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Einstein Finsler Metrics and Killing Vector Fields on Riemannian Manifolds

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Abstract

In this paper, we use a Killing form on a Riemannian manifold to construct a class of Finsler metrics. We find equations that characterize Einstein metrics among this class. In particular, we construct a family of Einstein metrics on S^3 with $\mathrm{Ric} = 2F^2$, $\mathrm{Ric} = 0$ and $\mathrm{Ric} = -2F^2$, respectively. This family of metrics provide an important class of Finsler metrics in dimension three, whose Ricci curvature is a constant, but the flag curvature is not.

Keywords: Killing vector field; Finsler metric; (α, β)-metric; Ricci curvature; Einstein metric; Ricci-flat metric

Mathematics Subject Classification 2010: 53B40, 53C60

1 Introduction

One of the most important problems in Finsler geometry is to study and characterize Einstein Finsler metrics. By definition, a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M is *of isotropic Ricci curvature* if

$$\mathrm{Ric} = (n-1)KF^2, \tag{1}$$

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where $K = K(x)$ is a scalar function on M. Recently, a new notion of Ricci (curvature) tensor Ric_{ij} has been studied [11] (See Section 2 below for the definition). A Finsler metric $F = F(x, y)$ is *of isotropic Ricci tensor* if

$$\mathrm{Ric}_{ij} = (n-1)K g_{ij}, \tag{2}$$

where $K = K(x)$ is a scalar function on M.

There is an important non-Riemannian quantity $H_{ij} = E_{ij|m}y^m$ defined as the covariant derivative of the mean Berwald curvature E_{ij} along a geodesic. It relates the Ricci curvature tensor Ric_{ij} and the Ricc curvature Ric as follows

$$H_{ij} = \mathrm{Ric}_{ij} - \frac{1}{2}[\mathrm{Ric}]_{y^i y^j}. \tag{3}$$

Therefore (1) and (2) are equivalent for Finsler metrics with $H_{ij} = 0$.

A Finsler metric is called an *Einstein metric* if it is of isotropic Ricci curvature and a *strong Einstein metric* if it is of *isotropic Ricci curvature tensor*. A famous question asked by S. S. Chern is: whether or not does every smooth manifold admit an Einstein Finsler metric with Ricci-constant?

Clearly, in dimension two, a Finsler metric is of isotropic Ricci curvature (tensor) if and only if it is of isotropic flag curvature. In dimension three, a Riemannian metric is of isotropic Ricci curvature (tensor) if and only if it is of isotropic sectional curvature, in this case, the sectional curvature must be a constant. Bao-Robles have proved that the conclusion as above still hold for Randers metrics, that is, a Randers metric on a manifold of dimension three is of isotropic Ricci curvature (tensor) if and only if it is of constant flag curvature ([2]). A natural question arises: is there any Finsler metric with isotropic Ricci curvature (tensor) but not isotropic flag curvature? The answer is yes. See the following

Theorem 1.1 *There are Einstein Finsler metrics on S^3 with $K = 1$, $K = 0$ and $K = -1$, respectively. The metrics take the following form $F = \alpha\phi(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a Killing 1-form of constant length b on S^3 satisfying*

$$\overline{\mathrm{Ric}} = 2\alpha^2 - 4(b^2\alpha^2 - \beta^2), \quad s_{0m}s^m{}_0 = -(b^2\alpha^2 - \beta^2), \quad s^m{}_{0;m} = 2\beta$$

and $\phi = \phi(s)$ satisfies

$$(b^2 - s^2)[(1 + sQ)Q_s - Q^2 + 2] - [2sQ + b^2Q^2 + 1] + K\phi^2 = 0, \tag{4}$$

where $Q := \phi'/(\phi - s\phi_s)$, $\overline{\mathrm{Ric}}$ denotes the Ricci curvature of α and $s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$ the anti-symmetric part of the covariant derivative of β with rexpect to α. These metrics are not of constant flag curvature in general, except for the case when $K = 1$ and $\phi = 1 + s$.

