# A $\lambda$-Cut and Goal-Programming-Based Algorithm for Fuzzy-Linear Multiple-Objective Bilevel Optimization 

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#### Abstract

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## Keywords

optimization, $\lambda$-cut, bilevel, algorithm, goal-programming-based, fuzzy-linear, multiple-objective

## Disciplines

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Ya Gao, Guangquan Zhang, Jun Ma, and Jie Lu


#### Abstract

Bilevel-programming techniques are developed to handle decentralized problems with two-level decision makers, which are leaders and followers, who may have more than one objective to achieve. This paper proposes a $\lambda$-cut and goal-programming-based algorithm to solve fuzzy-linear multipleobjective bilevel (FLMOB) decision problems. First, based on the definition of a distance measure between two fuzzy vectors using $\lambda$-cut, a fuzzy-linear bilevel goal (FLBG) model is formatted, and related theorems are proved. Then, using a $\lambda$-cut for fuzzy coefficients and a goal-programming strategy for multiple objectives, a $\lambda$-cut and goal-programming-based algorithm to solve FLMOB decision problems is presented. A case study for a newsboy problem is adopted to illustrate the application and executing procedure of this algorithm. Finally, experiments are carried out to discuss and analyze the performance of this algorithm.


Index Terms-Bilevel programming, decision making, fuzzy sets, goal programming, multiple-objective linear programming, optimization.

## I. InTRODUCTION

BILEVEL-PROGRAMMING techniques, which are initiated by Von Stackelberg [39], are mainly developed to solve decentralized management problems when decision makers are in a hierarchical organization, with the upper termed as the leader and the lower termed as the follower [4]. In a bilevel problem, the control of decision factors is partitioned among the leader and the follower who seek to optimize their individual objective functions, and the corresponding decisions do not control, but affect that of the other level [1]. A leader attempts to optimize his or her objective; however, he or she must anticipate all possible responses from the follower [22]. A follower observes the leader's decision and then responds to it in a way that is individually optimal. Because the set of feasible choices available to either decision makers is interdependent, a leader's choice will affect the follower's decision, and vice versa. The investigation of bilevel problems is strongly motivated by realworld applications, and bilevel-programming techniques have been applied with remarkable success in different domains, such

[^0]as decentralized resource planning [41], electronic power market [16], logistics [43], civil engineering [2], and road-network management [14].

A large part of the research on bilevel problems has centered on their linear version, i.e., the linear bilevel problems [40]. Nearly two dozen algorithms [5], [9], [13], [25], [34]-[36] for linear bilevel problems have been proposed. These computation algorithms can be roughly classified into three categories: the vertex-enumeration-based algorithms [5], [34], which use the important characteristic that, at least, one global optimal solution is attained at an extreme point of the constraints set; the KuhnTucker algorithms [9], [35], [36], in which a bilevel problem is transferred into a single-level problem that solves the upper level's problem while including the lower level's optimality conditions as extra constraints; and the heuristics [13], [25], which are known as global optimization techniques based on convergence analysis.

When using bilevel techniques to model real-world cases, two practical issues are frequently confronted.

First, when formulating a bilevel problem, the coefficients of objective functions and constraints are sometimes obtained through experiments or experts' understanding of the nature of those coefficients. It has been observed that, in most situations, the possible values of these coefficients are often only imprecisely or ambiguously known to the experts and cannot be described by precise values. With this observation, it would certainly be more appropriate to interpret the experts' understanding of the coefficients as fuzzy numerical data that can be represented by means of fuzzy sets [42]. Linear bilevel programming in which the coefficients are characterized by fuzzy numbers is called fuzzy-linear bilevel programming [44].

Lai [22] and Shih et al. [38] first applied a fuzzy approach to bilevel programming, although the bilevel problems addressed do not involve fuzzy coefficients. Sakawa et al. [33] have adopted the method suggested by Zimmermann [47] to make an overall satisfactory balance between both levels and developed an interactive fuzzy algorithm. This algorithm derives a satisfactory solution and updates the satisfactory degrees of decision makers with considerations of overall satisfactory balance among all levels. In our research laboratory, an approximation algorithm has been developed [15], [44], which is based on the framework building and models formatting [26], [27]. Solutions can be reached by solving associated multiple-objective bilevel (MOB) decision problem under different $\lambda$-cuts.

Second, for a bilevel decision problem, the decision makers from either level may have several objectives to be considered
simultaneously. Often, these objectives may be in conflict with each other, with any improvement in one achieved only at the expense of others. While multiobjective optimization has been well-studied in single-level decision making [3], [11], [12], [29], little research has been conducted in two levels' situations [40]. In a bilevel decision model, the selection of a solution by a leader is affected by the follower's optimal reactions at the same time. Therefore, a solution for a leader, who has multiple objectives, needs to consider both the solution for the leader's multiple objectives and the follower's decision.

For bilevel multiobjective problems, Shi and Xia [37] have presented an interactive algorithm. It first sets goals for a leader's objectives and then obtains many solutions those are close enough to the goals, which are set to be larger than some certain "satisfactoriness." Fixing the preferences from a leader, the follower's response will be obtained one by one. The final solution can be obtained once a follower's choice is near enough to that of the leader. In this method, to set a suitable "satisfactoriness" would be critical: If it were too big, there would be no solution at all, while a value that is too small would cause huge computation. However, to set a suitable "satisfactoriness" is neither easy, nor direct, which requires preliminary knowledge and profound understanding of the original problem.

When both these two practical issues are involved in bilevel decision making, the problems become fuzzy multiobjective bilevel decision problems for which only extremely limited research has been done. Zhang et al. [45] developed an algorithm to fuzzy-linear MOB (FLMOB) problems by using a $\lambda$-cut method to defuzzify fuzzy coefficients and a weighting method to combine multiple objectives into only one. As straightforward to understand and easy to implement as the weighting method is, setting a suitable weight to every individual objective is sometimes difficult. Usually, it is more rational and feasible for decision makers to set certain goals for their objectives than allocate weighting numbers to them. In such a situation, goal programming would be a suitable technique for FLMOB problems.

Goal programming, which was originally proposed by Charnes and Cooper [6] for a linear model, has been further developed by Charnes and Cooper [7], Ignizio [18], [19], and Lee [23]. For recent research on goal programming, see [17], [24], [28], [30], and [31]. Goal programming requests a decision maker to set a goal for the objective that he/she wishes to attain. A preferred solution is then defined to minimize the deviation from the goal. Therefore, goal programming seems to yield a satisfactory solution rather than an optimal one.

This research applies the idea of goal programming to FLMOB problems. Based on the formulation of an FLBG decision problem, it is proved that the solutions can be obtained by solving the corresponding linear bilevel decision problem, which can be handled easily by Kuhn-Tucker and simplex algorithms. Therefore, it is possible for the algorithm, which is developed in this research, to deal with FLMOB problems stably and effectively.

This paper is organized as follows. Following the introduction in Section I, Section II introduces related definitions and formulations. In Section III, after defining a $\lambda$-cut-based distance
measure between two fuzzy vectors and modeling an FLMOB decision problem, a $\lambda$-cut and goal-programming-based algorithm for FLMOB decision problems is presented. A case study for a newsboy problem is illustrated, and experiments are analyzed in Section IV. Conclusions and further studies are discussed in Section V.

## II. Preliminaries

In this section, some definitions and formulations used in subsequent sections are presented.

Throughout this paper, $R$ represents the set of all real numbers, $R^{n}$ is a $n$-dimensional Euclidean space, and $F^{*}(R)$ and $\left(F^{*}(R)\right)^{n}$ are the set of all finite fuzzy numbers and the set of all $n$-dimensional finite fuzzy numbers on $R^{n}$, respectively. A finite fuzzy number is a fuzzy number whose 0 -cut is an interval where ends are finite numbers.
In a bilevel decision problem, we suppose that the leader controls the vector $x \in X \subseteq R^{n}$, while the follower has the control over $y \in Y \subseteq R^{m}$. The leader moves first by selecting an $x$ in an attempt to minimize his or her objective function $F(x, y)$, which is subject to certain constraints. Then, the follower observes the leader's action and reacts by choosing a $y$ to minimize his or her own objective function $f(x, y)$ under some constraints as well. Thus, a bilevel decision problem is formatted as follows [4].

Definition 1: For $x \in X \subset R^{n}, y \in Y \subset R^{m}$, a bilevel decision problem is defined as

$$
\begin{align*}
& \min _{x \in X} F(x, y)  \tag{1a}\\
& \text { s.t. } G(x, y) \leq 0  \tag{1b}\\
& \min _{y \in Y} f(x, y)  \tag{1c}\\
& \text { s.t. } g(x, y) \leq 0 \tag{1d}
\end{align*}
$$

where $F: R^{n} \times R^{m} \rightarrow R^{s}, G: R^{n} \times R^{m} \rightarrow R^{p}, f: R^{n} \times$ $R^{m} \rightarrow R^{t}$, and $g: R^{n} \times R^{m} \rightarrow R^{q}$.

Definition 2 [32]: The $\lambda$-cut of a fuzzy set $\tilde{A}$ is defined as an ordinary set $A_{\lambda}$ so that

$$
A_{\lambda}=\left\{x \mid \mu_{\tilde{A}}(x) \geq \lambda\right\}, \quad \lambda \in[0,1] .
$$

If $A_{\lambda}$ is a nonempty bounded closed interval, it can be denoted by

$$
A_{\lambda}=\left[A_{\lambda}^{L}, A_{\lambda}^{R}\right]
$$

where $A_{\lambda}^{L}$ and $A_{\lambda}^{R}$ are the lower and upper bounds of the interval, respectively.

