Computing the Geometric Intersection Number of Curves*†

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Abstract

The geometric intersection number of a curve on a surface is the minimal number of self-intersections of any homotopic curve, i.e., of any curve obtained by continuous deformation. Given a curve c represented by a closed walk of length at most ℓ on a combinatorial surface of complexity n we describe simple algorithms to (1) compute the geometric intersection number of c in $O(n+\ell^2)$ time, (2) construct a curve homotopic to c that realizes this geometric intersection number in $O(n+\ell^4)$ time, (3) decide if the geometric intersection number of c is zero, i.e., if c is homotopic to a simple curve, in $O(n+\ell\log^2\ell)$ time.

To our knowledge, no exact complexity analysis had yet appeared on those problems. An optimistic analysis of the complexity of the published algorithms for problems (1) and (3) gives at best a $O(n+g^2\ell^2)$ time complexity on a genus g surface without boundary. No polynomial time algorithm was known for problem (2). Interestingly, our solution to problem (3) is the first quasi-linear algorithm since the problem was raised by Poincaré more than a century ago. Finally, we note that our algorithm for problem (1) extends to computing the geometric intersection number of two curves of length at most ℓ in $O(n+\ell^2)$ time.

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1 Introduction

Let S be a surface. Two closed curves $\alpha,\beta:\mathbb{R}/\mathbb{Z}\to S$ are **freely homotopic**, written $\alpha\sim\beta$, if there exists a continuous map $h:[0,1]\times\mathbb{R}/\mathbb{Z}$ such that $h(0,t)=\alpha(t)$ and $h(1,t)=\beta(t)$ for all $t\in\mathbb{R}/\mathbb{Z}$. Assuming the curves are in general position, their number of intersections is

$$|\alpha\cap\beta| = |\{(t,t') \mid t,t'\in\mathbb{R}/\mathbb{Z} \text{ and } \alpha(t)=\beta(t')\}|.$$

Their **geometric intersection number** only depends on their free homotopy classes and is defined as

$$i(\alpha,\beta) = \min_{\alpha'\sim\alpha,\beta'\sim\beta} |\alpha'\cap\beta'|.$$

Likewise, the number of self-intersections of α is given by

$$\frac{1}{2}|\{(t,t') \mid t\neq t'\in\mathbb{R}/\mathbb{Z} \text{ and } \alpha(t)=\alpha(t')\}|,$$

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and its minimum over all the curves freely homotopic to α is its **geometric self-intersection number** $i(\alpha)$. Note the one half factor that comes from the identification of (t, t') with (t', t).

The geometric intersection number is an important parameter that allows to stratify the set of homotopy classes of curves on a surface. The surface is usually endowed with a hyperbolic metric, implying that each homotopy class is identified by its unique geodesic representative. Extending a former result by Mirzakhani [24], Sapir [31, 25] has recently provided upper and lower bounds for the number of closed geodesics with bounded length and bounded geometric intersection number. Chas and Lalley [6] also proved that the distribution of the geometric intersection number with respect to the word length approaches the Gaussian distribution as the length grows to infinity. Other more experimental results were obtained with the help of a computer to show the existence of length-equivalent homotopy classes with distinct geometric intersection numbers [5]. Hence, for both theoretical and practical reasons various aspects of the computation of geometric intersection numbers have been studied in the past including the algorithmic ones. Nonetheless, all the previous approaches rely on rather complex mathematical arguments and to our knowledge no exact complexity analysis has yet appeared. In this paper, we make our own the words of Dehn who noted that the metric on words (on some basis of the fundamental group of the surface) can advantageously replace the hyperbolic metric [12]. We propose a combinatorial framework that leads to simple algorithms of low complexity to compute the geometric intersection number of curves or to test if this number is zero. Our approach is based on the computation of canonical forms as recently introduced in the purpose of testing whether two curves are homotopic [22, 15]. Canonical forms are instances of combinatorial geodesics who share nice properties with the geodesics of a hyperbolic surface. On such surfaces each homotopy class contains a unique geodesic that moreover minimizes the number of self-intersections. Although a combinatorial geodesic is generally not unique in its homotopy class, it must stay at distance one from its canonical representative and a careful analysis of its structure leads to the first result of the paper.

▶ **Theorem 1.** *Given two curves represented by closed walks of length at most ℓ on an orientable combinatorial surface of complexity n we can compute the geometric intersection number of each curve or of the two curves in $O(n + \ell^2)$ time.*

As usual the complexity of a combinatorial surface stands for its total number of vertices, edges and faces. A key point in our algorithm is the ability to compute the primitive root of a canonical curve c in linear time. This is a curve r that is not homotopic to a proper power of any other curve and such that $c \sim r^k$ for some integer k. We next provide an algorithm to compute an actual curve immersion – its combinatorial description is part of our combinatorial framework – that minimizes the number of self-intersections in its homotopy class.

▶ **Theorem 2.** *Let c be a closed walk of length ℓ in canonical form. We can compute a combinatorial immersion with $i(c)$ crossings in $O(\ell^4)$ time.*

We also propose a nearly optimal algorithm that answers an old problem studied by Poincaré [29, §4]: decide if the geometric intersection number of a curve is null, that is if the curve is homotopic to a simple curve.

▶ **Theorem 3.** *Given a curve represented by a closed walk of length ℓ on an orientable combinatorial surface of complexity n we can decide if the curve is homotopic to a simple curve in $O(n + \ell \log^2 \ell)$ time.*

We emphasize that our results represent significant progress with respect to the state of the art. No precise analysis appeared in the previously proposed algorithms [2, 7, 8, 9, 23, 11, 28, 18] concerning Theorems 1 or 3. An optimistic analysis of what seems the most efficient approach [23, Th. 3.7], although particularly complex, gives at best a quadratic time complexity for computing the geometric intersection number on a genus g surface without boundary, assuming that the curves are primitive. Schaefer et al. [32] propose an efficient computation of the geometric intersection number of curves represented by normal coordinates in a triangulated surface. However, their approach is limited to *simple* input curves. Apart from a recent algorithm by Aretinnes [1], which is restricted to surfaces with *nonempty* boundary, we know of no polynomial time algorithm for Theorem 2. Finally, Theorem 3 states the first quasi-linear algorithm for detecting homotopy classes of simple curves since the problem was raised by Poincaré more than a century ago [29, §4].

