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Partially Observable Total-Cost Markov Decision Processes with Weakly Continuous Transition Probabilities

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This paper describes sufficient conditions for the existence of optimal policies for partially observable Markov decision processes (POMDPs) with Borel state, observation, and action sets, when the goal is to minimize the expected total costs over finite or infinite horizons. For infinite-horizon problems, one-step costs are either discounted or assumed to be nonnegative. Action sets may be noncompact and one-step cost functions may be unbounded. The introduced conditions are also sufficient for the validity of optimality equations, semicontinuity of value functions, and convergence of value iterations to optimal values. Since POMDPs can be reduced to completely observable Markov decision processes (COMDPs), whose states are posterior state distributions, this paper focuses on the validity of the above-mentioned optimality properties for COMDPs. The central question is whether the transition probabilities for the COMDP are weakly continuous. We introduce sufficient conditions for this and show that the transition probabilities for a COMDP are weakly continuous, if transition probabilities of the underlying Markov decision process are weakly continuous and observation probabilities for the POMDP are continuous in total variation. Moreover, the continuity in total variation of the observation probabilities cannot be weakened to setwise continuity. The results are illustrated with counterexamples and examples.

Keywords: partially observable Markov decision processes; total cost; optimality inequality; optimal policy
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1. Introduction. Partially observable Markov decision processes (POMDPs) play an important role in operations research, electrical engineering, and computer science. They have a broad range of applications to various areas including sensor networks, artificial intelligence, target tracking, control and maintenance of complex systems, finance, and medical decision making. In principle, it is known how to solve POMDPs. A POMDP can be reduced to a completely observable Markov decision process (COMDP), which is a fully observable Markov decision process (MDP) whose states are belief (posterior state) probabilities for the POMDP; see Hinderer [23, §7.1] and Sawarigi and Yoshikawa [29] for countable state spaces and Rhenius [26], Yushkevich [36], Dynkin and Yushkevich [12, Chapter 8], Bertsekas and Shreve [8, Chapter 10], and Hernández-Lerma [20, Chapter 4] for Borel state spaces. After an optimal policy for the COMDP is found, it can be used to compute an optimal policy for the POMDP. However, except for finite state and action POMDPs (Sondik [33]), problems with a continuous filtering transition probability H described in Equation (6) (Hernández-Lerma [20, Chapter 4], Hernández-Lerma and Romera [22]), and a large variety of particular problems considered in the literature, little is known regarding the existence and properties of optimal policies for COMDPs and POMDPs.

This paper investigates the existence of optimal policies for COMDPs and therefore for POMDPs with the expected total discounted costs and, if the one-step costs are nonnegative, with the expected total costs. We provide conditions for the existence of optimal policies and for the validity of other properties of optimal values and optimal policies: they satisfy optimality equations, optimal values are lower semicontinuous functions, and value iterations converge to optimal infinite-horizon values.

Since a COMDP is an MDP with Borel state and action sets, it is natural to apply results on the existence of optimal policies for MDPs to COMDPs. Feinberg et al. [14] introduced a mild assumption, called Assumption (W*), for the existence of stationary optimal policies for infinite-horizon MDPs, lower semicontinuity of value functions, characterization of the sets of optimal actions via optimality equations, and convergence of value iterations to optimal values for the expected total discounted costs, if one-step costs are bounded below, and for the expected total costs, if the one-step costs are nonnegative (according to the main result in Feinberg et al. [14], if another mild assumption is added to Assumption (W*)), then there exist stationary optimal policies for average costs per

unit time). Assumption (**W***) consists of two conditions: transition probabilities are weakly continuous and one-step cost functions are $\mathbb{K}$-inf-compact. The notion of $\mathbb{K}$-inf-compactness (see the definition below) was introduced in Feinberg et al. [15], and it is slightly stronger than the lower semicontinuity of the cost function and its inf-compactness in the action parameter. In operations research applications, one-step cost functions are usually $\mathbb{K}$-inf-compact.

Here we consider a POMDP whose underlying MDP satisfies Assumption (**W***). According to Theorem 3.3, this implies $\mathbb{K}$-inf-compactness of the cost function for the COMDP. Theorem 3.6 states that weak continuity of transition probabilities and continuity of observation probabilities in total variation imply weak continuity of transition probabilities for the COMDP. Thus, Assumption (**W***) for the underlying MDP and continuity of observation probabilities in total variation imply that the COMDP satisfies Assumption (**W***) and therefore optimal policies exist for the COMDP and for the POMDP, value iterations converge to the optimal value, and other optimality properties hold; see Theorem 3.5. Example 4.1 demonstrates that continuity of observation probabilities in total variation cannot be relaxed to setwise continuity.

For problems with incomplete information, the filtering equation $z_{t+1} = H(z_t, a_t, y_{t+1})$ presented in Equation (7), that links the posterior state probabilities z_t, z_{t+1}, the selected action a_t, and the observation y_{t+1}, plays an important role. This equation presents a general form of Bayes's rule. Hernández-Lerma [20, Chapter 4] showed that the weak continuity of the stochastic kernel H in all three variables and weak continuity of transition and observation probabilities imply weak continuity of transition probabilities for the COMDP. In this paper we introduce another condition, Assumption (**H**), which is weaker than the weak continuity of the filtering kernel H in (z_t, a_t, y_{t+1}). We prove that Assumption (**H**) and setwise continuity of the stochastic kernel on the observation set, given a posterior state probability and prior action, imply weak continuity of the transition probability for the COMDP; see Theorem 3.4. Furthermore, weak continuity of transition probabilities and continuity of the observation kernel in total variation imply Assumption (**H**) and setwise continuity of the stochastic kernel described in the previous sentence; see Theorem 3.6. In particular, if either of these two assumptions or the weak continuity of H and observation probabilities are added to Assumption (**W***) for the underlying MDP of the POMDP, the COMDP satisfies Assumption (**W***) and therefore various optimality properties, including the existence of stationary optimal policies and convergence of value iterations; see Theorem 3.2.

If the observation set is countable and it is endowed with the discrete topology, convergence in total variation and weak convergence are equivalent. Thus, Theorem 3.6 implies weak continuity of the transition probability for the COMDP with a countable observation set endowed with the discrete topology and with weakly continuous transition and observation kernels; see Hernández-Lerma [20, p. 93]. However, as Example 4.2 demonstrates, under these conditions the filtering transition probability H may not be continuous. In other words, the statement in Hernández-Lerma [20, p. 93], that continuity of transition and observation kernels imply weak continuity of H, if the observation set is countable and endowed with discrete topology, is incorrect. Example 4.2 motivated us to introduce Assumption (**H**).

The main results of this paper are presented in §3. Section 4 contains three counterexamples. In addition to the two described examples, Example 4.3 demonstrates that setwise continuity of the stochastic kernel on the observation set, given a posterior state probability and prior action, is essential to ensure that Assumption (**H**) implies continuity of the transition probability for the COMDP. Section 5 describes properties of stochastic kernels used in the proofs of the main results presented in §6. Section 7 introduces a sufficient condition for the weak continuity of transition probabilities for the COMDP that combines Assumption (**H**) and the weak continuity of H. Combining these properties together is important because Assumption (**H**) may hold for some observations and weak continuity of H may hold for others. Section 8 contains three illustrative examples: (i) a model defined by stochastic equations including Kalman's filter; (ii) a model for inventory control with incomplete records (for particular inventory control problems of such type see Bensoussan et al. [4, 5, 6, 7] and references therein); and (iii) the classic Markov decision model with incomplete information studied by Aoki [1], Dynkin [11], Shiryaev [31], Hinderer [23, §7.1], Sawarigi and Yoshikawa [29], Rhenius [26], Yushkevich [36], and Dynkin and Yushkevich [12, Chapter 8], for which we provide a sufficient condition for the existence of optimal policies, convergence of value iterations to optimal values, and other optimality properties formulated in Theorems 3.1, 3.2, and 3.5.

Some of the results of §§5 and 8.3 have been further developed in Feinberg et al. [17, 18]. In particular, new results on convergence of probability measures and on continuity of stochastic kernels are described in Feinberg et al. [17, §§3, 4], and they are used to generalize in Feinberg et al. [17, Theorem 6.2] the main result of §8.3, Theorem 6.2, that states sufficient conditions for optimality for Markov decision models with incomplete information. The uniform Fatou's lemma is introduced in Feinberg et al. [18, Theorem 1.1]. It provides the natural necessary and sufficient condition for a stronger inequality than Fatou's lemma, and Theorem 5.2 presents a particular version of its necessary condition.

2. Model description. For a metric space $\mathbb{S}$, let $\mathscr{B}(\mathbb{S})$ be its Borel σ-field, that is, the σ-field generated by all open subsets of the metric space $\mathbb{S}$. For a Borel set $E \in \mathscr{B}(\mathbb{S})$, we denote by $\mathscr{B}(E)$ the σ-field whose elements are intersections of E with elements of $\mathscr{B}(\mathbb{S})$. Observe that E is a metric space with the same metric as on $\mathbb{S}$, and $\mathscr{B}(E)$ is its Borel σ-field. For a metric space $\mathbb{S}$, we denote by $\mathbb{P}(\mathbb{S})$ the *set of probability measures* on $(\mathbb{S}, \mathscr{B}(\mathbb{S}))$. A sequence of probability measures $\{\mu^{(n)}\}_{n=1,2,\ldots}$ from $\mathbb{P}(\mathbb{S})$ *converges weakly* (*setwise*) to $\mu \in \mathbb{P}(\mathbb{S})$ if for any bounded continuous (bounded Borel-measurable) function f on $\mathbb{S}$

$$\int_{\mathbb{S}} f(s)\mu^{(n)}(ds) \to \int_{\mathbb{S}} f(s)\mu(ds) \quad \text{as } n \to \infty.$$

A sequence of probability measures $\{\mu^{(n)}\}_{n=1,2,\ldots}$ from $\mathbb{P}(\mathbb{S})$ *converges in total variation* to $\mu \in \mathbb{P}(\mathbb{S})$ if

$$\sup\left\{ \left| \int_{\mathbb{S}} f(s)\mu^{(n)}(ds) - \int_{\mathbb{S}} f(s)\mu(ds) \right| \,\middle|\, f\colon \mathbb{S} \to [-1,1] \text{ is Borel-measurable} \right\} \to 0 \quad \text{as } n \to \infty;$$

see Feinberg et al. [18] for properties of these three types of convergence of probability measures. These types of convergence of measures are used in Yüksel and Linder [35] to describe convergence of observation channels. Note that $\mathbb{P}(\mathbb{S})$ is a separable metric space with respect to the weak convergence topology for probability measures, when $\mathbb{S}$ is a separable metric space; see Parthasarathy [25, Chapter II]. For metric spaces $\mathbb{S}_1$ and $\mathbb{S}_2$, a (Borel-measurable) *stochastic kernel* $R(ds_1 \mid s_2)$ on $\mathbb{S}_1$ given $\mathbb{S}_2$ is a mapping $R(\cdot \mid \cdot)\colon \mathscr{B}(\mathbb{S}_1) \times \mathbb{S}_2 \to [0,1]$, such that $R(\cdot \mid s_2)$ is a probability measure on $\mathbb{S}_1$ for any $s_2 \in \mathbb{S}_2$, and $R(B \mid \cdot)$ is a Borel-measurable function on $\mathbb{S}_2$ for any Borel set $B \in \mathscr{B}(\mathbb{S}_1)$. A stochastic kernel $R(ds_1 \mid s_2)$ on $\mathbb{S}_1$ given $\mathbb{S}_2$ defines a Borel-measurable mapping $s_2 \to R(\cdot \mid s_2)$ of $\mathbb{S}_2$ to the metric space $\mathbb{P}(\mathbb{S}_1)$ endowed with the topology of weak convergence. A stochastic kernel $R(ds_1 \mid s_2)$ on $\mathbb{S}_1$ given $\mathbb{S}_2$ is called *weakly continuous* (*setwise continuous, continuous in total variation*), if $R(\cdot \mid x^{(n)})$ converges weakly (setwise, in total variation) to $R(\cdot \mid x)$ whenever $x^{(n)}$ converges to x in $\mathbb{S}_2$. For one-point sets $\{s_1\} \subset \mathbb{S}_1$, we sometimes write $R(s_1 \mid s_2)$ instead of $R(\{s_1\} \mid s_2)$.

For a Borel subset S of a metric space $(\mathbb{S}, \rho)$, where ρ is a metric, consider the metric space (S, ρ). A set B is called open (closed, compact) in S if $B \subseteq S$ and B is open (closed, compact, respectively) in (S, ρ). Of course, if $S = \mathbb{S}$, we omit "in $\mathbb{S}$." Observe that, in general, an open (closed, compact) set in S may not be open (closed, compact, respectively).

Let $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{A}$ be Borel subsets of Polish spaces (a Polish space is a complete separable metric space), $P(dx' \mid x, a)$ be a stochastic kernel on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$, $Q(dy \mid a, x)$ be a stochastic kernel on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$, $Q_0(dy \mid x)$ be a stochastic kernel on $\mathbb{Y}$ given $\mathbb{X}$, p be a probability distribution on $\mathbb{X}$, $c\colon \mathbb{X} \times \mathbb{A} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a bounded below Borel function on $\mathbb{X} \times \mathbb{A}$, where $\mathbb{R}$ is a real line.

A POMDP is specified by a tuple $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c)$, where $\mathbb{X}$ is the *state space*, $\mathbb{Y}$ is the *observation set*, $\mathbb{A}$ is the *action set*, $P(dx' \mid x, a)$ is the *state transition law*, $Q(dy \mid a, x)$ is the *observation kernel*, $c\colon \mathbb{X} \times \mathbb{A} \to \bar{\mathbb{R}}$ is the *one-step cost*.

The partially observable Markov decision process evolves as follows:

- at time $t = 0$, the initial unobservable state x_0 has a given prior distribution p;
- the initial observation y_0 is generated according to the initial observation kernel $Q_0(\cdot \mid x_0)$;
- at each time epoch $t = 0, 1, \ldots,$ if the state of the system is $x_t \in \mathbb{X}$ and the decision maker chooses an action $a_t \in \mathbb{A}$, then the cost $c(x_t, a_t)$ is incurred;
- the system moves to a state x_{t+1} according to the transition law $P(\cdot \mid x_t, a_t)$, $t = 0, 1, \ldots;$
- an observation $y_{t+1} \in \mathbb{Y}$ is generated by the observation kernel $Q(\cdot \mid a_t, x_{t+1})$, $t = 0, 1, \ldots$.

Define the *observable histories*: $h_0 := (p, y_0) \in \mathbb{H}_0$ and $h_t := (p, y_0, a_0, \ldots, y_{t-1}, a_{t-1}, y_t) \in \mathbb{H}_t$ for all $t = 1, 2, \ldots,$ where $\mathbb{H}_0 := \mathbb{P}(\mathbb{X}) \times \mathbb{Y}$ and $\mathbb{H}_t := \mathbb{H}_{t-1} \times \mathbb{A} \times \mathbb{Y}$ if $t = 1, 2, \ldots$. A *policy* π for the POMDP is defined as a sequence $\pi = \{\pi_t\}_{t=0,1,\ldots}$ of stochastic kernels π_t on $\mathbb{A}$ given $\mathbb{H}_t$. A policy π is called *nonrandomized*, if each probability measure $\pi_t(\cdot \mid h_t)$ is concentrated at one point. The *set of all policies* is denoted by Π. The Ionescu Tulcea theorem (Bertsekas and Shreve [8, pp. 140–141] or Hernández-Lerma and Lassere [21, p. 178]) implies that a policy $\pi \in \Pi$ and an initial distribution $p \in \mathbb{P}(\mathbb{X})$, together with the stochastic kernels P, Q, and Q_0, determine a unique probability measure P_p^π on the set of all trajectories $(\mathbb{X} \times \mathbb{Y} \times \mathbb{A})^\infty$ endowed with the σ-field defined by the products of Borel σ-fields $\mathscr{B}(\mathbb{X})$, $\mathscr{B}(\mathbb{Y})$, and $\mathscr{B}(\mathbb{A})$. The expectation with respect to this probability measure is denoted by $\mathbb{E}_p^\pi$.

For a finite horizon $T = 0, 1, \ldots,$ the *expected total discounted costs* are

$$V_{T,\alpha}^\pi(p) := \mathbb{E}_p^\pi \sum_{t=0}^{T-1} \alpha^t c(x_t, a_t), \quad p \in \mathbb{P}(\mathbb{X}), \quad \pi \in \Pi, \tag{1}$$

where $\alpha \geq 0$ is the discount factor, $V_{0,\alpha}^\pi(p) = 0$. Consider the following assumptions.

ASSUMPTION (**D**). The function c is bounded below on $\mathbb{X}\times\mathbb{A}$ and $\alpha\in[0,1)$.

ASSUMPTION (**P**). The function c is nonnegative on $\mathbb{X}\times\mathbb{A}$ and $\alpha\in[0,1]$.

When $T=\infty$, formula (1) defines the *infinite horizon expected total discounted cost*, and we denote it by $V^{\pi}_{\alpha}(p)$. We use the notations **(D)** and **(P)** following Bertsekas and Shreve [8, p. 214], where cases (D), (N), and (P) are considered. However, Assumption **(D)** here is weaker than the conditions assumed in case **(D)** in Bertsekas and Shreve [8, p. 214], where one-step costs are assumed to be bounded.

Since the function c is bounded below by a constant M on $\mathbb{X}\times\mathbb{A}$, a discounted model can be converted into a model with nonnegative costs by replacing costs $c(x,a)$ with $c(x,a)+M$. Though Assumption **(P)** is more general, Assumption **(D)** is met in a wide range of applications. Thus we formulate the results for either of these assumptions.

For any function $g^{\pi}(p)$, including $g^{\pi}(p)=V^{\pi}_{T,\alpha}(p)$ and $g^{\pi}(p)=V^{\pi}_{\alpha}(p)$, define the *optimal values*

$$g(p):=\inf_{\pi\in\Pi}g^{\pi}(p),\quad p\in\mathbb{P}(\mathbb{X}).$$

A policy π is called *optimal* for the respective criterion, if $g^{\pi}(p)=g(p)$ for all $p\in\mathbb{P}(\mathbb{X})$. For $g^{\pi}=V^{\pi}_{T,\alpha}$, the optimal policy is called *T-horizon discount-optimal*; for $g^{\pi}=V^{\pi}_{\alpha}$, it is called *discount-optimal*.

In this paper, for the expected total costs and objective values, we use similar notations for POMDPs, MDPs, and COMDPs. However, we reserve the symbol V for POMDPs, the symbol v for MDPs, and the notation $\bar{v}$ for COMDPs. So, in addition to the notations $V^{\pi}_{T,\alpha}$, V^{π}_{α}, $V_{T,\alpha}$, and V_{α} introduced for POMDPs, we shall use the notations $v^{\pi}_{T,\alpha}$, v^{π}_{α}, $v_{T,\alpha}$, v_{α} and $\bar{v}^{\pi}_{T,\alpha}$, $\bar{v}^{\pi}_{\alpha}$, $\bar{v}_{T,\alpha}$, $\bar{v}_{\alpha}$ for the similar objects for MDPs and COMDPs, respectively.

We recall that a function c defined on $\mathbb{X}\times\mathbb{A}$ with values in $\bar{\mathbb{R}}$ is inf-compact if the set $\{(x,a)\in\mathbb{X}\times\mathbb{A}\colon c(x,a)\le\lambda\}$ is compact for any finite number λ. A function c defined on $\mathbb{X}\times\mathbb{A}$ with values in $\bar{\mathbb{R}}$ is called $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$, if for any compact set $K\subseteq\mathbb{X}$, the function $c\colon K\times\mathbb{A}\to\bar{\mathbb{R}}$ defined on $K\times\mathbb{A}$ is inf-compact; see Feinberg et al. [13, 15, Definition 1.1]. According to Feinberg et al. [15, Lemma 2.5], a bounded below function c is $\mathbb{K}$-inf-compact on the product of metric spaces $\mathbb{X}$ and $\mathbb{A}$ if and only if it satisfies the following two conditions:

(a) c is lower semicontinuous;

(b) if a sequence $\{x^{(n)}\}_{n=1,2,\dots}$ with values in $\mathbb{X}$ converges and its limit x belongs to $\mathbb{X}$, then any sequence $\{a^{(n)}\}_{n=1,2,\dots}$ with $a^{(n)}\in\mathbb{A}$, $n=1,2,\dots,$ satisfying the condition that the sequence $\{c(x^{(n)},a^{(n)})\}_{n=1,2,\dots}$ is bounded above, has a limit point $a\in\mathbb{A}$.

Various applications deal with $\mathbb{K}$-inf-compact cost functions that are not inf-compact. For example, the functions $c(x,a)=(x-a)^2$ and $c(x,a)=|x-a|$ defined on $\mathbb{R}\times\mathbb{R}$ are $\mathbb{K}$-inf-compact, but they are not inf-compact.

For a POMDP $(\mathbb{X},\mathbb{Y},\mathbb{A},P,Q,c)$, consider the MDP $(\mathbb{X},\mathbb{A},P,c)$, in which all the states are observable. An MDP can be viewed as a particular POMDP with $\mathbb{Y}=\mathbb{X}$ and $Q(B\mid a,x)=Q(B\mid x)=\mathbf{I}\{x\in B\}$ for all $x\in\mathbb{X}$, $a\in\mathbb{A}$, and $B\in\mathscr{B}(\mathbb{X})$. In addition, for an MDP an initial state is observable. Thus, for an MDP, an initial state x is considered instead of the initial distribution p. In fact, this MDP possesses the property that action sets at all the states are equal. For MDPs, Feinberg et al. [14] provides general conditions for the existence of optimal policies, validity of optimality equations, and convergence of value iterations. Here we formulate these conditions for an MDP whose action sets in all states are equal.