Definition 3 [46]: For any $n$-dimensional fuzzy vectors $\tilde{a}=$ $\left(\tilde{a_{1}}, \ldots, \tilde{a_{n}}\right), \tilde{b}=\left(\tilde{b_{1}}, \ldots, \tilde{b_{n}}\right), \tilde{a_{i}}, \tilde{b}_{i} \in F^{*}(R)$, under a certain satisfactory degree $\alpha \in[0,1]$, we define

$$
\begin{align*}
& \tilde{a} \preceq_{\alpha} \tilde{b}, \text { iff } a_{i \lambda}^{L} \leq b_{i \lambda}^{L} \quad \text { and } \quad a_{i \lambda}^{R} \leq b_{i \lambda}^{R} \\
& \quad i=1, \ldots, n \quad \forall \lambda \in[\alpha, 1] . \tag{2}
\end{align*}
$$

Definition 3 means that when comparing two fuzzy numbers, all values with membership grades smaller than $\alpha$ are neglected. When two fuzzy numbers cannot be compared under a certain
$\alpha$ by this ranking method, we can adjust $\alpha$ to a larger degree to achieve the comparison.

## III. $\lambda$-Cut and Goal-Programming-Based Algorithm for Fuzzy-Linear Multiple-Objective Bilevel Problems

## A. Definitions and Theorems

Based on the fuzzy-ranking method in Definition 3, an FLMOB decision problem is defined as follows.

Definition 4: For $x \in X \subset R^{n}, y \in Y \subset R^{m}, F: X \times Y \rightarrow$ $\left(F^{*}(R)\right)^{s}$, and $f: X \times Y \rightarrow\left(F^{*}(R)\right)^{t}$

$$
\begin{align*}
& \min _{x \in X} F(x, y)=\left(\tilde{c}_{11} x+\tilde{d}_{11} y, \ldots, \tilde{c}_{s 1} x+\tilde{d}_{s 1} y\right)^{T}  \tag{3a}\\
& \text { s.t. } \tilde{A}_{1} x+\tilde{B}_{1} y \preceq_{\alpha} \tilde{b}_{1}  \tag{3b}\\
& \min _{y \in Y} f(x, y)=\left(\tilde{c}_{12} x+\tilde{d}_{12} y, \ldots, \tilde{c}_{t 2} x+\tilde{d}_{t 2} y\right)^{T}  \tag{3c}\\
& \text { s.t. } \tilde{A}_{2} x+\tilde{B}_{2} y \preceq_{\alpha} \tilde{b}_{2} \tag{3d}
\end{align*}
$$

where $\tilde{c}_{h 1}, \quad \tilde{c}_{i 2} \in\left(F^{*}(R)\right)^{n}, \quad \tilde{d}_{h 1}, \quad \tilde{d}_{i 2} \in\left(F^{*}(R)\right)^{m}, \quad h=$ $1,2, \ldots, s, \quad i=1,2, \ldots, t, \quad \tilde{b}_{1} \in\left(F^{*}(R)\right)^{p}, \quad \tilde{b}_{2} \in\left(F^{*}(R)\right)^{q}$, $\tilde{A}_{1}=\left(\tilde{a}_{i j}\right)_{p \times n}, \quad \tilde{B}_{1}=\left(\tilde{b}_{i j}\right)_{p \times m}, \quad \tilde{A}_{2}=\left(\tilde{e}_{i j}\right)_{q \times n}, \quad$ and $\tilde{B}_{2}=$ $\left(\tilde{s}_{i j}\right)_{q \times m}, \tilde{a}_{i j}, \tilde{b}_{i j}, \tilde{e}_{i j}, \tilde{s}_{i j} \in F^{*}(R)$.

To build an FLBG model, a distance measure between two fuzzy vectors is needed. There are many important measures to compare two fuzzy numbers, such as Hausdorff distance [8], Hamming distance [10], Euclidean distance [10], and maximum distance [21]. In this paper, a certain number of $\lambda$-cuts will be used to approximate a fuzzy number. A final solution is considered to be reached when solutions under two adjacent $\lambda$-cuts are near enough. To help implement this strategy, a new distance measure between two fuzzy vectors by using $\lambda$-cuts is defined as follows.

Definition 5: Letting $\quad \tilde{a}=\left(\tilde{a}_{1}, \quad \tilde{a}_{2}, \ldots, \tilde{a}_{n}\right), \quad \tilde{b}=\left(\tilde{b}_{1}\right.$, $\left.\tilde{b}_{2}, \ldots, \tilde{b}_{n}\right)$ be $n$-dimensional fuzzy vectors, $\Phi=\left\{\alpha \leq \lambda_{0}<\right.$ $\left.\lambda_{1}<\cdots<\lambda_{l} \leq 1\right\}$ be a division of $[\alpha, 1]$, the distance between $\tilde{a}$ and $\tilde{b}$ under $\phi$ is defined as

$$
\begin{equation*}
D(\tilde{a}, \tilde{b}) \triangleq \frac{1}{l+1} \sum_{i=1}^{n} \sum_{j=0}^{l}\left\{\left|a_{i \lambda_{j}}^{L}-b_{i \lambda_{j}}^{L}\right|+\left|a_{i \lambda_{j}}^{R}-b_{i \lambda_{j}}^{R}\right|\right\} \tag{4}
\end{equation*}
$$

where $\alpha$ is a predefined satisfactory degree.
In this fuzzy-distance definition, a satisfactory degree $\alpha$ is used to give more flexibility to compare fuzzy vectors. It is possible that two fuzzy vectors might not be compared by Definition 5. For example, when we compare two fuzzy vectors $\tilde{a}$ and $\tilde{b}$, if some of the left $\lambda$-cuts of $\tilde{a}$ are less than those of $\tilde{b}$, while some right $\lambda$-cuts of $\tilde{a}$ are larger than those of $\tilde{b}$, there is no ranking relation between $\tilde{a}$ and $\tilde{b}$.

To solve this problem, we can enhance the aspiration levels of the attributes, i.e., we can adjust the satisfactory degree $\alpha$ to a point where all incomparable parts are discarded. It can be understood as a risk taken by a decision maker who neglects all values with the possibility of occurrence smaller than $\alpha$. In such a situation, a solution is supposed to be reached under this aspiration level. Therefore, normally, we take the same $\alpha$ for both objectives and constraints in one bilevel problem.

Lemma 1: For any $n$-dimensional fuzzy vectors $\tilde{a}, \tilde{b}$, and $\tilde{c}$, fuzzy distance $D$ defined earlier satisfies the following properties.

1) $D(\tilde{a}, \tilde{b})=0$, if $\tilde{a}_{i}=\tilde{b}_{i}, i=1,2, \ldots, n$.
2) $D(\tilde{a}, \tilde{b})=D(\tilde{b}, \tilde{a})$.
3) $D(\tilde{a}, \tilde{b}) \leq D(\tilde{a}, \tilde{c})+D(\tilde{c}, \tilde{b})$.

Goals set for the objectives of a leader $\left(\tilde{g}_{L}\right)$ and a follower $\left(\tilde{g}_{F}\right)$ in (3) are defined as

$$
\begin{align*}
& \tilde{g}_{L}=\left(\tilde{g}_{L 1}, \tilde{g}_{L 2}, \ldots, \tilde{g}_{L s}\right)^{T}  \tag{5a}\\
& \tilde{g}_{F}=\left(\tilde{g}_{F 1}, \tilde{g}_{F 2}, \ldots, \tilde{g}_{F t}\right)^{T} \tag{5b}
\end{align*}
$$

where $\tilde{g}_{L i}, i=1, \ldots, s, \tilde{g}_{F j}, j=1, \ldots, t$, are fuzzy numbers with membership functions of $\mu_{\tilde{g}_{L i}}$ and $\mu_{\tilde{g}_{F j}}$.

Our concern is to make the objectives of both a leader and the follower as near to their goals as possible. Using the distance measure defined in (4), we format an FLBG problem as follows.

For $x \in X \subset R^{n}, y \in Y \subset R^{m}, F: X \times Y \rightarrow\left(F^{*}(R)\right)^{s}$, and $f: X \times Y \rightarrow\left(F^{*}(R)\right)^{t}$, we have

$$
\begin{align*}
& \min _{x \in X} D\left(F(x, y), \tilde{g}_{L}\right)  \tag{6a}\\
& \text { s.t. } \tilde{A}_{1} x+\tilde{B}_{1} y \preceq_{\alpha} \tilde{b}_{1}  \tag{6b}\\
& \min _{y \in Y} D\left(f(x, y), \tilde{g}_{F}\right)  \tag{6c}\\
& \text { s.t. } \tilde{A}_{2} x+\tilde{B}_{2} y \preceq_{\alpha} \tilde{b}_{2} \tag{6d}
\end{align*}
$$

where $\tilde{A}_{1}=\left(\tilde{a}_{i j}\right)_{p \times n}, \tilde{B}_{1}=\left(\tilde{b}_{i j}\right)_{p \times m}, \tilde{A}_{2}=\left(\tilde{e}_{i j}\right)_{q \times n}, \tilde{B}_{2}=$ $\left(\tilde{s}_{i j}\right)_{q \times m}, \tilde{a}_{i j}, \tilde{b}_{i j}, \tilde{e}_{i j}, \tilde{s}_{i j} \in F^{*}(R)$, and $\alpha$ is a predefined satisfactory degree.