Section 2 presents our general simple strategy to compute the geometric intersection number. We introduce our combinatorial framework in Sections 3 - 5. The proof of Theorem 1 is given in the next three sections where the case of non-primitive curves is also treated. The computation of a minimally crossing immersion is presented in Section 9. We finally propose a simple algorithm to detect and embed curves that are homotopic to simple curves (Theorem 3) in Section 10. Due to space limitations most proofs are deferred to the full arXiv version of the paper.

2 Our strategy for counting intersections

Following Poincaré's original approach we represent the surface S as the hyperbolic quotient surface $\mathbb{D}/\Gamma$ where Γ is a discrete group of hyperbolic motions of the Poincaré disk $\mathbb{D}$. We denote by $p : \mathbb{D} \to \mathbb{D}/\Gamma = S$ its universal covering map. Any closed curve $\alpha : \mathbb{R}/\mathbb{Z} \to S$ gives rise to its infinite power $\alpha^\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to S$ that wraps around α infinitely many times. A **lift** of α is any curve $\tilde\alpha : \mathbb{R} \to \mathbb{D}$ such that $p \circ \tilde\alpha = \alpha^\infty$ where the parameter of $\tilde\alpha$ is defined up to an integer translation (we thus identify the curves $t \mapsto \tilde\alpha(t + k)$, $k \in \mathbb{Z}$). Note that $p^{-1}(\alpha)$ is the union of all the images $\Gamma \cdot \tilde\alpha$ of $\tilde\alpha$ by the motions in Γ. The curve $\tilde\alpha$ has two limit points on the boundary of $\mathbb{D}$ which can be joined by a unique hyperbolic line L. The projection $p(L)$ covers infinitely many times the unique geodesic homotopic to α. In particular, the set of pairs of limit points of all lifts of α only depends on the homotopy class of α.

No two motions of Γ have a limit point in common unless they are powers of the same motion. This can be used to show that when α is primitive its lifts are uniquely identified by their limit points [16]. Let α and β be two primitive curves. We fix a lift $\tilde\alpha$ of α and denote by $\tau \in \Gamma$ the hyperbolic motion sending $\tilde\alpha(0)$ to $\tilde\alpha(1)$. Let $\Gamma \cdot \tilde\beta$ be the set of lifts of β. We consider the subset of lifts

$$B = \{\tilde\beta' \in \Gamma \cdot \tilde\beta \mid \text{ the limit points of } \tilde\beta' \text{ and } \tilde\alpha \text{ alternate along } \partial\mathbb{D}\},$$

and we denote by B/τ the set of equivalence classes of lifts generated by the relations $\tilde\beta' \sim \tau(\tilde\beta')$.

▶ **Lemma 4** (Reinhart [30])**.** $i(\alpha, \beta) = |B/\tau|$.

In the ideal situation of hyperbolic geodesics, each intersection point of α and β corresponds precisely to a class in B/τ. When α and β are not geodesic the situation is more ambiguous and their lifts may have multiple intersection points. When dealing with combinatorial geodesics, the situation is more constrained and somehow intermediate between the hyperbolic case and the most general situation. See Figure 1.

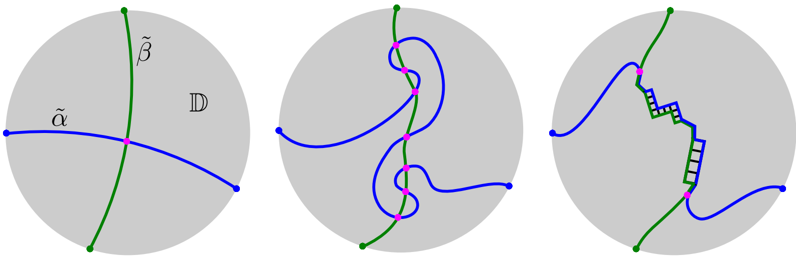

Figure 1 Left, two intersecting hyperbolic lines. Middle, two lifts of non-geodesic curves may intersect several times. Right, lifts of combinatorial geodesics.

Our strategy to compute the geometric intersection number consist of identifying B/τ with certain pairs of homotopic subpaths of α and β. We shall work in a combinatorial framework as described in the next section. In order to mimic at best hyperbolic geometry we restrict our framework to system of quads as introduced in Section 4. The structure of combinatorial geodesics in a system of quads is analysed in Section 5.

3 Combinatorial framework

Combinatorial surfaces. As usual in computational topology, we model a surface by a cellular embedding of a graph G in a compact topological surface S. Such a cellular embedding can be encoded by a **combinatorial surface** composed of the graph G itself together with a rotation system [26] that records for every vertex of the graph the clockwise cyclic order of the incident arcs. The **facial walks** are obtained from the rotation system by the face traversal procedure as described in [26, p.93]. In order to handle surfaces with boundaries we allow every face of G in S to be either an open disk or an annulus (open on one side). In other words G is a cellular embedding in the closure of S obtained by attaching a disk to every boundary of S. We record this information by storing a boolean for every facial walk of G indicating whether the associated face is perforated or not. All the considered graphs may have loop and multiple edges. A directed edge will be called an **arc** and each edge corresponds to two opposite arcs. We denote by a^{-1} the arc opposite to an arc a. **We only consider orientable surfaces in this paper**. Every combinatorial surface Σ can be reduced by first contracting the edges of a spanning tree and then deleting edges incident to distinct faces. The resulting **reduced surface** has a single vertex and a single face. The combinatorial surface Σ and its reduced version encode different cellular embeddings on a same topological surface.