The Finsler metrics that we constructed on S^3 are called (α, β)-metrics. Randers metrics $F = \alpha + \beta$ are among the simplest non-Riemannian Finsler metrics. In [2], Bao-Robles find two equations on α and β that characterize Einstein Randers metrics. Using the navigation idea, they can actually classify Einstein Randers metrics upto the classification of Einstein Riemann metrics and the homothetic vector fields. To search for new Einstein metrics, we shall consider *almost regular* (α, β)-metrics (see Section 2 below for definition). Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where " ; " denotes the covariant derivative with respect to the Levi-Civita connection of α. Further, put

$$r_j := b^m r_{mj}, \quad s_j := b^m s_{mj},$$

where $\left(a^{ij}\right) := \left(a_{ij}\right)^{-1}$ and $b^i := a^{ij} b_j$. We will denote $r_{i0} := r_{ij} y^j$, $s_{i0} := s_{ij} y^j$ and $r_{00} := r_{ij} y^i y^j$, $r_0 := r_i y^i$, $s_0 := s_i y^i$, etc. Clearly, β is a Killing form if and only if $r_{ij} = 0$. Thus β is a Killing form of constant length with respect to α if and only if it satisfies the following equations:

$$r_{ij} = 0, \quad s_j = 0. \tag{5}$$

The first author has proved that a regular (α, β)-metric of non-Randers type is of isotropic S-curvature if and only if (5) holds ([4]). Thus (5) implies that the S-curvature is a constant, hence $H_{ij} = 0$.

For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ with parallel β (i.e. $b_{i;j} = 0$), it is an Einstein metric if and only if it is Ricci-flat, and if and only if α is Ricci-flat , regardless of the choice of ϕ (see [8]).

In the following we shall consider (α, β)-metrics with non-parallel Killing form β of constant length. Here we allow the metric to be almost regular in the above sense. We prove the following

Theorem 1.2 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost regular (α, β)-metric on an n-dimensional manifold. Suppose that β is a non-parallel Killing form with non-zero constant length $b := \|\beta_x\|_\alpha$. Then F is an (strong) Einstein metric if and only if one of the following cases occurs:*

(i) *F is an (strong) Einstein (α, β)-metric of Randers type.*

(ii) *α and β satisfy*

$$\overline{\mathrm{Ric}} = (n-1)\tau\{K_1\alpha^2 + K_2(b^2\alpha^2 - \beta^2)\} - K_3 s_{0m} s^m{}_0, \tag{6}$$

$$s^i{}_m s^m{}_i = (n-1)\tau, \tag{7}$$

$$s^m{}_{0;m} = -(n-1)b^{-2}\tau\beta \tag{8}$$

and $\phi = \phi(s)$ satisfies

$$Q = -b^{-2}s \pm \sqrt{\delta}\sqrt{1 - b^{-2}s^2}, \tag{9}$$

where $Q := \phi'/(\phi - s\phi')$, $\delta > 0$ *is a constant,* $\tau = \tau(x)$ *is a scalar function and*

$$K_1 = -b^{-2}, \quad K_2 = b^{-2}(b^{-2} + \delta), \quad K_3 = -2(b^{-2} + \delta). \tag{10}$$

In this case, $\mathrm{Ric} = 0$ *and* F *is an* (α, β)*-metric of Randers type with singularuty.*

(iii) α *and* β *satisfy*

$$\overline{\mathrm{Ric}} = (n-1)\tau\{K_1\alpha^2 + K_2(b^2\alpha^2 - \beta^2)\}, \tag{11}$$

$$s_{0m}s^m_{\ 0} = b^{-2}\tau(b^2\alpha^2 - \beta^2), \tag{12}$$

$$s^m_{\ 0;m} = -(n-1)b^{-2}\tau\beta \tag{13}$$

and $\phi = \phi(s)$ *satisfies*

$$\frac{1}{n-1}(b^2 - s^2)[2(1 + sQ)Q_s - 2Q^2] - (2sQ + b^2Q^2) + k[\delta_1 s^2 + (b^2 - s^2)] + \delta_2 s^2 + (b^2 - s^2)\delta_3 = kb^2\phi^2, \tag{14}$$

where $\delta_1, \ \delta_2, \ \delta_3, \ k$ *are constants,* $\tau = \tau(x)(\neq 0)$ *is a scalar function and*

$$K_1 = \delta_1 k + \delta_2, \quad K_2 = b^{-2}[(1 - \delta_1)k + \delta_3 - \delta_2]. \tag{15}$$

In this case, $\mathrm{Ric} = (n-1)KF^2$ *with* $K = k\tau$ *and* F *can be of Randers type.*