From Definitions 3 to 5, we transfer problem (6) into

$$
\begin{align*}
& \min _{x \in X} \triangleq \frac{1}{l+1} \sum_{h=1}^{s} \sum_{j=0}^{l}\left\{\left|c_{h 1 \lambda_{j}}^{L} x+d_{h 1 \lambda_{j}}^{L} y-g_{L h \lambda_{j}}^{L}\right|\right. \\
& \left.+\left|c_{h 1 \lambda_{j}}^{R} x+d_{h 1 \lambda_{j}}^{R} y-g_{L h \lambda_{j}}^{R}\right|\right\}  \tag{7a}\\
& \text { s.t. } A_{1}{ }_{\lambda_{j}}^{L} x+B_{1}{ }_{\lambda_{j}}^{L} y \leq b_{1}{ }_{\lambda_{j}}^{L} \\
& A_{1}{ }_{\lambda_{j}}^{R} x+B_{1}{ }_{\lambda_{j}}^{R} y \leq b_{1}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l  \tag{7b}\\
& \min _{y \in Y} \triangleq \frac{1}{l+1} \sum_{i=1}^{t} \sum_{j=0}^{l}\left\{\left|c_{i 2 \lambda_{j}}^{L} x+d_{i 2 \lambda_{j}}^{L} y-g_{F i \lambda_{j}}^{L}\right|\right. \\
& \left.+\left|c_{i 2 \lambda_{j}}^{R} x+d_{i 2 \lambda_{j}}^{R} y-g_{F i \lambda_{j}}^{R}\right|\right\}  \tag{7c}\\
& \text { s.t. } A_{2}{ }_{\lambda}^{L}{ }_{j} x+B_{2}{ }_{\lambda}^{L} y \leq b_{2}{ }_{\lambda}^{L}{ }_{j} \\
& A_{2}{ }_{\lambda_{j}}^{R} x+B_{2}{ }_{\lambda}^{R} y \leq b_{2}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l \tag{7d}
\end{align*}
$$

where $\Phi=\left\{\alpha \leq \lambda_{0}<\lambda_{1}<\cdots<\lambda_{l} \leq 1\right\}$ is a division of $[\alpha, 1]$.

For a clear understanding of the idea that is adopted, let us define

$$
v_{h 1}^{L-}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right|\right.
$$

$$
\begin{align*}
& \left.-\left(\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right)\right] \\
& v_{h 1}^{L+}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right|\right. \\
& \left.+\left(\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right)\right] \\
& v_{h 1}^{R-}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right|\right. \\
& \left.-\left(\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right)\right] \\
& v_{h 1}^{R+}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right|\right. \\
& \left.+\left(\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right)\right] \\
& h=1,2, \ldots, s \\
& v_{i 2}^{L-}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}\right|\right. \\
& \left.-\left(\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}\right)\right] \\
& v_{i 2}^{L+}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}\right|\right. \\
& \left.+\left(\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}\right)\right] \\
& v_{i 2}^{R-}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R}\right|\right. \\
& \left.-\left(\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R}\right)\right] \\
& v_{i 2}^{R+}=\frac{1}{2}\left[\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R}\right|\right. \\
& \left.+\left(\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R}\right)\right] \\
& i=1,2, \ldots, t \tag{8}
\end{align*}
$$

where $v_{h 1}^{L-}$ and $v_{h 1}^{L+}$ are deviational variables representing the underachievement and overachievement of the $h$ th goal for a leader under the left $\lambda$-cut. Terms $v_{h 1}^{R-}$ and $v_{h 1}^{R+}$ are deviational variables representing the underachievement and overachieve-
ment of the $h$ th goal for a leader under the right $\lambda$-cut. Terms $v_{i 2}^{L-}, v_{i 2}^{L+}, v_{i 2}^{R-}$, and $v_{i 2}^{R+}$ are for a follower, respectively.

Associated with the linear bilevel problem (7), we now consider the following bilevel problem.

For $\left(v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s 1}^{L-}, v_{s 1}^{L+}, v_{s 1}^{R-}, v_{s 1}^{R+}\right) \in R^{4 s}$, $X^{\prime} \subseteq X \times R^{4 s},\left(v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t 2}^{L-}, v_{t 2}^{L+}, v_{t 2}^{R-}\right.$, $\left.v_{t 2}^{R+}\right) \in R^{4 t}, \quad Y^{\prime} \subseteq Y \times R^{4 t}$, letting $x=\left(x_{1}, \ldots, x_{n}\right) \in X$, $x^{\prime}=\left(x_{1}, \ldots, x_{n}, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s 1}^{L-}, v_{s 1}^{L+}, v_{s 1}^{R-}\right.$, $\left.v_{s 1}^{R+}\right) \in X^{\prime}, y=\left(y_{1}, \ldots, y_{m}\right) \in Y, y^{\prime}=\left(y_{1}, \ldots, y_{m}, v_{12}^{L-}\right.$, $\left.v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t 2}^{L-}, v_{t 2}^{L+}, v_{t 2}^{R-}, v_{t 2}^{R+}\right) \in Y^{\prime}$, and $v_{1}, v_{2}$ : $X^{\prime} \times Y^{\prime} \rightarrow R$.

$$
\begin{align*}
& \min _{x^{\prime} \in X^{\prime}} v_{1}=\sum_{h=1}^{s}\left(v_{h 1}^{L-}+v_{h 1}^{L+}+v_{h 1}^{R-}+v_{h 1}^{R+}\right)  \tag{9a}\\
& \text { s.t. } \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y+v_{h 1}^{L-}-v_{h 1}^{L+}=\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L} \\
& \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y+v_{h 1}^{R-}-v_{h 1}^{R+}=\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R} \\
& v_{h 1}^{L-}, v_{h 1}^{L+}, v_{h 1}^{R-}, v_{h 1}^{R+} \geq 0 \\
& v_{h 1}^{L-} \cdot v_{h 1}^{L+}=0, \quad v_{h 1}^{R-} \cdot v_{h 1}^{R+}=0 \\
& h=1,2, \ldots, s \\
& A_{1}{ }_{\lambda_{j}}^{L} x+B_{1}{ }_{\lambda_{j}}^{L} y \leq b_{1}{ }_{\lambda_{j}}^{L} \\
& A_{1}{ }_{\lambda_{j}}^{R} x+B_{1}{ }_{\lambda_{j}}^{R} y \leq b_{1}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l  \tag{9b}\\
& \min _{y^{\prime} \in Y^{\prime}} v_{2}=\sum_{i=1}^{t}\left(v_{i 2}^{L-}+v_{i 2}^{L+}+v_{i 2}^{R-}+v_{i 2}^{R+}\right)  \tag{9c}\\
& \text { s.t. } \sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y+v_{i 2}^{L-}-v_{i 2}^{L+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L} \\
& \sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{R} y+v_{i 2}^{R-}-v_{i 2}^{R+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R} \\
& v_{i 2}^{L-}, v_{i 2}^{L+}, v_{i 2}^{R-}, v_{i 2}^{R+} \geq 0 \\
& v_{i 2}^{L-} \cdot v_{i 2}^{L+}=0, \quad v_{i 2}^{R-} \cdot v_{i 2}^{R+}=0 \\
& i=1,2, \ldots, t \\
& A_{2}{ }_{\lambda_{j}}^{L} x+B_{2}{ }_{\lambda}^{L}{ }_{j}^{L} y \leq b_{2}{ }_{2}^{L}{ }_{j} \\
& A_{2}{ }_{\lambda_{j}}^{R} x+B_{2}{ }_{\lambda_{j}}^{R} y \leq b_{2}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l \text {. } \tag{9d}
\end{align*}
$$

Theorem 1: Letting $\left(x_{R, *}^{R+*} y^{\prime *}\right)=\left(x^{*}, v_{11}^{L-*}, v_{11}^{L+*}, v_{11}^{R-*}\right.$, $v_{11}^{R+*}, \ldots, v_{s 1}^{L-*}, v_{s 1}^{L+*}, v_{s 1}^{R-*}, v_{s 1}^{R+*}, y^{*}, v_{12}^{L-*}, v_{12}^{L+*}, v_{12}^{R-*}$, $\left.v_{12}^{R+*}, \ldots, v_{t 2}^{L-*}, v_{t 2}^{L+*}, v_{t 2}^{R-*}, v_{t 2}^{R+*}\right)$ be the optimal solution to bilevel problem (9), $\left(x^{*}, y^{*}\right)$ is the then optimal solution to the bilevel problem defined by (7).

Proof: See the proof of Theorem 1 in the Appendix.

