-k+1$, so we assume $q \geq d-k+2$.

We now sketch the proof of the upper bound $O_{d,k}(\alpha^{d-k})$ if $q \geq d-k+2$. Let $\Lambda^*$ be the dual lattice of $\Lambda$. That is, $\Lambda^*$ is the set of vectors $y$ from $\mathbb{R}^d$ that satisfy $\langle x,y\rangle \in \mathbb{Z}$ for every $x \in \Lambda$. In the rest of the section, we use $\mu_i$ to denote $\lambda_i(\Lambda^*, B^d)$ for every $i \in [d]$. It follows from the results of Mahler [16] and Banaszczyk [4] that $1 \leq \lambda_i \cdot \mu_{d-i+1} \leq d$ holds for every $i \in [d]$. This together with the assumption $\lambda_d \leq 1$ implies $\mu_1 \geq 1$ and $\alpha = \Theta_{d,k}((\mu_1\cdots\mu_q)^{1/(q-1)})$.

We now proceed by induction on $d-k$. The case $d-k=1$ is treated similarly as in the proof of Theorem 1 by Bárány et al. [5]. Using the pigeonhole principle, we can construct a set $D'$ of primitive points from $\Lambda^* \setminus \{0\}$ such that $|D'| \leq O_d(\alpha)$ and such that for every $x \in \Lambda \cap B^d$ there is $z \in D'$ with $\langle x,z\rangle = 0$. We let $\mathcal{S}$ to be the set of hyperplanes that contain the origin and have normal vectors from $D'$. Observe that $\mathcal{S}$ is a set of $O_d(\alpha) = O_d(\alpha^{d-k})$ $(d-1)$-dimensional linear subspaces that cover $\Lambda \cap B^d$.

For the inductive step, assume that $d-k \geq 2$. We consider the set $\mathcal{S}$ of hyperplanes in $\mathbb{R}^d$ that has been constructed in the base of the induction. For every hyperplane $H(z) \in \mathcal{S}$ with the normal vector $z \in D'$, we let $\Lambda_{H(z)}$ be the set $\Lambda \cap H(z)$. Note that $\Lambda_{H(z)}$ is a lattice of dimension at most $d-1$. We now proceed inductively and cover each set $\Lambda_{H(z)} \cap B^d$ using the inductive hypothesis for $\Lambda_{H(z)}$ and $k$. To do so, we employ the fact that, for every $z \in D'$, the larger $\|z\|$ is, the fewer $k$-dimensional linear subspaces we need to cover $\Lambda_{H(z)} \cap B^d$. In particular, we prove that if $z$ is a point from $D'$ and $q \geq d-k+2$, then $\Lambda_{H(z)} \cap B^d$ can be covered with $O_{d,k}\left(((\mu_1\cdots\mu_q)/\|z\|)^{(d-k-1)/(q-2)}\right)$ $k$-dimensional linear subspaces.

Then we partition $D'$ into subsets $S_1,\dots,S_q$ such that all vectors from $S_i$ have approximately the same Euclidean norm. Then, for every $i \in [q]$, we sum the number $c_i$ of $k$-dimensional linear subspaces needed to cover $\Lambda_{H(z)} \cap B^d$ for $z \in S_i$ and show that $c_1 + \cdots + c_q \leq O_{d,k}(\alpha^{d-k})$.

## 3.2 The general case

Here, we finish the proof of Theorem 4 by extending Theorem 14 to arbitrary convex bodies from $\mathcal{K}^d$. This is done by approximating a given body $K$ from $\mathcal{K}^d$ with ellipsoids. A *$d$-dimensional ellipsoid* in $\mathbb{R}^d$ is an image of $B^d$ under a nonsingular affine map. Such approximation exists by the following classical result, called *John's lemma* [14].