We are primarily concerned about the existence of α and β satisfying the conditions in Theorem 1.2. At this moment we have not found any pair (α, β) satisfying (5), (6), (7) and (8) with $K_3 \neq 0$ and $\tau \neq 0$. However, by using Maple program, we can get a solution of (9) as follows:

$$\phi(s) = C_1\phi(s) = C_1(\sqrt{b^2 - s^2} \pm \sqrt{\delta}bs).$$

In this case, $F = C_1(\sqrt{b^2\alpha^2 - \beta^2} \pm \sqrt{\delta}b\beta)$ is an (α, β)-metric of Randers type. Obviously, the Riemannian metric $\sqrt{b^2\alpha^2 - \beta^2}$ is singular at any point $(x, y) \in TM$ satisfying $b\alpha = \beta$.

In Theorem 1.2 (iii), if we let $c_1 := K_1 + K_2b^2$ and $c_2 := -K_2$ with $\delta_2 = -b^{-2}$ and $k = 0$, then (11), (12) and (14) are listed in [13] as sufficient conditions for F being Ricci-flat. As shown in [13], the equation (13) can be derived from (11) and (12).

There are many pairs (α, β) satisfying (5), (11), (12) and (13). We can construct them using Einstein Randers metrics. Let $\bar{F} = \alpha + \beta$ be an Einstein Randers metric with $\mathrm{Ric}_{\bar{F}} = (n-1)\sigma\bar{F}^2$. If β is a Killing form of constant length b satisfying $s_{0m}s^m_{\ 0} = -\sigma(b^2\alpha^2 - \beta^2)$, then, by (7.16) and (7.29) in [7], α and β satisfy (5), (11), (12) and (13) with

$$\tau = -b^2\sigma, \quad K_1 = -b^{-2}, \quad K_2 = \frac{n+1}{n-1}b^{-2}.$$

Taking constants $\delta_1, \delta_2, \delta_3, k$ in (15) such that $K_1 = -b^{-2}$ and $K_2 = \frac{n+1}{n-1}b^{-2}$ and letting $\phi = \phi(s)$ satisfy (14), we obtain an Einstein metric $F = \alpha\phi(\beta/\alpha)$ with $\mathrm{Ric} = (n-1)KF^2$, where $K = -k\tau$. One can easily verify that $\phi = 1 + s$ satisfies (14) with $\delta_2 = -b^{-2}(1 - \delta_1)$, $\delta_3 = \frac{n+1}{n-1}$ and $k = -b^{-2}$. On the other hand, it is surprising that, for a famous Finsler metric with singularity – Kropina metric in the form $F = \alpha^2/\beta$, its $\phi(s) = \frac{1}{s}$ satisfies differential equation (14) with $k = -\frac{1}{4}$, $\delta_2 = \frac{\delta_1}{4} - \frac{1}{b^2}$, $\delta_3 = \frac{1}{4} - \frac{1}{b^2}$.

The paper is organized as follows. In Section 2, we give some definitions and notations which are necessary for the present paper, and some lemmas and propositions are contained. The proof of the sufficient condition in Theorem 1.2 is also given in this section. Then the proof of the necessary condition in Theorem 1.2 is given in Section 3. Further, based on Theorem 1.2, we construct Einstein (α, β)-metrics with $Ric = 2F, Ric = 0$ and $Ric = -2F$ on S^3, respectively in Section 4. Here we view S^3 as a Lie group. We prove that these constructed metrics are not of constant flag curvature, which means that Theorem 1.1 holds. Finally, we give the conclusion and describe a related reseaech for our research work in Section 5.

2 Preliminaries

Let $F = F(x, y)$ be a Finsler metric on a manifold M. The geodesic coefficients G^i of F are given by

$$G^i = \frac{1}{4}g^{il}\left\{\left[F^2\right]_{x^k y^l}y^k - \left[F^2\right]_{x^l}\right\}, \tag{16}$$

where $g_{ij}(x, y) := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x, y)$ and $(g^{ij}) := (g_{ij})^{-1}$.

The well-known non-Riemannian quantity, S-curvature, is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}[\ln \sigma_F], \tag{17}$$

where $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is the Busemann-Hausdorff volume form. By definition, F is said to be of *isotropic S-curvature* if there exists a scalar function $c(x)$ on M such that $\mathbf{S}(x, y) = (n + 1)c(x)F(x, y)$. If $c(x)$ =constant, we say that F has *constant S-curvature*.