Adopting weighting method, (9) can be further transferred as follows:

$$
\begin{align*}
& \min _{x^{\prime} \in X^{\prime}} v_{1}^{-}+v_{1}^{+}  \tag{10a}\\
& \text {s.t. } c_{1} x+d_{1} y+v_{1}^{-}-v_{1}^{+}=\sum_{h=1}^{s} \sum_{j=0}^{l}\left(g_{L h \lambda_{j}}^{L}+g_{L h \lambda_{j}}^{R}\right) \\
& v_{1}^{-}, v_{1}^{+} \geq 0 \\
& v_{1}^{-} \cdot v_{1}^{+}=0 \\
& A_{1}{ }_{\lambda_{j}}^{L} x+B_{1}{ }_{\lambda_{j}}^{L} y \leq b_{1}{ }_{\lambda_{j}}^{L} \\
& A_{1}{ }_{\lambda_{j}}^{R} x+B_{1}{ }_{\lambda_{j}}^{R} y \leq b_{1}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l  \tag{10b}\\
& \min _{y^{\prime} \in Y^{\prime}} v_{2}^{-}+v_{2}^{+}  \tag{10c}\\
& \text {s.t. } c_{2} x+d_{2} y=\sum_{i=1}^{t} \sum_{j=0}^{l}\left(g_{F i \lambda_{j}}^{L}+g_{F i \lambda_{j}}^{R}\right) \\
& v_{2}^{-}, v_{2}^{+} \geq 0 \\
& v_{2}^{-} \cdot v_{2}^{+}=0 \\
& A_{2}^{L}{ }_{\lambda}^{j} \\
& A_{2}^{R}{ }_{\lambda}^{R} x+B_{2} \\
& \lambda_{j}^{L} y \leq B_{2}{ }_{\lambda_{j}}^{R} y \leq b_{2}{ }_{\lambda_{j}}^{R}  \tag{10d}\\
& j=0,1, \ldots, l
\end{align*}
$$

where $v_{1}^{-}=\sum_{h=1}^{s}\left(v_{h 1}^{L-}+v_{h 1}^{R-}\right), v_{1}^{+}=\sum_{h=1}^{s}\left(v_{h 1}^{L+}+v_{h 1}^{R+}\right)$, $v_{2}^{-}=\sum_{i=1}^{t}\left(v_{i 2}^{L-}+v_{i 2}^{R-}\right), \quad v_{2}^{+}=\sum_{i=1}^{t}\left(v_{i 2}^{L+}+v_{i 2}^{R+}\right), \quad c_{1}=$ $\sum_{h=1}^{s} \sum_{j=0}^{l}\left(c_{h 1 \lambda_{j}}^{L}+c_{h 1 \lambda_{j}}^{R}\right), \quad d_{1}=\sum_{h=1}^{s} \sum_{j=0}^{l}\left(d_{h 1 \lambda_{j}}^{L}+\right.$ $\left.d_{h 1 \lambda_{j}}^{R}\right), \quad c_{2}=\sum_{i=1}^{t} \sum_{j=0}^{l}\left(c_{i 2 \lambda_{j}}^{L}+c_{i 2 \lambda_{j}}^{R}\right), \quad$ and $\quad d_{2}=\sum_{i=1}^{t}$ $\sum_{j=0}^{l}\left(d_{i 2 \lambda_{j}}^{L}+d_{i 2 \lambda_{j}}^{R}\right)$. In this formula, $v_{1}^{-}$and $v_{1}^{+}$are deviational variables representing the underachievement and overachievement of goals for a leader, and $v_{2}^{-}$and $v_{2}^{+}$are deviational variables representing the underachievement and overachievement of goals for a follower, respectively.

The nonlinear conditions of $v_{1}^{-} \cdot v_{1}^{+}=0$, and $v_{2}^{-} \cdot v_{2}^{+}=0$ need not be maintained if the Kuhn-Tucker algorithm [35] together with the simplex algorithm are adopted, since only equivalence at an optimum is required. For further explanation, see [6]. Thus, problem (10) is further transformed into the following.

For $\left(v_{1}^{-}, v_{1}^{+}\right) \in R^{2}, \bar{X}^{\prime} \subseteq X \times R^{2},\left(v_{2}^{-}, v_{2}^{+}\right) \in R^{2}, \bar{Y}^{\prime} \subseteq$ $Y \times R^{2}$, letting $x=\left(x_{1}, \cdots, x_{n}\right) \in X, \overline{x^{\prime}}=\left(x_{1}, \cdots, x_{n}, v_{1}^{-}\right.$, $\left.v_{1}^{+}\right) \in \bar{X}^{\prime}, y=\left(y_{1}, \ldots, y_{m}\right) \in Y, \overline{y^{\prime}}=\left(y_{1}, \ldots, y_{m}, v_{2}^{-}, v_{2}^{+}\right) \in$ $\bar{Y}^{\prime}$, and $v_{1}, v_{2}: \bar{X}^{\prime} \times \bar{Y}^{\prime} \rightarrow F^{*}(R)$

$$
\begin{equation*}
\min _{\left(x, v_{1}^{-}, v_{1}^{+}\right) \in \bar{X}^{\prime}} v_{1}=v_{1}^{-}+v_{1}^{+} \tag{11a}
\end{equation*}
$$

$$
\begin{aligned}
& \text { s.t. } c_{1} x+d_{1} y+v_{1}^{-}-v_{1}^{+}=\sum_{h=1}^{s} \sum_{j=0}^{l}\left(g_{L h \lambda_{j}}^{L}+g_{L h \lambda_{j}}^{R}\right) \\
& A_{1}{ }_{\lambda_{j}}^{L} x+B_{1}{ }_{\lambda_{j}}^{L} y \leq b_{1}{ }_{\lambda_{j}}^{L}
\end{aligned}
$$

$$
\begin{align*}
& \quad A_{1}{ }_{\lambda_{j}}^{R} x+B_{1}{ }_{\lambda_{j}}^{R} y \leq b_{1}{ }_{\lambda_{j}}^{R} \\
& j=0,1, \ldots, l  \tag{11b}\\
& \min _{\left(y, v_{2}^{-}, v_{2}^{+}\right) \in \bar{Y}^{\prime}} v_{2}=v_{2}^{-}+v_{2}^{+}  \tag{11c}\\
& \text {s.t. } c_{2} x+d_{2} y=\sum_{i=1}^{t} \sum_{j=0}^{l}\left(g_{F i \lambda_{j}}^{L}+g_{F i \lambda_{j}}^{R}\right) \\
& A_{2}{ }_{\lambda_{j}}^{L} x+B_{2}{ }_{\lambda_{j}}^{L} y \leq b_{2}{ }_{\lambda_{j}}^{L} \\
& A_{2}{ }_{\lambda}^{R} x+B_{2}{ }_{\lambda}^{R} y \leq b_{2}{ }_{\lambda}^{R} \\
& j=0,1, \ldots, l . \tag{11d}
\end{align*}
$$

Problem (11) is a standard linear bilevel problem that can be solved by the Kuhn-Tucker algorithm [35].

## B. $\lambda$-Cut and Goal-Programming-Based Algorithm

Based on the previous analysis, the $\lambda$-cut and goal-programming-based algorithm is detailed as follows.
[Step 1] (Input):
Obtain relevant coefficients, which include

1) coefficients of (3);
2) coefficients of (5);
3) satisfactory degree: $\alpha$;
4) $\varepsilon>0$.
[Step 2] (Initialize):
Letting $k=1$, which is the counter to record current loop.
In (7), where $\lambda_{j} \in[\alpha, 1]$, letting $\lambda_{0}=\alpha$, and $\lambda_{1}=1$, respectively, then each objective will be transferred into four nonfuzzy objective functions, and each fuzzy constraint is converted into four nonfuzzy constraints.
[Step 3] (Compute):
By introducing auxiliary variables $v_{1}^{-}, v_{1}^{+}, v_{2}^{-}$, and $v_{2}^{+}$, we obtain the format of (11).

The solution $\left(x, v_{1}^{-}, v_{1}^{+}, y, v_{2}^{-}, v_{2}^{+}\right)_{2}$ of (11) is obtained by the Kuhn-Tucker algorithm.
[Step 4] (Compare):
IF $(k=1)$
THEN $\left(x, v_{1}^{-}, v_{1}^{+}, y, v_{2}^{-}, v_{2}^{+}\right)_{1}=\left(x, v_{1}^{-}, v_{1}^{+}, y, v_{2}^{-}, v_{2}^{+}\right)_{2}$;
goto [Step 5];
Else
IF $\left\|\left(x, v_{1}^{-}, v_{1}^{+}, y, v_{2}^{-}, v_{2}^{+}\right)_{2}-\left(x, v_{1}^{-}, v_{1}^{+}, y, v_{2}^{-}, v_{2}^{+}\right)_{1}\right\|<\varepsilon$ THEN goto [Step 7];

## EndIf

[Step 5] (Split):
Suppose there are $(L+1)$ nodes $\lambda_{j},(j=0,1, \ldots, L)$ in the interval $[\alpha, 1]$, insert $L$ new nodes $\delta_{t}(t=1,2 \ldots, L)$ in $[\alpha, 1]$ so that $\delta_{t}=\left(\lambda_{t-1}+\lambda_{t}\right) / 2$.
[Step 6] (Loop):
$k=k+1 ;$
goto [Step 3].
[Step 7] (Output):
$(x, y)_{2}$ is obtained as the final solution.

## IV. CASE Study and Experiments

In this section, we apply the $\lambda$-cut and goal-programmingbased algorithm proposed in this paper on a real-world "newsboy problem" to illustrate its operation and application. Experiments are then carried out on some numerical examples with different scales to test the algorithm's performance.