▶ **Lemma 17** (John's lemma [17, see Theorem 13.4.1]). *For every positive integer $d$ and every $K \in \mathcal{K}^d$, there is a $d$-dimensional ellipsoid $E$ with the center in the origin that satisfies*

$$E/\sqrt{d} \subseteq K \subseteq E.$$

Let $\Lambda \in \mathcal{L}^d$ be a given lattice and let $\lambda_i := \lambda_i(\Lambda, K)$ for every $i \in [d]$. From our assumptions, we know that $\lambda_d \leq 1$. Let $E$ be the ellipsoid from Lemma 17. Since $E$ is an ellipsoid, there is a nonsingular affine map $h\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $E = h(B^d)$. Since $E$ is centered in the origin, we see that $h$ is in fact a linear map. Thus $\Lambda' := h^{-1}(\Lambda) \in \mathcal{L}^d$. Observe that we have $\lambda_i = \lambda_i(\Lambda', h^{-1}(K))$ for every $i \in [d]$.

For every $i \in [d]$, we use $\lambda'_i$ to denote the $i$th successive minimum $\lambda_i(\Lambda', B^d) = \lambda_i(\Lambda, E)$. From the choice of $E$, we have $\lambda_i/\sqrt{d} \leq \lambda'_i \leq \lambda_i$. In particular, $\lambda'_d \leq 1$. Thus, by Theorem 14, the set $\Lambda' \cap B^d$ can be covered with $O_{d,k}((\alpha')^{d-k})$ $k$-dimensional linear subspaces, where $\alpha' := \min_{1 \leq j \leq k}(\lambda'_j \cdots \lambda'_d)^{-1/(d-j)}$.

Since $\lambda_i = \Theta_d(\lambda'_i)$ for every $i \in [d]$, we see that the set $\Lambda' \cap h^{-1}(K)$ can be covered with $O_{d,k}(\alpha^{d-k})$ $k$-dimensional linear subspaces, where $\alpha := \min_{1 \leq j \leq k}(\lambda_j \cdots \lambda_d)^{-1/(d-j)}$. Since every nonsingular linear transformation preserves incidences and successive minima and maps a $k$-dimensional linear subspace to a $k$-dimensional linear subspace, the set $\Lambda \cap K$ can be covered with $O_{d,k}(\alpha^{d-k})$ $k$-dimensional linear subspaces.

## 4 Proof of Theorem 5

Let $d$ and $k$ be positive integers satisfying $1 \leq k \leq d-1$ and let $K$ be a body from $\mathcal{K}^d$ with $\lambda_d(\mathbb{Z}^d, K) \leq 1$. For every $i \in [d]$, we let $\lambda_i$ be the $i$th successive minimum $\lambda_i(\mathbb{Z}^d, K)$. Let $\varepsilon$ be a number from $(0,1)$. We use a probabilistic approach to show that there is a set $S \subseteq \mathbb{Z}^d \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, where $\beta := \min_{1 \leq j \leq d-1}(\lambda_j \cdots \lambda_d)^{-1/(d-j)}$, such that every $k$-dimensional linear subspace contains at most $r-1$ points from $S$.

Note that it is sufficient to prove the statement only for the lattice $\mathbb{Z}^d$. For a general lattice $\Lambda \in \mathcal{L}^d$ we can apply a linear transformation $h$ such that $h(\Lambda) = \mathbb{Z}^d$ and then use the result for $\mathbb{Z}^d$ and $h(K)$, since $\lambda_i(\Lambda, K) = \lambda_i(\mathbb{Z}^d, h(K))$ for every $i \in [d]$. We also remark that in the case $k = d-1$ the stronger lower bound $\Omega_d((1-\lambda_d)\beta)$ from Theorem 1 by Bárány et al. [5] applies.

The proof is based on the following two results, first of which is by Bárány et al. [5].

▶ **Lemma 18** ([5]). *For an integer $d \geq 2$ and $K \in \mathcal{K}^d$, if $\lambda_d < 1$ and $p$ is an integer satisfying $1 < p < (1-\lambda_d)\beta/(8d^2)$, then, for every $v \in \mathbb{R}^d$, there exist an integer $1 \leq j < p$ and a point $w \in \mathbb{Z}^d$ with $jv + pw \in K$.*

For a prime number $p$, let $\mathbb{F}_p$ be the finite field of size $p$. The second main ingredient in the proof of Theorem 5 is the following lemma.