ASSUMPTION (**W***) (CP. FEINBERG ET AL. [14, 15, LEMMA 2.5]).

(i) the function c is $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$;

(ii) the transition probability $P(\cdot\mid x,a)$ is weakly continuous in $(x,a)\in\mathbb{X}\times\mathbb{A}$.

For an MDP, a nonrandomized policy is called *Markov* if all decisions depend only on the current state and time. A Markov policy is called *stationary* if all decisions depend only on current states.

THEOREM 2.1 (CP. FEINBERG ET AL. [14, THEOREM 2]). *Let MDP $(\mathbb{X},\mathbb{A},P,c)$ satisfy Assumption* **(W*****)**. *Let either Assumptions* **(P)** *or* **(D)** *hold. Then,*

(i) *the functions $v_{t,\alpha}$, $t=0,1,\dots,$ and v_{α} are lower semicontinuous on $\mathbb{X}$, and $v_{t,\alpha}(x)\to v_{\alpha}(x)$ as $t\to\infty$ for all $x\in\mathbb{X}$;*

(ii) *for each $x\in\mathbb{X}$ and $t=0,1,\dots,$*

$$v_{t+1,\alpha}(x)=\min_{a\in\mathbb{A}}\left\{c(x,a)+\alpha\int_{\mathbb{X}}v_{t,\alpha}(y)P(dy\mid x,a)\right\},\tag{2}$$

where $v_{0,\alpha}(x)=0$ for all $x\in\mathbb{X}$, and the nonempty sets

$$A_{t,\alpha}(x):=\left\{a\in\mathbb{A}\colon v_{t+1,\alpha}(x)=c(x,a)+\alpha\int_{\mathbb{X}}v_{t,\alpha}(y)P(dy\mid x,a)\right\},\quad x\in\mathbb{X},\quad t=0,1,\dots,$$

satisfy the following properties: (*a*) *the graph* $\mathrm{Gr}(A_{t,\alpha}) = \{(x,a)\colon x \in \mathbb{X},\ a \in A_{t,\alpha}(x)\}$, $t = 0, 1, \ldots,$ *is a Borel subset of* $\mathbb{X} \times \mathbb{A}$, *and* (*b*) *if* $v_{t+1,\alpha}(x) = +\infty$, *then* $A_{t,\alpha}(x) = \mathbb{A}$ *and, if* $v_{t+1,\alpha}(x) < +\infty$, *then* $A_{t,\alpha}(x)$ *is compact*;

(iii) *for each* $T = 1, 2, \ldots,$ *there exists an optimal Markov* T*-horizon policy* $(\phi_0, \ldots, \phi_{T-1})$, *and if for a* T*-horizon Markov policy* $(\phi_0, \ldots, \phi_{T-1})$ *the inclusions* $\phi_{T-1-t}(x) \in A_{t,\alpha}(x)$, $x \in \mathbb{X}$, $t = 0, \ldots, T-1$, *hold, then this policy is* T*-horizon optimal*;

(iv) *for each* $x \in \mathbb{X}$

$$v_\alpha(x) = \min_{a \in \mathbb{A}} \left\{ c(x,a) + \alpha \int_{\mathbb{X}} v_\alpha(y) P(dy \mid x, a) \right\}, \tag{3}$$

and the nonempty sets

$$A_\alpha(x) := \left\{ a \in \mathbb{A}\colon v_\alpha(x) = c(x,a) + \alpha \int_{\mathbb{X}} v_\alpha(y) P(dy \mid x, a) \right\}, \quad x \in \mathbb{X},$$

satisfy the following properties: (a) *the graph* $\mathrm{Gr}(A_\alpha) = \{(x,a)\colon x \in \mathbb{X},\ a \in A_\alpha(x)\}$ *is a Borel subset of* $\mathbb{X} \times \mathbb{A}$, *and* (b) *if* $v_\alpha(x) = +\infty$, *then* $A_\alpha(x) = \mathbb{A}$ *and, if* $v_\alpha(x) < +\infty$, *then* $A_\alpha(x)$ *is compact*;

(v) *for infinite-horizon problems there exists a stationary discount-optimal policy* ϕ_α, *and a stationary policy* ϕ_α^* *is optimal if and only if* $\phi_\alpha^*(x) \in A_\alpha(x)$ *for all* $x \in \mathbb{X}$;

(vi) (*Feinberg and Lewis* [19, *Proposition* 3.1(iv)]) *if* c *is inf-compact on* $\mathbb{X} \times \mathbb{A}$, *then the functions* $v_{t,\alpha}$, $t = 1, 2, \ldots,$ *and* v_α *are inf-compact on* $\mathbb{X}$.

3. Reduction of POMDPs to COMDPs and main results. In this section we formulate the main results of the paper, Theorems 3.2, 3.5, and the relevant statements. These theorems provide sufficient conditions for the existence of optimal policies for COMDPs and therefore for POMDPs with expected total costs, as well as optimality equations and convergence of value iterations for COMDPs. These conditions consist of two major components: the conditions for the existence of optimal policies for the underlying MDP and additional conditions on the POMDP. Theorem 3.5 states that the continuity of the observation kernel Q in total variation is the additional sufficient condition under which there is a stationary optimal policy for the COMDP, and this policy satisfies the optimality equations and can be found by value iterations. In particular, the continuity of Q in total variation and the weak continuity of P imply the setwise continuity of the stochastic kernel R' defined in (5) and the validity of Assumption **(H)** introduced in this section; see Theorem 3.6. These two additional properties imply the weak continuity of the transition probability q for the COMDP (Theorem 3.4) and eventually the desired properties of the COMDP; see Theorem 3.2.

This section starts with the description of known results on the general reduction of a POMDP to the COMDP; see Bertsekas and Shreve [8, §10.3], Dynkin and Yushkevich [12, Chapter 8], Hernández-Lerma [20, Chapter 4], Rhenius [26], and Yushkevich [36]. To simplify notations, we sometimes drop the time parameter. Given a posterior distribution z of the state x at time epoch $t = 0, 1, \ldots$ and given an action a selected at epoch t, denote by $R(B \times C \mid z, a)$ the joint probability that the state at time $(t+1)$ belongs to the set $B \in \mathscr{B}(\mathbb{X})$ and the observation at time $t+1$ belongs to the set $C \in \mathscr{B}(\mathbb{Y})$,

$$R(B \times C \mid z, a) := \int_{\mathbb{X}} \int_B Q(C \mid a, x') P(dx' \mid x, a) z(dx), \quad B \in \mathscr{B}(\mathbb{X}),\ C \in \mathscr{B}(\mathbb{Y}),\ z \in \mathbb{P}(\mathbb{X}),\ a \in \mathbb{A}. \tag{4}$$

Observe that R is a stochastic kernel on $\mathbb{X} \times \mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$; see Bertsekas and Shreve [8, §10.3], Dynkin and Yushkevich [12, Chapter 8], Hernández-Lerma [20, p. 87], Yushkevich [36], or Rhenius [26] for details. The probability that the observation y at time $t+1$ belongs to the set $C \in \mathscr{B}(\mathbb{Y})$, given that at time t the posterior state probability is z and selected action is a, is

$$R'(C \mid z, a) := \int_{\mathbb{X}} \int_{\mathbb{X}} Q(C \mid a, x') P(dx' \mid x, a) z(dx), \quad C \in \mathscr{B}(\mathbb{Y}),\ z \in \mathbb{P}(\mathbb{X}),\ a \in \mathbb{A}. \tag{5}$$

Observe that R' is a stochastic kernel on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$. By Bertsekas and Shreve [8, Proposition 7.27], there exists a stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ such that

$$R(B \times C \mid z, a) = \int_C H(B \mid z, a, y) R'(dy \mid z, a), \quad B \in \mathscr{B}(\mathbb{X}),\ C \in \mathscr{B}(\mathbb{Y}),\ z \in \mathbb{P}(\mathbb{X}),\ a \in \mathbb{A}. \tag{6}$$

The stochastic kernel $H(\cdot \mid z, a, y)$ defines a measurable mapping $H\colon \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \to \mathbb{P}(\mathbb{X})$, where $H(z,a,y)(\cdot) = H(\cdot \mid z, a, y)$. For each pair $(z,a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$, the mapping $H(z,a,\cdot)\colon \mathbb{Y} \to \mathbb{P}(\mathbb{X})$ is defined $R'(\cdot \mid z, a)$-almost surely uniquely in $y \in \mathbb{Y}$; see Bertsekas and Shreve [8, Corollary 7.27.1] or Dynkin and

Yushkevich [12, Appendix 4.4]. For a posterior distribution $z_t \in \mathbb{P}(\mathbb{X})$, action $a_t \in \mathbb{A}$, and an observation $y_{t+1} \in \mathbb{Y}$, the posterior distribution $z_{t+1} \in \mathbb{P}(\mathbb{X})$ is

$$z_{t+1} = H(z_t, a_t, y_{t+1}). \tag{7}$$

However, the observation y_{t+1} is not available in the COMDP model, and therefore y_{t+1} is a random variable with the distribution $R'(\cdot \mid z_t, a_t)$, and the right-hand side of (7) maps $(z_t, a_t) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$ to $\mathbb{P}(\mathbb{P}(\mathbb{X}))$. Thus, z_{t+1} is a random variable with values in $\mathbb{P}(\mathbb{X})$ whose distribution is defined uniquely by the stochastic kernel

$$q(D \mid z, a) := \int_{\mathbb{Y}} \mathbf{I}\{H(z, a, y) \in D\} R'(dy \mid z, a), \quad D \in \mathscr{B}(\mathbb{P}(\mathbb{X})), \quad z \in \mathbb{P}(\mathbb{X}), \quad a \in \mathbb{A}; \tag{8}$$

see Hernández-Lerma [20, p. 87]. The particular choice of a stochastic kernel H satisfying (6) does not effect the definition of q from (8), since for each pair $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$, the mapping $H(z, a, \cdot)\colon \mathbb{Y} \to \mathbb{P}(\mathbb{X})$ is defined $R'(\cdot \mid z, a)$ almost surely uniquely in $y \in \mathbb{Y}$; see Bertsekas and Shreve [8, Corollary 7.27.1], and Dynkin and Yushkevich [12, Appendix 4.4].

Similar to the stochastic kernel R, consider a stochastic kernel R_0 on $\mathbb{X} \times \mathbb{Y}$ given $\mathbb{P}(\mathbb{X})$ defined by

$$R_0(B \times C \mid p) := \int_B Q_0(C \mid x) p(dx), \quad B \in \mathscr{B}(\mathbb{X}), \quad C \in \mathscr{B}(\mathbb{Y}), \quad p \in \mathbb{P}(\mathbb{X}).$$

This kernel can be decomposed as

$$R_0(B \times C \mid p) = \int_C H_0(B \mid p, y) R_0'(dy \mid p), \quad B \in \mathscr{B}(\mathbb{X}), \quad C \in \mathscr{B}(\mathbb{Y}), \quad p \in \mathbb{P}(\mathbb{X}), \tag{9}$$

where $R_0'(C \mid p) = R_0(\mathbb{X} \times C \mid p)$, $C \in \mathscr{B}(\mathbb{Y})$, $p \in \mathbb{P}(\mathbb{X})$, is a stochastic kernel on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})$ and $H_0(dx \mid p, y)$ is a stochastic kernel on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{Y}$. Any initial prior distribution $p \in \mathbb{P}(\mathbb{X})$ and any initial observation y_0 define the initial posterior distribution $z_0 = H_0(p, y_0)$ on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$. Similar to (7), the observation y_0 is not available in the COMDP and this equation is stochastic. In addition, $H_0(p, y)$ is defined $R_0'(dy \mid p)$ almost surely uniquely in $y \in \mathbb{Y}$ for each $p \in \mathbb{P}(\mathbb{X})$.

Similar to (8), the stochastic kernel

$$q_0(D \mid p) := \int_{\mathbb{Y}} \mathbf{I}\{H_0(p, y) \in D\} R_0'(dy \mid p), \quad D \in \mathscr{B}(\mathbb{P}(\mathbb{X})) \text{ and } p \in \mathbb{P}(\mathbb{X}), \tag{10}$$

on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})$ defines the initial distribution on the set of posterior probabilities. Define $q_0(p)(D) = q_0(D \mid p)$, where $D \in \mathscr{B}(\mathbb{P}(\mathbb{X}))$. Then $q_0(p)$ is the initial distribution of $z_0 = H_0(p, y_0)$ corresponding to the initial state distribution p.

The COMDP is defined as an MDP with parameters $(\mathbb{P}(\mathbb{X}), \mathbb{A}, q, \bar{c})$, where

(i) $\mathbb{P}(\mathbb{X})$ is the state space;

(ii) $\mathbb{A}$ is the action set available at all states $z \in \mathbb{P}(\mathbb{X})$;

(iii) the one-step cost function $\bar{c}\colon \mathbb{P}(\mathbb{X}) \times \mathbb{A} \to \bar{\mathbb{R}}$, defined as

$$\bar{c}(z, a) := \int_{\mathbb{X}} c(x, a) z(dx), \quad z \in \mathbb{P}(\mathbb{X}), \quad a \in \mathbb{A}; \tag{11}$$

(iv) transition probabilities q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ defined in (8).

Denote by i_t, $t = 0, 1, \ldots,$ a t-horizon history for the COMDP, also called an *information vector*,

$$i_t := (z_0, a_0, \ldots, z_{t-1}, a_{t-1}, z_t) \in I_t, \quad t = 0, 1, \ldots,$$

where z_0 is the initial posterior distribution and $z_t \in \mathbb{P}(\mathbb{X})$ are recursively defined by Equation (7), $I_t := \mathbb{P}(\mathbb{X}) \times (\mathbb{A} \times \mathbb{P}(\mathbb{X}))^t$ for all $t = 0, 1, \ldots,$ with $I_0 := \mathbb{P}(\mathbb{X})$. An *information policy* (I-policy) is a policy in a COMDP, i.e., I-policy is a sequence $\delta = \{\delta_t\colon t = 0, 1, \ldots\}$ such that $\delta_t(\cdot \mid i_t)$ is a stochastic kernel on $\mathbb{A}$ given I_t for all $t = 0, 1, \ldots$; see Bertsekas and Shreve [8, Chapter 10], and Hernández-Lerma [20, p. 88]. Denote by Δ the set of all I-policies. We also consider Markov I-policies and stationary I-policies.

For an I-policy $\delta = \{\delta_t\colon t = 0, 1, \ldots\}$, define a policy $\pi^\delta = \{\pi_t^\delta\colon t = 0, 1, \ldots\}$ in Π as

$$\pi_t^\delta(\cdot \mid h_t) := \delta_t(\cdot \mid i_t(h_t)) \quad \text{for all } h_t \in H_t \text{ and } t = 0, 1, \ldots, \tag{12}$$

where $i_t(h_t) \in I_t$ is the information vector determined by the observable history h_t via (7). Thus δ and π^δ are equivalent in the sense that π_t^δ assigns the same conditional probability on $\mathbb{A}$ given the observed history h_t as δ_t

for the history $i_t(h_t)$. If δ is an optimal policy for the COMDP then π^δ is an optimal policy for the POMDP. This follows from the facts that $V_{t,\alpha}(p)=\bar v_{t,\alpha}(q_0(p))$, $t=0,1,\dots,$ and $V_\alpha(p)=\bar v_\alpha(q_0(p))$; see Hernández-Lerma [20, p. 89] and references therein. Let $z_t(h_t)$ be the last element of the information vector $i_t(h_t)$. With a slight abuse of notations, by using the same notations for a measure concentrated at a point and a function at this point, if δ is Markov, then (12) becomes $\pi_t^\delta(h_t)=\delta_t(z_t(h_t))$, and if δ is stationary, then $\pi_t^\delta(h_t)=\delta(z_t(h_t))$, $t=0,1,\dots$.

Thus, an optimal policy for a COMDP defines an optimal policy for the POMDP. However, very little is known for the conditions on POMDPs that lead to the existence of optimal policies for the corresponding COMDPs. For the COMDP, Assumption (W*) has the following form:

(i) $\bar c$ is $\mathbb{K}$-inf-compact on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$;

(ii) the transition probability $q(\cdot\mid z,a)$ is weakly continuous in $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$.

Recall that the notation $\bar v$ has been reserved for the expected total costs for COMDPs. The following theorem follows directly from Theorem 2.1 applied to the COMDP $(\mathbb{P}(\mathbb{X}),\mathbb{A},q,\bar c)$.

THEOREM 3.1. *Let either Assumption* (D) *or Assumption* (P) *hold. If the COMDP* $(\mathbb{P}(\mathbb{X}),\mathbb{A},q,\bar c)$ *satisfies Assumption* (W*), *then*

(i) *the functions* $\bar v_{t,\alpha}$, $t=0,1,\dots,$ *and* $\bar v_\alpha$ *are lower semicontinuous on* $\mathbb{P}(\mathbb{X})$, *and* $\bar v_{t,\alpha}(z)\to\bar v_\alpha(z)$ *as* $t\to\infty$ *for all* $z\in\mathbb{P}(\mathbb{X})$;

(ii) *for each* $z\in\mathbb{P}(\mathbb{X})$ *and* $t=0,1,\dots,$

$$\begin{aligned}\bar v_{t+1,\alpha}(z)&=\min_{a\in\mathbb{A}}\left\{\bar c(z,a)+\alpha\int_{\mathbb{P}(\mathbb{X})}\bar v_{t,\alpha}(z')q(dz'\mid z,a)\right\}\\&=\min_{a\in\mathbb{A}}\left\{\int_{\mathbb{X}}c(x,a)z(dx)+\alpha\int_{\mathbb{X}}\int_{\mathbb{X}}\int_{\mathbb{Y}}\bar v_{t,\alpha}(H(z,a,y))Q(dy\mid a,x')P(dx'\mid x,a)z(dx)\right\},\end{aligned}\tag{13}$$

where $\bar v_{0,\alpha}(z)=0$ *for all* $z\in\mathbb{P}(\mathbb{X})$, *and the nonempty sets*

$$A_{t,\alpha}(z):=\left\{a\in\mathbb{A}\colon \bar v_{t+1,\alpha}(z)=\bar c(z,a)+\alpha\int_{\mathbb{P}(\mathbb{X})}\bar v_{t,\alpha}(z')q(dz'\mid z,a)\right\},\quad z\in\mathbb{P}(\mathbb{X}),\quad t=0,1,\dots,$$

satisfy the following properties: (a) *the graph* $\mathrm{Gr}(A_{t,\alpha})=\{(z,a)\colon z\in\mathbb{P}(\mathbb{X}),\ a\in A_{t,\alpha}(z)\}$, $t=0,1,\dots,$ *is a Borel subset of* $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, *and* (b) *if* $\bar v_{t+1,\alpha}(z)=+\infty$, *then* $A_{t,\alpha}(z)=\mathbb{A}$ *and, if* $\bar v_{t+1,\alpha}(z)<+\infty$, *then* $A_{t,\alpha}(z)$ *is compact*;

(iii) *for each* $T=1,2,\dots,$ *there exists an optimal Markov* T*-horizon* I*-policy* $(\phi_0,\dots,\phi_{T-1})$, *and if for a* T*-horizon Markov* I*-policy* $(\phi_0,\dots,\phi_{T-1})$ *the inclusions* $\phi_{T-1-t}(z)\in A_{t,\alpha}(z)$, $z\in\mathbb{P}(\mathbb{X})$, $t=0,\dots,T-1$, *hold, then this* I*-policy is* T*-horizon optimal*;

(iv) *for each* $z\in\mathbb{P}(\mathbb{X})$

$$\begin{aligned}\bar v_\alpha(z)&=\min_{a\in\mathbb{A}}\left\{\bar c(z,a)+\alpha\int_{\mathbb{P}(\mathbb{X})}\bar v_\alpha(z')q(dz'\mid z,a)\right\}\\&=\min_{a\in\mathbb{A}}\left\{\int_{\mathbb{X}}c(x,a)z(dx)+\alpha\int_{\mathbb{X}}\int_{\mathbb{X}}\int_{\mathbb{Y}}\bar v_\alpha(H(z,a,y))Q(dy\mid a,x')P(dx'\mid x,a)z(dx)\right\},\end{aligned}\tag{14}$$

and the nonempty sets

$$A_\alpha(z):=\left\{a\in\mathbb{A}\colon \bar v_\alpha(z)=\bar c(z,a)+\alpha\int_{\mathbb{P}(\mathbb{X})}\bar v_\alpha(z')q(dz'\mid z,a)\right\},\quad z\in\mathbb{P}(\mathbb{X}),$$

satisfy the following properties: (a) *the graph* $\mathrm{Gr}(A_\alpha)=\{(z,a)\colon z\in\mathbb{P}(\mathbb{X}),\ a\in\mathbb{A}_\alpha(z)\}$ *is a Borel subset of* $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, *and* (b) *if* $\bar v_\alpha(z)=+\infty$, *then* $A_\alpha(z)=\mathbb{A}$ *and, if* $\bar v_\alpha(z)<+\infty$, *then* $A_\alpha(z)$ *is compact*;

(v) *for infinite-horizon problems there exists a stationary discount-optimal* I*-policy* ϕ_α, *and a stationary* I*-policy* ϕ_α^* *is optimal if and only if* $\phi_\alpha^*(z)\in A_\alpha(z)$ *for all* $z\in\mathbb{P}(\mathbb{X})$; *and*

(vi) *if* $\bar c$ *is inf-compact on* $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, *then the functions* $\bar v_{t,\alpha}$, $t=1,2,\dots,$ *and* $\bar v_\alpha$ *are inf-compact on* $\mathbb{P}(\mathbb{X})$.