Combinatorial curves. Consider a combinatorial surface with its graph G. A combinatorial curve (or path) c is a walk in G, i.e., an alternating sequence of vertices and arcs, starting and ending with a vertex, such that each vertex in the sequence is the target vertex of the previous arc and the source vertex of the next arc. We generally omit the vertices in the sequence. A combinatorial curve is closed when additionally the first and last vertex are equal. When no confusion is possible we shall drop the adjective combinatorial. The **length** of c is its total number of arc occurrences, which we denote by $|c|$. If c is closed, we write $c(i)$, $i \in \mathbb{Z}/|c|\mathbb{Z}$, for the vertex of index i of c and $c[i, i+1]$ for the arc joining $c(i)$ to $c(i+1)$. For convenience we set $c[i+1, i] = c[i, i+1]^{-1}$ to allow the traversal of c in reverse direction. In order to differentiate the arcs with their occurrences we denote by $[i, i\pm 1]_c$ the corresponding occurrence of the arc $c[i, i\pm 1]$ in $c^{\pm 1}$, where c^{-1} is obtained by traversing $c^1 := c$ in the opposite direction. More generally, for any non-negative integer ℓ and any sign $\varepsilon \in \{-1, 1\}$, the sequence of indices $(i, i+\varepsilon, i+2\varepsilon, \dots, i+\varepsilon\ell)$ is called an **index path** of c of length ℓ. The index path can be **forward** ($\varepsilon = 1$) or **backward** ($\varepsilon = -1$) and can be longer than c, so that an index may appear more than once in the sequence. We denote this path by $[i \xrightarrow{\varepsilon\ell}]_c$. Its **image path** is given by the arc sequence

$$c[i \xrightarrow{\varepsilon\ell}] := (c[i, i+\varepsilon], c[i+\varepsilon, i+2\varepsilon], \dots, c[\varepsilon(\ell-1), \varepsilon\ell]).$$

The image path of a length zero index path is just a vertex. A **spur** of c is a subsequence of arcs of the form (a, a^{-1}). A closed curve is **contractible** if it is homotopic to a trivial curve (i.e., a curve reduced to a single vertex). We will implicitly assume that a homotopy has fixed endpoints when applied to paths and is free when applied to closed curves.

Combinatorial immersions. A combinatorial curve may be seen as a continuous curve in general position snapped to the graph of the combinatorial surface. When doing so several parts of the continuous curve may be mapped to the same edge. In order not to lose information, one needs to record their ordering. Following the notion of a combinatorial set of loops as in [10], we thus define a **combinatorial immersion** of a set $\mathcal{C}$ of combinatorial curves as the data for each arc a in G of a left-to-right order $\preceq_a$ over all the occurrences of a or a^{-1} in the curves of $\mathcal{C}$. The only requirement is that opposite arcs should be associated with inverse orders. Let A_v be the set of occurrences of a or a^{-1} in the curves of $\mathcal{C}$, where a runs over all arcs of G with origin v. A combinatorial immersion induces for each vertex v of G a circular order $\preceq_v$ over A_v; if $a_1, \dots, a_k$ is the clockwise-ordered list of arcs of G with origin v, then $\preceq_v$ is the cyclic concatenation of the orders $\preceq_{a_1}, \dots, \preceq_{a_k}$.

Combinatorial crossings. Given an immersion $\mathcal{I}$ of two combinatorial closed curves c and d we define a **double point** of (c, d) as a pair of indices $(i, j) \in \mathbb{Z}/|c|\mathbb{Z} \times \mathbb{Z}/|d|\mathbb{Z}$ such that $c(i) = d(j)$. Likewise, a double point of c is a pair $(i, j) \in \mathbb{Z}/|c|\mathbb{Z} \times \mathbb{Z}/|c|\mathbb{Z}$ with $i \neq j$ and $c(i) = c(j)$. The double point (i, j) is a **crossing** in $\mathcal{I}$ if the pairs of arc occurrences $([i-1, i]_c, [i, i+1]_c)$ and $([j-1, j]_d, [j, j+1]_d)$ are linked in the $\preceq_{c(i)}$-order, i.e., if they appear in the cyclic order

$$\cdots [i, i-1]_c \cdots [j, j-1]_d \cdots [i, i+1]_c \cdots [j, j+1]_d \cdots,$$

with respect to $\preceq_{c(i)}$ or the opposite order. An analogous definition holds for a self-crossing of a single curve, taking $c = d$ in the above definition. The number of crossings of c and d and of self-crossings of c in $\mathcal{I}$ is denoted respectively by $i_{\mathcal{I}}(c, d)$ and $i_{\mathcal{I}}(c)$. Note that every combinatorial immersion can be realized by continuous curves with the same number of crossings. The **combinatorial self-crossing number** of c, denoted by $i(c)$, is the minimum of $i_{\mathcal{I}}(c')$ over all the combinatorial immersions $\mathcal{I}$ of any combinatorial curve c' freely homotopic[1] to c. The **combinatorial crossing number** of two combinatorial curves c and d is defined the same way taking into account crossings between c and d only. It is easily proved that $i(c)$ and $i(c, d)$ coincide with the geometric (self-)intersection number of continuous realizations of c and d.

[1] Homotopy of combinatorial curves can be defined equivalently via their continuous realizations or thanks to combinatorial homotopies based on elementary moves. See Appendix A in the arXiv version.

4 Systems of quads

Reduction to a system of quads. Let Σ be a combinatorial surface with negative Euler characteristic. We describe the construction of a system of quads for a surface without boundary. A similar construction applies when Σ has perforated faces. Following Lazarus and Rivaud [22] we start putting Σ into a standard form called a **system of quads** by Erickson and Whittlesey [15]. After reducing Σ to a surface Σ' with a single vertex v and a single face f this system of quads is obtained by adding a vertex w at the center of f, adding edges between w and all occurrences of v in the facial walk of f, and finally deleting the edges of Σ'. The graph of the resulting system of quads, called the **radial graph** [22], is bipartite. It contains two vertices, namely v and w, and $4g$ edges, where g is the genus of Σ. All its faces are quadrilaterals. Note that this system of quads is deduced from Σ by a sequence of edge contractions, deletions or insertions, including one edge subdivision to insert w. Every cycle of Σ can be modified accordingly to give a homotopic cycle in the system of quads.