Using the S-curvature, one obtains another quantity $H = H_{ij}dx^i \otimes dx^j$ given by

$$H_{ij} = \frac{1}{2}\mathbf{S}_{ij;m}y^m.$$

For any $x \in M$ and $y \in T_xM\backslash\{0\}$, the *Riemann curvature* $\mathbf{R}_y = R^i{}_k\frac{\partial}{\partial x^i} \otimes dx^k$ of F is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{18}$$

The Riemann curvature tensor $R_j{}^i{}_{kl}$ is given by

$$R_j{}^i{}_{kl} = \frac{1}{3}\left\{R^i{}_{k\cdot l\cdot j} - R^i{}_{l\cdot k\cdot j}\right\}.$$

It is easy to see that $R^i{}_k = R_j{}^i{}_{kl}y^jy^l$. The two curvature tensors $R_j{}^i{}_{kl}$ and $R^i{}_k$ essentially contain the same geometric data. The *Ricci curvature tensor* Ric_{ij} is defined by

$$\mathrm{Ric}_{ij} := \frac{1}{2}\left\{R_i{}^m{}_{mj} + R_j{}^m{}_{mi}\right\}. \tag{19}$$

The *Ricci curvature* is the trace of the Riemann curvature, which is defined by

$$\mathrm{Ric} = R^m_{\ m}. \tag{20}$$

We have $\mathrm{Ric} = \mathrm{Ric}_{ij}y^iy^j = R_i{}^m{}_{mj}y^iy^j$. A Finsler metric F on an n-dimensional manifold M is called an *Einstein metric* if the Ricci curvature satisfies

$$\mathrm{Ric} = (n-1)KF^2, \tag{21}$$

where $K = K(x)$ is a scalar function on M. F is said to be of *Ricci constant* if F satisfies (21) and $K =$ constant. We call F the *Ricci flat* Finsler metric if $\mathrm{Ric} = 0$.

We have the following important identity ([11]):

$$\mathrm{Ric}_{ij} - \frac{1}{2}[\mathrm{Ric}]_{y^iy^j} = H_{ij}. \tag{22}$$

We now focus on (α, β)-metrics. By definition, an (α, β)-metric on a manifold M is expressed in the following form

$$F = \alpha\phi(s),\ \ s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. It is proved ([1]) that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric for any α, β with $\|\beta\|_\alpha(x) < b_o,\ x \in M$, if and only if the function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

For the above function $\phi = \phi(s)$, if the 1-form β satisfies $b(x) := \|\beta\|_\alpha(x) \le b_0$, then $F = \alpha\phi(\beta/\alpha)$ might be singular at the points $x \in M$ with $b(x) = b_0$. Such metrics are called *almost regular (α, β)-metrics*. When $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is just the Randers metric. When $\phi = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called *Kropina metric*. Randers metrics and Kropina metrics are both C-reducible ([12]). However, Randers metrics are regular Finsler metrics but Kropina metrics are Finsler metrics with singularity.

We have the following

Lemma 2.1 ([1][9]) *For an (α,β)-metric $F=\alpha\phi(s), s=\beta/\alpha$, the geodesic coefficients G^i of F and the geodesic coefficients $\bar{G}^i$ of α are related by*

$$G^i=\bar{G}+\alpha Q s^i{}_0+\alpha^{-1}\Theta\left\{r_{00}-2Q\alpha s_0\right\}y^i+\Psi\left\{r_{00}-2Q\alpha s_0\right\}b^i, \tag{23}$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi-s\phi'}, \\ \Theta &:= \frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi\left[(\phi-s\phi')+(b^2-s^2)\phi''\right]}, \\ \Psi &:= \frac{\phi''}{2\left[(\phi-s\phi')+(b^2-s^2)\phi''\right]}. \end{aligned}$$

In [5], we have proved that a Randers metric $F=\alpha+\beta$ is of isotropic S-curvature, $\mathbf{S}=(n+1)c(x)F$, if and only if

$$r_{ij}+b_is_j+b_js_i=2c(x)(a_{ij}-b_ib_j). \tag{24}$$

Further, we considered Finsler metrics of Randers type in the following form

$$F=k_1\sqrt{\alpha^2+k_2\beta^2}+k_3\beta,$$

where $k_1>0, k_2$ and $k_3\neq 0$ are constants. We obtained the sufficient and necessary condition that a Finsler metric of Randers type to be of isotropic S-curvature ([6]). More general, we characterized almost regular (α,β)-metrics of non-Randers type with isotropic S-curvature ([6]). For regular (α,β)-metrics, the first author has proved the following theorem.