▶ **Lemma 19.** *Let $d$ and $k$ be integers satisfying $2 \leq k \leq d-2$ and let $\varepsilon \in (0,1)$. Then there is a positive integer $p_0 = p_0(d, \varepsilon, k)$ such that for every prime number $p \geq p_0$ there exists a subset $R$ of $\mathbb{F}_p^{d-1}$ of size at least $p^{d-k-\varepsilon}/2$ such that every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from $R$ for $r := \lceil k(d-k+1)/\varepsilon \rceil$.*

**Proof.** We assume that $p$ is large enough with respect to $d$, $\varepsilon$, and $k$ so that $p^{k-1} > r$. We set $P := p^{1-k-\varepsilon}$ and we let $X$ be a subset of $\mathbb{F}_p^{d-1}$ obtained by choosing every point from $\mathbb{F}_p^{d-1}$ independently at random with the probability $P$.

Let $A$ be a $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$. Then $|A| = p^{k-1}$. It is well-known that the number of $(k-1)$-dimensional linear subspaces of $\mathbb{F}_p^{d-1}$ is exactly the *Gaussian binomial coefficient*

$$\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p := \frac{(p^{d-1}-1)(p^{d-1}-p)\cdots(p^{d-1}-p^{k-2})}{(p^{k-1}-1)(p^{k-1}-p)\cdots(p^{k-1}-p^{k-2})}$$
$$\leq \frac{p^{d-1}\cdot p^{d-2}\cdots p^{d-k+1}}{(p^{k-1}-1)(p^{k-2}-1)\cdots(p-1)} \leq p^{(k-1)d-(k-1)k/2-(k-1)(k-2)/2} = p^{(k-1)(d-k+1)}. \quad (4)$$

We used the fact $p^{k-i} - 1 \geq p^{k-i-1}$ for $k > i$ in the last inequality.

Since every $(k-1)$-dimensional affine subspace $A$ of $\mathbb{F}_p^{d-1}$ is of the form $A = x + L$ for some $x \in \mathbb{F}_p^{d-1}$ and a $(k-1)$-dimensional linear subspace $L$ of $\mathbb{F}_p^{d-1}$ and $x + L = y + L$ if and only if $x - y \in L$, the total number of $(k-1)$-dimensional affine subspaces of $\mathbb{F}_p^{d-1}$

is $p^{d-k}\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p$. This is because by considering pairs $(x, L)$, where $x \in \mathbb{F}_p^{d-1}$ and $L$ is a $(k-1)$-dimensional linear subspace of $\mathbb{F}_p^{d-1}$, every $(k-1)$-dimensional affine subspace $A$ is counted $p^{k-1}$ times.

We use the following Chernoff-type bound (see the last bound of [12]) to estimate the probability that $A$ contains at least $r$ points of $X$. Let $q \in [0, 1]$ and let $Y_1, \dots, Y_m$ be independent 0-1 random variables with $\Pr[Y_i = 1] = q$ for every $i \in [m]$. Then, for $mq \le s < m$, we have

$$\Pr[Y_1 + \cdots + Y_m \ge s] \le \left(\frac{mq}{s}\right)^s e^{s-mq}. \tag{5}$$

Choosing $Y_x$ as the indicator variable for the event $x \in A \cap X$ for each $x \in A$, we have $m = |A| = p^{k-1}$ and $q = P$. Since $p, r \ge 1$ and $p^{k-1} > r$, we have $p^{-\varepsilon} = mq \le r < m = p^{k-1}$ and thus the bound (5) implies

$$\Pr[|A \cap X| \ge r] \le \left(\frac{p^{k-1}P}{r}\right)^r e^{r-p^{k-1}P} = \left(\frac{p^{-\varepsilon}}{r}\right)^r e^{r-p^{-\varepsilon}} = p^{-\varepsilon r} e^{r(1-\ln r)-p^{-\varepsilon}} < p^{-\varepsilon r},$$

where the last inequality follows from $r \ge e$, as then $1 - \ln r \le 0$.