## A. Case Study

A classical newsboy problem is to find a newspaper's order quantity to maximize the profit of a newsboy (newspaper retailer) [20]. In a real-world situation, both a newspaper manufacturer and a retailer have more than one concern. Using an FLMOB model, a newsboy problem is expressed as follows: The leader, which is a manufacturer, controls the decision variable of the wholesale price $(x)$, while the follower, which is a retailer, decides his or her order quantity ( $y$ ). The manufacturer has two main objectives: to maximize the net profits, which is represented by $F_{1}(x, y)$, and to maximize the newspaper quality, by $F_{2}(x, y)$, but subject to some constraints, including the requirements of material, marketing cost, and labor cost. The retailer also has two objectives to achieve: to minimize his or her purchase cost, which is represented by $f_{1}(x, y)$, and to minimize the working hours, by $f_{2}(x, y)$ under his own constraints. Meanwhile, both the manufacturer and the retailer will set goals ( $g_{L 1}, g_{L 2}, g_{F 1}, g_{F 1}$ ) for each of their two objectives.

When modeling this multiobjective bilevel decision problem, the main difficulty is to establish coefficients of the objectives and constraints for both the leader and the follower. We can only estimate some values for material cost, labor cost, etc., according to our experience and previous data. For some items, the values can only be assigned by linguistic terms as about $\$ 1000$. This is a common case in any organizational decision practice. By using fuzzy numbers to describe these uncertain values in coefficients, an FLMOB model can be established for this decision problem.

To illustrate the $\lambda$-cut and goal-programming-based algorithm, which is introduced in Section III, this newsboy problem will be solved step by step as follows.
[Step 1] (Input the relevant coefficients):

1) Coefficients of (3).

The newsboy problem is formatted as

$$
\begin{aligned}
& \text { Leader }: \max _{x \in X} F_{1}(x, y)=\tilde{6} x+\tilde{3} y \\
& \qquad \max _{x \in X} F_{2}(x, y)=\tilde{-} 3 x+\tilde{6} y \\
& \text { s.t. } \tilde{-1} x+\tilde{3} y \leq \tilde{21} \\
& \text { Follower : } \min _{y \in Y} f_{1}(x, y)=\tilde{4} x+\tilde{3} y \\
& \qquad \min _{y \in Y} f_{2}(x, y)=\tilde{3} x+\tilde{1} y \\
& \text { s.t. } \tilde{-1} x \tilde{-3} y \leq \tilde{27}
\end{aligned}
$$

where $x \in R^{1}, y \in R^{1}$, and $X=x \geq 0, Y=y \geq 0$.

The membership functions for this FMOLB are as follows:

$$
\begin{aligned}
& \mu_{\tilde{6}}(x)= \begin{cases}0, & x<5 \\
\frac{x^{2}-25}{11}, & 5 \leq x<8 \\
1, & x=6 \\
\frac{64-x^{2}}{28}, & 6<x \leq 8 \\
0, & x>8\end{cases} \\
& \mu_{\tilde{3}}(x)= \begin{cases}0, & x<2 \\
\frac{x^{2}-4}{5}, & 2 \leq x<3 \\
1, & x=3 \\
\frac{25-x^{2}}{16}, & 3<x \leq 5 \\
0, & x>5\end{cases} \\
& \mu_{-3}(x)= \begin{cases}0, & x<-4 \\
\frac{16-x^{2}}{7}, & -4 \leq x<-3 \\
1, & x=-3 \\
\frac{x^{2}-1}{8}, & -3<x \leq-1 \\
0, & x>-1\end{cases} \\
& \mu_{\tilde{4}}(x)= \begin{cases}0, & x<3 \\
\frac{x^{2}-9}{7}, & 3 \leq x<4 \\
1, & x=4 \\
\frac{36-x^{2}}{20}, & 4<x \leq 6 \\
0, & x>6\end{cases} \\
& \mu_{\tilde{1}}(x)= \begin{cases}0, & x<0.5 \\
\frac{x^{2}-0.25}{0.75}, & 0.5 \leq x<1 \\
1, & x=1 \\
\frac{4-x^{2}}{3}, & 1<x \leq 2 \\
0, & x>2\end{cases} \\
& \mu_{-1}(x)= \begin{cases}0, & x<-2 \\
\frac{4-x^{2}}{3}, & -2 \leq x<-1 \\
1 & x=-1 \\
\left(x^{2}-0.25\right) / 0.75 & -1<x \leq-0.5 \\
0 & x>-0.5\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
& \mu_{\tilde{2} 1}(x)= \begin{cases}0, & x<19 \\
\frac{x^{2}-361}{80}, & 19 \leq x<21 \\
1, & x=21 \\
\frac{625-x^{2}}{184}, & 21<x \leq 25 \\
0, & x>25\end{cases} \\
& \mu_{\tilde{2} 7}(x)= \begin{cases}0, & x<25 \\
\frac{x^{2}-625}{104}, & 25 \leq x<27 \\
1, & x=27 \\
\frac{961-x^{2}}{232}, & 27<x \leq 31 \\
0, & x>31\end{cases}
\end{aligned}
$$

2) Let us suppose that the membership functions of the fuzzy goals set for the leader are

$$
\begin{aligned}
& \mu_{\tilde{g}_{L 1}}(x)= \begin{cases}\frac{0,}{\frac{x^{2}-225}{175},} & x<15 \\
1, & x=20 \\
\frac{900-x^{2}}{500}, & 20<x \leq 30 \\
0, & x>30\end{cases} \\
& \mu_{\tilde{g}_{L 2}}(x)= \begin{cases}0, & x<4 \\
\frac{x^{2}-16}{48}, & 4 \leq x<8 \\
1, & x=8 \\
\frac{225-x^{2}}{161}, & 8<x \leq 15 \\
0, & x>15 .\end{cases}
\end{aligned}
$$

The membership functions of the fuzzy goals set for the follower are

$$
\begin{aligned}
& \mu_{\tilde{g}_{F 1}}(x)= \begin{cases}\frac{0,}{\frac{x^{2}-100}{225},} & x<10 \\
1, & x=15 \\
\frac{400-x^{2}}{175}, & 15<x \leq 20 \\
0, & x>20\end{cases} \\
& \mu_{\tilde{g}_{F 2}}(x)= \begin{cases}0, & x<7 \\
\frac{x^{2}-49}{32}, & 7 \leq x<9 \\
1, & x=9 \\
\frac{121-x^{2}}{40}, & 9<x \leq 11 \\
0, & x>11 .\end{cases}
\end{aligned}
$$

3) Satisfactory degree: $\alpha=0.2$.
4) $\varepsilon=0.15$.
[Step 2] (Initialization): Letting $k=1$. Associated with this example, the corresponding $\mathrm{MOB}_{\lambda}$ problem is given by

$$
\begin{aligned}
& \min _{x \in X}|\sqrt{11 \lambda+25} x+\sqrt{5 \lambda+4} y-\sqrt{175 \lambda+225}| \\
& \quad+|\sqrt{64-28 \lambda} x+25-\sqrt{25-16 \lambda} y-\sqrt{900-500 \lambda}| \\
& \min _{x \in X}|-\sqrt{16-7 \lambda} x+\sqrt{11 \lambda+25} y-\sqrt{48 \lambda+16}| \\
& \quad+|-\sqrt{8 \lambda+1}+\sqrt{64-28 \lambda}-\sqrt{225-161 \lambda}| \\
& \text { s.t. }-\sqrt{4-2 \lambda} x+\sqrt{5 \lambda+4} y \leq \sqrt{80 \lambda+361} \\
& \quad-\sqrt{-0.75 \lambda+0.25} x+\sqrt{25-16 \lambda} y \leq \sqrt{625-184 \lambda} \\
& \min _{y \in Y}|\sqrt{7 \lambda+9} x+\sqrt{5 \lambda+4} y-\sqrt{225 \lambda+100}| \\
& \quad+|\sqrt{36-20 \lambda} x+25-\sqrt{25-16 \lambda} y-\sqrt{400-175 \lambda}| \\
& \min _{y \in Y}|-\sqrt{5 \lambda+4} x+\sqrt{0.75 \lambda+0.25} y-\sqrt{32 \lambda+49}| \\
& \quad+|-\sqrt{25-16 \lambda} x+\sqrt{4-3 \lambda} y-\sqrt{121-40 \lambda}| \\
& \text { s.t. } \sqrt{0.75 \lambda+0.25} x+\sqrt{5 \lambda+4} y \leq \sqrt{104 \lambda+625} \\
& \sqrt{4-3 \lambda} x+\sqrt{25}-16 \lambda y \leq \sqrt{901-232 \lambda}
\end{aligned}
$$

where $\lambda \in[0.2,1]$.
Referring to the algorithm, only $\lambda_{0}=0.2$, and $\lambda_{1}=1$ are considered initially. Thus, four nonfuzzy objective functions and four nonfuzzy constraints for the leader and follower are generated, respectively, as follows:

$$
\begin{aligned}
& \min _{x \in X} \frac{1}{4}\{|\sqrt{27.2} x+\sqrt{5} y-\sqrt{260}|+|6 x+3 y-20| \\
& \quad+|\sqrt{58.4} x+\sqrt{21.8} y-20 \sqrt{2}|+|6 x+3 y-20| \\
& \quad+|-\sqrt{14.6} x+\sqrt{27.2} y-\sqrt{25.6}|+|-3 x+6 y-8| \\
& \quad+|-\sqrt{2.6}+\sqrt{58.4} y-\sqrt{192.8}|+|-3 x+6 y-8|\}
\end{aligned}
$$