▶ **Lemma 5** ([13, 22]). *Let n be the number of edges of Σ. The above construction of a system of quads can be performed in $O(n)$ time so that for every closed curve c of length ℓ in Σ, we can compute in $O(\ell)$ time a homotopic curve of length at most 2ℓ in the system of quads.*

For the rest of the paper we shall assume that **all surfaces have negative Euler characteristic**. The case of surfaces with non-negative Euler characteristic is handled in the proof of Theorem 1.

Diagrams. A **disk diagram** over the combinatorial surface Σ is a combinatorial sphere Δ with one perforated face together with a labelling of the arcs of Δ by the arcs of Σ such that

1. opposite arcs receive opposite labels,
2. the facial walk of each non-perforated face of Δ is labelled by the facial walk of some non-perforated face of Σ.

The diagram is **reduced** when no edge of Δ is incident to two non-perforated faces labelled by the same facial walk (with opposite orientations) of Σ. An **annular diagram** is defined similarly by a combinatorial sphere with two distinct perforated faces. A vertex of a diagram that is not incident to any perforated face is said **interior**.

▶ **Lemma 6** (van Kampen, See [15, Sec. 2.4]). *A cycle of Σ is contractible if and only if it is the label of the facial walk of the perforated face of a reduced disk diagram over Σ. Two cycles are freely homotopic if and only if the facial walks of the two perforated faces of a reduced annular diagram over Σ are labelled by these two cycles respectively.*

Note that two non-perforated faces that are adjacent and consistently oriented in a reduced diagram are labelled by adjacent faces that are consistently oriented in Σ. Moreover, the degree of an interior vertex of the diagram is a multiple of the degree of the corresponding vertex in Σ. In the sequel, **all the considered diagrams will be supposed reduced**.

Spurs, brackets and canonical curves. Thanks to Lemma 5 we may assume that our combinatorial surface Σ is a system of quads. Moreover, the construction of this system of quads with the assumption on the Euler characteristic implies that **all interior vertices have degree at least 8**. Following the terminology of Erickson and Whittlesey [15], we define the **turn** of a pair of arcs (a_1, a_2) sharing their origin vertex v as the number of face

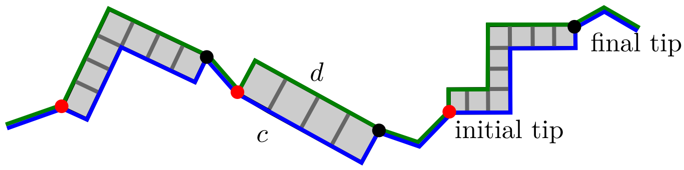

Figure 2 A disk diagram for two homotopic paths c and d composed of paths and staircases.

corners between a_1 and a_2 in clockwise order around v. Hence, if v is a vertex of degree d in Σ, the turn of (a_1, a_2) is an integer modulo d that is zero when $a_1 = a_2$. The **turn sequence** of a subpath $(a_i, a_{i+1}, \ldots, a_{i+j-1})$ of a closed curve of length ℓ is the sequence of $j+1$ turns of $(a_{i+k}^{-1}, a_{i+k+1})$ for $-1 \leq k < j$, where indices are taken modulo ℓ. The subpath may have length ℓ, thus leading to a sequence of $\ell+1$ turns. Note that the turn of $(a_{i+k}^{-1}, a_{i+k+1})$ is zero precisely when (a_{i+k}, a_{i+k+1}) is a spur. A **bracket** is any subpath whose turn sequence has the form 12^*1 or $\bar{1}\bar{2}^*\bar{1}$, where t^* stands for a possibly empty sequence of turns t and $\bar{x}$ stands for $-x$. It follows from Lemma 6 and a simple combinatorial Gauss-Bonnet theorem [17] that

▶ **Theorem 7** ([17, 15]). *A nontrivial contractible closed curve on a system of quads must have either a spur or four brackets. Moreover, if the curve is the label of the boundary walk (i.e., of the facial walk of the perforated face) of a disk diagram with at least one interior vertex, then the curve must have either a spur or five brackets.*

Lazarus and Rivaud [22] have introduced a canonical form for every nontrivial free homotopy class of closed curves in a system of quads. In particular, two curves are freely homotopic if and only if their **canonical forms** are equal (up to a circular shift of their vertex indices). It was further characterized by Erickson and Whittlesey [15] in terms of turns and brackets. It is the unique homotopic curve that contains no spurs or brackets and whose turning sequence contains no -1's and contains at least one turn that is not -2.

▶ **Theorem 8** ([22, 15]). *The canonical form of a closed curve of length ℓ on a system of quads can be computed in $O(\ell)$ time.*

5 Geodesics

The canonical form is an instance of a **combinatorial geodesic**, i.e., a curve that contains no spurs or brackets. The canonical form is the rightmost homotopic geodesic. The definitions of a geodesic and of a canonical form extend trivially to paths. In particular, the canonical form of a path is the unique homotopic path that contains no spurs or brackets and whose turning sequence contains no -1's. Although we cannot claim in general the uniqueness of geodesics in a homotopy class, homotopic geodesics are almost equal and have the same length. Specifically, define a **(quad) staircase** as a planar sequence of quads obtained by stitching an alternating sequence of rows and columns of quads to get the shape of a staircase. See Fig. 2. Assuming that the staircase goes up from left to right, we define the **initial tip** of a quad staircase as the lower left vertex of the first quad in the sequence. The **final tip** is defined as the upper right vertex of the last quad.

A **closed staircase** is obtained by identifying the two vertical arcs incident to the initial and final tips of a staircase.

▶ **Theorem 9.** *Let c, d be two non-trivial homotopic combinatorial geodesics. If c, d are closed curves, then they label the two boundary cycles of an annular diagram composed of a* *unique closed staircase or of an alternating sequence of paths (possibly reduced to a vertex) and quad staircases connected through their tips. Likewise, if c, d are paths, then the closed curve $c \cdot d^{-1}$ labels the boundary of a disk diagram composed of an alternating sequence of paths (possibly reduced to a vertex) and quad staircases connected through their tips.*

▶ **Corollary 10.** *With the hypothesis of Theorem 9, c and d have equal length which is minimal among homotopic curves. Moreover, c and d have no index path whose image path is contractible.*

The next two lemmas follow directly from the characterization of geodesics and canonical forms in terms of spurs, brackets and turns.