By the Union bound, the probability that there is a $(k-1)$-dimensional affine subspace $A$ of $\mathbb{F}_p^{d-1}$ with $|A \cap X| \ge r$ is less than

$$p^{d-k}\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_p \cdot p^{-\varepsilon r} \le p^{(d-k)+(k-1)(d-k+1)-\varepsilon r} \le p^{k(d-k+1)-1-k(d-k+1)} = p^{-1},$$

where the first inequality follows from (4) and the second inequality is due to the choice of $r$. From $p \ge 2$, we see that this probability is less than $1/2$.

The expected size of $X$ is $\mathbb{E}[|X|] = |\mathbb{F}_p^{d-1}| \cdot P = p^{d-1}p^{1-k-\varepsilon} = p^{d-k-\varepsilon}$. Since $|X| \sim \mathrm{Bi}(p^{d-1}, P)$, the variance of $|X|$ is $p^{d-1}P(1-P) < p^{d-k-\varepsilon}$ and Chebyshev's inequality implies $\Pr[||X| - \mathbb{E}[|X|]| \ge \sqrt{2p^{d-k-\varepsilon}}] < p^{d-k-\varepsilon}/(2p^{d-k-\varepsilon}) = 1/2$.

Thus there is a set $R$ of size at least $p^{d-k-\varepsilon} - \sqrt{2p^{d-k-\varepsilon}} \ge p^{d-k-\varepsilon}/2$ such that every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from $R$. ◀

Let $\varepsilon \in (0, 1)$ be given. To derive Theorem 5, we combine Lemma 18 with Lemma 19. This is a similar approach as in [5], where the authors derive a lower bound for the case $k = d-1$ by combining Lemma 18 with a construction found by Erdős in connection with Heilbronn's triangle problem [20].

Let $p$ be the largest prime number that satisfies the assumptions of Lemma 18. If such $p$ does not exist, then the statement of the theorem is trivial. By Bertrand's postulate, we have $p > (1-\lambda_d)\beta/(16d^2)$. We may assume that $p \ge p_0$, where $p_0 = p_0(d, \varepsilon, k)$ is the constant from Lemma 19, since otherwise the statement of Theorem 5 is trivial.

For $k \ge 2$ and $t := \lceil p^{d-k-\varepsilon}/2 \rceil$, let $R = \{v_1, \dots, v_t\} \subseteq \mathbb{F}_p^{d-1}$ be the set of points from Lemma 19. That is, every $(k-1)$-dimensional affine subspace of $\mathbb{F}_p^{d-1}$ contains at most $r-1$ points from $R$ for $r := \lceil k(d-k+1)/\varepsilon \rceil$. In particular, every $r$-tuple of points from $R$ contains $k+1$ affinely independent points over the field $\mathbb{F}_p$. For $k = 1$, we can set $r := 2$ and let $R$ be the whole set $\mathbb{F}_p^{d-1}$ of size $t := p^{d-k} = p^{d-1}$. Then every $r$-tuple of points from $R$ contains two affinely independent points over the field $\mathbb{F}_p$.

For $i = 1, \dots, t$, let $u_i \in \mathbb{Z}^d$ be the vector obtained from $v_i$ by adding 1 as the last coordinate. From the choice of $R$, every $r$-tuple of points from $\{u_1, \dots, u_t\}$ contains $k+1$ points that are linearly independent over the field $\mathbb{F}_p$.

By Lemma 18, there exist an integer $1 \le j_i < p$ and a point $w_i \in \mathbb{Z}^d$ for every $i \in [t]$ such that $u_i' := j_i u_i + p w_i$ lies in $K$. We have $u_i' \equiv j_i u_i \pmod{p}$ for every $i \in [t]$ and thus every

$r$-tuple of vectors from $S := \{u'_1, \dots, u'_t\} \subseteq \mathbb{Z}^d$ contains $k+1$ linearly independent vectors over the field $\mathbb{F}_p$, and hence over $\mathbb{R}$. In other words, every $k$-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from $S$. Since $|S| = t = \lceil p^{d-k-\varepsilon}/2 \rceil$ and $p > (1-\lambda_d)\beta/(16d^2)$, we have $l(d,k,n,r) \geq \Omega_{d,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$. This completes the proof of Theorem 5.