s.t. $-\sqrt{3.4} x+\sqrt{5} y \leq \sqrt{377}$
$-x+3 y \leq 21$
$-\sqrt{0.4}+\sqrt{5} y \leq \sqrt{645.8}$
$-x+3 y \leq 21$
$\min _{y \in Y} \frac{1}{4}\{|3 x+2 y-12.04|+|4 x+3 y-19.1|$
$+|6 x-5 y-7.4|+|4 x-3 y-10.63|$
$+|-2 x+0.5 y-18.03|+|-3 x+y-15|$
$+|-5 x+2 y-9|+|-3 x+y-9|\}$
s.t. $\sqrt{0.4} x+\sqrt{5} y \leq \sqrt{645.8}$

$$
x+3 y \leq 27
$$

$$
\sqrt{3.4} x+\sqrt{21.8} y \leq \sqrt{914.6}
$$

$$
x+3 y \leq 27
$$

TABLE I
Summary of the Running Solution

| $k$ | $x$ | $y$ | $v_{1 \lambda}^{+}$ | $v_{1 \lambda}^{-}$ | $v_{2 \lambda}^{+}$ | $v_{2 \lambda}^{-}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1.901 | 2.434 | 0 | 0 | 0 | 0 |
| 2 | 2.011 | 2.356 | 0 | 0 | 0 | 0 |
| 3 | 1.872 | 2.446 | 0 | 0 | 0 | 0 |
| 4 | 1.957 | 2.388 | 0 | 0 | 0 | 0 |

[Step 3] (Compute):
By introducing auxiliary variables $v_{1}^{-}, v_{1}^{+}, v_{2}^{-}$, and $v_{2}^{+}$, we have

$$
\begin{aligned}
& \quad \min _{\left(x, v_{1}^{-}, v_{1}^{+}\right) \in \bar{X}^{\prime}} v_{1}^{-}+v_{1}^{+} \\
& \text {s.t. } 3.083 x+20.076 y+v_{1}^{-}-v_{1}^{+}=54.73 \\
& \quad-1.8 x+2.2 y \leq 19.4 \\
& \quad-x+3 y \leq 21 \\
& \quad-0.6 x+4.7 y \leq 24.3 \\
& \quad-x+3 y \leq 21 \\
& \quad \min _{\left(y, v_{2}^{-}, v_{2}^{+}\right) \in \bar{Y}^{\prime}} v_{2}^{-}+v_{2}^{+} \\
& \text {s.t. } 16.498 x+8.205 y+v_{2}^{-}-v_{2}^{+}=51.337 \\
& 0.6 x+2.2 y \leq 25.4 \\
& x+3 y \leq 7 \\
& \quad 1.8 x+4.7 y \leq 30.2 \\
& x+3 y \leq 27 .
\end{aligned}
$$

Using branch-and-bound algorithm [5], the current solution is $(1.901,0,0,2.434,0,0)$.
[Step 4] (Compare): Because $k=1$, goto [Step 5].
[Step 5] (Split): By inserting a new node $\lambda_{1}=(0.2+1) / 2=$ 0.6 , there are a total of three nodes of $\lambda_{0}=0.2, \lambda_{1}=0.6$, and $\lambda_{2}=1$. Then, a total of six nonfuzzy objective functions for the leader and follower, together with six nonfuzzy constraints for the leader and follower, respectively, are generated.
[Step 6] (Loop): Because $k=1+1=2$, goto [Step 3], and a current solution of $(2.011,0,0,2.356,0,0)$ is obtained. As $|2.011-1.901|+|2.356-2.434|=0.188>\varepsilon=$ 0.15 , the algorithm continues until the solution of $(1.957,0,0$, $2.388,0,0$ ) is obtained. The computing results are listed in Table I.
[Step 7] (Output): As $|1.957-1.872|+|2.388-2.2 .446|=$ $0.14<\varepsilon=0.15,\left(x^{*}, y^{*}\right)=(1.957,2.388)$ is the final solution of this FLMOB problem. The objectives for the leader and follower under $\left(x^{*}, y^{*}\right)=(1.957,2.388)$ are

$$
\left\{\begin{array}{l}
F_{1}\left(x^{*}, y^{*}\right)=F(1.957,2.388)=1.957 \tilde{c_{11}}+2.388 \tilde{d_{11}} \\
F_{2}\left(x^{*}, y^{*}\right)=F(1.957,2.388)=1.957 \tilde{c_{12}}+2.388 \tilde{d_{12}} \\
f_{1}\left(x^{*}, y^{*}\right)=F(1.957,2.388)=1.957 \tilde{c_{21}}+2.388 \tilde{d_{21}} \\
f_{2}\left(x^{*}, y^{*}\right)=F(1.957,2.388)=1.957 \tilde{c_{22}}+2.388 \tilde{d_{22}}
\end{array}\right.
$$

Under this solution, the membership functions for the leader's objectives are shown in Fig. 1, and the membership functions for the follower's objectives are shown in Fig. 2.

These fuzzy values shown in Figs. 1 and 2 describe the achievements of objectives from both the manufacturer and the retailer under the solutions. From Fig. 1, we can see that if the manufacturer chooses his or her decision variable as 1.957 , the most possible net profit will be 18.9025 , which is very close to the goal set for this objective. The other objective values can be interpreted in the same way.

## B. Experiments and Evaluation

The algorithm, which is proposed in this study, was implemented by Visual Basic 6.0, and run on a desktop computer with CPU Pentium $42.8-\mathrm{GHz}$, RAM 1 G , and Windows XP. To test the performance of the proposed algorithm, the following experiments are carried out.

1) To test the efficiency of the proposed algorithm, we employ ten numerical examples and enlarge the problem scales by changing the numbers of decision variables, objective functions, and constraints for both leaders and followers from two to ten simultaneously. For each of these examples, the final solution has been obtained within 5 s .
2) To test the performance of the fuzzy-distance measure in Definition 5, we adjust the satisfactory degree values from 0 to 0.5 on the ten numerical examples again. At the same time, we change some of the fuzzy coefficients in the constraints by moving the points whose membership values equal 0 by $10 \%$ from the left and right, respectively. Experiments reveal that, when a satisfactory degree is set as 0 , the average solution will change by about $6 \%$ if some of the constraint coefficients are moved, as discussed earlier. When we increase satisfactory degrees, the average solution change decreases. At the point where satisfactory degrees are equal to 0.5 , the average solution change is 0 .
From experiment 1), we can see that this proposed algorithm is quite efficient. The reason is the fact that final solutions can be reached by solving corresponding linear bilevel-programming problems, which can be handled by the Kuhn-Tucker and the simplex algorithms.
From experiment 2), we can see that if we change some coefficients of fuzzy numbers within a small range, solutions will be less sensitive to this change under a higher satisfactory degree. The reason is that, when the satisfactory degree is set to 0 , every $\lambda$-cut of fuzzy coefficients from 0 to 1 will be considered. Thus, the decision maker can certainly be influenced by minor information.

For a decision-making process involved with fuzzy coefficients, decision makers may sometimes make small adjustment on the uncertain information about the preference or circumstances. If the change occurs to the minor information, i.e., with smaller satisfactory degrees, there should normally be no tremendous change to the final solution. For example, when estimating future profit, the manufacturer may adjust the possibility of $\$ 5000$ profit from $2 \%$ to $3 \%$, while the


Fig. 1. Membership functions of $F_{1}\left(x^{*}, y^{*}\right)$ and $F_{2}\left(x^{*}, y^{*}\right)$.


Fig. 2. Membership functions of $f_{1}\left(x^{*}, y^{*}\right)$ and $f_{2}\left(x^{*}, y^{*}\right)$.
possibility of $\$ 100000$ profit remains $100 \%$. In such a situation, there should be no outstanding change for his or her final decision on the device investment. Therefore, to increase the satisfactory degrees is an acceptable strategy for a feasible solution.

From the earlier analysis, the advantages and disadvantages of the algorithm proposed in this study are as follows.

1) This algorithm is quite efficient, as it adopts strategies to transform a nonlinear bilevel problem into a linear problem.
2) When pursuing optimality, the negative effect from conflicting objectives can be avoided, and a leader can finally reach his or her satisfactory solution by setting goals for the objectives.
3) The information of the original fuzzy numbers are considered adequately by using a certain number of $\lambda$-cuts to approximate the final precise solution.
4) In some situations, this algorithm might suffer from expensive calculation, as the size of $\lambda$-cuts will increase exponentially with respect to iteration counts.

## V. Conclusion and Future Study

Many organizational decision problems can be formulated by bilevel decision models. In a bilevel decision model, the leader and/or the follower may have more than one objective to achieve, which is different from simple bilevel optimization problems. This kind of bilevel decision problem was studied by goal programming in this paper. Meanwhile, we take into consideration the situation where coefficients to formulate a bilevel decision model are not precisely known to us. Fuzzy-set method was applied to handle these coefficients.

This paper proposed a $\lambda$-cut and goal-programming-based algorithm for FLMOB decision problems and presented a real case study on a newsboy problem to explain this algorithm. Experiments reveal that the algorithm is quite effective and efficient. In the future, we will focus on situations that involve multiple followers.

## APPENDIX

The model of a general bilevel decision problem with multiple objectives for both the leader and follower is given in [37], which is reformulated in this paper as follows.