▶ **Lemma 11.** *The image path of any index path of a combinatorial geodesic is geodesic. If the combinatorial geodesic is in canonical form, so is the image path.*

▶ **Lemma 12.** *Likewise, any power c^k of a combinatorial closed geodesic c is also a combinatorial geodesic. Moreover, if c is in canonical form, so is c^k.*

6 Crossing Double-paths

Let c, d be two combinatorial closed curves on a combinatorial surface. A **double-path** of (c, d) of length ℓ is a pair of forward index paths $([i \xrightarrow{\ell}]_c, [j \xrightarrow{\ell}]_d)$ with the same image path $c[i \xrightarrow{\ell}] = d[j \xrightarrow{\ell}]$. If $\ell = 0$ then the double path is just a double point. A double path of c is defined similarly, taking $c = d$ and assuming $i \neq j$. The next Lemma follows from Lemma 11.

▶ **Lemma 13.** *Let $[i \xrightarrow{\ell}]_c$ and $[j \xrightarrow{k}]_d$ be forward index paths of two canonical curves c and d such that the image paths $c[i \xrightarrow{\ell}]$ and $d[j \xrightarrow{k}]$ are homotopic. Then $k = \ell$ and $([i \xrightarrow{\ell}]_c, [j \xrightarrow{\ell}]_d)$ is a double path.*

A double path $([i \xrightarrow{\ell}]_c, [j \xrightarrow{\ell}]_d)$ gives rise to a sequence of $\ell + 1$ double points $(i + k, j + k)$ for $k \in [0, \ell]$. A priori a double point could occur several times in this sequence. The next two lemmas claim that this is not possible when the curves are primitive. Recall that a curve is **primitive** if its homotopy class cannot be expressed as a proper power of another class.

▶ **Lemma 14.** *A double path of a primitive combinatorial curve c cannot contain a double point more than once in its sequence. In particular, a double path of c must be strictly shorter than c.*

▶ **Lemma 15.** *Let c and d be two non-homotopic primitive combinatorial curves. A double path of (c, d) cannot contain a double point more than once in its sequence. Moreover, the length of a double path of (c, d) must be less than $|c| + |d| - 1$.*

A double path whose index paths cannot be extended is said **maximal**. As an immediate consequence of Lemmas 14 and 15 we have:

▶ **Corollary 16.** *The maximal double paths of a primitive curve or of two primitive curves in canonical form induce a partition of the double points of the curves.*

Let (i, j) and $(i + \ell, j + \ell)$ be the first and the last double points of a maximal double path of (c, d), possibly with $c = d$. When $\ell \geq 1$ the arcs $c[i, i - 1]$, $d[j, j - 1]$, $c[i, i + 1]$ must be pairwise distinct because canonical curves have no spurs, and similarly for the three arcs $c[i + \ell, i + \ell + 1]$, $d[j + \ell, j + \ell + 1]$, $c[i + \ell, i + \ell - 1]$. We declare the maximal double

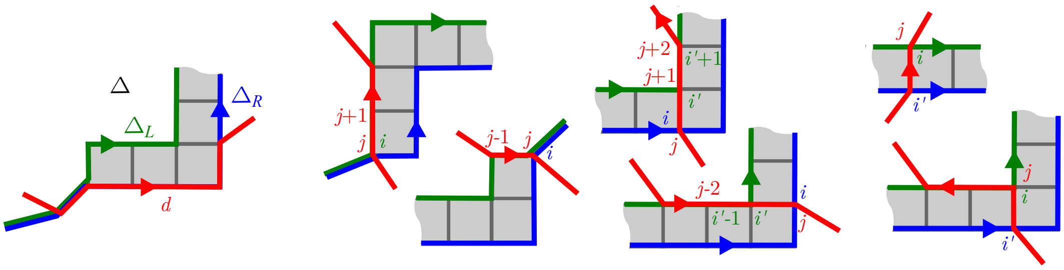

Figure 3 Left, a typical crossing double path in $\mathcal{D}_+$. Middle, four configurations in $\mathcal{D}_0$. Right, two configurations in $\mathcal{D}_-$.

path to be a **crossing double path** if the circular ordering of the first three arcs at $c(i)$ and the circular ordering of the last three arcs at $c(i+\ell)$ are either both clockwise or both counterclockwise with respect to the rotation system of the system of quads. When $\ell = 0$, that is when the maximal double path is reduced to the double point (i, j), we require that the arcs $c[i, i-1], d[j, j-1], c[i, i+1], d[j, j+1]$ are pairwise distinct and appear in this circular order, or its opposite, around the vertex $c(i) = d(j)$.

7 Counting intersections combinatorially: the primitive case

Let c, d be primitive combinatorial curves such that d is canonical and let c_R and c_L^{-1} be the canonical curves homotopic to c and c^{-1} respectively. We denote by Δ the annular diagram with left and right boundaries Δ_L and Δ_R corresponding to c_L and c_R given by Theorem 9. We consider the following set of double paths:

- $\mathcal{D}_+$ is the set of crossing double paths of positive length of c_R and d,
- $\mathcal{D}_0$ is the set of crossing double paths (i, j) of zero length of c_R and d such that either
  - the two boundaries of Δ coincide at $\Delta_L(i) = \Delta_R(i)$ and $d[j-1, j] = c_L[i-1, i]$ or $d[j, j+1] = c_L[i, i+1]$, or
  - one of $d[j, j-1]$ or $d[j, j+1]$ is the label of a spoke $(\Delta_R(i), \Delta_L(i'))$ of Δ and $d[j-2, j-1] = c_L[i'-1, i']$ in the first case or $d[j+1, j+2] = c_L[i', i'+1]$ in the other case.
- $\mathcal{D}_-$ is the set of crossing double paths $([i \xrightarrow{\ell}]_{c_L^{-1}}, [j \xrightarrow{\ell}]_d)$ $(\ell \geq 0)$ of c_L^{-1} and d such that *none* of the following situations occurs:
  - the two boundaries of Δ coincide at $\Delta_L^{-1}(i) = \Delta_R(i')$ and $d[j-1, j] = c_R[i'-1, i']$,
  - the two boundaries of Δ coincide at $\Delta_L^{-1}(i+\ell) = \Delta_R(i')$ and $d[j+\ell, j+\ell+1] = c_R[i', i'+1]$,
  - $d[j-1, j]$ is the label of a spoke $(\Delta_L^{-1}(i), \Delta_R(i'))$ of Δ and $d[j-2, j-1] = c_R[i'-1, i']$,
  - $d[j+\ell, j+\ell+1]$ is the label of a spoke $(\Delta_L^{-1}(i+\ell), \Delta_R(i'))$ of Δ and $d[j+\ell+1, j+\ell+2] = c_R[i', i'+1]$.