## 5 Proof of Theorem 8

Let $d$ and $k$ be integers with $1 \leq k \leq d-1$ and let $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$. We let $\lambda_i := \lambda_i(\Lambda, K)$ for every $i \in [d]$ and assume that $\lambda_d \leq 1$. First, we observe that it is sufficient to prove the statement only for $K = B^d$, as we can then strengthen the statement to an arbitrary $K \in \mathcal{K}^d$ using John's lemma (Lemma 17) analogously as in the proof of Theorem 4.

First, we prove the upper bound. That is, we show that $\Lambda \cap B^d$ can be covered with $O_{d,k}((\lambda_{k+1}\cdots\lambda_d)^{-1})$ $k$-dimensional affine subspaces of $\mathbb{R}^d$. By Lemma 15, there is a positive integer $r = r(d,k)$ and a projection $p$ of $\mathbb{R}^d$ along $k$ vectors $b_1, \dots, b_k$ from $\Lambda$ onto a $(d-k)$-dimensional linear subspace $N$ of $\mathbb{R}^d$ such that $\Lambda \cap B^d$ is mapped to $\Lambda \cap N \cap B^d(r)$ and such that $\lambda'_i := \lambda_i(\Lambda \cap N, B^d(r) \cap N) = \Theta_{d,k}(\lambda_{i+k})$ for every $i \in [d-k]$.

For each point $z$ of $\Lambda \cap N \cap B^d(r)$, we define $A(z)$ to be the affine hull of the set $\{z, b_1 + z, \dots, b_k + z\}$. Every $A(z)$ is then a $k$-dimensional affine subspace of $\mathbb{R}^d$ and the set $\mathcal{A} := \{A(z) \colon z \in \Lambda \cap N \cap B^d(r)\}$ covers $\Lambda \cap B^d$, since $p(z) \in \Lambda \cap N \cap B^d(r)$ for every $z \in \Lambda \cap B^d$. We have $|\mathcal{A}| = |\Lambda \cap N \cap B^d(r)|$ and, since $\lambda_d \leq 1$ and $\lambda'_1 \leq \cdots \leq \lambda'_{d-k} \leq O_{d,k}(\lambda_d)$, Theorem 12 implies $|\Lambda \cap N \cap B^d(r)| \leq O_{d,k}((\lambda'_1 \cdots \lambda'_{d-k})^{-1})$. The bound $\lambda'_i \geq \Omega_{d,k}(\lambda_{i+k})$ for every $i \in [d-k]$ then gives $|\mathcal{A}| \leq O_{d,k}((\lambda_{k+1}\cdots\lambda_d)^{-1})$.

To show the lower bound, we prove that we need at least $\Omega_{d,k}((\lambda_{k+1}\cdots\lambda_d)^{-1})$ $k$-dimensional affine subspaces of $\mathbb{R}^d$ to cover $\Lambda \cap B^d$.

Let $A$ be a $k$-dimensional affine subspace of $\mathbb{R}^d$. We show that $A$ contains at most $O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$ points from $\Lambda \cap B^d$. Let $y$ be an arbitrary point from $\Lambda \cap A \cap B^d$. Then $A = L + y$, where $L$ is a $k$-dimensional linear subspace of $\mathbb{R}^d$, and $(\Lambda \cap A) - y = \Lambda \cap L$. For every $i \in [k]$, we let $\lambda'_i := \lambda_i(\Lambda \cap L, B^d(2))$ and we observe that $\lambda'_i \geq \lambda_i/2$. By Theorem 12, we have $|\Lambda \cap L \cap B^d(2)| \leq O_{d,k}((\lambda'_1 \cdots \lambda'_s)^{-1})$, where $s$ is the maximum integer $j$ from $[k]$ with $\lambda'_j \leq 1$. Since $\lambda'_i \geq \lambda_i/2$ for every $i \in [k]$, we have $|\Lambda \cap L \cap B^d(2)| \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$. For every $x \in A \cap B^d$, we have $\|x - y\| \leq \|x\| + \|y\| \leq 2$ and thus $x - y \in L \cap B^d(2)$. It follows that $(\Lambda \cap A \cap B^d) - y \subseteq \Lambda \cap L \cap B^d(2)$ and thus $|\Lambda \cap A \cap B^d| \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$.