Theorem 2.2 ([4]) *A regular (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ of non-Randers type on an n-dimensional manifold M is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, if and only if β satisfies*

$$r_{ij}=0, \quad s_j=0. \tag{25}$$

In this case, $\mathbf{S}=0$, regardless of the choice of a particular $\phi=\phi(s)$.

For an almost regular (α,β)-metric, if (25) holds, then $\mathbf{S}=0$. The converse might not be true.

We need the following lemma.

Lemma 2.3 *Assume that (25) holds, then*

$$b^m s_{jm;k}=-s_{jm}s^m{}_k. \tag{26}$$

In particular, we have

$$b^i s^m{}_{i;m}=-s^i{}_m s^m{}_i. \tag{27}$$

Proof:

$$\begin{aligned} 0 &= s_{j;k} = b^m{}_{;k} s_{mj} + b^m s_{mj;k} \\ &= b_{m;k} s^m{}_j - b^m s_{jm;k} \\ &= s_{mk} s^m{}_j - b^m s_{jm;k} \\ &= -s_{jm} s^m{}_k - b^m s_{jm;k}. \end{aligned}$$

Q.E.D.

Let $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ be an (α, β)-metric. In the following, we always assume that β is a Killing form of constant length b, namely it satisfies (5).

$$r_{ij} = 0, \quad s_j = 0. \tag{28}$$

By (23), we can rewrite the geodesic coefficients of an (α, β)-metric as

$$G^i = \bar{G} + T^i, \tag{29}$$

where $T^i := \alpha Q s^i{}_0$. From (18) and by use of a technique for computing Riemannian curvature, we have

$$R^i{}_j = \bar{R}^i{}_j + RT^i{}_j,$$

where $\bar{R}^i{}_j$ denote the Riemann curvature of α and

$$RT^i{}_j := 2T^i{}_{;j} - (T^i{}_{;k})_{y^j} y^k + 2T^k (T^i)_{y^j y^k} - (T^i)_{y^k} (T^k)_{y^j}.$$

After a series of complex computations and by use of Maple program, we can obtain

Proposition 2.4 ([8]) *For any (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the Riemannian curvature is given by*

$$R^i{}_j = \bar{R}^i{}_j + RT^i{}_j, \tag{30}$$

where

$$\begin{aligned} RT^i{}_j &= C_{331} s^i{}_0 s_{0j} + \alpha C_{332} s^i{}_k s^k{}_0 l_j + \alpha C_{333} s^i{}_k s^k{}_0 b_j - (Q^2\alpha^2) s^i{}_k s^k{}_j \\ &\quad + (2Q\alpha) s^i{}_{0|j} - (Q\alpha) s^i{}_{j|0} + [C_{311} s^i{}_{0|0} l_j - Q_s s^i{}_{0|0} b_j] \end{aligned}$$

and

$$\begin{aligned} C_{331} &= -3Q^2 + 3sQQ_s + 3Q_s, \\ C_{340} &= -2[Q_s(B - s^2) + sQ + 1]\Psi + Q_s, \\ C_{332} &= (Q - sQ_s)Q, \\ C_{333} &= QQ_s, \\ C_{311} &= sQ_s - Q. \end{aligned}$$

By Proposition 3.3 in [8], we get the following

Proposition 2.5 *For an (α,β)-metric $F=\alpha\phi(s), s=\beta/\alpha$, if (25) holds, then the Ricci curvature of F is related to the Ricci curvature $\overline{\mathrm{Ric}}$ of α by*

$$\mathrm{Ric}=\overline{\mathrm{Ric}}+s_{0m}s^m{}_0c_{19}+\alpha s^m{}_{0;m}c_{24}+\alpha^2 s^i{}_m s^m{}_i c_{26}, \tag{31}$$

where

$$\begin{aligned} c_{19} &= -2Q^2+2(1+sQ)Q_s,\\ c_{24} &= 2Q,\\ c_{26} &= -Q^2. \end{aligned}$$

Proof of the sufficient condition in Theorem 1.2:
(ii) Assume that α and β satisfy (6)-(8) with

$$K_1=-b^{-2},\qquad K_2=b^{-2}(b^{-2}+\delta),\qquad K_3=-2(b^{-2}+\delta)$$

and ϕ satisfies (9). Then Q satisfies the following two equations:

$$\begin{aligned} &c_{19}=K_3,\\ &K_1+K_2(b^2-s^2)-2b^{-2}sQ-Q^2=0. \end{aligned}$$