Those definitions allow the case $c \sim d$, recalling that the index paths of a double path of c must be distinct by definition. Figure 3 depicts some configurations.

Referring to Section 2, we view the underlying surface of the system of quads Σ as a quotient $\mathbb{D}/\Gamma$ of the Poincaré disk. The system of quads lifts to a quadrangulation of $\mathbb{D}$ and the lifts of a combinatorial curve in Σ are combinatorial bi-infinite paths in this quadrangulation. By Lemma 12, if the combinatorial curve is geodesic (resp. canonical) so are its lifts. In this case, each lift is simple by Corollary 10. We fix a lift $\tilde{c_R}$ of c_R and consider the set B/τ of Lemma 4 corresponding to the classes of lifts of d whose limit points alternate with the limit points of $\tilde{c_R}$ along $\partial\mathbb{D}$.

▶ **Proposition 17.** *B/τ is in 1-1 correspondence with the disjoint union $\mathcal{D}_+ \cup \mathcal{D}_0 \cup \mathcal{D}_-$.*

The proof relies on a careful analysis of canonical paths inside a disk diagram.

8 Non-primitive curves and proof of Theorem 1

In order to finish the proof of Theorem 1, we need to handle the case of non-primitive curves. Thanks to canonical forms, computing the primitive root of a curve becomes extremely simple[2].

▶ **Lemma 18.** *Let c be a combinatorial curve of length $\ell > 0$ in canonical form. A primitive curve d such that c is homotopic to d^k for some integer k can be computed in $O(\ell)$ time.*

Proof. By Theorem 8, we may assume that c and d are in canonical form. By Lemma 12, the curve d^k is also in canonical form. The uniqueness of the canonical form implies that $c = d^k$, possibly after some circular shift of d. It follows that d is the smallest prefix of c such that c is a power of this prefix. It can be found in $O(\ell)$ time using a variation of the Knuth-Morris-Pratt algorithm to find the smallest period of a word [21]. ◀

The geometric intersection number of non-primitive curves is related to the geometric intersection number of their primitive roots. The next result is part of the folklore although we could only find references in some relatively recent papers.

▶ **Proposition 19** ([11, 18]). *Let c and d be primitive curves and let p, q be positive integers. Then,*

$$i(c^p) = p^2 \times i(c) + p - 1 \qquad \textit{and} \qquad i(c^p, d^q) = \begin{cases} 2pq \times i(c) & \textit{if } c \sim d \textit{ or } c \sim d^{-1}, \\ pq \times i(c,d) & \textit{otherwise.} \end{cases}$$

Proof of Theorem 1. Let c, d and Σ be the two combinatorial curves and the combinatorial surface as in the Theorem. We first assume that the Σ has negative Euler characteristic. We can compute the canonical forms of c, c^{-1} and d in $O(\ell)$ time after $O(n)$ time preprocessing by Lemma 5. Thanks to Lemma 18 we can further determine primitive curves c' and d' and integers p, q such that $c \sim c'^p$ and $d \sim d'^p$ in $O(\ell)$ time. We then use the formulas in Proposition 19 to deduce $i(c, d)$ and $i(c)$ from $i(c', d')$ and $i(c')$. We can thus assume that c and d are primitive. According to Proposition 17 and Lemma 4, we have

$$i(c, d) = |\mathcal{D}_+| + |\mathcal{D}_0| + |\mathcal{D}_-|$$

The set $\mathcal{D}_+$ can be constructed in $O(\ell^2)$ time. Indeed, since the maximal double paths of c and d form disjoint sets of double points by Corollary 16, we just need to traverse the grid $\mathbb{Z}/|c|\mathbb{Z} \times \mathbb{Z}/|d|\mathbb{Z}$ and group the double points into maximal double paths. Those correspond to diagonal segments in the grid that can be computed in time proportional to the size of the grid. We can also determine which double paths are crossing in the same amount of time. Likewise, we can construct the sets $\mathcal{D}_0, \mathcal{D}_-$ in $O(\ell^2)$ time. We can also compute $i(c)$ in quadratic time using that $i(c, c) = 2i(c)$.

If Σ is a sphere or a disk, then every curve is contractible and $i(c, d) = i(c) = 0$. If Σ is a cylinder, then every two curves can be made non crossing so that $i(c, d) = 0$ while $i(c) = p - 1$. Finally, if Σ is a torus, the radial graph of the system of quads can be decomposed into two loops α, β such that $c \sim \alpha^x \cdot \beta^y$ and $d \sim \alpha^{x'} \cdot \beta^{y'}$. We may then use the classical formulas: $i(c) = \gcd(x, y) - 1$ and $i(c, d) = |\det\big((x, y), (x', y')\big)|$. ◀

[2] Compare with the short-lex straight normal form and its use by Epstein and Holt [14, Sec. 3.2].

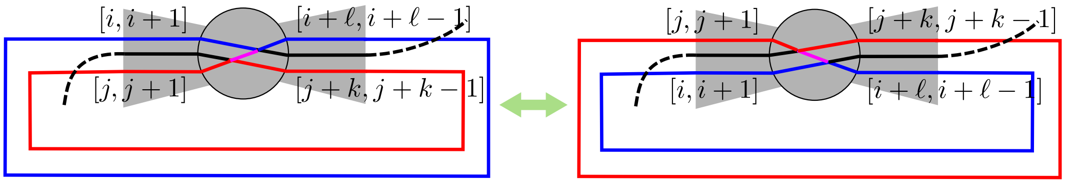

Figure 4 Right, the realization of the bigon $([i \xrightarrow{\ell}], [j \xrightarrow{k}])$ when $j = i + \ell$ and Condition 2 in the definition of a singular bigon is not satisfied. The small purple part is at the same time the beginning of the red side of the bigon and the end of the blue side. Swapping this bigon does not reduce the number of crossings.