Let $\mathcal{A}$ be a collection of $k$-dimensional affine subspaces of $\mathbb{R}^d$ that covers $\Lambda \cap B^d$. We have $|\mathcal{A}| \geq |\Lambda \cap B^d|/m$, where $m$ is the maximum of $|\Lambda \cap A \cap B^d|$ taken over all subspaces $A$ from $\mathcal{A}$. We know that $m \leq O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})$. It is a well-known fact that follows from Minkowski's second theorem (Theorem 11) that $|\Lambda \cap B^d| \geq \Omega_{d,k}((\lambda_1 \cdots \lambda_d)^{-1})$. Thus we obtain

$$|\mathcal{A}| \geq \frac{|\Lambda \cap B^d|}{m} \geq \frac{\Omega_{d,k}((\lambda_1 \cdots \lambda_d)^{-1})}{O_{d,k}((\lambda_1 \cdots \lambda_k)^{-1})} \geq \Omega_{d,k}((\lambda_{k+1} \cdots \lambda_d)^{-1}),$$

which finishes the proof of Theorem 8.

## 6 Proofs of Theorems 9 and 10

We now improve the lower bounds from Theorem 3 on the number of point-hyperplane incidences. We use essentially the same construction as Brass and Knauer [6].

Assume that we are given integers $d$ and $k$ with $0 \leq k \leq d-2$ and let $\varepsilon$ be a real number in $(0,1)$. Let $\delta = \delta(d, \varepsilon, k) \in (0,1)$ be a sufficiently small constant. By (1), there is a positive integer $r_1 = r_1(d, \delta, k)$ and a constant $c_1 = c_1(d, \delta, k)$ such that for every $s \in \mathbb{N}$ there is a subset $P$ of $\mathbb{Z}^d \cap B^d(s)$ of size $c_1 \cdot s^{d-k-\delta}$ such that every $k$-dimensional affine subspace of $\mathbb{R}^d$ contains at most $r_1 - 1$ points from $P$. In the case $k = 0$, we can clearly obtain the stronger bound $c_1 \cdot s^d$.

By Corollary 6, there is a positive integer $r_2 = r_2(d, \delta, k)$ and a constant $c_2 = c_2(d, \delta, k)$ such that for every $t \in \mathbb{N}$ there is a subset $N'$ of $\mathbb{Z}^d \cap B^d(t)$ of size $c_2 \cdot t^{d(k+1-\delta)/(d-1)}$ such that every $(d-k-1)$-dimensional linear subspace contains at most $r_2 - 1$ points from $N'$. In particular, every 1-dimensional linear subspace contains at most $r_2 - 1$ points from $N'$ and thus there is a set $N \subseteq N'$ of size $|N| = |N'|/(r_2 - 1) = c_2 \cdot t^{d(k+1-\delta)/(d-1)}/(r_2 - 1)$ containing only primitive vectors. We note that for $k = 0$ we can apply Theorem 1 instead of Corollary 6 and obtain the stronger bound $|N| = c_2 \cdot t^{d/(d-1)}/(r_2 - 1)$. We let $\mathcal{H}$ be the set of hyperplanes in $\mathbb{R}^d$ with normal vectors from $N$ such that every hyperplane from $\mathcal{H}$ contains at least one point of $P$.