Thus, in view of Theorem 3.1, the important question is under which conditions on the original POMDP, the COMDP satisfies the conditions under which there are optimal policies for MDPs. Monograph Hernández-Lerma [20, p. 90] provides the following conditions for this: (a) $\mathbb{A}$ is compact, (b) the cost function c is bounded and continuous, (c) the transition probability $P(dx'\mid x,a)$ and the observation kernel $Q(dy\mid a,x)$ are weakly continuous stochastic kernels, and (d) there exists a weakly continuous stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}\times\mathbb{Y}$ satisfying (6). Consider the following relaxed version of assumption (d) that does not require that H is continuous in y. We introduce this assumption, called Assumption (H), because it holds in many important situations when a weakly continuous stochastic kernel H satisfying (6) does not exist; see Example 4.2 and Theorem 3.6.

Assumption (**H**). There exists a stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}\times\mathbb{Y}$ satisfying (6) such that if a sequence $\{z^{(n)}\}_{n=1,2,\dots}\subseteq\mathbb{P}(\mathbb{X})$ converges weakly to $z\in\mathbb{P}(\mathbb{X})$, and a sequence $\{a^{(n)}\}_{n=1,2,\dots}\subseteq\mathbb{A}$ converges to $a\in\mathbb{A}$ as $n\to\infty$, then there exists a subsequence $\{(z^{(n_k)},a^{(n_k)})\}_{k=1,2,\dots}\subseteq\{(z^{(n)},a^{(n)})\}_{n=1,2,\dots}$ and a measurable subset C of $\mathbb{Y}$ such that $R'(C\mid z,a)=1$ and for all $y\in C$

$$H(z^{(n_k)},a^{(n_k)},y)\text{ converges weakly to }H(z,a,y). \tag{15}$$

In other words, (15) holds $R'(\cdot\mid z,a)$ almost surely.

Theorem 3.2. *Let the following assumptions hold*:
(a) *either Assumption* (**D**) *or Assumption* (**P**) *holds*;
(b) *the function c is $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$*;
(c) *either*
(i) *the stochastic kernel $R'(dy\mid z,a)$ on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is setwise continuous and Assumption* (**H**) *holds*; *or*
(ii) *the stochastic kernels $P(dx'\mid x,a)$ on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ and $Q(dy\mid a,x)$ on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ are weakly continuous and there exists a weakly continuous stochastic kernel $H(dx\mid z,a,y)$ on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}\times\mathbb{Y}$ satisfying* (6).
Then the COMDP $(\mathbb{P}(\mathbb{X}),\mathbb{A},q,\bar{c})$ satisfies Assumption (**W***) *and therefore statements* (i)–(vi) *of Theorem* 3.1 *hold.*

Remark 3.1. Assumption (**H**) is weaker than weak continuity of H stated in Assumption (ii) of Theorem 3.2. Assumption (**H**) is introduced because, according to Theorem 3.6, this assumption and setwise continuity of R' stated in assumption (i) hold, if the stochastic kernel $P(dx'\mid x,a)$ on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ is weakly continuous and the stochastic kernel $Q(dy\mid a,x)$ on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ is continuous in total variation, whereas a weakly continuous version of H may not exist; see Example 4.2.

Remark 3.2. Throughout this paper we follow the terminology according to which finite sets are countable. If $\mathbb{Y}$ is countable, then Equation (13) transforms into

$$\bar{v}_{t+1,\alpha}(z)=\min_{a\in\mathbb{A}}\left\{\int_{\mathbb{X}}c(x,a)z(dx)+\alpha\sum_{y\in\mathbb{Y}}\bar{v}_{t,\alpha}(H(z,a,y))R'(y\mid z,a)\right\},\quad z\in\mathbb{P}(\mathbb{X}),\quad t=0,1,\dots,$$

and Equation (14) transforms into

$$\bar{v}_{\alpha}(z)=\min_{a\in\mathbb{A}}\left\{\int_{\mathbb{X}}c(x,a)z(dx)+\alpha\sum_{y\in\mathbb{Y}}\bar{v}_{\alpha}(H(z,a,y))R'(y\mid z,a)\right\},\quad z\in\mathbb{P}(\mathbb{X}).$$

Theorem 3.2 follows from Theorems 3.1, 3.3, and 3.4. In particular, Theorem 3.3 implies that, if Assumption (**D**) or (**P**) holds for a POMDP, then it also holds for the corresponding COMDP.

Theorem 3.3. *If the function $c\colon\mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ is bounded below and $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$, then the cost function $\bar{c}\colon\mathbb{P}(\mathbb{X})\times\mathbb{A}\to\bar{\mathbb{R}}$ defined for the COMDP in* (11) *is bounded from below by the same constant as c and $\mathbb{K}$-inf-compact on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.*

Theorem 3.4. *The stochastic kernel $q(dz'\mid z,a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is weakly continuous if condition* (c) *from Theorem* 3.2 *holds.*

The following theorem provides sufficient conditions for the existence of optimal policies for the COMDP and therefore for the POMDP in terms of the initial parameters of the POMDP.

Theorem 3.5. *Let assumptions* (a) *and* (b) *of Theorem* 3.2 *hold, the stochastic kernel $P(dx'\mid x,a)$ on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ be weakly continuous, and the stochastic kernel $Q(dy\mid a,x)$ on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ be continuous in total variation. Then the COMDP $(\mathbb{P}(\mathbb{X}),\mathbb{A},q,\bar{c})$ satisfies Assumption* (**W***) *and therefore statements* (i)–(vi) *of Theorem* 3.1 *hold.*

Theorem 3.5 follows from Theorem 3.3 and from the following statement.

Theorem 3.6. *The weak continuity of the stochastic kernel $P(dx'\mid x,a)$ on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ and continuity in total variation of the stochastic kernel $Q(dy\mid a,x)$ on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ imply that condition* (i) *from Theorem* 3.2 *holds* (*that is, R' is setwise continuous and Assumption* (**H**) *holds*) *and therefore the stochastic kernel $q(dz'\mid z,a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is weakly continuous.*

Example 4.1 demonstrates that, if the stochastic kernel $Q(dy \mid a, x)$ on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ is setwise continuous, then the transition probability q for the COMDP may not be weakly continuous in $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$. In this example the state set consists of two points. Therefore, even if the stochastic kernel $P(dx' \mid x, a)$ on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ is setwise continuous (even if it is continuous in total variation) in $(x, a) \in \mathbb{X} \times \mathbb{A}$ then the setwise continuity of the stochastic kernel $Q(dy \mid a, x)$ on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ is not sufficient for the weak continuity of q.

COROLLARY 3.1 (CP. HERNÁNDEZ-LERMA [20, p. 93]). *If the stochastic kernel $P(dx' \mid x, a)$ on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ is weakly continuous, $\mathbb{Y}$ is countable, and for each $y \in \mathbb{Y}$ the function $Q(y \mid a, x)$ is continuous on $\mathbb{A} \times \mathbb{X}$, then the following statements hold*:

(a) *for each $y \in \mathbb{Y}$ the function $R'(y \mid z, a)$ is continuous on $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ with respect to the topology of weak convergence on $\mathbb{P}(\mathbb{X})$, and Assumption* **(H)** *holds*;

(b) *the stochastic kernel $q(dz' \mid z, a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is weakly continuous*; *and*

(c) *if, in addition to the above conditions, assumptions* (a) *and* (b) *from Theorem 3.2 hold, then the COMDP $(\mathbb{P}(\mathbb{X}), \mathbb{A}, q, \bar{c})$ satisfies Assumption* **(W*)** *and therefore statements* (i)–(vi) *of Theorem 3.1 hold.*

PROOF. For a countable $\mathbb{Y}$, the continuity in total variation of the stochastic kernel $Q(\cdot \mid a, x)$ on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ follows from the continuity of $Q(y \mid a, x)$ for each $y \in \mathbb{Y}$ in $(a, x) \in \mathbb{A} \times \mathbb{X}$ and from $Q(\mathbb{Y} \mid a, x) = 1$ for all $(a, x) \in \mathbb{A} \times \mathbb{X}$ (the similar fact holds for the stochastic kernel $R'(\cdot \mid z, a)$). Indeed, let $(a^{(n)}, x^{(n)}) \to (a, x)$ as $n \to \infty$. For an arbitrary $\epsilon > 0$ choose a finite set $Y_\epsilon \subseteq \mathbb{Y}$ such that $Q(Y_\epsilon \mid a, x) \geq 1 - \epsilon$, which is equivalent to $Q(\mathbb{Y} \setminus Y_\epsilon \mid a, x) \leq \epsilon$. Let us choose a natural number N_ϵ such that $Q(Y_\epsilon \mid a^{(n)}, x^{(n)}) \geq 1 - 2\epsilon$ and, therefore, $Q(\mathbb{Y} \setminus Y_\epsilon \mid a^{(n)}, x^{(n)}) \leq 2\epsilon$ for $n > N_\epsilon$. Denote $\Delta_n := \sum_{y \in \mathbb{Y}} |Q(y \mid a^{(n)}, x^{(n)}) - Q(y \mid a, x)|$. Observe that

$$\Delta_n \leq Q(\mathbb{Y} \setminus Y_\epsilon \mid a^{(n)}, x^{(n)}) + Q(\mathbb{Y} \setminus Y_\epsilon \mid a, x) + \sum_{y \in Y_\epsilon} |Q(y \mid a^{(n)}, x^{(n)}) - Q(y \mid a, x)|.$$

Since $Q(y \mid a, x)$ is continuous in $(a, x) \in \mathbb{A} \times \mathbb{X}$ for each $y \in \mathbb{Y}$ and the set Y_ϵ is finite, the limit of the sum in the right-hand side of the last inequality is 0 as $n \to \infty$. This implies that $\limsup_{n \to \infty} \Delta_n \leq 3\epsilon$. Since $\epsilon > 0$ is arbitrary, then $\Delta_n \to 0$ as $n \to \infty$. Thus, the continuity of $Q(\cdot \mid a, x)$ in total variation takes place. Statements (a) and (b) follow from Theorem 3.6, and statement (c) follows from Theorem 3.5. □

4. Counterexamples. In this section we provide three counterexamples. Example 4.1 demonstrates that the assumption in Theorems 3.5 and 3.6, that the stochastic kernel Q is continuous in total variation, cannot be weakened to the assumption that Q is setwise continuous. Example 4.2 shows that, under conditions of Corollary 3.1, a weakly continuous mapping H satisfying (7) may not exist. The existence of such a mapping is mentioned in Hernández-Lerma [20, p. 93]. Example 4.3 illustrates that the setwise continuity of the stochastic kernel $R'(dy \mid z, a)$ on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is essential in condition (i) of Theorem 3.2. Without this assumption, Assumption **(H)** alone is not sufficient for the weak continuity of the stochastic kernel $q(dz' \mid z, a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ and therefore for the correctness of the conclusions of Theorems 3.2 and 3.4.

We would like to mention that before the authors constructed Example 4.1, Huizhen (Janey) Yu provided them with an example when the weak continuity of the observation kernel Q is not sufficient for the weak continuity of the stochastic kernel $q(\cdot \mid z, a)$. In her example, $\mathbb{X} = \{1, 2\}$, the system does not move, $\mathbb{Y} = \mathbb{A} = [0, 1]$, at state 1 the observation is 0 for any action a and at state 2, under an action $a \in \mathbb{A}$, the observation is uniformly distributed on $[0, a]$. The initial belief distribution is $z = (0.5, 0.5)$.

EXAMPLE 4.1. *Continuity of Q in total variation cannot be relaxed to setwise continuity in Theorems 3.5 and 3.6.* Let $\mathbb{X} = \{1, 2\}$, $\mathbb{Y} = [0, 1]$, and $\mathbb{A} = \{0\} \cup \{1/n\colon n = 1, 2, \dots\}$. The system does not move. This means that $P(x \mid x, a) = 1$ for all $x = 1, 2$ and $a \in \mathbb{A}$. This stochastic kernel P is weakly continuous and, since $\mathbb{X}$ is finite, it is setwise continuous and continuous in total variation. The observation kernel Q is $Q(dy \mid a, 1) = Q(dy \mid 0, 2) = m(dy)$, $a \in \mathbb{A}$, with m being the Lebesgue measure on $\mathbb{Y} = [0, 1]$, and $Q(dy \mid 1/n, 2) = m^{(n)}(dy)$, $n = 1, 2, \dots$, where $m^{(n)}$ is the absolutely continuous measure on $\mathbb{Y} = [0, 1]$ with the density $f^{(n)}$,

$$f^{(n)}(y) = \begin{cases} 0, & \text{if } 2k/2^n < y < (2k+1)/2^n \quad \text{for } k = 0, 1, \dots, 2^{n-1} - 1; \\ 2, & \text{otherwise.} \end{cases} \tag{16}$$

First we show that $Q(dy \mid a, x)$ on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ is setwise continuous in (a, x). In our case, this means that the probability distributions $Q(dy \mid 1/n, i)$ converge setwise to $Q(dy \mid 0, i)$ as $n \to \infty$, where $i = 1, 2$. For $i = 1$ this statement is trivial, because $Q(dy \mid a, 1) = m(dy)$ for all $a \in \mathbb{A}$. For $i = 2$ we need to verify that $m^{(n)}$ converge setwise to m as $n \to \infty$. According to Bogachev [10, Theorem 8.10.56], which is Pfanzagl's generalization

of the Fichtenholz-Dieudonné-Grothendiek theorem, measures $m^{(n)}$ converge setwise to the measure m, if $m^{(n)}(C) \to m(C)$ for each open set C in $[0,1]$. Since $m^{(n)}(0) = m(0) = m^{(n)}(1) = m(1)$, $n = 1, 2, \ldots$, then $m^{(n)}(C) \to m(C)$ for each open set C in $[0,1]$ if and only if $m^{(n)}(C) \to m(C)$ for each open set C in $(0,1)$. Choose an arbitrary open set C in $(0,1)$. Then C is a union of a countable set of open disjoint intervals (a_i, b_i). Therefore, for any $\varepsilon > 0$ there is a finite number n_ε of open intervals $\{(a_i, b_i)\colon i = 1, \ldots, n_\varepsilon\}$ such that $m(C \setminus C_\varepsilon) \le \varepsilon$, where $C_\varepsilon = \bigcup_{i=1}^{n_\varepsilon}(a_i, b_i)$. Since $f^{(n)} \le 2$, this implies that $m^{(n)}(C \setminus C_\varepsilon) \le 2\varepsilon$ for any $n = 1, 2, \ldots$. Since $|m^{(n)}((a,b)) - m((a,b))| \le 1/2^{n-1}$, $n = 1, 2, \ldots$, for any interval $(a,b) \subset [0,1]$, this implies that $|m(C_\varepsilon) - m^{(n)}(C_\varepsilon)| \le \varepsilon$ if $n \ge N_\varepsilon$, where N_ε is any natural number satisfying $1/2^{N_\varepsilon - 1} \le \varepsilon$. Therefore, if $n \ge N_\varepsilon$, then $|m^{(n)}(C) - m(C)| \le |m^{(n)}(C_\varepsilon) - m(C_\varepsilon)| + m(C \setminus C_\varepsilon) + m^{(n)}(C \setminus C_\varepsilon) \le 4\varepsilon$. This implies that $m^{(n)}(C) \to m(C)$ as $n \to \infty$. Thus $m^{(n)}$ converge setwise to m as $n \to \infty$.

Second, we verify that the transition kernel q does not satisfy the weak continuity property. Consider the posterior probability distribution $z = (z(1), z(2)) = (0.5, 0.5)$ of the state at the current step. Since the system does not move, this is the prior probability distribution at the next step. If the action 0 is selected at the current step, then nothing new can be learned about the state during the next step. Thus $q(z \mid z, 0) = 1$. Let y be an observation at the next step, and let D be the event that the state is 2. At the next step, the prior probability of the event D is 0.5, because $z(2) = 0.5$. Now let an action $1/n$ be selected at the current step. The new posterior state probabilities depend on the event $A = \{f^{(n)}(y) = 2\}$. If the event D takes place (the state is 2), then the probability of the event A is 1 and the probability of the event $\bar{A} = \{f^{(n)}(y) = 0\}$ is 0. If the event $\bar{D}$ takes place (the new state is 1), then the probabilities of the events A and $\bar{A}$ are 0.5. Bayes's formula implies that the posterior probabilities are $(1/3, 2/3)$, if $f^{(n)}(y) = 2$, and $(1, 0)$, if $f^{(n)}(y) = 0$. Since $f^{(n)}(2) = 2$ with probability $3/4$ and $f^{(n)}(y) = 0$ with probability $1/4$, then $q((1/3, 2/3) \mid z, 1/n) = 3/4$ and $q((1,0) \mid z, 1/n) = 1/4$. So, all the measures $q(\cdot \mid z, 1/n)$ are constants and they are not equal to the measure $q(\cdot \mid z, 0)$, which is concentrated at the point $z = (0.5, 0.5)$. Thus the transition kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is not weakly continuous. □

Example 4.2. Under conditions of Corollary 3.1 there is no weakly continuous stochastic kernel $H(\cdot \mid z, a, y)$ on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ satisfying (7). Consider a POMDP with the state and observation spaces $\mathbb{X} = \mathbb{Y} = \{1, 2\}$; the action space $\mathbb{A} = [-1, 1]$; the system does not move, that is $P(1 \mid 1, a) = P(2 \mid 2, a) = 1$ for all $a \in \mathbb{A}$; for each $y \in \mathbb{Y}$ the observation kernel $Q(y \mid a, x)$ is continuous in $a \in \mathbb{A}$,

$$Q(1 \mid a, 1) = \begin{cases} |a|, & a \in [-1, 0), \\ a^2, & a \in [0, 1], \end{cases} \qquad Q(1 \mid a, 2) = \begin{cases} a^2, & a \in [-1, 0), \\ |a|, & a \in [0, 1], \end{cases}$$

$$Q(2 \mid a, 1) = \begin{cases} 1 - |a|, & a \in [-1, 0), \\ 1 - a^2, & a \in [0, 1], \end{cases} \qquad Q(2 \mid a, 2) = \begin{cases} 1 - a^2, & a \in [-1, 0), \\ 1 - |a|, & a \in [0, 1]; \end{cases}$$

and $z = (z(1), z(2)) = (\frac{1}{2}, \frac{1}{2})$ is the probability distribution on $\mathbb{X} = \{1, 2\}$.

Formula (4) with $B = \{1\}$ and $C = \{1\}$ implies

$$R((1,1) \mid z, a) = \frac{1}{2} Q(1 \mid a, 1) = \begin{cases} \dfrac{|a|}{2}, & a \in [-1, 0), \\ \dfrac{a^2}{2}, & a \in [0, 1]. \end{cases} \tag{17}$$

Setting $C = \{1\}$ in (5), we obtain

$$R'(1 \mid z, a) = \frac{1}{2} Q(1 \mid a, 1) + \frac{1}{2} Q(1 \mid a, 2) = \frac{|a| + a^2}{2}, \quad a \in [-1, 1]. \tag{18}$$

Formulas (17) and (18) imply that, if H satisfies (6), then

$$H(1 \mid z, a, 1) = \frac{R((1,1) \mid z, a)}{R'(1 \mid z, a)} = \begin{cases} \dfrac{|a|}{|a| + a^2}, & a \in [-1, 0), \\ \dfrac{a^2}{|a| + a^2}, & a \in (0, 1]. \end{cases}$$

Therefore,

$$\lim_{a \uparrow 0} H(1 \mid z, a, 1) = 1 \qquad \text{and} \qquad \lim_{a \downarrow 0} H(1 \mid z, a, 1) = 0.$$

Thus, the stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ is not weakly continuous in a, that is, $H\colon \mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y} \to \mathbb{P}(\mathbb{X})$ is not a continuous mapping. In view of Corollary 3.1, Assumption **(H)** holds. □

EXAMPLE 4.3. Stochastic kernels P on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ and Q on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ are weakly continuous, the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, defined by formula (5), is weakly continuous, but it is not setwise continuous. Though Assumption (**H**) holds, the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, defined by formula (8), is not weakly continuous.