Then it follows from (31) that

$$\begin{aligned} \mathrm{Ric} &= (n-1)\tau\{K_1\alpha^2+K_2(b^2\alpha^2-\beta^2)\}-K_3s_{0m}s^m_{\ 0}+c_{19}s_{0m}s^m_{\ 0}\\ &\quad -(n-1)b^{-2}\tau c_{24}\alpha\beta+(n-1)\tau c_{26}\alpha^2\\ &= (n-1)\tau\alpha^2\Big\{K_1+K_2(b^2-s^2)-b^{-2}sc_{24}+c_{26}\Big\}=0. \end{aligned}$$

This proves the sufficient condition in Theorem 1.2 (ii).
(iii) Assume that α and β satisfy (11)-(13) with K_1, K_2 given by

$$K_1=\delta_1k+\delta_2,\qquad K_2=b^{-2}[(1-\delta_1)k+(\delta_3-\delta_2)],$$

and ϕ satisfies (14). Then it follows from (31) that

$$\begin{aligned} \mathrm{Ric} &= (n-1)\tau\{K_1\alpha^2+K_2(b^2\alpha^2-\beta^2)\}\\ &\quad +c_{19}b^{-2}\tau(b^2\alpha^2-\beta^2)-(n-1)c_{24}b^{-2}\tau\alpha\beta+(n-1)c_{26}\tau\alpha^2\\ &= \Big\{(n-1)b^2[k+\delta_3]-(n-1)[(1-\delta_1)k+(\delta_3-\delta_2)]s^2\\ &\quad +c_{19}(b^2-s^2)-(n-1)c_{24}s+(n-1)c_{26}b^2\Big\}b^{-2}\tau\alpha^2\\ &= (n-1)\Big\{k[\delta_1s^2+(b^2-s^2)]+\delta_2s^2+(b^2-s^2)\delta_3\\ &\quad +\frac{1}{n-1}(b^2-s^2)c_{19}-(c_{24}s-b^2c_{26})\Big\}b^{-2}\tau\alpha^2\\ &= (n-1)k\tau\phi^2\alpha^2=(n-1)k\tau F^2. \end{aligned}$$

This proves the sufficient condition in Theorem 1.2 (iii).

3 Proof of Theorem 1.2

In this section, we shall prove the necessary condition in Theorem 1.2.

Let $F=\alpha\phi(s),\ s=\beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M. Firstly, we take an orthonormal basis on T_xM with respect to α at any fixed point x such that

$$\alpha=\sqrt{\sum (y^i)^2},\quad \beta=by^1. \tag{32}$$

To simplify the computations, we take the following coordinate transformation $\psi:(s,u^A)\to(y^i)$:

$$y^1=\frac{s}{\sqrt{b^2-s^2}}\bar{\alpha},\qquad y^A=u^A, \tag{33}$$

where $\bar{\alpha}=\sqrt{\sum_{A=2}^n(u^A)^2}$. Here, our index conventions are as follows:

$$1\le i,j,k,\cdots\le n,\quad 2\le A,B,C,\cdots\le n.$$

Then

$$\alpha=\frac{b}{\sqrt{b^2-s^2}}\bar{\alpha},\qquad \beta=\frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}. \tag{34}$$

Thus

$$F=\alpha\phi(\beta/\alpha)=\frac{b\phi(s)}{\sqrt{b^2-s^2}}\bar{\alpha}.$$

Further,

$$\begin{aligned} s_{0m}s^m_{\ 0} &= \frac{s^2s_{1m}s^m_{\ 1}}{b^2-s^2}\bar{\alpha}^2+\frac{2s_{1m}\bar{s}^m_{\ 0}}{\sqrt{b^2-s^2}}s\bar{\alpha}+\bar{s}_{0m}\bar{s}^m_{\ 0},\\ s^m_{\ 0;m} &= \frac{s^m_{\ 1;m}}{\sqrt{b^2-s^2}}s\bar{\alpha}+\bar{s}^m_{\ 0;m}, \end{aligned}$$

where $\bar{s}^m_{\ 0}:=s^m_{\ A}y^A,\ \bar{s}_{0m}\bar{s}^m_{\ 0}:=s_{Am}s^m_{\ B}y^Ay^B$, etc..