We show that the graph $G(P, \mathcal{H})$ does not contain $K_{r_1,r_2}$. If there is an $r_2$-tuple of hyperplanes from $\mathcal{H}$ with a nonempty intersection, then these hyperplanes have distinct normal vectors that span a linear subspace of dimension at least $d - k$ by the choice of $N$. The intersection of these hyperplanes is thus an affine subspace of dimension at most $k$. From the definition of $P$, it contains at most $r_1 - 1$ points from $P$.

We set $n := c_1 \cdot s^{d-k-\delta}$ and $m := \frac{3c_2}{r_2-1} \cdot s \cdot t^{d(k+2-1/d-\delta)/(d-1)}$. Then we have $|P| = n$. For every $p \in P$ and $z \in N$, we have $\langle p, z\rangle \in \mathbb{Z}$ and $|\langle p, z\rangle| \le \|p\|\|z\| \le st$ by the Cauchy–Schwarz inequality. Thus every point $z$ from $N$ is the normal vector of at most $2st + 1 \le 3st$ hyperplanes from $\mathcal{H}$. It follows that

$$|\mathcal{H}| \le 3st|N| = 3st\frac{c_2 \cdot t^{d(k+1-\delta)/(d-1)}}{r_2 - 1} = \frac{3c_2}{r_2 - 1} \cdot s \cdot t^{d(k+2-1/d-\delta)/(d-1)} = m.$$

From the definition of $\mathcal{H}$, the number of incidences between $P$ and $\mathcal{H}$ is at least

$$\begin{aligned} |P||N| = n \cdot \frac{c_2 \cdot t^{d(k+1-\delta)/(d-1)}}{r_2 - 1} &= \Omega_{d,\varepsilon,k}\left(n \cdot (m/s)^{(k+1-\delta)/(k+2-1/d-\delta)}\right) \\ &= \Omega_{d,\varepsilon,k}\left(n^{1-(k+1-\delta)/((k+2-1/d-\delta)(d-k-\delta))} m^{(k+1-\delta)/(k+2-1/d-\delta)}\right) \\ &\ge \Omega_{d,\varepsilon,k}\left(n^{1-(k+1)/((k+2-1/d)(d-k))-\varepsilon} m^{(k+1)/(k+2-1/d)-\varepsilon}\right), \end{aligned} \tag{6}$$

where the last inequality holds for $\delta$ sufficiently small with respect to $d$, $\varepsilon$, and $k$. This finishes the proof of Theorem 10.

To maximize the number of incidences in the diagonal case, we choose $k := \lfloor \frac{d-2}{2} \rfloor$. For $d$ odd, we then have at least

$$\Omega_{d,\varepsilon}\left(n^{1-2(d-1)/((d+1-2/d)(d+3))-\varepsilon} m^{(d-1)/(d+1-2/d)-\varepsilon}\right)$$

incidences by (6). By duality, we may obtain a symmetrical expression by averaging the exponents. Then we obtain

$$\mathrm{I}(P, \mathcal{H}) \ge \Omega_{d,\varepsilon}\left((mn)^{(d^2+3d+3)/(d^2+5d+6)-\varepsilon}\right) = \Omega_{d,\varepsilon}\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right).$$

For $d$ even, the choice of $k$ implies that the number of incidences is at least

$$\Omega_{d,\varepsilon}\left(n^{1-2d/((d+2-2/d)(d+2))-\varepsilon} m^{d/(d+2-2/d)-\varepsilon}\right)$$

by (6). Using the averaging argument, we obtain

$$\begin{aligned} \mathrm{I}(P,\mathcal{H}) &\geq \Omega_{d,\varepsilon}\left((mn)^{(d^3+2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) \\ &= \Omega_{d,\varepsilon}\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right). \end{aligned}$$

This completes the proof of Theorem 9. For $d \leq 3$, we have $k = 0$ and thus we can get rid of the $\varepsilon$ in the exponent by applying the stronger bounds on $m$ and $n$.