Let $\mathbb{X}=\{1,2\}$, $\mathbb{Y}=\mathbb{A}=\{1,\frac{1}{2},\frac{1}{3},\dots\}\cup\{0\}$ with the metric $\rho(a,b)=|a-b|$, $a,b\in\mathbb{Y}$, and $P(x\mid x,a)=1$, $x\in\mathbb{X}$, $a\in\mathbb{A}$. Let also $Q(0\mid 0,x)=1$, $Q(0\mid 1/m,x)=Q(1/n\mid 0,x)=0$, $x\in\mathbb{X}$, and $Q(1/n\mid 1/m,1)=a_{m,n}\sin^2(\pi n/(2m))$, $Q(1/n\mid 1/m,2)=a_{m,n}\cos^2(\pi n/(2m))$, $m,n=1,2,\dots$, where $a_{m,2mk+\ell}=1/(2^{k+1}m)$ for $k=0,1,\dots$, $\ell=1,2,\dots,2m$. Since $\sum_{\ell=1}^{2m}\sin^2(\pi\ell/(2m))=\sum_{\ell=1}^{2m}\cos^2(\pi\ell/(2m))=\sum_{\ell=1}^{m}(\sin^2(\pi\ell/(2m))+\cos^2(\pi\ell/(2m)))=m$, then $\sum_{n=1}^{\infty}Q(1/n\mid 1/m,x)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{2m}Q(1/(2mk+\ell)\mid 1/m,x)=\sum_{k=0}^{\infty}(1/2^{k+1})=1$, $x\in\mathbb{X}$, and Q is a stochastic kernel on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$. The stochastic kernels P on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ and Q on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ are weakly continuous. The former is true for the same reasons as in Example 4.1. The latter is true because $\limsup_{m\to\infty}Q(C\mid a_m,x)\le Q(C\mid 0,x)$ for any closed set C in $\mathbb{Y}$. Indeed, a set C is closed in $\mathbb{Y}$ if and only if either (i) $0\in C$ or (ii) $0\notin C$ and C is finite. Let $C\subseteq\mathbb{Y}$ be closed. In case (i), $\limsup_{m\to\infty}Q(C\mid a_m,x)\le 1=Q(C\mid 0,x)$ as $a_m\to 0$, $x\in\mathbb{X}$. In case (ii), $\lim_{m\to\infty}Q(C\mid a_m,x)=0=Q(C\mid 0,x)$ as $a_m\to 0$, since $\lim_{m\to\infty}Q(1/n\mid a_m,x)=0=Q(1/n\mid 0,x)$ for $n=1,2,\dots$ and for $x\in\mathbb{X}$.

Formula (4) implies that $R(1,1/n\mid z,1/m)=z(1)a_{m,n}\sin^2(\pi n/(2m))$, $R(2,1/n\mid z,1/m)=z(2)a_{m,n}\cos^2(\pi n/(2m))$, $R(1,0\mid z,1/m)=0$, $R(2,0\mid z,1/m)=0$, and $R(1,1/n\mid z,0)=0$, $R(2,1/n\mid z,0)=0$, $R(1,0\mid z,0)=z(1)$, $R(2,0\mid z,0)=z(2)$ for $m,n=1,2,\dots$, $z=(z(1),z(2))\in\mathbb{P}(\mathbb{X})$. Formula (5) yields $R'(0\mid z,1/m)=0$, $R'(1/n\mid z,1/m)=z(1)a_{m,n}\sin^2(\pi n/(2m))+z(2)a_{m,n}\cos^2(\pi n/(2m))$, and $R'(0\mid z,0)=1$, $R'(1/n\mid z,0)=0$ for $m,n=1,2,\dots$, $z=(z(1),z(2))\in\mathbb{P}(\mathbb{X})$. Therefore, $R'(0\mid z,1/m)\not\to R'(0\mid z,0)$ as $m\to\infty$. Thus the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ *is not setwise continuous*. However, stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is weakly continuous.

Observe that $\mathbb{P}(\mathbb{X})=\{(z(1),z(2))\colon z(1),z(2)\ge 0,\ z(1)+z(2)=1\}\subset\mathbb{R}^2$. Let $z=(z(1),z(2))\in\mathbb{P}(\mathbb{X})$. If $R'(y\mid z,a)>0$, in view of (6), $H(x'\mid z,a,y)=R((x',y)\mid z,a)/R'(y\mid z,a)$ for all $x'\in\mathbb{X}$, $a\in\mathbb{A}$, and $y\in\mathbb{Y}$. Thus, if $R'(y\mid z,a)>0$, then

$$H(z,a,y)=\begin{cases}\left(\dfrac{z(1)\sin^2(\pi n/(2m))}{z(1)\sin^2(\pi n/(2m))+z(2)\cos^2(\pi n/(2m))},\ \dfrac{z(2)\cos^2(\pi n/(2m))}{z(1)\sin^2(\pi n/(2m))+z(2)\cos^2(\pi n/(2m))}\right),\\ \quad\text{if } a=\dfrac{1}{m},\quad y=\dfrac{1}{n},\quad m,n=1,2,\dots,\\ (z(1),z(2)),\quad \text{if } a=y=0.\end{cases}$$

If $R'(y\mid z,a)=0$, we set $H(z,a,y)=z=(z(1),z(2))$. In particular, $H(z,1/m,0)=z$ for all $m=1,2,\dots$.

Observe that Assumption (**H**) holds because, if $R'(y\mid z,a)>0$ and if sequences $\{z^{(N)}\}_{N=1,2,\dots}\subseteq\mathbb{P}(\mathbb{X})$ and $\{a^{(N)}\}_{N=1,2,\dots}\subseteq\mathbb{A}$ converge to $z\in\mathbb{P}(\mathbb{X})$ and $a\in\mathbb{A}$, respectively, as $N\to\infty$, then $H(z^{(N)},a^{(N)},y)\to H(z,a,y)$ as $N\to\infty$. Indeed, it is sufficient to verify this property only for the following two cases: (i) $y=1/n$, $a=1/m$, and $R'(1/n\mid z,1/m)>0$, where $m,n=1,2,\dots$, and (ii) $y=a=0$. In case (i), $a^{(N)}=1/m$, when N is large enough, and $H(z^{(N)},1/m,1/n)\to H(z,1/m,1/n)$ as $N\to\infty$ because the function $H(z,1/m,1/n)$ is continuous in z, when $R'(1/n\mid z,1/m)>0$. For case (ii), $H(z^{(N)},a^{(N)},0)=z^{(N)}\to z$ as $N\to\infty$.

Fix $z=(\frac{1}{2},\frac{1}{2})$. According to the above formulae, $H(z,1/m,1/n)=(\sin^2(\pi n/2m),\cos^2(\pi n/2m))$ and $R'(1/n\mid z,1/m)=a_{m,n}/2$. Consider a closed subset $D=\{(z'(1),z'(2))\in\mathbb{P}(\mathbb{X})\colon z'(1)\ge\frac{3}{4}\}$ in $\mathbb{P}(\mathbb{X})$. Then

$$\begin{aligned}q(D\mid z,1/m)&=\sum_{n=1,2,\dots}\{\sin^2(\pi n/2m)\ge 3/4\}(a_{m,n}/2)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{2m}\mathbf{I}\{\sin(\pi\ell/(2m))\ge\sqrt{3}/2\}((a_{m,2mk}+\ell)/2)\\&=\sum_{\ell=1}^{2m}\mathbf{I}\{\sin(\pi\ell/2m)\ge\sqrt{3}/2\}1/2m\sum_{k=0}^{\infty}(1/2^{k+1})\to\tfrac{1}{3}>0\end{aligned}$$

as $m\to\infty$, where the limit takes place because $|[2m/3]-\sum_{\ell=1}^{2m}\mathbf{I}\{\sin(\pi\ell/2m)\ge\sqrt{3}/2\}|\le 1$, where $[\cdot]$ is the integer part of a number, and $\sum_{k=0}^{\infty}(1/2^{k+1})=1$. In addition, $q(D\mid z,0)=0$ since $z\notin D$ and $q(z\mid z,0)=\mathbf{I}\{H(z,0,0)=z\}R'(0\mid z,0)=1$. Thus, $\lim_{m\to\infty}q(D\mid z,1/m)=\frac{1}{3}>0=q(D\mid z,0)$ for a closed set D in $\mathbb{P}(\mathbb{X})$. This implies that the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ *is not weakly continuous*. □

5. Continuity of transition kernels for posterior probabilities. This section contains the proofs of Theorems 3.4 and 3.6. The following two versions of Fatou's lemma for a sequence of measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ are used in the proofs provided below.

Lemma 5.1 (Generalized Fatou's Lemma). *Let $\mathbb{S}$ be an arbitrary metric space, $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathbb{P}(\mathbb{S})$, and $\{f^{(n)}\}_{n=1,2,\dots}$ be a sequence of measurable nonnegative $\bar{\mathbb{R}}$-valued functions on $\mathbb{S}$. Then*

(i) (Royden [28, p. 231]) *if $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathbb{P}(\mathbb{S})$ converges setwise to $\mu \in \mathbb{P}(\mathbb{S})$, then*

$$\int_{\mathbb{S}} \liminf_{n\to\infty} f^{(n)}(s)\mu(ds) \le \liminf_{n\to\infty} \int_{\mathbb{S}} f^{(n)}(s)\mu^{(n)}(ds); \tag{19}$$

(ii) (Schäl [30, Lemma 2.3(ii)], Jaśkiewicz and Nowak [24, Lemma 3.2], Feinberg et al. [14, Lemma 4], Feinberg et al. [16, Theorem 1.1]) *if $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathbb{P}(\mathbb{S})$ converges weakly to $\mu \in \mathbb{P}(\mathbb{S})$, then*

$$\int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f^{(n)}(s')\mu(ds) \le \liminf_{n\to\infty} \int_{\mathbb{S}} f^{(n)}(s)\mu^{(n)}(ds). \tag{20}$$

Proof of Theorem 3.4. According to Parthasarathy [25, Theorem 6.1, p. 40], Shiryaev [32, p. 311], or Billingsley [9, Theorem 2.1], the stochastic kernel $q(dz' \mid z, a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is weakly continuous if and only if $q(D \mid z, a)$ is lower semicontinuous in $(z, a) \in (\mathbb{P}(\mathbb{X}) \times \mathbb{X})$ for every open set D in $\mathbb{P}(\mathbb{X})$, that is,

$$\liminf_{n\to\infty} q(D \mid z^{(n)}, a^{(n)}) \ge q(D \mid z, a), \tag{21}$$

for all $z, z^{(n)} \in \mathbb{P}(\mathbb{X})$, and $a, a^{(n)} \in \mathbb{A}$, $n = 1, 2, \dots$, such that $z^{(n)} \to z$ weakly and $a^{(n)} \to a$.

To prove (21), suppose that

$$\liminf_{n\to\infty} q(D \mid z^{(n)}, a^{(n)}) < q(D \mid z, a).$$

Then there exists $\varepsilon^* > 0$ and a subsequence $\{(z^{(n,1)}, a^{(n,1)})\}_{n=1,2,\dots} \subseteq \{(z^{(n)}, a^{(n)})\}_{n=1,2,\dots}$ such that

$$q(D \mid z^{(n,1)}, a^{(n,1)}) \le q(D \mid z, a) - \varepsilon^*, \quad n = 1, 2, \dots. \tag{22}$$

If condition (ii) of Theorem 3.2 holds, then formula (8), the weak continuity of the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ (this weak continuity is proved in Hernández-Lerma [20, p. 92]), and Lemma 5.1(ii) contradict (22). If condition (i) of Theorem 3.2 holds, then there exists a subsequence $\{(z^{(n,2)}, a^{(n,2)})\}_{n=1,2,\dots} \subseteq \{(z^{(n,1)}, a^{(n,1)})\}_{n=1,2,\dots}$ such that $H(z^{(n,2)}, a^{(n,2)}, y) \to H(z, a, y)$ weakly as $n \to \infty$, $R'(\cdot \mid z, a)$ almost surely in $y \in \mathbb{Y}$. Therefore, since D is an open set in $\mathbb{P}(\mathbb{X})$,

$$\liminf_{n\to\infty} \mathbf{I}\{H(z^{(n,2)}, a^{(n,2)}, y) \in D\} \ge \mathbf{I}\{H(z, a, y) \in D\}, \quad R'(\cdot \mid z, a) \text{ almost surely in } y \in \mathbb{Y}. \tag{23}$$

Formulas (8), (23), the setwise continuity of the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$, and Lemma 5.1(i) imply $\liminf_{n\to\infty} q(D \mid z^{(n,2)}, a^{(n,2)}) \ge q(D \mid z, a)$, which contradicts (22). Thus (21) holds. □

To prove Theorem 3.6, we need to formulate and prove several auxiliary facts. Let $\mathbb{S}$ be a metric space, $\mathbb{F}(\mathbb{S})$ and $\mathbb{C}(\mathbb{S})$ be, respectively, the spaces of all real-valued functions and all bounded continuous functions defined on $\mathbb{S}$. A subset $\mathscr{A}_0 \subseteq \mathbb{F}(\mathbb{S})$ is said to be *equicontinuous at a point $s \in \mathbb{S}$*, if $\sup_{f\in\mathscr{A}_0} |f(s') - f(s)| \to 0$ as $s' \to s$. A subset $\mathscr{A}_0 \subseteq \mathbb{F}(\mathbb{S})$ is said to be *uniformly bounded*, if there exists a constant $M < +\infty$ such that $|f(s)| \le M$, for all $s \in \mathbb{S}$ and for all $f \in \mathscr{A}_0$. Obviously, if a subset $\mathscr{A}_0 \subseteq \mathbb{F}(\mathbb{S})$ is equicontinuous at all the points $s \in \mathbb{S}$ and uniformly bounded, then $\mathscr{A}_0 \subseteq \mathbb{C}(\mathbb{S})$.

Theorem 5.1. *Let $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_3$ be arbitrary metric spaces, $\Psi(ds_2 \mid s_1)$ be a weakly continuous stochastic kernel on $\mathbb{S}_2$ given $\mathbb{S}_1$, and a subset $\mathscr{A}_0 \subseteq \mathbb{C}(\mathbb{S}_2 \times \mathbb{S}_3)$ be equicontinuous at all the points $(s_2, s_3) \in \mathbb{S}_2 \times \mathbb{S}_3$ and uniformly bounded. If $\mathbb{S}_2$ is separable, then for every open set $\mathscr{O}$ in $\mathbb{S}_2$ the family of functions defined on $\mathbb{S}_1 \times \mathbb{S}_3$,*

$$\mathscr{A}_{\mathscr{O}} = \left\{(s_1, s_3) \to \int_{\mathscr{O}} f(s_2, s_3)\Psi(ds_2 \mid s_1)\colon f \in \mathscr{A}_0\right\},$$

is equicontinuous at all the points $(s_1, s_3) \in \mathbb{S}_1 \times \mathbb{S}_3$ and uniformly bounded.

Proof. The family $\mathscr{A}_{\varnothing}$ consists of a single function, which is identically equal to 0. Thus, the statement of the theorem holds when $\mathscr{O} = \varnothing$. Let $\mathscr{O} \ne \varnothing$. Since $\mathscr{A}_0 \subseteq \mathbb{C}(\mathbb{S}_2 \times \mathbb{S}_3)$ is uniformly bounded, then

$$\mathscr{M} = \sup_{f\in\mathscr{A}_0} \sup_{s_2\in\mathbb{S}_2} \sup_{s_3\in\mathbb{S}_3} |f(s_2, s_3)| < \infty, \tag{24}$$

and, since $\Psi(ds_2 \mid s_1)$ is a stochastic kernel, the family of functions $\mathscr{A}_{\mathscr{O}}$ is uniformly bounded by $\mathscr{M}$.

Let us fix an arbitrary nonempty open set $\mathscr{O} \subseteq \mathbb{S}_2$ and an arbitrary point $(s_1, s_3) \in \mathbb{S}_1 \times \mathbb{S}_3$. We shall prove that $\mathscr{A}_{\mathscr{O}} \subset \mathbb{F}(\mathbb{S}_1 \times \mathbb{S}_3)$ is equicontinuous at the point (s_1, s_3). For any $s \in \mathbb{S}_2$ and $\delta > 0$ denote by $B_\delta(s)$ and $\bar{B}_\delta(s)$, respectively, the open and closed balls in the metric space $\mathbb{S}_2$ of radius δ with center s and by $S_\delta(s)$ the sphere in $\mathbb{S}_2$ of radius δ with center s. Note that $S_\delta(s) = \bar{B}_\delta(s) \backslash B_\delta(s)$ is the boundary of $B_\delta(s)$. Every ball $B_\delta(s)$ contains a ball $B_{\delta'}(s)$, $0 < \delta' \leq \delta$, such that

$$\Psi(\bar{B}_{\delta'}(s) \backslash B_{\delta'}(s) \mid s_1) = \Psi(S_{\delta'}(s) \mid s_1) = 0,$$

that is, $B_{\delta'}(s)$ is a continuity set for the probability measure $\Psi(\cdot \mid s_1)$; Parthasarathy [25, p. 50]. Since $\mathscr{O}$ is an open set in $\mathbb{S}_2$, for any $s \in \mathscr{O}$ there exists $\delta_s > 0$ such that $B_{\delta_s}(s)$ is a continuity set for a probability measure $\Psi(\cdot \mid s_1)$ and $B_{\delta_s}(s) \subseteq \mathscr{O}$. The family $\{B_{\delta_s}(s)\colon s \in \mathscr{O}\}$ is a cover of $\mathscr{O}$. Since $\mathbb{S}_2$ is a separable metric space, by Lindelöf's lemma, there exists a sequence $\{s^{(j)}\}_{j=1,2,\ldots} \subset \mathscr{O}$ such that $\{B_{\delta_{s^{(j)}}}(s^{(j)})\colon j = 1, 2, \ldots\}$ is a cover of $\mathscr{O}$. The sets

$$A^{(1)} := B_{\delta_{s^{(1)}}}(s^{(1)}), \qquad A^{(2)} := B_{\delta_{s^{(2)}}}(s^{(2)}) \backslash B_{\delta_{s^{(1)}}}(s^{(1)}), \ldots, \qquad A^{(j)} := B_{\delta_{s^{(j)}}}(s^{(j)}) \Bigg\backslash \left(\bigcup_{i=1}^{j-1} B_{\delta_{s^{(i)}}}(s^{(i)}) \right), \ldots,$$

are continuity sets for the probability measure $\Psi(\cdot \mid s_1)$. In view of Parthasarathy [25, Theorem 6.1(e), p. 40],

$$\Psi(A^{(j)} \mid s_1') \to \Psi(A^{(j)} \mid s_1), \quad \text{as } s_1' \to s_1, \quad j = 1, 2, \ldots. \tag{25}$$

Moreover,

$$\bigcup_{j=1,2,\ldots} A^{(j)} = \mathscr{O} \qquad \text{and} \qquad A^{(i)} \cap A^{(j)} = \varnothing, \quad \text{for all } i \neq j. \tag{26}$$

The next step of the proof is to show that for each $j = 1, 2, \ldots$

$$\sup_{f \in \mathscr{A}_0} \left| \int_{A^{(j)}} f(s_2, s_3') \Psi(ds_2 \mid s_1') - \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2 \mid s_1) \right| \to 0 \quad \text{as } (s_1', s_3') \to (s_1, s_3). \tag{27}$$

Fix an arbitrary $j = 1, 2, \ldots$. If $\Psi(A^{(j)} \mid s_1) = 0$, then formula (27) directly follows from (25) and (24). Now let $\Psi(A^{(j)} \mid s_1) > 0$. Formula (25) implies the existence of such $\delta > 0$ that $\Psi(A^{(j)} \mid s_1') > 0$ for all $s_1' \in B_\delta(s_1)$. We endow $A^{(j)}$ with the induced topology from $\mathbb{S}_2$ and set

$$\Psi_j(C \mid s_1') := \frac{\Psi(C \mid s_1')}{\Psi(A^{(j)} \mid s_1')}, \quad s_1' \in B_\delta(s_1), \quad C \in \mathscr{B}(A^{(j)}).$$

Formula (25) yields

$$\Psi_j(ds_2 \mid s_1') \text{ converges weakly to } \Psi_j(ds_2 \mid s_1), \quad \text{in } \mathbb{P}(A^{(j)}) \text{ as } s_1' \to s_1. \tag{28}$$

According to Parthasarathy [25, Theorem 6.8, p. 51],

$$\sup_{f \in \mathscr{A}_0} \left| \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2 \mid s_1') - \int_{A^{(j)}} f(s_2, s_3) \Psi(ds_2 \mid s_1) \right| \to 0, \quad \text{as } s_1' \to s_1. \tag{29}$$

Equicontinuity of $\mathscr{A}_0 \subseteq \mathbb{C}(\mathbb{S}_2 \times \mathbb{S}_3)$ at all the points $(s_2, s_3) \in \mathbb{S}_2 \times \mathbb{S}_3$ and the inequality $|f(s_2', s_3') - f(s_2', s_3)| \leq |f(s_2', s_3') - f(s_2, s_3)| + |f(s_2', s_3) - f(s_2, s_3)|$ imply

$$\limsup_{(s_2', s_3') \to (s_2, s_3)} \sup_{f \in \mathscr{A}_0} |f(s_2', s_3') - f(s_2', s_3)| = 0, \quad \text{for all } s_2 \in \mathbb{S}_2. \tag{30}$$

Thus, formulas (30) and (28) and Lemma 5.1(ii) imply

$$\begin{aligned} &\limsup_{(s_1', s_3') \to (s_1, s_3)} \sup_{f \in \mathscr{A}_0} \left| \int_{A^{(j)}} (f(s_2, s_3') - f(s_2, s_3)) \Psi(ds_2 \mid s_1') \right| \\ &\qquad \leq \int_{A^{(j)}} \limsup_{(s_2', s_3') \to (s_2, s_3)} \sup_{f \in \mathscr{A}_0} |f(s_2', s_3') - f(s_2', s_3)| \Psi(ds_2 \mid s_1) = 0. \end{aligned} \tag{31}$$

Formula (27) follows from (29) and (31).