9 Computing a minimal immersion

In the subsequent sections we deal with the self-intersection number of a single curve. We thus drop the subscript c to denote an index path $[i \xrightarrow{\ell}]$ or an arc occurrence $[i, i+1]$.

Bigons and monogons. A **bigon** of an immersion $\mathcal{I}$ of c is a pair of index paths $([i \xrightarrow{\ell}], [j \xrightarrow{k}])$ whose **sides** $c[i \xrightarrow{\ell}]$ and $c[j \xrightarrow{k}]$ have strictly positive lengths, are homotopic, and whose **tips** (i, j) and $(i + \ell, j + k)$ are combinatorial crossings for $\mathcal{I}$. A **monogon** of $\mathcal{I}$ is an index path $[i \xrightarrow{\ell}]$ of strictly positive length such that $(i, i + \ell)$ is a combinatorial crossing and the image path $c[i \xrightarrow{\ell}]$ is contractible. In agreement with the terminology of Hass and Scott [19], a bigon $([i \xrightarrow{\ell}], [j \xrightarrow{k}])$ is said **singular** if

1. its two index paths have disjoint interiors, i.e., they do not share any arc occurrence;
2. when $j = i + \ell$ the following arc occurrences

$$[i, i-1], [j, j-1], [i, i+1], [j+k, j+k+1], [j, j+1], [j+k, j+k-1]$$

do *not* appear in this order or its opposite in the circular ordering induced by $\mathcal{I}$ at $c(j)$;
3. when $i = j + k$ the following arc occurrences

$$[i, i-1], [j, j-1], [i+\ell, i+\ell-1], [i, i+1], [i+\ell, i+\ell+1], [j, j+1]$$

do *not* appear in this order or its opposite in the circular ordering induced by $\mathcal{I}$ at $c(i)$.

Remark that when the above conditions 2 and 3 are not satisfied, the bigon maps to a non-singular bigon in the continuous realization of $\mathcal{I}$. See Figure 4.

When the bigon is singular we can **swap** its two sides by exchanging the two arc occurrences $[i + p, i + p + 1]$ and $[j + \varepsilon p, i + \varepsilon(p + 1)]$ in $\mathcal{I}$, for $0 \leq p < \ell$ and $k = \varepsilon\ell$. By performing the swap on a continuous realization of $\mathcal{I}$, then projecting back to a combinatorial version, we obtain the following

▶ **Lemma 20.** *Swapping the two sides of a singular bigon of an immersion of a geodesic primitive curve decreases its number of crossings by at least two.*

Hence, by swapping singular bigons we may decrease the number of crossings until there is no more singular bigons. Since a combinatorial immersion of a primitive geodesic c cannot have a monogon by Corollary 10, it follows from the next theorem that the resulting immersion has no excess crossing.

▶ **Theorem 21** (Hass and Scott [19, Th. 4.2])**.** *An immersion of a primitive curve has excess crossing if and only if it contains a monogon or a singular bigon.*

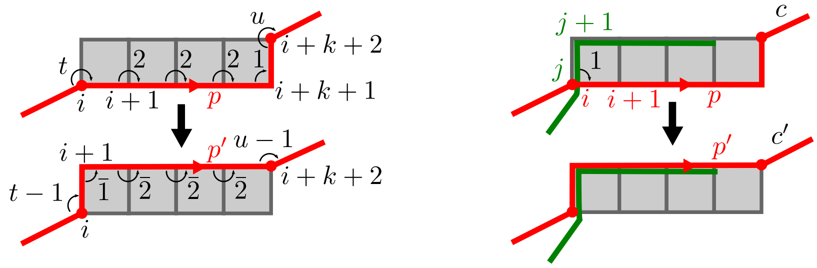

Figure 5 Left, the arc $[i, i+1]$ is switchable. Right, a switch may avoid a crossing.

In the full version of the paper it is shown that we can find a singular bigon of a given combinatorial immersion of a curve of length ℓ with excess intersection in $O(\ell^2)$ time. This allows to conclude the proof of Theorem 2 since such an immersion contains $O(\ell^2)$ crossings.

10 The unzip algorithm

We now turn to the original problem of Poincaré [29, §4], deciding whether a given curve c is homotopic to a simple curve. In the affirmative we know by Lemma 20 and Theorem 21 that some geodesic homotopic to c must have a (combinatorial) **embedding**, i.e., an immersion without crossings. Rather than swapping the sides of a singular bigon as in Lemma 20 we can choose to switch one side along the other side. This will also decrease the number of crossings if the bigon contains no other interior bigons. By considering interior-most bigons only, we can thus enforce a given edge of c to stay fixed as we remove crossings. This suggests an incremental computation of an embedding in which the image of the first arc occurrence is left unchanged: we assume that c is canonical and consider the trivial embedding of its first arc occurrence $[0, 1]$. We next insert the successive arc occurrences incrementally to obtain an embedding of the path formed by the already inserted arcs. When inserting the occurrence $[i, i+1]$ we need to compare its left-to-right order with each already inserted arc occurrence β of its supporting arc. If $\beta \neq [0, 1]$ we can use the comparison of the occurrence $[i-1, i]$ with the occurrence γ preceding β (or succeeding β if it is a backward occurrence). If $[i-1, i]$ and γ have the same supporting arc, we just propagate their relative order to $[i, i+1]$ and β. Otherwise, we use the circular ordering of the supporting arcs of $[i-1, i]$, γ and $[i, i+1]$ in order to conclude. When $\beta = [0, 1]$, we cannot use the occurrence preceding $[0, 1]$ as it is not yet inserted. We rather compare $[i, i+1]$ and $[0, 1]$ as follows. In the Poincaré disk, we consider two lifts $\tilde{d}_i$ and $\tilde{d}_0$ of c such that $\tilde{d}_i[i, i+1] = \tilde{d}_0[0, 1]$. We decide to insert $[i, i+1]$ to the left (right) of $[0, 1]$ if one of the limit points of $\tilde{d}_i$ lies to the left (right) of $\tilde{d}_0$.