▶ Remark. An upper bound similar to (2) holds in a much more general setting, where we bound the maximum number of edges in $K_{r,r}$-free *semi-algebraic bipartite graphs* $G = (P \cup Q, E)$ *in* $(\mathbb{R}^d, \mathbb{R}^d)$ *with bounded description complexity* $t$ (see [11] for definitions). Fox et al. [11] showed that the maximum number of edges in such graphs with $|P| = n$ and $|Q| = m$ is at most $O_{d,\varepsilon,r,t}((mn)^{1-1/(d+1)+\varepsilon} + m + n)$ for any $\varepsilon > 0$. Theorem 9 provides the best known lower bound for this problem, as every incidence graph $G(P, \mathcal{H})$ of $P$ and $\mathcal{H}$ in $\mathbb{R}^d$ is a semi-algebraic graph in $(\mathbb{R}^d, \mathbb{R}^d)$ with bounded description complexity.

## References


1 E. Ackerman. On topological graphs with at most four crossings per edge. `http://arxiv.org/abs/1509.01932`, 2015.

2 R. Apfelbaum and M. Sharir. Large complete bipartite subgraphs in incidence graphs of points and hyperplanes. *SIAM J. Discrete Math.*, 21(3):707–725, 2007.

3 M. Balko, J. Cibulka, and P. Valtr. Covering lattice points by subspaces and counting point-hyperplane incidences. `http://arxiv.org/abs/1703.04767`, 2017.

4 W. Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Math. Ann.*, 296(4):625–635, 1993.

5 I. Bárány, G. Harcos, J. Pach, and G. Tardos. Covering lattice points by subspaces. *Period. Math. Hungar.*, 43(1–2):93–103, 2001.

6 P. Brass and C. Knauer. On counting point-hyperplane incidences. *Comput. Geom.*, 25(1–2):13–20, 2003.

7 P. Brass, W. Moser, and J. Pach. *Research problems in discrete geometry.* Springer, New York, 2005.

8 B. Chazelle. Cutting hyperplanes for Divide-and-Conquer. *Discrete Comput. Geom.*, 9(2):145–158, 1993.

9 P. Erdős. On sets of distances of $n$ points. *Amer. Math. Monthly*, 53:248–250, 1946.

10 J. Erickson. New lower bounds for hopcroft's problem. *Discrete Comput. Geom.*, 16(4):389–418, 1996.

11 J. Fox, J. Pach, A. Sheffer, and A. Suk. A semi-algebraic version of Zarankiewicz's problem. `http://arxiv.org/abs/1407.5705`, 2014.

12 T. Hagerup and C. Rüb. A guided tour of Chernoff bounds. *Inform. Process. Lett.*, 33(6):305–308, 1990.

13 M. Henk. Successive minima and lattice points. *Rend. Circ. Mat. Palermo (2) Suppl.*, 70(I):377–384, 2002.

14 F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays, presented to R. Courant on his 60th birthday, January 8, 1948*, pages 187–204. Interscience Publ., New York, 1948.

15 H. Lefmann. Extensions of the No-Three-In-Line Problem. `www.tu-chemnitz.de/informatik/ThIS/downloads/publications/lefmann_no_three_submitted.pdf`, 2012.

16 K. Mahler. Ein Übertragungsprinzip für konvexe Körper. *Časopis Pěst. Mat. Fys.*, 68:93–102, 1939.

17 J. Matoušek. *Lectures on Discrete Geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

18 H. Minkowski. *Geometrie der Zahlen.* Leipzig, Teubner, 1910.

19 János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17:427–439, 1997.

20 K. F. Roth. On a problem of Heilbronn. *J. London Math. Soc.*, 26:198–204, 1951.

21 Adam Sheffer. Lower bounds for incidences with hypersurfaces. *Discrete Anal.*, 2016. Paper No. 16, 14.

22 C. L. Siegel and K. Chandrasekharan. *Lectures on the geometry of numbers.* Springer-Verlag, Berlin, 1989.

23 E. Szemerédi and W. T. Trotter Jr. Extremal problems in discrete geometry. *Combinatorica*, 3(3–4):381–392, 1983.