Since, for all $j=1,2,\dots$ and for all $(s_1',s_3')\in\mathbb{S}_1\times\mathbb{S}_3$,

$$\sup_{f\in\mathscr{A}_0}\left|\int_{A^{(j)}} f(s_2,s_3')\Psi(ds_2\mid s_1')-\int_{A^{(j)}} f(s_2,s_3)\Psi(ds_2\mid s_1)\right|\le 2\mathscr{M}\Psi(A^{(j)}\mid s_1)$$

and $\sum_{j=1}^{\infty}\Psi(A^{(j)}\mid s_1)=\Psi(\mathscr{O}\mid s_1)\le 1$, then equicontinuity of $\mathscr{A}_{\mathscr{O}}$ at the point (s_1,s_3) follows from (26) and (27). Indeed,

$$\begin{aligned}&\sup_{f\in\mathscr{A}_0}\left|\int_{\mathscr{O}} f(s_2,s_3')\Psi(ds_2\mid s_1')-\int_{\mathscr{O}} f(s_2,s_3)\Psi(ds_2\mid s_1)\right|\\&\quad\le\sum_{j=1,2,\dots}\sup_{f\in\mathscr{A}_0}\left|\int_{A^{(j)}} f(s_2,s_3')\Psi(ds_2\mid s_1')-\int_{A^{(j)}} f(s_2,s_3)\Psi(ds_2\mid s_1)\right|\to 0,\quad\text{as } (s_1',s_3')\to(s_1,s_3).\end{aligned}$$

As $(s_1,s_3)\in\mathbb{S}_1\times\mathbb{S}_3$ is arbitrary, the above inequality implies that $\mathscr{A}_{\mathscr{O}}$ is equicontinuous at all the points $(s_1,s_3)\in\mathbb{S}_1\times\mathbb{S}_3$. In particular, $\mathscr{A}_{\mathscr{O}}\subseteq\mathbb{C}(\mathbb{S}_1\times\mathbb{S}_3)$. □

For a set $B\in\mathscr{B}(\mathbb{X})$, let $\mathscr{R}_B$ be the following family of functions defined on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$:

$$\mathscr{R}_B=\{(z,a)\to R(B\times C\mid z,a)\colon C\in\mathscr{B}(\mathbb{Y})\}. \tag{32}$$

Lemma 5.2. *Let the stochastic kernel $P(dx'\mid x,a)$ on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ be weakly continuous and the stochastic kernel $Q(dy\mid a,x)$ on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ be continuous in total variation. Consider the stochastic kernel $R(\cdot\mid z,a)$ on $\mathbb{X}\times\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ defined in formula (4). Then, for every pair of open sets $\mathscr{O}_1$ and $\mathscr{O}_2$ in $\mathbb{X}$, the family of functions $\mathscr{R}_{\mathscr{O}_1\setminus\mathscr{O}_2}$ defined on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is uniformly bounded and is equicontinuous at all the points $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$, that is, for all $z,z^{(n)}\in\mathbb{P}(\mathbb{X})$, $a,a^{(n)}\in\mathbb{A}$, $n=1,2,\dots$, such that $z^{(n)}\to z$ weakly and $a^{(n)}\to a$,*

$$\sup_{C\in\mathscr{B}(\mathbb{Y})}|R((\mathscr{O}_1\setminus\mathscr{O}_2)\times C\mid z^{(n)},a^{(n)})-R((\mathscr{O}_1\setminus\mathscr{O}_2)\times C\mid z,a)|\to 0. \tag{33}$$

Proof. Since R is a stochastic kernel, all the functions in the family $\mathscr{R}_{\mathscr{O}_1\setminus\mathscr{O}_2}$ are nonnegative and bounded above by 1. Thus, this family is uniformly bounded. The remaining proof establishes the equicontinuity of $\mathscr{R}_{\mathscr{O}_1\setminus\mathscr{O}_2}$ at all the points $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. First we show that $\mathscr{R}_{\mathscr{O}}$ is equicontinuous at all the points (z,a) when $\mathscr{O}$ is an open set in $\mathbb{X}$. Theorem 5.1, with $\mathbb{S}_1=\mathbb{X}\times\mathbb{A}$, $\mathbb{S}_2=\mathbb{X}$, $\mathbb{S}_3=\mathbb{A}$, $\mathscr{O}=\mathscr{O}$, $\Psi=P$, and $\mathscr{A}_0=\{(a,x)\to Q(C\mid a,x)\colon C\in\mathscr{B}(\mathbb{Y})\}\subseteq\mathbb{C}(\mathbb{A}\times\mathbb{X})$, implies that the family of functions $\mathscr{A}_{\mathscr{O}}^1=\{(x,a)\to\int_{\mathscr{O}}Q(C\mid a,x')P(dx'\mid x,a)\colon C\in\mathscr{B}(\mathbb{Y})\}$ is equicontinuous at all the points $(x,a)\in\mathbb{X}\times\mathbb{A}$. In particular, $\mathscr{A}_{\mathscr{O}}^1\subseteq\mathbb{C}(\mathbb{A}\times\mathbb{X})$. Thus, Theorem 5.1, with $\mathbb{S}_1=\mathbb{P}(\mathbb{X})$, $\mathbb{S}_2=\mathbb{X}$, $\mathbb{S}_3=\mathbb{A}$, $\mathscr{O}=\mathbb{X}$, $\Psi(B\mid z)=z(B)$, $B\in\mathscr{B}(\mathbb{X})$, $z\in\mathbb{P}(\mathbb{X})$, and $\mathscr{A}_0=\mathscr{A}_{\mathscr{O}}^1$, implies that the family $\mathscr{R}_{\mathscr{O}}$ is equicontinuous at all the points $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. Second, let $\mathscr{O}_1$ and $\mathscr{O}_2$ be arbitrary open sets in $\mathbb{X}$. Then the families of functions $\mathscr{R}_{\mathscr{O}_1}$, $\mathscr{R}_{\mathscr{O}_2}$, and $\mathscr{R}_{\mathscr{O}_1\cup\mathscr{O}_2}$ are equicontinuous at all the points $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. Thus, for all $z,z^{(n)}\in\mathbb{P}(\mathbb{X})$, $a,a^{(n)}\in\mathbb{A}$, $n=1,2,\dots$, such that $z^{(n)}\to z$ weakly and $a^{(n)}\to a$,

$$\begin{aligned}&\sup_{C\in\mathscr{B}(\mathbb{Y})}\left|R((\mathscr{O}_1\setminus\mathscr{O}_2)\times C\mid z^{(n)},a^{(n)})-R((\mathscr{O}_1\setminus\mathscr{O}_2)\times C\mid z,a)\right|\\&\quad\le\sup_{C\in\mathscr{B}(\mathbb{Y})}\left|R((\mathscr{O}_1\cup\mathscr{O}_2)\times C\mid z^{(n)},a^{(n)})-R((\mathscr{O}_1\cup\mathscr{O}_2)\times C\mid z,a)\right|\\&\qquad+\sup_{C\in\mathscr{B}(\mathbb{Y})}\left|R(\mathscr{O}_2\times C\mid z^{(n)},a^{(n)})-R(\mathscr{O}_2\times C\mid z,a)\right|\to 0,\end{aligned} \tag{34}$$

that is, the family of functions $\mathscr{R}_{\mathscr{O}_1\setminus\mathscr{O}_2}$ is equicontinuous at all the points $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. □

Corollary 5.1. *Let assumptions of Lemma 5.2 hold. Then the stochastic kernel $R'(dy\mid z,a)$ on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, defined in formula (5), is continuous in total variation.*

Proof. This corollary follows from Lemma 5.2 applied to $\mathscr{O}_1=\mathbb{X}$ and $\mathscr{O}_2=\varnothing$. □

Theorem 5.2 (cp. Feinberg et al. [18, Corollary 1.8]). *Let $\mathbb{S}$ be an arbitrary metric space, $\{h,h^{(n)}\}_{n=1,2,\dots}$ be Borel-measurable uniformly bounded real-valued functions on $\mathbb{S}$, $\{\mu^{(n)}\}_{n=1,2,\dots}\subset\mathbb{P}(\mathbb{S})$ converges in total variation to $\mu\in\mathbb{P}(\mathbb{S})$, and*

$$\sup_{S\in\mathscr{B}(\mathbb{S})}\left|\int_S h^{(n)}(s)\mu^{(n)}(ds)-\int_S h(s)\mu(ds)\right|\to 0,\quad\text{as } n\to\infty. \tag{35}$$

Then $\{h^{(n)}\}_{n=1,2,\dots}$ converges in probability μ to h, and therefore there is a subsequence $\{h^{(n_k)}\}_{k=1,2,\dots}\subseteq\{h^{(n)}\}_{n=1,2,\dots}$ such that $\{h^{(n_k)}\}_{k=1,2,\dots}$ converges μ almost surely to h.

Proof. Fix an arbitrary $\varepsilon > 0$ and set

$$S^{(n,+)} := \{s \in \mathbb{S}\colon h^{(n)}(s) - h(s) \geq \varepsilon\}, \qquad S^{(n,-)} := \{s \in \mathbb{S}\colon h(s) - h^{(n)}(s) \geq \varepsilon\},$$
$$S^{(n)} := \{s \in \mathbb{S}\colon |h^{(n)}(s) - h(s)| \geq \varepsilon\} = S^{(n,+)} \cup S^{(n,-)}, \quad n = 1, 2, \ldots.$$

Note that for all $n = 1, 2, \ldots$

$$\begin{aligned}\varepsilon\mu^{(n)}(S^{(n,+)}) &\leq \int_{S^{(n,+)}} h^{(n)}(s)\mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s)\mu^{(n)}(ds)\\ &\leq \left|\int_{S^{(n,+)}} h^{(n)}(s)\mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s)\mu(ds)\right| + \left|\int_{S^{(n,+)}} h(s)\mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s)\mu(ds)\right|. \qquad (36)\end{aligned}$$

The convergence in total variation of $\mu^{(n)}$ to $\mu \in \mathbb{P}(\mathbb{S})$ implies that

$$\left|\int_{S^{(n,+)}} h(s)\mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s)\mu(ds)\right| \to 0 \qquad \text{and} \qquad \left|\mu^{(n)}(S^{(n,+)}) - \mu(S^{(n,+)})\right| \to 0, \quad \text{as } n \to \infty. \qquad (37)$$

Formula (35) yields $|\int_{S^{(n,+)}} h^{(n)}(s)\mu^{(n)}(ds) - \int_{S^{(n,+)}} h(s)\mu(ds)| \to 0$ as $n \to \infty$. Thus, in view of (36) and (37),

$$\mu(S^{(n,+)}) \to 0, \quad \text{as } n \to \infty. \qquad (38)$$

Being applied to the functions $\{-h, -h^{(n)}\}_{n=1,2,\ldots}$, formula (38) implies that $\mu(S^{(n,-)}) \to 0$ as $n \to \infty$. This fact and (38) yield $\mu(S^{(n)}) = \mu(S^{(n,+)}) + \mu(S^{(n,-)}) \to 0$ as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, $\{h^{(n)}\}_{n=1,2,\ldots}$ converges to h in probability μ and, therefore, $\{h^{(n)}\}_{n=1,2,\ldots}$ contains a subsequence $\{h^{(n_k)}\}_{k=1,2,\ldots}$ that converges μ almost surely to h. □

Lemma 5.3. *If the topology on $\mathbb{X}$ has a countable base $\tau_b = \{\mathcal{O}^{(j)}\}_{j=1,2,\ldots}$ such that, for each finite intersection $\mathcal{O} = \bigcap_{i=1}^{N} \mathcal{O}^{(j_i)}$ of its elements $\mathcal{O}^{(j_i)} \in \tau_b$, $i = 1, 2, \ldots, N$, the family of functions $\mathscr{R}_{\mathcal{O}}$ defined in (32) is equicontinuous at all the points $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$, then for any sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$, such that $\{z^{(n)}\}_{n=1,2,\ldots} \subseteq \mathbb{P}(\mathbb{X})$ converges weakly to $z \in \mathbb{P}(\mathbb{X})$ and $\{a^{(n)}\}_{n=1,2,\ldots} \subseteq \mathbb{A}$ converges to a, there exists a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots}$ and a set $C^* \in \mathscr{B}(\mathbb{Y})$ such that*

$$R'(C^* \mid z, a) = 1 \qquad \text{and} \qquad H(\cdot \mid z^{(n_k)}, a^{(n_k)}, y) \text{ converges weakly to } H(\cdot \mid z, a, y), \quad \text{for all } y \in C^*, \qquad (39)$$

and, therefore, Assumption (**H**) *holds.*

As explained in Feinberg et al. [17, Remark 4.5], the intersection assumption in Lemma 5.3 is equivalent to the similar assumption for finite unions.

Proof of Lemma 5.3. According to Billingsley [9, Theorem 2.1] or Shiryaev [32, p. 311], (39) holds if there exists a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots}$ of the sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ and a set $C^* \in \mathscr{B}(\mathbb{Y})$ such that

$$R'(C^* \mid z, a) = 1 \qquad \text{and} \qquad \liminf_{k\to\infty} H(\mathcal{O} \mid z^{(n_k)}, a^{(n_k)}, y) \geq H(\mathcal{O} \mid z, a, y), \quad \text{for all } y \in C^*, \qquad (40)$$

for all open sets $\mathcal{O}$ in $\mathbb{X}$. The rest of the proof establishes the existence of a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots}$ of the sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ and a set $C^* \in \mathscr{B}(\mathbb{Y})$ such that (40) holds for all open sets $\mathcal{O}$ in $\mathbb{X}$.

Let $\mathscr{A}_1$ be a family of all the subsets of $\mathbb{X}$ that are finite unions of sets from τ_b, and let $\mathscr{A}_2$ be a family of all subsets B of $\mathbb{X}$ such that $B = \tilde{\mathcal{O}} \backslash \mathcal{O}'$ with $\tilde{\mathcal{O}} \in \tau_b$ and $\mathcal{O}' \in \mathscr{A}_1$. Observe that (i) both $\mathscr{A}_1$ and $\mathscr{A}_2$ are countable, (ii) any open set $\mathcal{O}$ in $\mathbb{X}$ can be represented as

$$\mathcal{O} = \bigcup_{j=1,2,\ldots} \mathcal{O}^{(j,1)} = \bigcup_{j=1,2,\ldots} B^{(j,1)}, \quad \text{for some } \mathcal{O}^{(j,1)} \in \tau_b, \quad j = 1, 2, \ldots, \qquad (41)$$

where $B^{(j,1)} = \mathcal{O}^{(j,1)} \backslash (\bigcup_{i=1}^{j-1} \mathcal{O}^{(i,1)})$ are disjoint elements of $\mathscr{A}_2$ (it is allowed that $\mathcal{O}^{(j,1)} = \varnothing$ or $B^{(j,1)} = \varnothing$ for some $j = 1, 2, \ldots$).

To prove (40) for all open sets $\mathcal{O}$ in $\mathbb{X}$, we first show that there exists a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\ldots}$ of the sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\ldots}$ and a set $C^* \in \mathscr{B}(\mathbb{Y})$ such that (40) holds for all $\mathcal{O} \in \mathscr{A}_2$.

Consider an arbitrary $\mathcal{O}^* \in \mathscr{A}_1$. Then $\mathcal{O}^* = \bigcup_{i=1}^n \mathcal{O}^{(j_i)}$ for some $n = 1, 2, \dots$, where $\mathcal{O}^{(j_i)} \in \tau_b$, $i = 1, \dots, n$. Let $\mathscr{A}^{(n)} = \{\bigcap_{m=1}^k \mathcal{O}^{(i_m)}\colon \{i_1, i_2, \dots, i_k\} \subseteq \{j_1, j_2, \dots, j_n\}\}$ be the finite set of possible intersections of $\mathcal{O}^{(j_1)}, \dots, \mathcal{O}^{(j_n)}$. The principle of inclusion exclusion implies that for $\mathcal{O}^* = \bigcup_{i=1}^n \mathcal{O}^{(j_i)}$, $C \in \mathscr{B}(\mathbb{Y})$, $z, z' \in \mathbb{P}(\mathbb{X})$, and $a, a' \in \mathbb{A}$,

$$\bigl|R(\mathcal{O}^* \times C \mid z, a) - R(\mathcal{O}^* \times C \mid z', a')\bigr| \le \sum_{B \in \mathscr{A}^{(n)}} \bigl|R(B \times C \mid z, a) - R(B \times C \mid z', a')\bigr|. \tag{42}$$

In view of the assumption of the lemma regarding finite intersections of the elements of the base τ_b, for each $\mathcal{O}^* \in \mathscr{A}_1$ the family $\mathscr{R}_{\mathcal{O}^*}$ is equicontinuous at all the points $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$. Inequality (34) implies that for each $B \in \mathscr{A}_2$ the family $\mathscr{R}_B$ is equicontinuous at all the points $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$, that is, (33) holds with arbitrary $\mathcal{O}_1 \in \tau_b$ and $\mathcal{O}_2 \in \mathscr{A}_1$. This fact along with the definition of H (see (6)) means that

$$\lim_{n\to\infty} \sup_{C \in \mathscr{B}(\mathbb{Y})} \left| \int_C H(B \mid z^{(n)}, a^{(n)}, y) R'(dy \mid z^{(n)}, a^{(n)}) - \int_C H(B \mid z, a, y) R'(dy \mid z, a) \right| = 0 \tag{43}$$

for all $B \in \mathscr{A}_2$.

Since the set $\mathscr{A}_2$ is countable, let $\mathscr{A}_2 := \{B^{(j)}\colon j = 1, 2, \dots\}$. Denote $z^{(n,0)} = z^{(n)}$, $a^{(n,0)} = a^{(n)}$ for all $n = 1, 2, \dots$. For $j = 1, 2, \dots$, from (43) and Theorem 5.2 with $\mathbb{S} = \mathbb{Y}$, $s = y$, $h^{(n)}(s) = H(B^{(j)} \mid z^{(n,j-1)}, a^{(n,j-1)}, y)$, $\mu^{(n)}(\cdot) = R'(\cdot \mid z^{(n,j-1)}, a^{(n,j-1)})$, $h(s) = H(B^{(j)} \mid z, a, y)$, and $\mu(\cdot) = R'(\cdot \mid z, a)$, there exists a subsequence $\{(z^{(n,j)}, a^{(n,j)})\}_{n=1,2,\dots}$ of the sequence $\{(z^{(n,j-1)}, a^{(n,j-1)})\}_{n=1,2,\dots}$ and a set $C^{(*,j)} \in \mathscr{B}(\mathbb{Y})$ such that

$$R'(C^{(*,j)} \mid z, a) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} H(B^{(j)} \mid z^{(n,j)}, a^{(n,j)}, y) = H(B^{(j)} \mid z, a, y), \quad \text{for all } y \in C^{(*,j)}. \tag{44}$$

Let $C^* := \bigcap_{j=1}^\infty C^{(*,j)}$. Observe that $R'(C^* \mid z, a) = 1$. Let $z^{(n_k)} = z^{(k,k)}$ and $a^{(n_k)} = a^{(k,k)}$, $k = 1, 2, \dots$. As follows from Cantor's diagonal argument, (40) holds with $\mathcal{O} = B^{(j)}$ for all $j = 1, 2, \dots$. In other words, (40) holds for all $\mathcal{O} \in \mathscr{A}_2$.

Let $\mathcal{O}$ be an arbitrary open set in $\mathbb{X}$ and $B^{(1,1)}, B^{(2,1)}, \dots$ be disjoint elements of $\mathscr{A}_2$ satisfying (41). Countable additivity of probability measures $H(\cdot \mid \cdot, \cdot)$ implies that for all $y \in C^*$

$$\begin{aligned} \liminf_{k\to\infty} H(\mathcal{O} \mid z^{(n_k)}, a^{(n_k)}, y) &= \liminf_{k\to\infty} \sum_{j=1}^\infty H(B^{(j,1)} \mid z^{(n_k)}, a^{(n_k)}, y) \\ &\ge \sum_{j=1}^\infty \liminf_{k\to\infty} H(B^{(j,1)} \mid z^{(n_k)}, a^{(n_k)}, y) = \sum_{j=1}^\infty H(B^{(j,1)} \mid z, a, y) = H(\mathcal{O} \mid z, a, y), \end{aligned}$$

where the inequality follows from Fatou's lemma. Since $R'(C^* \mid z, a) = 1$, (40) holds. □

Proof of Theorem 3.6. The setwise continuity of the stochastic kernel R' follows from Corollary 5.1 that states the continuity of R' in total variation. The validity of Assumption **(H)** follows from Lemmas 5.2 and 5.3. In particular, τ_b is any countable base of the state space $\mathbb{X}$, and, in view of Lemma 5.2, the family $\mathscr{R}_{\mathcal{O}}$ is equicontinuous at all the points $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$ for each open set $\mathcal{O}$ in $\mathbb{X}$, which implies that the assumptions of Lemma 5.3 hold. □

6. Preservation of properties of one-step costs and proof of Theorem 3.3. As shown in this section, the reduction of a POMDP to the COMDP preserves properties of one-step cost functions that are needed for the existence of optimal policies. These properties include inf-compactness and $\mathbb{K}$-inf-compactness. In particular, in this section we prove Theorem 3.3 and thus complete the proof of Theorem 3.2.

We recall that an $\bar{\mathbb{R}}$-valued function f, defined on a nonempty subset U of a metric space $\mathbb{U}$, is called *inf-compact on* U if all the level sets $\{y \in U\colon f(y) \le \lambda\}$, $\lambda \in \mathbb{R}$, are compact. A function f is called *lower semicontinuous* if all the level sets are closed.