For $x \in X \subset R^{n}$, and $y \in Y \subset R^{m}$, an MOB model is given by

$$
\begin{align*}
& \min _{x \in X} F(x, y)  \tag{12a}\\
& \text { s.t. } G(x, y) \leq 0  \tag{12b}\\
& \min _{y \in Y} f(x, y)  \tag{12c}\\
& \text { s.t. } g(x, y) \leq 0 \tag{12d}
\end{align*}
$$

where $F: R^{n} \times R^{m} \rightarrow R^{s}, G: R^{n} \times R^{m} \rightarrow R^{p}, f: R^{n} \times$ $R^{m} \rightarrow R^{t}$, and $g: R^{n} \times R^{m} \rightarrow R^{q}$.

Associated with the MOB problem (12), some definitions are listed as follows.

Definition 6:

1) Constraint region of the MOB (12) is given by

$$
S \triangleq\{(x, y): x \in X, y \in Y, G(x, y) \leq 0, g(x, y) \leq 0\}
$$

It refers to all possible combination of choices that the leader and follower may make.
2) Projection of $S$ onto the leader's decision space is given by

$$
S(X) \triangleq\{x \in X: \exists y \in Y, G(x, y) \leq 0, g(x, y) \leq 0\}
$$

3) The feasible set for the follower $\forall x \in S(X)$ is given by

$$
S(x) \triangleq\{y \in Y:(x, y) \in S\}
$$

4) The follower's rational reaction set for $x \in S(X)$ is given by

$$
P(x) \triangleq\{y \in Y: y \in \operatorname{argmin}[f(x, \hat{y}): \hat{y} \in S(x)]\}
$$

where $\operatorname{argmin}[f(x, \hat{y}): \hat{y} \in S(x)]=\{y \in S(x): f(x, y)$ $\leq f(x, \hat{y}), \hat{y} \in S(x)\}$.
The follower observes the leader's action and reacts by selecting $y$ from his or her feasible set to minimize his or her objective function.
5) The inducible region is given by

$$
\mathrm{IR} \triangleq\{(x, y):(x, y) \in S, y \in P(x)\}
$$

which represents the set over which a leader may optimize his or her objectives.
To ensure that (12) is well posed, it is assumed that $S$ is nonempty and compact and that for all decisions taken by the leader, the follower has some room to respond, i.e., $P(x) \neq \emptyset$.

Thus, in terms of the previous notation, the MOB can be written as

$$
\begin{equation*}
\min \{F(x, y):(x, y) \in \mathrm{IR}\} \tag{13}
\end{equation*}
$$

Proof of Theorem 1: By Definition 6, let the notations associated with problem (7) be denoted by

$$
\begin{align*}
& S=\left\{(x, y): A_{k}{ }_{\lambda_{j}}^{L} x+B_{k}{ }_{\lambda_{j}}^{L} y \leq b_{k}{ }_{\lambda_{j}}^{L}\right. \\
& A_{k}{ }_{\lambda_{j}}^{R} x+B_{k}{ }_{\lambda_{j}}^{R} y \leq b_{k}{ }_{\lambda_{j}}^{R} \\
& k=1,2, j=0,1, \ldots, l,\}  \tag{14a}\\
& S(X)=\left\{x \in X: \exists y \in Y, A_{k}{ }_{\lambda}^{L} \lambda_{j} x+B_{k}{ }_{\lambda_{j}}^{L} y \leq b_{k}{ }_{\lambda}^{L}{ }_{j}\right. \\
& A_{k}{ }_{\lambda_{j}}^{R} x+B_{k}{ }_{\lambda_{j}}^{R} y \leq b_{k}{ }_{\lambda_{j}}^{R} \\
& k=1,2, \quad j=0,1 \ldots, l,\}  \tag{14b}\\
& S(x)=\{y \in Y:(x, y) \in S\}  \tag{14c}\\
& P(x)=\{y \in Y: y \in \operatorname{argmin} \Psi\} \tag{14d}
\end{align*}
$$

where

$$
\begin{align*}
\Psi= & (1 /(l+1)) \sum_{i=1}^{t} \sum_{j=0}^{l}\left\{\left|c_{i 2 \lambda_{j}}^{L} x+d_{i 2 \lambda_{j}}^{L} \hat{y}-g_{F i \lambda_{j}}^{L}\right|\right. \\
& \left.+\left|c_{i 2 \lambda_{j}}^{R} x+d_{i 2 \lambda_{j}}^{R} \hat{y}-g_{F i \lambda_{j}}^{R}\right|, \hat{y} \in S(x)\right\} \\
\operatorname{IR}= & \{(x, y):(x, y) \in S, y \in P(x)\} . \tag{14e}
\end{align*}
$$

Problem (7) can be written as

$$
\begin{align*}
& \min _{x \in X} \frac{1}{l+1} \sum_{h=1}^{s} \sum_{j=0}^{l}\left\{\left|c_{h 1 \lambda_{j}}^{L} x+d_{h 1 \lambda_{j}}^{L} y-g_{L h \lambda_{j}}^{L}\right|\right. \\
& \left.\quad+\left|c_{h 1 \lambda_{j}}^{R} x+d_{h 1 \lambda_{j}}^{R} y-g_{L h \lambda_{j}}^{R}\right|\right\}  \tag{15}\\
& \text { s.t. }(x, y) \in \mathrm{IR} \tag{16}
\end{align*}
$$

and those of problem (9) are denoted by

$$
\left.\begin{array}{rl}
S^{\prime}=\{ & \left(x^{\prime}, y^{\prime}\right): A_{k}{\lambda_{j}}_{j}^{L} x+B_{k}{\lambda_{j}}_{j}^{L} y \leq b_{k}{ }_{\lambda_{j}}^{L} \\
& A_{k} \lambda_{\lambda_{j}}^{R} x+B_{k}{ }_{\lambda_{j}}^{R} y \leq b_{k}{ }_{\lambda_{j}}^{R}, \quad k=1,2, \quad j=0,1 \ldots, l
\end{array}\right\}
$$

$$
\begin{align*}
& \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y+v_{h 1}^{R-}-v_{h 1}^{R+}=\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R} \\
& v_{h 1}^{L-}, v_{h 1}^{L+}, v_{h 1}^{R-}, v_{h 1}^{R+} \geq 0 \\
& v_{h 1}^{L-} \cdot v_{h 1}^{L+}=0 \\
& v_{h 1}^{R-} \cdot v_{h 1}^{R+}=0 \\
& h=1,2, \ldots, s \\
& \sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y+v_{i 2}^{L-}-v_{i 2}^{L+}=\sum_{j=0}^{l} g_{F i}^{L} \lambda_{j} \\
& \sum_{j=0}^{l} c_{i 2}^{R} \lambda_{j} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{R} y+v_{i 2}^{R-}-v_{i 2}^{R+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R} \\
& v_{i 2}^{L-}, v_{i 2}^{L+}, v_{i 2}^{R-}, v_{i 2}^{R+} \geq, 0 \\
& v_{i 2}^{L-} \cdot v_{i 2}^{L+}=0 \\
& v_{i 2}^{R-} \cdot v_{i 2}^{R+}=0 \\
& i=1,2, \ldots, t,\} \\
& S\left(X^{\prime}\right)=\left\{x^{\prime} \in X^{\prime}: \exists y^{\prime} \in Y^{\prime}, A_{k} \lambda_{j}^{L} x+B_{k} \lambda_{j}^{L} y \leq b_{k}{ }_{\lambda_{j}}^{L}\right. \\
& A_{k}{ }_{\lambda_{j}}^{R} x+B_{k}{ }_{\lambda}^{R} y \leq b_{k}{ }_{\lambda_{j}}^{R} \\
& k=1,2, \quad j=0,1 \ldots, l \\
& \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y+v_{h 1}^{L-}-v_{h 1}^{L+}=\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L} \\
& \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y+v_{h 1}^{R-}-v_{h 1}^{R+}=\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R} \\
& v_{h 1}^{L-}, v_{h 1}^{L+}, v_{h 1}^{R-}, v_{h 1}^{R+} \geq 0 \\
& v_{h 1}^{L-} \cdot v_{h 1}^{L+}=0 \\
& v_{h 1}^{R-} \cdot v_{h 1}^{R+}=0 \\
& h=1,2, \ldots, s \\
& \sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y+v_{i 2}^{L-}-v_{i 2}^{L+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L} \\
& \sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{R} y+v_{i 2}^{R-}-v_{i 2}^{R+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R} \\
& v_{i 2}^{L-}, v_{i 2}^{L+}, v_{i 2}^{R-}, v_{i 2}^{R+} \geq 0 \\
& v_{i 2}^{L-} \cdot v_{i 2}^{L+}=0 \\
& v_{i 2}^{R-} \cdot v_{i 2}^{R+}=0 \\
& i=1,2, \ldots, t,\} \tag{17c}
\end{align*}
$$

$S\left(x^{\prime}\right)=\left\{y^{\prime} \in Y^{\prime}:\left(x^{\prime}, y^{\prime}\right) \in S^{\prime}\right\}$
$P\left(x^{\prime}\right)=\left\{y^{\prime} \in Y^{\prime}:\right.$

$$
\begin{equation*}
\left.y^{\prime} \in \operatorname{argmin}\left[\sum_{i=1}^{t}\left(\hat{v}_{i 2}^{L-}+\hat{v}_{i 2}^{L+}+\hat{v}_{i 2}^{R-}+\hat{v}_{i 2}^{R+}\right): \hat{y}^{\prime} \in S\left(x^{\prime}\right)\right]\right\} \tag{17e}
\end{equation*}
$$