After comparing $[i, i+1]$ with all the occurrences of its supporting arc, we can insert it in the correct place. If no crossings were introduced this way, we proceed with the next occurrence $[i+1, i+2]$. It may happen, however, that no matter how we insert $[i, i+1]$ in the left-to-right order of its supporting arc, the resulting immersion of $[0 \overset{i+1}{\to}]$ will have a combinatorial crossing. In order to handle this case, we first check if $[i, i+1]$ is **switchable**, i.e., if for some $k \geq 0$ and some turns t, u the subpath $p := c[i \overset{k+2}{\to}]$ has turn sequence $t2^k1u$ and the index path $[i \overset{k+2}{\to}]$ does not contain the arc occurrence $[0, 1]$. When $[i, i+1]$ is switchable we can switch p to a new subpath p' with turn sequence $(t-1)\bar{1}\bar{2}^k(u-1)$ such that p and p' bound a diagram composed of a single horizontal staircase. See Figure 5.

We then insert the arc occurrence $[i, i+1]$ and proceed with the algorithm using c' in place of c. The successive switches in the course of the computation untangle c incrementally and we call our embedding procedure the **unzip algorithm**.

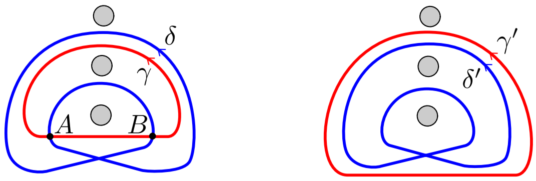

Figure 6 The plain circles represent non-contractible curves. The two curves γ and δ on the left have homotopic disjoint curves γ' and δ'. They thus have excess intersection although there is no singular bigon between the two. If $A = \gamma(0) = \delta(0)$ and $B = \gamma(u) = \delta(v)$ we nonetheless have $\delta|_{[0,v]} \sim \gamma|_{[0,1+u]}$ where $\gamma|_{[0,1+u]}$ is the concatenation of γ with $\gamma|_{[0,u]}$. In particular, $\gamma|_{[0,1+u]}$ wraps more that once around γ.

▶ **Lemma 22.** *The unzip algorithm applied to a canonical primitive curve c of length ℓ can be implemented to run in $O(\ell \log^2 \ell)$ time.*

▶ **Proposition 23.** *If $i(c) = 0$ the unzip algorithm returns an embedding of a geodesic homotopic to c.*

The proof of the proposition is far from trivial. Assuming that the the unzip algorithm returns an immersion with crossings, the rough idea is to show that c has two lifts in $\mathbb{D}$ whose limit points alternate along $\partial\mathbb{D}$.

Proof of Theorem 3. Let c be a combinatorial curve of length ℓ on a combinatorial surface of size n. We compute its canonical form in $O(n+\ell)$ time and check in linear time that c is primitive. In the negative, we conclude that either c is contractible, hence reduced to a vertex, or that c has no embedding by Proposition 19. In the affirmative, we apply the unzip algorithm to compute an immersion $\mathcal{I}$ of some geodesic c' homotopic to c. According to Proposition 23, we have $i(c) = 0$ if and only if $\mathcal{I}$ has no crossings. This is easily verified in $O(n+\ell)$ time by checking for each vertex v of the system of quads that the set of paired arc occurrences with v as middle vertex form a well-parenthesized sequence with respect to the local ordering $\prec_v$ induced by $\mathcal{I}$. We conclude the proof thanks to Lemma 22. ◀

A related problem was tackled by Chang *et al.* [4, Th. 8.2] who can decide if a given closed walk on a combinatorial surface has an embedding. In their formulation, though, the combinatorial path is fixed and they only look for the existence of a combinatorial immersion without crossing. They suggest a linear time complexity for this problem and it seems likely that we could also eliminate the $\log^2 \ell$ factor in our complexity.

11 Concluding remarks

The existence of a singular bigon claimed in Theorem 21 relies on Theorem 4.2 of Hass and Scott [19]. As noted by the authors themselves this result is "surprisingly difficult to prove". Except for this result and the recourse to some hyperbolic geometry in the general strategy of Section 2, our algorithms and proofs are purely combinatorial. Concerning Theorem 21, the existence of an immersion without bigon could be achieved in our combinatorial viewpoint by showing that if an immersion has bigons, then one of them can be swapped to reduce the number of crossings. One difficulty is that such bigons need not be singular as shown by our example in Figure 6.

If those swappable bigons could be found easily this would provide an algorithm to compute a minimally crossing immersion of two curves by iteratively swapping bigons as in Section 9. Note that in the approaches based on Reidemeister-like moves by de Graff and Schrijver [11] or by Paterson [28], the number of moves required to reach a minimal configuration is unknown. Even though Chang and Erickson [3] conjectured that a minimal configuration could be reached in a quadratic number of moves, it would remain to construct the corresponding sequence of moves efficiently. Comparatively, the number of bigon swapping would be just half the excess crossing of a given immersion. We would thus obtain a polynomial time algorithm for computing a minimally crossing immersion of two curves (there is an exponential time algorithm by a result of Neumann-Coto [27, Prop. 2.2]).

It would be interesting to see if the unzip algorithm of Section 10 yields minimally crossing curves even with curves that are not homotopic to simple curves, thus improving Theorem 2. It is also tempting to check whether the unzip algorithm applies to compute the geometric intersection number of two curves rather than a single curve. Finally, say that two curves are in the **same configuration** if there is an ambient isotopy of the surface where they live that brings one curve to the other. It was shown by Neumann-Coto [27] that every minimally crossing immersion is in the configuration of shortest geodesics for some Riemaniann metric μ, but Hass and Scott [20] gave counterexamples to the fact that we could always choose μ to be hyperbolic. *Is there an algorithm to construct or detect combinatorial immersions that have a realization in the configuration of geodesics for some hyperbolic metric?*

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