The notion of a $\mathbb{K}$-inf-compact function $c(x, a)$, defined in section 2 for a function $c\colon \mathbb{X} \times \mathbb{A} \to \bar{\mathbb{R}}$, is also applicable to a function $f\colon \mathbb{S}_1 \times \mathbb{S}_2 \to \bar{\mathbb{R}}$, where $\mathbb{S}_1$ and $\mathbb{S}_2$ are metric spaces, or certain more general toplogical spaces; see Feinberg et al. [15, 13] for details, where the properties of $\mathbb{K}$-inf-compact functions are described. In particular, according to Feinberg et al. [15, Lemma 2.1], if a function f is inf-compact on $\mathbb{S}_1 \times \mathbb{S}_2$ then it is $\mathbb{K}$-inf-compact on $\mathbb{S}_1 \times \mathbb{S}_2$. According to Feinberg et al. [15, Lemmas 2.2, 2.3], a $\mathbb{K}$-inf-compact function f on $\mathbb{S}_1 \times \mathbb{S}_2$ is lower semicontinuous on $\mathbb{S}_1 \times \mathbb{S}_2$, and, in addition, for each $s_1 \in \mathbb{S}_1$, the function $f(s_1, \cdot)$ is inf-compact on $\mathbb{S}_2$.

Lemma 6.1. *If the function $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ is bounded below and lower semicontinuous on $\mathbb{X}\times\mathbb{A}$, then the function $\bar{c}\colon \mathbb{P}(\mathbb{X})\times\mathbb{A}\to\bar{\mathbb{R}}$ defined in (11) is bounded below and lower semicontinuous on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.*

Proof. The statement of this lemma directly follows from the generalized Fatou's Lemma 5.1(ii). □

The inf-compactness of c on $\mathbb{X}\times\mathbb{A}$ implies the inf-compactness of $\bar{c}$ on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$. We recall that an inf-compact function on $\mathbb{X}\times\mathbb{A}$ with values in $\bar{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is bounded below on $\mathbb{X}\times\mathbb{A}$.

Theorem 6.1. *If $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ is an inf-compact function on $\mathbb{X}\times\mathbb{A}$, then the function $\bar{c}\colon \mathbb{P}(\mathbb{X})\times\mathbb{A}\to\bar{\mathbb{R}}$ defined in (11) is inf-compact on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.*

Proof. Let $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ be an inf-compact function on $\mathbb{X}\times\mathbb{A}$. Fix an arbitrary $\lambda\in\mathbb{R}$. To prove that the level set $\mathscr{D}_{\bar{c}}(\lambda;\mathbb{P}(\mathbb{X})\times\mathbb{A})=\{(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}\colon \bar{c}(z,a)\le\lambda\}$ is compact, consider an arbitrary sequence $\{(z^{(n)},a^{(n)})\}_{n=1,2,\ldots}\subset\mathscr{D}_{\bar{c}}(\lambda;\mathbb{P}(\mathbb{X})\times\mathbb{A})$. It is enough to show that $\{(z^{(n)},a^{(n)})\}_{n=1,2,\ldots}$ has a limit point $(z,a)\in\mathscr{D}_{\bar{c}}(\lambda;\mathbb{P}(\mathbb{X})\times\mathbb{A})$.

Let us show that the sequence of probability measures $\{z^{(n)}\}_{n=1,2,\ldots}$ has a limit point $z\in\mathbb{P}(\mathbb{X})$. Define $\mathbb{X}_{<+\infty}:=\mathbb{X}\backslash\mathbb{X}_{+\infty}$, where $\mathbb{X}_{+\infty}:=\{x\in\mathbb{X}\colon c(x,a)=+\infty \text{ for all } a\in\mathbb{A}\}$.

The inequalities

$$\int_{\mathbb{X}} c(x,a^{(n)})z^{(n)}(dx)\le\lambda,\quad n=1,2,\ldots, \tag{45}$$

imply that $z^{(n)}(\mathbb{X}_{+\infty})=0$ for each $n=1,2,\ldots$. Thus (45) transforms into

$$\int_{\mathbb{X}_{<+\infty}} c(x,a^{(n)})z^{(n)}(dx)\le\lambda,\quad n=1,2,\ldots. \tag{46}$$

By definition of inf-compactness, the function $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ is inf-compact on $\mathbb{X}_{<+\infty}\times\mathbb{A}$. According to Feinberg et al. [14, Corollary 3.2], the real-valued function $\psi(x)=\inf_{a\in\mathbb{A}} c(x,a)$, $x\in\mathbb{X}_{<+\infty}$, with values in $\mathbb{R}$, is inf-compact on $\mathbb{X}_{<+\infty}$. Furthermore, (46) implies that $\int_{\mathbb{X}_{<+\infty}}\psi(x)z^{(n)}(dx)\le\lambda$, $n=1,2,\ldots$. Thus, Hernández-Lerma and Lassere [21, Proposition E.8] and Prohorov's Theorem (Hernández-Lerma and Lassere [21, Theorem E.7]) yield relative compactness of the sequence $\{z^{(n)}\}_{n=1,2,\ldots}$ in $\mathbb{P}(\mathbb{X}_{<+\infty})$. Thus there exists a subsequence $\{z^{(n_k)}\}_{k=1,2,\ldots}\subseteq\{z^{(n)}\}_{n=1,2,\ldots}$ and a probability measure $z\in\mathbb{P}(\mathbb{X}_{<+\infty})$ such that $z^{(n_k)}$ converges to z in $\mathbb{P}(\mathbb{X}_{<+\infty})$. Let us set $z(\mathbb{X}_{+\infty})=0$. As $z^{(n)}(\mathbb{X}_{+\infty})=0$ for all $n=1,2,\ldots$, then the sequence of probability measures $\{z^{(n_k)}\}_{k=1,2,\ldots}$ converges weakly and its limit point z belongs to $\mathbb{P}(\mathbb{X})$.

The sequence $\{a^{(n_k)}\}_{k=1,2,\ldots}$ has a limit point $a\in\mathbb{A}$. Indeed, inequality (45) implies that for any $k=1,2,\ldots$ there exists at least one $x^{(k)}\in\mathbb{X}$ such that $c(x^{(k)},a^{(n_k)})\le\lambda$. The inf-compactness of $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ on $\mathbb{X}\times\mathbb{A}$ implies that $\{a^{(k)}\}_{k=1,2,\ldots}$ has a limit point $a\in\mathbb{A}$. To finish the proof, note that Lemma 6.1, the generalized Fatou's Lemma 5.1(ii), and (45) imply that $\int_{\mathbb{X}} c(x,a)z(dx)\le\lambda$. □

Proof of Theorem 3.3. If c is bounded below on $\mathbb{X}\times\mathbb{A}$, then formula (11) implies that $\bar{c}$ is bounded below on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ by the same lower bound as c. Thus, it is enough to prove the $\mathbb{K}$-inf-compactness of $\bar{c}$ on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.

Let a sequence of probability measures $\{z^{(n)}\}_{n=1,2,\ldots}$ on $\mathbb{X}$ weakly converges to $z\in\mathbb{P}(\mathbb{X})$. Consider an arbitrary sequence $\{a^{(n)}\}_{n=1,2,\ldots}\subset\mathbb{A}$ satisfying the condition that the sequence $\{\bar{c}(z^{(n)},a^{(n)})\}_{n=1,2,\ldots}$ is bounded above. Observe that $\{a^{(n)}\}_{n=1,2,\ldots}$ has a limit point $a\in\mathbb{A}$. Indeed, boundedness below of the $\bar{\mathbb{R}}$-valued function c on $\mathbb{X}\times\mathbb{A}$ and the generalized Fatou's Lemma 5.1(ii) imply that for some $\lambda<+\infty$

$$\int_{\mathbb{X}}\underline{c}(x)z(dx)\le\liminf_{n\to\infty}\int_{\mathbb{X}} c(x,a^{(n)})z^{(n)}(dx)\le\lambda, \tag{47}$$

where

$$\underline{c}(x):=\liminf_{y\to x,\,n\to\infty} c(y,a^{(n)}). \tag{48}$$

Inequalities (47) imply the existence of $x^{(0)}\in\mathbb{X}$ such that $\underline{c}(x^{(0)})\le\lambda$. Therefore, formula (48) implies the existence of a subsequence $\{a^{(n_k)}\}_{k=1,2,\ldots}\subseteq\{a^{(n)}\}_{n=1,2,\ldots}$ and a sequence $\{y^{(k)}\}_{k=1,2,\ldots}\subset\mathbb{X}$ such that $y^{(k)}\to x^{(0)}$ as $k\to\infty$ and $c(y^{(k)},a^{(n_k)})\le\lambda+1$ for $k=1,2,\ldots$. Since $c\colon \mathbb{X}\times\mathbb{A}\to\bar{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$, the sequence $\{a^{(n_k)}\}_{k=1,2,\ldots}$ has a limit point $a\in\mathbb{A}$, which is the limit point of the initial sequence $\{a^{(n)}\}_{n=1,2,\ldots}$. Thus, the function $\bar{c}$ is $\mathbb{K}$-inf-compact on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$. □

Arguments similar to the proof of Theorem 3.3 imply the inf-compactness of $\bar{c}(z,a)$ in $a\in\mathbb{A}$ for any $z\in\mathbb{P}(\mathbb{X})$, if $c(x,a)$ is inf-compact in $a\in\mathbb{A}$ for any $x\in\mathbb{X}$.

Theorem 6.2. *If the function $c(x,a)$ is inf-compact in $a\in\mathbb{A}$ for each $x\in\mathbb{X}$ and bounded below on $\mathbb{X}\times\mathbb{A}$, then the function $\bar{c}(z,a)$ is inf-compact in $a\in\mathbb{A}$ for each $z\in\mathbb{P}(\mathbb{X})$ and bounded below on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.*

PROOF. Fix $z \in \mathbb{P}(\mathbb{X})$ and consider a sequence $\{a^{(n)}\}_{n=1,2,\dots}$ in $\mathbb{A}$ such that $c(z, a^{(n)}) \le \lambda$ for some $\lambda < \infty$, $n = 1, 2, \dots$. The classic Fatou's lemma implies that (47) holds with $z^{(n)} = z$, $n = 1, 2, \dots$, and $\underline{c}(x) = \liminf_{n\to\infty} c(x, a^{(n)})$, $x \in \mathbb{X}$. Thus, there exists $x^{(0)} \in \mathbb{X}$ such that $\liminf_{n\to\infty} c(x^{(0)}, a^{(n)}) \le \lambda$. This together with the inf-compactness of $c(x^{(0)}, a)$ in $a \in \mathbb{A}$ implies that the sequence $\{a^{(n)}\}_{n=1,2,\dots}$ has a limit point in $\mathbb{A}$. □

PROOF OF THEOREM 3.2. Theorem 3.2 follows from Theorems 3.1, 3.3, and 3.4. □

7. Combining Assumption (H) and the weak continuity of H. Theorem 3.2 assumes either the weak continuity of H or Assumption (H) together with the setwise continuity of R'. For some applications, see, e.g., §8.2 that deals with inventory control, the filtering kernel H satisfies Assumption (H) for some observations and it is weakly continuous for other observations. The following theorem is applicable to such situations.

THEOREM 7.1. *Let the observation space $\mathbb{Y}$ be partitioned into two disjoint subsets $\mathbb{Y}_1$ and $\mathbb{Y}_2$ such that $\mathbb{Y}_1$ is open in $\mathbb{Y}$. If the following assumptions hold*:

(a) *the stochastic kernels P on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ and Q on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$ are weakly continuous*;

(b) *the measure $R'(\cdot \mid z, a)$ on $(\mathbb{Y}_2, \mathcal{B}(\mathbb{Y}_2))$ is setwise continuous in $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$, that is, for every sequence $\{(z^{(n)}, a^{(n)})\}_{n=1,2,\dots}$ in $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ converging to $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$ and for every $C \in \mathcal{B}(\mathbb{Y}_2)$, we have $R'(C \mid z^{(n)}, a^{(n)}) \to R'(C \mid z, a)$*;

(c) *there exists a stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ satisfying (6) such that*

(i) *the stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}_1$ is weakly continuous*;

(ii) *Assumption (H) holds on $\mathbb{Y}_2$, that is, if a sequence $\{z^{(n)}\}_{n=1,2,\dots} \subseteq \mathbb{P}(\mathbb{X})$ converges weakly to $z \in \mathbb{P}(\mathbb{X})$ and a sequence $\{a^{(n)}\}_{n=1,2,\dots} \subseteq \mathbb{A}$ converges to $a \in \mathbb{A}$, then there exists a subsequence $\{(z^{(n_k)}, a^{(n_k)})\}_{k=1,2,\dots} \subseteq \{(z^{(n)}, a^{(n)})\}_{n=1,2,\dots}$ and a measurable subset C of $\mathbb{Y}_2$ such that $R'(\mathbb{Y}_2 \setminus C \mid z, a) = 0$ and $H(z^{(n_k)}, a^{(n_k)}, y)$ converges weakly to $H(z, a, y)$ for all $y \in C$*;

then the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is weakly continuous. If, in addition to the above conditions, assumptions (a) *and* (b) *from Theorem 3.2 hold, then the COMDP $(\mathbb{P}(\mathbb{X}), \mathbb{A}, q, \bar{c})$ satisfies Assumption (W*) and therefore statements* (i)–(vi) *of Theorem 3.1 hold.*

PROOF. The stochastic kernel $q(dz' \mid z, a)$ on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is weakly continuous if and only if for every open set D in $\mathbb{P}(\mathbb{X})$ the function $q(D \mid z, a)$ is lower semicontinuous in $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$; see Billingsley [9, Theorem 2.1]. Thus, if q is not weakly continuous, there exist an open set D in $\mathbb{P}(\mathbb{X})$ and sequences $z^{(n)} \to z$ weakly and $a^{(n)} \to a$, where $z, z^{(n)} \in \mathbb{P}(\mathbb{X})$ and $a, a^{(n)} \in \mathbb{A}$, $n = 1, 2, \dots$, such that

$$\liminf_{n\to\infty} q(D \mid z^{(n)}, a^{(n)}) < q(D \mid z, a).$$

Then there exists $\varepsilon^* > 0$ and a subsequence $\{(z^{(n,1)}, a^{(n,1)})\}_{n=1,2,\dots} \subseteq \{(z^{(n)}, a^{(n)})\}_{n=1,2,\dots}$ such that for all $n = 1, 2, \dots$

$$\begin{aligned}
&\int_{\mathbb{Y}_1} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y) \in D\} R'(dy \mid z^{(n,1)}, a^{(n,1)}) + \int_{\mathbb{Y}_2} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y) \in D\} R'(dy \mid z^{(n,1)}, a^{(n,1)}) \\
&\quad = q(D \mid z^{(n,1)}, a^{(n,1)}) \le q(D \mid z, a) - \varepsilon^* \qquad (49) \\
&\quad = \int_{\mathbb{Y}_1} \mathbf{I}\{H(z, a, y) \in D\} R'(dy \mid z, a) + \int_{\mathbb{Y}_2} \mathbf{I}\{H(z, a, y) \in D\} R'(dy \mid z, a) - \varepsilon^*,
\end{aligned}$$

where the stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}$ satisfies (6) and assumption (c) of Theorem 7.1.

Since $\mathbb{Y}_1$ is an open set in $\mathbb{Y}$ and the stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}_1$ is weakly continuous, for all $y \in \mathbb{Y}_1$

$$\liminf_{\substack{n\to\infty \\ y'\to y}} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} = \liminf_{\substack{n\to\infty \\ y'\to y,\, y'\in\mathbb{Y}_1}} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} \ge \mathbf{I}\{H(z, a, y) \in D\}. \qquad (50)$$

The weak continuity of the stochastic kernels P and Q on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ and on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$, respectively, imply the weak continuity of the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$; see Hernández-Lerma [20, p. 92]. Therefore,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{Y}_1} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y) \in D\} R'(dy \mid z^{(n,1)}, a^{(n,1)}) \\
&\quad \ge \int_{\mathbb{Y}_1} \liminf_{n\to\infty,\, y'\to y} \mathbf{I}\{H(z^{(n,1)}, a^{(n,1)}, y') \in D\} R'(dy \mid z^{(n,1)}, a^{(n,1)}) \\
&\quad \ge \int_{\mathbb{Y}_1} \mathbf{I}\{H(z, a, y) \in D\} R'(dy \mid z, a), \qquad (51)
\end{aligned}$$

where the first inequality follows from Lemma 5.1(ii) and the second one follows from formula (50).

The inequality

$$\limsup_{n\to\infty}\int_{\mathbb{Y}_2}\mathbf{I}\{H(z^{(n,1)},a^{(n,1)},y)\in D\}R'(dy\mid z^{(n,1)},a^{(n,1)})\ge\int_{\mathbb{Y}_2}\mathbf{I}\{H(z,a,y)\in D\}R'(dy\mid z,a) \tag{52}$$

together with (51) contradicts (49). This contradiction implies that $q(\cdot\mid z,a)$ is a weakly continuous stochastic kernel on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$.

To complete the proof of Theorem 7.1, we prove inequality (52). If $R'(\mathbb{Y}_2\mid z,a)=0$, then inequality (52) holds. Now let $R'(\mathbb{Y}_2\mid z,a)>0$. Since $R'(\mathbb{Y}_2\mid z^{(n,1)},a^{(n,1)})\to R'(\mathbb{Y}_2\mid z,a)$ as $n\to\infty$, there exists $N=1,2,\dots$ such that $R'(\mathbb{Y}_2\mid z^{(n,1)},a^{(n,1)})>0$ for any $n\ge N$. We endow $\mathbb{Y}_2$ with the same metric as in $\mathbb{Y}$ and set

$$R'_1(C\mid z',a'):=\frac{R'(C\mid z',a')}{R'(\mathbb{Y}_2\mid z',a')},\quad z'\in\mathbb{P}(\mathbb{X}),\quad a'=a\in\mathbb{A},\quad C\in\mathscr{B}(\mathbb{Y}_2).$$

Assumption (b) of Theorem 7.1 means that the stochastic kernel $R'_1(dy\mid z,a)$ on $\mathbb{Y}_2$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is setwise continuous. Assumption (ii) of Theorem 7.1 implies the existence of a subsequence $\{(z^{(n,2)},a^{(n,2)})\}_{n=1,2,\dots}\subseteq\{(z^{(n,1)},a^{(n,1)})\}_{n=1,2,\dots}$ and a measurable subset C of $\mathbb{Y}_2$ such that $R'_1(\mathbb{Y}_2\backslash C\mid z,a)=0$ and $H(z^{(n,2)},a^{(n,2)},y)$ converges weakly to $H(z,a,y)$ as $n\to\infty$ for all $y\in C$. Therefore, since D is an open set in $\mathbb{P}(\mathbb{X})$, we have

$$\liminf_{k\to\infty}\mathbf{I}\{H(z^{(n,2)},a^{(n,2)},y)\in D\}\ge\mathbf{I}\{H(z,a,y)\in D\},\quad y\in C. \tag{53}$$

Formula (53), the setwise continuity of the stochastic kernel R'_1 on $\mathbb{Y}_2$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$, and Lemma 5.1(i) imply

$$\begin{aligned}&\frac{1}{R'(\mathbb{Y}_2\mid z,a)}\liminf_{k\to\infty}\int_{\mathbb{Y}_2}\mathbf{I}\{H(z^{(n,2)},a^{(n,2)},y)\in D\}R'(dy\mid z^{(n,2)},a^{(n,2)})\\&\quad\ge\liminf_{k\to\infty}\frac{\int_{\mathbb{Y}_2}\mathbf{I}\{H(z^{(n,2)},a^{(n,2)},y)\in D\}R'(dy\mid z^{(n,2)},a^{(n,2)})}{R'(\mathbb{Y}_2\mid z^{(n,2)},a^{(n,2)})}\ge\frac{\int_{\mathbb{Y}_2}\mathbf{I}\{H(z,a,y)\in D\}R'(dy\mid z,a)}{R'(\mathbb{Y}_2\mid z,a)},\end{aligned}$$

and thus (52) holds. □

Corollary 7.1. *Let the observation space $\mathbb{Y}$ be partitioned into two disjoint subsets $\mathbb{Y}_1$ and $\mathbb{Y}_2$ such that $\mathbb{Y}_1$ is open in $\mathbb{Y}$ and $\mathbb{Y}_2$ is countable. If the following assumptions hold*:

(a) *the stochastic kernels P on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ and Q on $\mathbb{Y}$ given $\mathbb{A}\times\mathbb{X}$ are weakly continuous*;

(b) *$Q(y\mid a,x)$ is a continuous function on $\mathbb{A}\times\mathbb{X}$ for each $y\in\mathbb{Y}_2$*;

(c) *there exists a stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}\times\mathbb{Y}$ satisfying (6) such that the stochastic kernel H on $\mathbb{X}$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}\times\mathbb{Y}_1$ is weakly continuous*;

then assumptions (b) *and* (ii) *of Theorem 7.1 hold, and the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is weakly continuous. If, in addition to the above conditions, assumptions* (a) *and* (b) *from Theorem 3.2 hold, then the COMDP $(\mathbb{P}(\mathbb{X}),\mathbb{A},q,\bar{c})$ satisfies Assumption* (**W***) *and therefore statements* (i)–(vi) *of Theorem 3.1 hold.*

Proof. To prove the corollary, it is sufficient to verify conditions (b) and (ii) of Theorem 7.1. For each $B\in\mathscr{B}(\mathbb{X})$ and for each $y\in\mathbb{Y}_2$, Hernández-Lerma [20, Appendix C, Proposition C.2(b)], being repeatedly applied to formula (4) with $C=\{y\}$, implies the continuity of $R(B\times\{y\}\mid z,a)$ in $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. In particular, the function $R'(y\mid\cdot,\cdot)$ is continuous on $\mathbb{P}(\mathbb{X})\times\mathbb{A}$. If $R'(y\mid z,a)>0$ then, in view of (6), $H(B\mid z,a,y)=R(B\times\{y\}\mid z,a)/R'(y\mid z,a)$, and, if y is fixed, this function is continuous at the point (z,a). Thus, condition (ii) of Theorem 7.1 holds. Since the set $\mathbb{Y}_2$ is closed in $\mathbb{Y}$, the function $Q(\mathbb{Y}_2\mid a,x)$ is upper semicontinuous in $(a,x)\in\mathbb{A}\times\mathbb{X}$. The generalized Fatou's Lemma 5.1, being repeatedly applied to (5) with $C=\mathbb{Y}_2$, implies that $R'(\mathbb{Y}_2\mid z,a)$ is upper semicontinuous in $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$. This implies that, for every $Y\subseteq\mathbb{Y}_2$ and for every sequence $\{(z^{(n)},a^{(n)})\}_{n=1,2,\dots}\subset\mathbb{P}(\mathbb{X})\times\mathbb{A}$ converging to $(z,a)\in\mathbb{P}(\mathbb{X})\times\mathbb{A}$,

$$|R'(Y\mid z^{(n)},a^{(n)})-R'(Y\mid z,a)|\le\sum_{y\in\mathbb{Y}_2}|R'(y\mid z^{(n)},a^{(n)})-R'(y\mid z,a)|\to0\quad\text{as } n\to\infty,$$

where the convergence takes place because of the same arguments as in the proof of Corollary 3.8. Thus, condition (b) of Theorem 7.1 holds. □

Remark 7.1. All the statements in this paper, that deal with finite-horizon problems under Assumption (**D**), including the corresponding statements in Theorems 2.1 and 3.1, hold for all $\alpha\ge0$ rather than just for $\alpha\in[0,1)$.