$\operatorname{IR}^{\prime}=\left\{\left(x^{\prime}, y^{\prime}\right):\left(x^{\prime}, y^{\prime}\right) \in S^{\prime}, y^{\prime} \in P\left(x^{\prime}\right)\right\}$.
Problem (9) can be written as

$$
\begin{equation*}
\min _{x^{\prime} \in X^{\prime}}\left\{\sum_{h=1}^{l}\left(v_{h 1}^{L-}+v_{h 1}^{L+}+v_{h 1}^{R-}+v_{h 1}^{R+}\right):\left(x^{\prime}, y^{\prime}\right) \in \mathrm{IR}^{\prime}\right\} . \tag{18}
\end{equation*}
$$

As $\left(x^{\prime *}, y^{\prime *}\right)$ is the optimal solution to problem (9), from (18), it can be seen that $\forall\left(x^{\prime}, y^{\prime}\right) \in \mathrm{I}^{\prime}$, we have

$$
\begin{aligned}
& \sum_{h=1}^{l}\left(v_{h 1}^{L-}+v_{h 1}^{L+}+v_{h 1}^{R-}+v_{h 1}^{R+}\right) \\
& \quad \geq \sum_{h=1}^{l}\left(v_{h 1}^{L-*}+v_{h 1}^{L+*}+v_{h 1}^{R-*}+v_{h 1}^{R+*}\right) .
\end{aligned}
$$

As $\quad \sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y+v_{h 1}^{L-}-v_{h 1}^{L+}=\sum_{j=0}^{l}$ $g_{L h \lambda_{j}}^{L}$, and $v_{h 1}^{L-} \cdot v_{h 1}^{L+}=0, h=1,2, \ldots, s$, we have

$$
\begin{aligned}
v_{h 1}^{-}+v_{h 1}^{+} & =\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right| \\
v_{h 1}^{L-*}+v_{h 1}^{L+*} & =\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x^{*}+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y^{*}-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right|
\end{aligned}
$$

for $h=1,2, \ldots, s$.
Similarly, we have

$$
\begin{array}{r}
v_{h 1}^{R-}+v_{h 1}^{R+}=\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right| \\
v_{h 1}^{R-*}+v_{h 1}^{R+*}=\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x^{*}+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y^{*}-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right|
\end{array}
$$

for $h=1,2, \ldots, s$.
Therefore, $\forall\left(x^{\prime}, y^{\prime}\right) \in \mathrm{IR}^{\prime}$

$$
\begin{aligned}
& \left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right| \\
& \quad+\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right| \\
& \geq\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{L} x^{*}+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{L} y^{*}-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{L}\right|
\end{aligned}
$$

$$
\begin{array}{r}
+\left|\sum_{j=0}^{l} c_{h 1 \lambda_{j}}^{R} x^{*}+\sum_{j=0}^{l} d_{h 1 \lambda_{j}}^{R} y^{*}-\sum_{j=0}^{l} g_{L h \lambda_{j}}^{R}\right| \\
h=1,2, \ldots, s \tag{19}
\end{array}
$$

We now prove that the projection of $S^{\prime}$ onto the $X \times Y$ space, which is denoted by $\left.S^{\prime}\right|_{X, Y}$, is equal to $S$.

On the one hand, $\left.\forall(x, y) \in S^{\prime}\right|_{X, Y}$, from constraints $A_{k} \frac{L}{\lambda_{j}} x+B_{k} \frac{L}{\lambda_{j}} y \leq b_{k} \stackrel{L}{\lambda_{j}}, A_{k}{ }_{\lambda}^{R} x+B_{k}{ }^{R}{ }_{\lambda_{j}} y \leq b_{k}{ }_{\lambda}^{R}, k=1,2$, $j=0,1 \ldots, l$, in $S^{\prime}$, we have $(x, y) \in S$; therefore, $\left.S^{\prime}\right|_{X, Y} \subseteq S$.

On the other hand, $\forall(x, y) \in S$, by (8), we can always find $v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s 1}^{L-}, v_{s 1}^{L+}, v_{s 1}^{R-}, v_{s 1}^{R+}, v_{12}^{L-}, v_{12}^{L+}$, $v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t 2}^{L-}, v_{t 2}^{L+}, v_{t 2}^{R-}$, and $v_{t 2}^{R+}$, which satisfies the constraints of (9b) and (9d). Together with the inequations of $A_{k}{ }_{\lambda}^{L} \lambda_{j} x+B_{k}{ }_{\lambda}^{L}{ }_{j} y \leq b_{k}{ }_{\lambda_{j}}^{L}$, and $A_{k}{ }_{\lambda}^{R} x+B_{k}{ }_{\lambda}^{R} y \leq b_{k}{ }_{\lambda}^{R}, k=$ $1,2, j=0,1 \ldots, l$, requested by $S$, we have $\left(x, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}\right.$, $v_{11}^{R+}, \ldots, v_{s 1}^{L-}, v_{s 1}^{L+}, v_{s 1}^{R-}, v_{s 1}^{R+}, y, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t 2}^{L-}$, $\left.v_{t 2}^{L+}, v_{t 2}^{R-}, v_{t 2}^{R+}\right) \in S^{\prime}$, thus $\left.(x, y) \in S^{\prime}\right|_{X, Y},\left.S \subseteq S^{\prime}\right|_{X, Y}$.

Therefore, we can prove that

$$
\begin{equation*}
\left.S^{\prime}\right|_{X, Y}=S \tag{20}
\end{equation*}
$$

Similarly, we have

$$
\begin{align*}
\left.S(x)^{\prime}\right|_{X, Y} & =S(x)  \tag{21a}\\
\left.S(X)^{\prime}\right|_{X, Y} & =S(X) \tag{21b}
\end{align*}
$$

In addition, from

$$
\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y+v_{i 2}^{L-}-v_{i 2}^{L+}=\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}
$$

and

$$
v_{i 2}^{L-} \cdot v_{i 2}^{L+}=0
$$

for $i=1,2, \ldots, t$, we have

$$
\begin{equation*}
v_{i 2}^{L-}+v_{i 2}^{L+}=\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{L} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{L} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{L}\right| \tag{22a}
\end{equation*}
$$

for $i=1,2, \ldots, t$. Similarly, we have

$$
\begin{equation*}
v_{i 2}^{R-}+v_{i 2}^{R+}=\left|\sum_{j=0}^{l} c_{i 2 \lambda_{j}}^{R} x+\sum_{j=0}^{l} d_{i 2 \lambda_{j}}^{R} y-\sum_{j=0}^{l} g_{F i \lambda_{j}}^{R}\right| \tag{22b}
\end{equation*}
$$

for $i=1,2, \ldots, t$.
Thus

$$
\begin{equation*}
P\left(x^{\prime}\right)=\left\{y^{\prime} \in Y^{\prime}: y^{\prime} \in \operatorname{argmin} \Psi^{\prime}\right\} \tag{23}
\end{equation*}
$$

where $\Psi^{\prime}=\sum_{i=1}^{t} \sum_{j=0}^{l}\left\{\left|c_{i 2 \lambda_{j}}^{L} x+d_{i 2 \lambda_{j}}^{L} \hat{y}-g_{F i \lambda_{j}}^{L}\right|+\mid c_{i 2 \lambda_{j}}^{R}\right.$ $\left.x+d_{i 2 \lambda_{j}}^{R} \hat{y}-g_{F i \lambda_{j}}^{R} \mid, \hat{y} \in S\left(x^{\prime}\right)\right\}$.

From (20) and (23), we have

$$
\begin{equation*}
\left.P\left(x^{\prime}\right)\right|_{X \times Y}=P(x) \tag{24}
\end{equation*}
$$

From (14e), (17f), (20), and (24), we have

$$
\begin{equation*}
\left.\mathrm{IR}^{\prime}\right|_{X \times Y}=\mathrm{IR} \tag{25}
\end{equation*}
$$

which means, in $X \times Y$ space, the leaders of problem (7) and (9) have the same optimizing space.

Thus, from (19) and (25), it can be obtained that $\forall(x, y) \in \mathrm{IR}$, we have

$$
\begin{aligned}
& \frac{1}{l+1} \sum_{h=1}^{s} \sum_{j=0}^{l}\left\{\left|c_{h 1 \lambda_{j}}^{L} x+d_{h 1 \lambda_{j}}^{L} y-g_{L h \lambda_{j}}^{L}\right|\right. \\
& \left.\quad+\left|c_{h 1 \lambda_{j}}^{R} x+d_{h 1 \lambda_{j}}^{R} y-g_{L h \lambda_{j}}^{R}\right|\right\} \\
& \geq \\
& \quad \frac{1}{l+1} \sum_{h=1}^{s} \sum_{j=0}^{l}\left\{\left|c_{h 1 \lambda_{j}}^{L} x^{*}+d_{h 1 \lambda_{j}}^{L} y^{*}-g_{L h \lambda_{j}}^{L}\right|\right. \\
& \left.\quad+\left|c_{h 1}^{R} \lambda_{j} x^{*}+d_{h 1 \lambda_{j}}^{R} y^{*}-g_{L h \lambda_{j}}^{R}\right|\right\} .
\end{aligned}
$$

Therefore, $\left(x^{*}, y^{*}\right)$ is the optimal solution of the problem (7).

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