8. Examples of applications. To illustrate theoretical results, they are applied in this section to three particular models: (i) problems defined by stochastic equations; see Striebel [34], Bensoussan [3], and Hernández-Lerma [20, p. 83]; (ii) inventory control; and (iii) Markov decision model with incomplete information.

8.1. Problems defined by stochastic equations. Let $\{\xi_t\}_{t=0,1,\dots}$ be a sequence of identically distributed random variables with values in $\mathbb{R}$ and with the distribution μ. Let $\{\eta_t\}_{t=0,1,\dots}$ be a sequence of random variables uniformly distributed on $(0,1)$. An initial state x_0 is a random variable with values in $\mathbb{R}$. It is assumed that the random variables $x_0, \xi_0, \eta_0, \xi_1, \eta_1, \dots$ are defined on the same probability space and mutually independent. Consider a stochastic partially observable control system

$$x_{t+1} = F(x_t, a_t, \xi_t), \quad t = 0, 1, \dots, \tag{54}$$

$$y_{t+1} = G(a_t, x_{t+1}, \eta_{t+1}), \quad t = 0, 1, \dots, \tag{55}$$

where F and G are given measurable functions from $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ and from $\mathbb{R}\times\mathbb{R}\times(0,1)$ to $\mathbb{R}$, respectively. The initial observation is $y_0 = G_0(x_0, \eta_0)$, where G_0 is a measurable function from $\mathbb{R}\times(0,1)$ to $\mathbb{R}$. The states x_t are not observable, whereas the states y_t are observable. The goal is to minimize the expected total discounted costs.

Instead of presenting formal definitions of the functions a_t, we describe the above problem as a POMDP with the state space $\mathbb{X} = \mathbb{R}$, observation space $\mathbb{Y} = \mathbb{R}$, and action space $\mathbb{A} = \mathbb{R}$. The transition law is

$$P(B \mid x, a) = \int_{\mathbb{R}} \mathbf{I}\{F(x, a, s) \in B\}\mu(ds), \quad B \in \mathscr{B}(\mathbb{R}), \quad x \in \mathbb{R}, \quad a \in \mathbb{R}. \tag{56}$$

The observation kernel is

$$Q(C \mid a, x) = \int_{(0,1)} \mathbf{I}\{G(a, x, s) \in C\}\lambda(ds), \quad C \in \mathscr{B}(\mathbb{R}), \quad a \in \mathbb{R}, \quad x \in \mathbb{R},$$

where $\lambda \in \mathbb{P}((0,1))$ is the Lebesgue measure on $(0,1)$. The initial state distribution p is the distribution of the random variable x_0, and the initial observation kenel $Q_0(C \mid x) = \int_{(0,1)} \mathbf{I}\{G_0(x, s) \in C\}\lambda(ds)$ for all $C \in \mathscr{B}(\bar{\mathbb{R}})$ and for all $x \in \mathbb{X}$.

Assume that $(x, a) \to F(x, a, s)$ is a continuous mapping on $\mathbb{R}\times\mathbb{R}$ for μ-a.e. $s \in \mathbb{R}$. Then the stochastic kernel $P(dx' \mid x, a)$ on $\mathbb{R}$ given $\mathbb{R}\times\mathbb{R}$ is weakly continuous; see Hernández-Lerma [20, p. 92].

Assume that (i) G is a continuous mapping on $\mathbb{R}\times\mathbb{R}\times(0,1)$, (ii) the partial derivative $g(x, y, s) = \partial G(x, y, s)/\partial s$ exists everywhere and is continuous, and (iii) there exists a constant $\beta > 0$ such that $|g(a, x, s)| \geq \beta$ for all $a \in \mathbb{R}$, $x \in \mathbb{R}$, and $s \in (0,1)$. Denote by $\mathscr{G}$ the inverse function for G with respect the last variable. Assume that $\mathscr{G}$ is continuous.

Let us prove that under these assumptions the observation kernel Q on $\mathbb{R}$ given $\mathbb{R}\times\mathbb{R}$ is continuous in total variation. For each $\varepsilon \in (0, \frac{1}{2})$, for each Borel set $C \in \mathscr{B}(\mathbb{R})$, and for all $(a', x'), (a, x) \in \mathbb{R}\times\mathbb{R}$

$$\begin{aligned}
|Q(C \mid a', x') - Q(C \mid a, x)| &= \left| \int_0^1 \mathbf{I}\{G(a', x', s) \in C\}\lambda(ds) - \int_0^1 \mathbf{I}\{G(a, x, s) \in C\}\lambda(ds) \right| \\
&\leq 4\varepsilon + \left| \int_\varepsilon^{1-\varepsilon} \mathbf{I}\{G(a', x', s) \in C\}\lambda(ds) - \int_\varepsilon^{1-\varepsilon} \mathbf{I}\{G(a, x, s) \in C\}\lambda(ds) \right| \\
&= 4\varepsilon + \left| \int_{G(a', x', [\varepsilon, 1-\varepsilon])} \frac{\mathbf{I}\{\tilde{s} \in C\}}{g(a', x', \mathscr{G}(a', x', \tilde{s}))}\tilde{\lambda}(d\tilde{s}) - \int_{G(a, x, [\varepsilon, 1-\varepsilon])} \frac{\mathbf{I}\{\tilde{s} \in C\}}{g(a, x, \mathscr{G}(a, x, \tilde{s}))}\tilde{\lambda}(d\tilde{s}) \right| \\
&\leq 4\varepsilon + \frac{|G(a', x', \varepsilon) - G(a, x, \varepsilon)| + |G(a', x', 1-\varepsilon) - G(a, x, 1-\varepsilon)|}{\beta} \\
&\quad + \frac{1}{\beta^2} \int_{G(a, x, [\varepsilon, 1-\varepsilon]) \cap G(a', x', [\varepsilon, 1-\varepsilon])} \left| g(a', x', \mathscr{G}(a', x', \tilde{s})) - g(a, x, \mathscr{G}(a, x, \tilde{s})) \right| \tilde{\lambda}(d\tilde{s}),
\end{aligned}$$

where $\tilde{\lambda}$ is the Lebesgue measure on $\mathbb{R}$, the second equality holds because of the changes $\tilde{s} = G(a', x', s)$ and $\tilde{s} = G(a, x, s)$ in the corresponding integrals, and the second inequality follows from direct estimations. Since, the function G is continuous, $G(a', x', \varepsilon) \to G(a, x, \varepsilon)$ and $G(a', x', 1-\varepsilon) \to G(a, x, 1-\varepsilon)$ as $(a', x') \to (a, x)$, for any $(a, x, \varepsilon) \in \mathbb{R}\times\mathbb{R}\times(0, \frac{1}{2})$. Thus, if

$$\int_{\mathbb{R}} D(a, x, a', x', \varepsilon, \tilde{s})\tilde{\lambda}(d\tilde{s}) \to 0 \quad \text{as } (a', x') \to (x, a), \tag{57}$$

where $D(a,x,a',x',\varepsilon,\tilde{s}) := |g(a',x',\mathcal{G}(a',x',\tilde{s})) - g(a,x,\mathcal{G}(a,x,\tilde{s}))|$, when $\tilde{s} \in G(a,x,[\varepsilon,1-\varepsilon]) \cap G(a',x',[\varepsilon,1-\varepsilon])$, and $D(a,x,a',x',\varepsilon,\tilde{s}) = 0$ otherwise, then

$$\lim_{(a',x')\to(a,x)} \sup_{C\in\mathscr{B}(\mathbb{R})} |Q(C \mid a',x') - Q(C \mid a,x)| = 0.$$

So, to complete the proof of the continuity in total variation of the observation kernel Q on $\mathbb{R}$ given $\mathbb{R}\times\mathbb{R}$, it is sufficient to verify (57). We fix an arbitrary vector $(a,x,\varepsilon) \in \mathbb{R}\times\mathbb{R}\times(0,\frac{1}{2})$ and consider arbitrary converging sequences $a^{(n)} \to a$ and $x^{(n)} \to x$. Let $(a',x') = (a^{(n)},x^{(n)})$, $n = 1,2,\dots$. Since the sets $K := \{(a^{(n)},x^{(n)})\colon n = 1,2,\dots\} \cup \{(a,x)\}$ and $[\varepsilon,1-\varepsilon]$ are compact and the function g is continuous on $\mathbb{R}\times\mathbb{R}\times(0,1)$, the function $|g|$ is bounded above on the compact set $K\times[\varepsilon,1-\varepsilon]$ by a positive constant M. Thus, the integrand in (57) is bounded above by $2M$ on the compact set $G(K\times[\varepsilon,1-\varepsilon])$ and is equal to 0 on its complement. Since G, g, and $\mathcal{G}$ are continuous functions, for each $\tilde{s}\in\mathbb{R}$ the integrand in (57) converges to 0 as $(a',x') \to (a,x)$. Therefore, (57) follows from the dominated convergence theorem, because the Lebeasgue measure of the set $G(K\times[\varepsilon,1-\varepsilon])$ is finite since this set is compact.

Finally, we assume that the one-period cost function $c\colon \mathbb{R}\times\mathbb{R}\to\bar{\mathbb{R}}$ is bounded below and $\mathbb{K}$-inf-compact. Thus, the assumptions of Theorem 3.5 are satisfied. Therefore, for this COMDP there exists a stationary optimal policy, the optimality equations hold, and value iterations converge to the optimal value.

We remark that the one-dimensional Kalman filter in discrete time satisfies the above assumptions. In this case, $F(x_t,a_t,\xi_t) = d^*x_t + b^*a_t + \xi_t$ and $G(a_t,x_{t+1},\eta_{t+1}) = h^*x_{t+1} + c^*\Phi^{-1}(\eta_{t+1})$, where $c^* \neq 0$ and Φ^{-1} is the inverse to the cumulative distribution function of a standard normal distribution ($\Phi^{-1}(\eta_{t+1})$ is a standard normal random variable). In particular, $|g(a,x,s)| = |c^*|(2\pi)^{1/2}e^{\Phi^{-1}(s)^2/2} \geq |c^*|(2\pi)^{1/2} > 0$ for all $s\in[0,1]$. Thus, if the cost function $c(x,a)$ is $\mathbb{K}$-inf-compact, then the conclusions of Theorem 3.5 hold for the Kalman filter. In particular, the quadratic cost function $c(x,a) = c_1x^2 + c_2a^2$ is $\mathbb{K}$-inf-compact if $c_1 \geq 0$ and $c_2 > 0$. Thus, the linear quadratic Gaussian control problem is a particular case of this model. The one-step cost functions $c(x,a) = (a-x)^2$ and $c(x,a) = |x-a|$, which are typically used for identification problems, are also $\mathbb{K}$-inf-compact. However, these two functions are not inf-compact. This illustrates the usefulness of the notion of $\mathbb{K}$-inf-compactness.

8.2. Inventory control with incomplete records. This example is motivated by Bensoussan et al. [4, 5, 6, 7], where several inventory control problems for periodic review systems, when the inventory manager (IM) may not have complete information about inventory levels, are studied. In Bensoussan et al. [6, 7], problems with backorders are considered. In the model considered in Bensoussan et al. [7], the IM does not know the inventory level, if it is nonnegative, and the IM knows the inventory level, if it is negative. In the model considered in Bensoussan et al. [6], the IM only knows whether the inventory level is negative or nonnegative. In Bensoussan et al. [4] a problem with lost sales is studied, when the IM only knows whether a lost sale took place or not. The underlying mathematical analysis is summarized in Bensoussan et al. [5], where additional references can be found. The analysis includes transformations of density functions of demand distributions.

The current example studies periodic review systems with backorders and lost sales, when some inventory levels are observable and some are not. The goal is to minimize the expected total costs. Demand distribution may not have densities.

In the case of full observations, we model the problem as an MDP with a state space $\mathbb{X} = \mathbb{R}$ (the current inventory level), action space $\mathbb{A} = \mathbb{R}$ (the ordered amount of inventory), and action sets $\mathbb{A}(x) = \mathbb{A}$ available at states $x\in\mathbb{X}$. If in a state x the amount of inventory a is ordered, then the holding/backordering cost $h(x)$, ordering cost $C(a)$, and lost sale cost $G(x,a)$ are incurred, where it is assumed that h, C, and G are nonnegative lower semicontinuous functions with values in $\bar{\mathbb{R}}$ and $C(a)\to+\infty$ as $|a|\to\infty$. Observe that the one-step cost function $c(x,a) = h(x) + C(a) + G(x,a)$ is $\mathbb{K}$-inf-compact on $\mathbb{X}\times\mathbb{A}$. Typically $G(x,a) = 0$ for $x \geq 0$.

Let D_t, $t = 0,1,\dots$, be i.i.d. random variables with the distribution function F_D, where D_t is the demand at epoch $t = 0,1,\dots$. The dynamics of the system is defined by $x_{t+1} = F(x_t,a_t,D_t)$, where x_t is the current inventory level and a_t is the ordered (or scrapped) inventory at epoch $t = 0,1,\dots$. For problems with backorders $F(x_t,a_t,D_t) = x_t + a_t - D_t$ and for problems with lost sales $F(x_t,a_t,D_t) = (x_t + a_t - D_t)^+$. In both cases, F is a continuous function defined on $\mathbb{R}^3$. To simplify and unify the presentation, we do not follow the common agreement that $\mathbb{X} = [0,\infty)$ for models with lost sales. However, for problems with lost sales, it is assumed that the initial state distribution p is concentrated on $[0,\infty)$, and this implies that states $x < 0$ will never be visited. We assume that the distribution function F_D is atomless (an equivalent assumption is that the function F_D is continuous). The state transition law P on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ is

$$P(B \mid x,a) = \int_{\mathbb{R}} \mathbf{I}\{F(x,a,s)\in B\}\,dF_D(s), \quad B\in\mathscr{B}(\mathbb{X}),\ \ x\in\mathbb{X},\ \ a\in\mathbb{A}. \tag{58}$$

Since we do not assume that demands are nonnegative, this model also covers cash balancing problems and problems with returns; see Feinberg and Lewis [19] and references therein. In a particular case, when $C(a) = +\infty$ for $a < 0$, orders with negative sizes are infeasible, and, if an order is placed, the ordered amount of inventory should be positive.

As mentioned previously, some states (inventory levels) $x \in \mathbb{X} = \mathbb{R}$ are observable and some are not. Let the inventory be stored in containers. From a mathematical prospective, containers are elements of a finite or countably infinite partition of $\mathbb{X} = \mathbb{R}$ into disjoint convex sets, and each of these sets is not a singleton. In other words, each container B_{i+1} is an interval (possibly open, closed, or semiopen) with ends d_i and d_{i+1} such that $-\infty \le d_i < d_{i+1} \le +\infty$, and the union of these disjoint intervals is $\mathbb{R}$. In addition, we assume that $d_{i+1} - d_i \ge \gamma$ for some constant $\gamma > 0$ for all containers, that is, the sizes of all the containers are uniformly bounded below by a positive number. We also follow an agreement that the 0-inventory level belongs to a container with end points d_0 and d_1, and a container with end points d_i and d_{i+1} is labeled as the $(i+1)$th container B_{i+1}. Thus, container B_1 is the interval in the partition containing point 0. Containers' labels can be nonpositive. If there is a container with the smallest (or largest) finite label n, then $d_{n-1} = -\infty$ (or $d_n = +\infty$, respectively). If there are containers with labels i and j, then there are containers with all the labels between i and j. In addition, each container is either transparent or nontransparent. If the inventory level x_t belongs to a nontransparent container, the IM only knows which container the inventory level belongs to. If an inventory level x_t belongs to a transparent container, the IM knows that the amount of inventory is exactly x_t.

For each nontransparent container with end points d_i and d_{i+1}, we fix an arbitrary point b_{i+1} satisfying $d_i < b_{i+1} < d_{i+1}$. For example, it is possible to set $b_{i+1} = 0.5d_i + 0.5d_{i+1}$, when $\max\{|d_i|, |d_{i+1}|\} < \infty$. If an inventory level belongs to a nontransparent container B_i, the IM observes $y_t = b_i$. Let L be the set of labels of the nontransparent containers. We set $Y_L = \{b_i\colon i \in L\}$ and define the observation set $\mathbb{Y} = \mathbb{T} \cup Y_L$, where $\mathbb{T}$ is the union of all transparent containers B_i (transparent elements of the partition). If the observation y_t belongs to a transparent container (in this case, $y_t \in \mathbb{T}$), then the IM knows that the inventory level $x_t = y_t$. If $y_t \in Y_L$ (in this case, $y_t = b_i$ for some i), then the IM knows that the inventory level belongs to the container B_i, and this container is nontransparent. Of course, the distribution of this level can be computed.

Let ρ be the Euclidean distance on $\mathbb{R}$: $\rho(a, b) = |a - b|$ for $a, b \in \mathbb{Y}$. On the state space $\mathbb{X} = \mathbb{R}$ we consider the metric $\rho_{\mathbb{X}}(a, b) = |a - b|$, if a and b belong to the same container, and $\rho_{\mathbb{X}}(a, b) = |a - b| + 1$ otherwise, where $a, b \in \mathbb{X}$. The space $(\mathbb{X}, \rho_{\mathbb{X}})$ is a Borel subset of a Polish space (this Polish space consists of closed containers, that is, each finite point d_i is represented by two points: one belonging to the container B_i and another one to the container B_{i+1}). We notice that $\rho_{\mathbb{X}}(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $|x^{(n)} - x| \to 0$ as $n \to \infty$ and the sequence $\{x^{(n)}\}_{n=N, N+1, \ldots}$ belongs to the same container as x for a sufficiently large N. Thus, convergence on $\mathbb{X}$ in the metric $\rho_{\mathbb{X}}$ implies convergence in the Euclidean metric. In addition, if $x \ne d_i$ for all containers i, then $\rho_{\mathbb{X}}(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $|x^{(n)} - x| \to 0$ as $n \to \infty$. Therefore, for any open set B in $(\mathbb{X}, \rho_{\mathbb{X}})$, the set $B \setminus (\bigcup_i \{d_i\})$ is open in $(\mathbb{X}, \rho)$. We notice that each container B_i is an open and closed set in $(\mathbb{X}, \rho_{\mathbb{X}})$.

Observe that the state transition law P given by (58) is weakly continuous in $(x, a) \in \mathbb{X} \times \mathbb{A}$. Indeed, let B be an open set in $(\mathbb{X}, \rho_{\mathbb{X}})$ and $\rho_{\mathbb{X}}(x^{(n)}, x) \to 0$ and $|a^{(n)} - a| \to 0$ as $n \to \infty$. The set $B^\circ := B \setminus (\bigcup_i \{d_i\}) = B \cap (\bigcup_i (d_i, d_{i+1}))$ is open in $(\mathbb{X}, \rho)$. Since F (as a function from $(\mathbb{X}, \rho_{\mathbb{X}}) \times (\mathbb{A}, \rho) \times (\mathbb{R}, \rho)$ into $(\mathbb{X}, \rho_{\mathbb{X}})$) is a continuous function in the both models, with backorders and lost sales, Fatou's lemma yields

$$\liminf_{n\to\infty} P(B^\circ \mid x^{(n)}, a^{(n)}) = \liminf_{n\to\infty} \int_{\mathbb{R}} \mathbf{I}\{F(x^{(n)}, a^{(n)}, s) \in B^\circ\}\, dF_D(s)$$

$$\ge \int_{\mathbb{R}} \liminf_{n\to\infty} \mathbf{I}\{F(x^{(n)}, a^{(n)}, s) \in B^\circ\}\, dF_D(s) \ge \int_{\mathbb{R}} \mathbf{I}\{F(x, a, s) \in B^\circ\}\, dF_D(s) = P(B^\circ \mid x, a),$$

where the second inequality holds because $\liminf_{n\to\infty} \mathbf{I}\{F(x^{(n)}, a^{(n)}, s) \in B^\circ\} \ge \mathbf{I}\{F(x, a, s) \in B^\circ\}$. Therefore, $\liminf_{n\to\infty} P(B \mid x^{(n)}, a^{(n)}) \ge P(B \mid x, a)$ because for the model with backorders $P(x^* \mid x', a') = 0$ for all $x^*, x', a' \in \mathbb{R}$ in view of the continuity of the distribution function F_D, and, for the model with lost sales, $P(x^* \mid x', a') = 0$ for any $x', a' \in \mathbb{R}$ and $x^* \ne 0$, and $P(0 \mid x', a') = 1 - F_D(x' + a')$ is continuous in $(x', a') \in \mathbb{X} \times \mathbb{A}$. Since B is an arbitrary open set in $(\mathbb{X}, \rho_{\mathbb{X}})$, the stochastic kernel P on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ is weakly continuous. Therefore, $\limsup_{n\to\infty} P(B \mid x^{(n)}, a^{(n)}) \le P(B \mid x, a)$, for any closed set B in $(\mathbb{X}, \rho_{\mathbb{X}})$. Since any container B_i is simultaneously open and closed in $(\mathbb{X}, \rho_{\mathbb{X}})$, we have $P(B_i \mid x^{(n)}, a^{(n)}) \to P(B_i \mid x, a)$ as $n \to \infty$.

Set $\Psi(x) = x$, if the inventory level x belongs to a transparent container, and $\Psi(x) = b_i$, if the inventory level belongs to a nontransparent container B_i with a label i. As follows from the definition of the metric $\rho_{\mathbb{X}}$, the function $\Psi\colon (\mathbb{X}, \rho_{\mathbb{X}}) \to (\mathbb{Y}, \rho)$ is continuous. Therefore, the observation kernels Q_0 on $\mathbb{Y}$ given $\mathbb{X}$ and Q on $\mathbb{Y}$ given $\mathbb{A} \times \mathbb{X}$, $Q_0(C \mid x) := Q(C \mid a, x) := \mathbf{I}\{\Psi(x) \in C\}$, $C \in \mathscr{B}(\mathbb{Y})$, $a \in \mathbb{A}$, $x \in \mathbb{X}$, are weakly continuous.

If all the containers are nontransparent, the observation set $\mathbb{Y} = Y_L$ is countable, and conditions of Corollary 3.1 hold. In particular, the function $Q(b_i \mid a, x) = \mathbf{I}\{x \in B_i\}$ is continuous, if the metric $\rho_{\mathbb{X}}$ is considered on $\mathbb{X}$. If some containers are transparent and some are not, the conditions of Corollary 7.1 hold. To verify this, we set $\mathbb{Y}_1 := \mathbb{T}$ and $\mathbb{Y}_2 := Y_L$ and note that $\mathbb{Y}_2$ is countable and the function $Q(b_i \mid x) = \mathbf{I}\{x \in B_i\}$ is continuous for each $b_i \in Y_L$ because B_i is open and closed in $(\mathbb{X}, \rho_{\mathbb{X}})$. Note that $H(B \mid z, a, y) = P(B \mid y, a)$ for any $B \in \mathscr{B}(\mathbb{X})$, $C \in \mathscr{B}(\mathbb{Y})$, $z \in \mathbb{P}(\mathbb{X})$, $a \in \mathbb{A}$, and $y \in \mathbb{T}$. The kernel H is weakly continuous on $\mathbb{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{Y}_1$. In addition, $\mathbb{T} = \bigcup_i B_i^t$, where B_i^t are transparent containers, is an open set in $(\mathbb{X}, \rho_{\mathbb{X}})$. Thus, if either Assumption **(D)** or Assumption **(P)** holds, then POMDP $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c)$ satisfies the assumptions of Corollary 7.1. Thus, for the corresponding COMDP, there are stationary optimal policies, optimal policies satisfy the optimality equations, and value iterations converge to the optimal value.

The models studied in Bensoussan et al. [4, 6, 7] correspond to the partition $B_1 = (-\infty, 0]$ and $B_2 = (0, +\infty)$ with the container B_2 being nontransparent and with the container B_1 being either nontransparent (backordered amounts are not known Bensoussan et al. [6]) or transparent (models with lost sales (Bensoussan et al. [4]), backorders are observable (Bensoussan et al. [7])). Note that, since F_D is atomless, the probability that $x_t + a_t - D_t = 0$ is 0, $t = 1, 2, \ldots$.

The model provided in this subsection is applicable to other inventory control problems, and the conclusions of Corollary 7.1 hold for them too. For example, for problems with backorders, a nontransparent container $B_0 = (-\infty, 0)$ and a transparent container $B_1 = [0, +\infty)$ model a periodic review inventory control system for which nonnegative inventory levels are known, and, when the inventory level is negative, it is known that they are backorders, but their values are unknown.

8.3. Markov decision model with incomplete information (MDMII). An MDMII is a particular version of a POMDP studied primarily before the POMDP model was introduced in its current formulation. The reduction of MDMIIs with Borel state and action sets to MDPs was described in Rhenius [26] and Yushkevich [36]; see also Dynkin and Yushkevich [12, Chapter 8]. MDMIIs with transition probabilities having densities were studied in Rieder [27]; see also Bäuerle and Rieder [2, Part II]. An MDMII is defined by an *observed state space* $\mathbb{Y}$, an *unobserved state space* $\mathbb{W}$, an *action space* $\mathbb{A}$, nonempty *sets of available actions* $A(y)$, where $y \in \mathbb{Y}$, a stochastic kernel P on $\mathbb{Y} \times \mathbb{W}$ given $\mathbb{Y} \times \mathbb{W} \times \mathbb{A}$, and a one-step cost function c: $G \to \bar{\mathbb{R}}$, where $G = \{(y, w, a) \in \mathbb{Y} \times \mathbb{W} \times \mathbb{A}: a \in A(y)\}$ is the graph of the mapping $A(y, w) = A(y)$, $(y, w) \in \mathbb{Y} \times \mathbb{W}$. Assume that

(i) $\mathbb{Y}$, $\mathbb{W}$, and $\mathbb{A}$ are Borel subsets of Polish spaces. For each $y \in \mathbb{Y}$ a nonempty Borel subset $A(y)$ of $\mathbb{A}$ represents the *set of actions* available at y;

(ii) the graph of the mapping A: $\mathbb{Y} \to 2^{\mathbb{A}}$, defined as $\mathrm{Gr}(A) = \{(y, a)$: $y \in \mathbb{Y}$, $a \in A(y)\}$ is measurable, that is, $\mathrm{Gr}(A) \in \mathscr{B}(\mathbb{Y} \times \mathbb{A})$, and this graph allows a measurable selection, that is, there exists a measurable mapping ϕ: $\mathbb{Y} \to \mathbb{A}$ such that $\phi(y) \in A(y)$ for all $y \in \mathbb{Y}$;

(iii) the transition kernel P on $\mathbb{X}$ given $\mathbb{Y} \times \mathbb{W} \times \mathbb{A}$ is weakly continuous in $(y, w, a) \in \mathbb{Y} \times \mathbb{W} \times \mathbb{A}$;

(iv) the one-step cost c is $\mathbb{K}$-inf-compact on G, that is, for each compact set $K \subseteq \mathbb{Y} \times \mathbb{W}$ and for each $\lambda \in \mathbb{R}$, the set $\mathscr{D}_{K,c}(\lambda) = \{(y, w, a) \in G$: $(y, w) \in K, c(y, w, a) \le \lambda\}$ is compact.

Let us define $\mathbb{X} = \mathbb{Y} \times \mathbb{W}$, and for $x = (y, w) \in \mathbb{X}$ let us define $Q(C \mid x) = \mathbf{I}\{y \in C\}$ for all $C \in \mathscr{B}(\mathbb{Y})$. Observe that this Q corresponds to the continuous function $y = F(x)$, where $F(y, w) = y$ for all $x = (y, w) \in \mathbb{X}$ (here F is a projection of $\mathbb{X} = \mathbb{Y} \times \mathbb{W}$ on $\mathbb{Y}$). Thus, as explained in Example 4.1, the stochastic kernel $Q(dy \mid x)$ is weakly continuous in $x \in \mathbb{X}$. Then by definition, an MDMII is a POMDP with the state space $\mathbb{X}$, observation set $\mathbb{Y}$, action space $\mathbb{A}$, available action sets $A(y)$, transition kernel P, observation kernel $Q(dy \mid a, x) := Q(dy \mid x)$, and one-step cost function c. However, this model differs from our basic definition of a POMDP because the action sets $A(y)$ depend on observations and the one-step costs $c(x, a) = c(y, w, a)$ are not defined when $a \notin A(y)$. To avoid this difficulty, we set $c(y, w, a) = +\infty$ when $a \notin A(y)$. The extended function c is $\mathbb{K}$-inf-compact on $\mathbb{X} \times \mathbb{A}$ because the set $\mathscr{D}_{K,c}(\lambda)$ remains unchanged for each $K \subseteq \mathbb{Y} \times \mathbb{W}$ and for each $\lambda \in \mathbb{R}$.

Thus, an MDMII is a special case of a POMDP $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, P, Q, c)$, when $\mathbb{X} = \mathbb{Y} \times \mathbb{W}$ and observation kernels Q and Q_0 are defined by the projection of $\mathbb{X}$ on $\mathbb{Y}$. The observation kernel $Q(\cdot \mid x)$ is weakly continuous in $x \in \mathbb{X}$. As Example 4.1 demonstrates, in general this is not sufficient for the weak continuity of q and therefore for the existence of optimal policies. The following example confirms this conclusion for MDMIIs by demonstrating even the stronger assumption, that P is setwise continuous, is not sufficient for the weak continuity of the transition probability q.

Example 8.1. Setwise continuity of a transition probability P on $\mathbb{X}$ given $\mathbb{X} \times \mathbb{A}$ for an MDMII is not sufficient for the weak continuity of the transition probability q for the corresponding COMDP. Set $\mathbb{W} = \{1, 2\}$, $\mathbb{Y} = [0, 1]$, $\mathbb{X} = \mathbb{Y} \times \mathbb{W}$, and $\mathbb{A} = \{0\} \cup \{1/n$: $n = 1, 2, \ldots\}$. Let m be the Lebesgue measure on $\mathbb{Y} = [0, 1]$

and $m^{(n)}$ be an absolutely continuous measure on $\mathbb{Y}=[0,1]$ with the density $f^{(n)}$ defined in (16). As shown in Example 4.1, the sequence of probability measures $\{m^{(n)}\}_{n=1,2,\dots}$ converges setwise to the Lebesgue measure m on $\mathbb{Y}=[0,1]$. Recall that $Q(C\mid a,y,w)=\mathbf{I}\{y\in C\}$ for $C\in\mathscr{B}(\mathbb{Y})$. In this example, the setwise continuous transition probability P is chosen to satisfy the following properties: $P(B\mid y,w,a)=P(B\mid w,a)$ for all $B\in\mathscr{B}(\mathbb{X})$, $y\in\mathbb{Y}$, $w\in\mathbb{W}$, $a\in\mathbb{A}$, that is, the transition probabilities do not depend on observable states, and $P(\mathbb{Y}\times\{w'\}\mid w,a)=0$, when $w'\neq w$ for all $w,w'\in\mathbb{W}$, $a\in\mathbb{A}$, that is, the unobservable states do not change. For $C\in\mathscr{B}(\mathbb{Y})$, $w\in\mathbb{W}$, and $a\in\mathbb{A}$, we set

$$P(C\times\{w\}\mid w,a)=\begin{cases}m^{(n)}(C), & w=2,\ \ a=\dfrac{1}{n},\ \ n=1,2,\dots;\\ m(C), & \text{otherwise.}\end{cases}$$

Fix $z\in\mathbb{P}(\mathbb{X})$ defined by

$$z(C\times\{w\})=0.5\big(\mathbf{I}\{w=1\}+\mathbf{I}\{w=2\}\big)m(C),\quad w\in\mathbb{W},\ \ C\in\mathscr{B}(\mathbb{Y}).$$

Direct calculations according to formulas (4)–(8) imply that for $C,C'\in\mathscr{B}(\mathbb{Y})$ and $w\in\mathbb{W}$

$$R(C\times\{w\}\times C'\mid z,a)=\begin{cases}0.5m^{(n)}(C\cap C'), & \text{if } w=2 \text{ and } a=\dfrac{1}{n};\\ 0.5m(C\cap C'), & \text{otherwise,}\end{cases}$$

which implies $R'(C'\mid z,1/n)=0.5(m(C')+m^{(n)}(C'))$, $R'(C'\mid z,0)=m(C')$, and therefore we can choose

$$H(C\times\{w\}\mid z,a,y)=\begin{cases}0.5\mathbf{I}\{y\in C\}, & \text{if } a=0;\\ \mathbf{I}\{y\in C, f^{(n)}(y)=0\}+\frac{1}{3}\mathbf{I}\{y\in C, f^{(n)}(y)=2\}, & \text{if } w=1,\ \ a=1/n;\\ \frac{2}{3}\mathbf{I}\{y\in C, f^{(n)}(y)=2\}, & \text{if } w=2,\ \ a=1/n;\end{cases}$$

$y\in\mathbb{Y}$ and $n=1,2,\dots$. The subset of atomic probability measures on $\mathbb{X}$

$$D:=\big\{z^{(y)}\in\mathbb{P}(\mathbb{X})\colon z^{(y)}(y,1)=\tfrac{1}{3},\ z^{(y)}(y,2)=\tfrac{2}{3},\ y\in\mathbb{Y}\big\}$$

is closed in $\mathbb{P}(\mathbb{X})$. Indeed, an integral of any bounded continuous function g on $\mathbb{X}$ with respect to a measure $z^{(y)}\in D$ equals $\frac{1}{3}g(y,1)+\frac{2}{3}g(y,2)$, $y\in\mathbb{Y}$. Therefore, a sequence $\{z^{(y^{(n)})}\}_{n=1,2,\dots}$ of measures from D weakly converges to $z'\in\mathbb{P}(\mathbb{X})$ if and only if $y^{(n)}\to y\in\mathbb{Y}$ as $n\to\infty$ for some $y\in Y$, and thus $z'=z^{(y)}\in D$. Since D is a closed set in $\mathbb{P}(\mathbb{X})$, if the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times A$ is weakly continuous then $\limsup_{n\to\infty}q(D\mid z,1/n)\le q(D\mid z,0)$; see Billingsley [9, Theorem 2.1(iii)]. However, $q(D\mid z,1/n)=z(f^{(n)}(y)=2)=0.5[m(f^{(n)}(y)=2)+m^{(n)}(f^{(n)}(y)=2)]=\frac{3}{4}$, $n=1,2,\dots$, and $q(D\mid z,0)=0$. Thus, the stochastic kernel q on $\mathbb{P}(\mathbb{X})$ given $\mathbb{P}(\mathbb{X})\times\mathbb{A}$ is not weakly continuous. □

Thus, the natural question is which conditions are needed for the existence of optimal policies for the COMDP corresponding to an MDMII? The first author of this paper learned about this question from Alexander A. Yushkevich around the time when Yushkevich was working on Yushkevich [36]. The following theorem provides such a condition. For each open set $\mathscr{O}$ in $\mathbb{W}$ and for any $C\in\mathscr{B}(\mathbb{Y})$, consider a family of functions $\mathscr{P}^*_{\mathscr{O}}=\{(x,a)\to P(C\times\mathscr{O}\mid x,a)\colon C\in\mathscr{B}(\mathbb{Y})\}$ mapping $\mathbb{X}\times\mathbb{A}$ into $[0,1]$. Observe that equicontinuity at all the points $(x,a)\in\mathbb{X}\times\mathbb{A}$ of the family of functions $\mathscr{P}^*_{\mathscr{O}}$ is a weaker assumption, than the continuity of the stochastic kernel P on $\mathbb{X}$ given $\mathbb{X}\times\mathbb{A}$ in total variation.

THEOREM 8.1 (CP. FEINBERG ET AL. [17, THEOREM 6.2]). *Consider the expected discounted cost criterion with the discount factor $\alpha\in[0,1)$ and, if the cost function c is nonnegative, then $\alpha=1$ is also allowed. If for each nonempty open set $\mathscr{O}$ in $\mathbb{W}$ the family of functions $\mathscr{P}^*_{\mathscr{O}}$ is equicontinuous at all the points $(x,a)\in\mathbb{X}\times\mathbb{A}$, then the POMDP $(\mathbb{X},\mathbb{Y},\mathbb{A},P,Q,c)$ satisfies assumptions* (a), (b), *and* (i) *of Theorem 3.2, and therefore the conclusions of that theorem hold.*

PROOF. Assumptions (a) and (b) of Theorem 3.2 are obviously satisfied, and the rest of the proof verifies assumption (i). From (4) and (5),

$$R(C_1\times B\times C_2\mid z,a)=\int_{\mathbb{X}}P((C_1\cap C_2)\times B\mid x,a)z(dx),\quad B\in\mathscr{B}(\mathbb{W}),\ \ C_1,C_2\in\mathscr{B}(\mathbb{Y}),\ \ z\in\mathbb{P}(\mathbb{X}),\ \ a\in\mathbb{A},$$

$$R'(C\mid z,a)=\int_{\mathbb{X}}P(C\times\mathbb{W}\mid x,a)z(dx),\quad C\in\mathscr{B}(\mathbb{Y}),\ \ z\in\mathbb{P}(\mathbb{X}),\ \ a\in\mathbb{A}.$$

For any nonempty open sets $\mathcal{O}_1$ in $\mathbb{Y}$ and $\mathcal{O}_2$ in $\mathbb{W}$, respectively, Theorem 5.1, with $\mathbb{S}_1 = \mathbb{P}(\mathbb{X})$, $\mathbb{S}_2 = \mathbb{X}$, $\mathbb{S}_3 = \mathbb{A}$, $\mathcal{O} = \mathbb{X}$, $\Psi(B \mid z) = z(B)$, and $\mathcal{A}_0 = \{(x, a) \to P((\mathcal{O}_1 \cap C) \times \mathcal{O}_2 \mid x, a)\colon C \in \mathcal{B}(\mathbb{Y})\}$, implies the equicontinuity of the family of functions

$$\mathcal{R}_{\mathcal{O}_1 \times \mathcal{O}_2} = \{(z, a) \to R(\mathcal{O}_1 \times \mathcal{O}_2 \times C \mid z, a)\colon C \in \mathcal{B}(\mathbb{Y})\},$$

defined on $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$, at all the points $(z, a) \in \mathbb{P}(\mathbb{X}) \times \mathbb{A}$. Being applied to $\mathcal{O}_1 = \mathbb{Y}$ and $\mathcal{O}_2 = \mathbb{W}$, this fact implies that the stochastic kernel R' on $\mathbb{Y}$ given $\mathbb{P}(\mathbb{X}) \times \mathbb{A}$ is continuous in total variation. In particular, the stochastic kernel R' is setwise continuous.

Now, we show that Assumption **(H)** holds. Since the metric spaces $\mathbb{Y}$ and $\mathbb{W}$ are separable, there exist countable bases $\tau_b^{\mathbb{Y}}$ and $\tau_b^{\mathbb{W}}$ of the topologies for the separable metric spaces $\mathbb{Y}$ and $\mathbb{W}$, respectively. Then $\tau_b = \{\mathcal{O}^{\mathbb{Y}} \times \mathcal{O}^{\mathbb{W}}\colon \mathcal{O}^{\mathbb{Y}} \in \tau_b^{\mathbb{Y}}, \mathcal{O}^{\mathbb{W}} \in \tau_b^{\mathbb{W}}\}$ is a countable base of the topology of the separable metric space $\mathbb{X} = \mathbb{Y} \times \mathbb{W}$. Therefore, Assumption **(H)** follows from Lemma 5.3, the equicontinuity of the family of functions $\mathcal{R}_{\mathcal{O}_1 \times \mathcal{O}_2}$ for any open sets $\mathcal{O}_1$ in $\mathbb{Y}$ and $\mathcal{O}_2$ in $\mathbb{W}$, and the property that, for any finite subset N of $\{1, 2, \dots\}$,

$$\bigcap_{j \in N} (\mathcal{O}_j^{\mathbb{Y}} \times \mathcal{O}_j^{\mathbb{W}}) = \left(\bigcap_{j \in N} \mathcal{O}_j^{\mathbb{Y}}\right) \times \left(\bigcap_{j \in N} \mathcal{O}_j^{\mathbb{W}}\right) = \mathcal{O}_1 \times \mathcal{O}_2, \quad \mathcal{O}_j^{\mathbb{Y}} \in \tau_b^{\mathbb{Y}}, \quad \mathcal{O}_j^{\mathbb{W}} \in \tau_b^{\mathbb{W}}, \quad \text{for all } j \in N,$$

where $\mathcal{O}_1 = \bigcap_{j \in N} \mathcal{O}_j^{\mathbb{Y}}$ and $\mathcal{O}_2 = \bigcap_{j \in N} \mathcal{O}_j^{\mathbb{W}}$ are open subsets of $\mathbb{Y}$ and $\mathbb{W}$, respectively. □

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