Since $\bar{G}=\frac{1}{2}{}^{\alpha}\Gamma^i_{jk}(x)y^jy^k$, where ${}^{\alpha}\Gamma^i_{jk}$ denote the Christoffel symbols of α, the Ricci curvature $\overline{\mathrm{Ric}}=G_{ij}(x)y^iy^j$ is quadratic polynomial in y. Hence, it is easy to prove the following lemma.

Lemma 3.1 *The Ricci curvature of α can be expressed as*

$$\overline{\mathrm{Ric}}=\frac{s^2G_{11}}{b^2-s^2}\bar{\alpha}^2+(\bar{G}_{10}+\bar{G}_{01})\frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}+\bar{G}_{00}.$$

Assume that $F=\alpha\phi(\beta/\alpha)$ is an Einstein metric, $\mathrm{Ric}=(n-1)KF^2$, and satisfies (5). Then, by (31) and Lemma 3.1, we obtain

$$\Xi_4\bar{\alpha}^2+\Xi_3\bar{\alpha}+\Xi_2=0, \tag{35}$$

where

$$\begin{aligned}
\Xi_4 &= G_{11}s^2 + s^m{}_{1;m}c_{24}bs + b^2 s^i{}_m s^m{}_i c_{26} - (n-1)Kb^2\phi^2, \\
\Xi_3 &= \sqrt{b^2-s^2}\left\{(\bar{G}_{10}+\bar{G}_{01})s + \bar{s}^m{}_{0;m}c_{24}b\right\}, \\
\Xi_2 &= (b^2-s^2)\left\{\bar{s}_{0m}\bar{s}^m{}_0 c_{19} + \bar{G}_{00}\right\}.
\end{aligned}$$

From (35), we obtain the following fundamental equations

$$\Xi_4\bar{\alpha}^2 + \Xi_2 = 0, \tag{36}$$

$$\Xi_3 = 0. \tag{37}$$

We may assume that $c_{24}/s = 2Q/s \neq constant$. Otherwise $\phi(s) = k_1\sqrt{1+k_2s^2}$ for some constants k_1, k_2, and hence F is Riemannian, which is excluded in the theorem. Then it follows from (37) that

$$\bar{G}_{10} + \bar{G}_{01} = 0, \qquad \bar{s}^m{}_{0;m} = 0. \tag{38}$$

For simplicity, we let

$$C(s) := \frac{\Xi_4}{b^2-s^2}.$$

Then we can rewrite (36) as

$$c_{19}\bar{s}_{0m}\bar{s}^m{}_0 + \bar{G}_{00} + C(s)\bar{\alpha}^2 = 0 \tag{39}$$

and get

$$c'_{19}\bar{s}_{0m}\bar{s}^m{}_0 + C'(s)\bar{\alpha}^2 = 0. \tag{40}$$

Case 1: $c'_{19}(s) = 0$.

By (40), one can easily see that $c_{19}(s) = C_3$ and $C(s) = C_4$ are independent of s. C_3 is a constant since c_{19} is independent of $x \in M$ while C_4 might be a function of $x \in M$. Let

$$C_1 := G_{11}, \qquad M_1 := s^m{}_{1;m}, \qquad M_2 := s^i{}_m s^m{}_i.$$

By the definitions of $c_{19}(s)$ and $C(s)$, we obtain two ODEs on ϕ:

$$-2Q^2 + 2(1+sQ)Q_s = C_3, \tag{41}$$

$$C_1s^2 + 2bQM_1s - b^2M_2Q^2 - (n-1)Kb^2\phi^2 = C_4(b^2-s^2). \tag{42}$$

We rewrite (39) as follows

$$\bar{G}_{00} = -C_4\bar{\alpha}^2 - C_3\bar{s}_{0m}\bar{s}^m{}_0. \tag{43}$$

In general coordinates, we have

$$\overline{\mathrm{Ric}} = C_1\alpha^2 + C_2(b^2\alpha^2 - \beta^2) - C_3 s_{0m}s^m{}_0, \tag{44}$$

where C_2 is determined by

$$C_1 + b^2 C_2 = -C_4. \tag{45}$$

By (38),

$$s^m{}_{0;m} = b^{-1} M_1 \beta. \tag{46}$$

By (27), we have

$$M_1 = -b^{-1} M_2. \tag{47}$$

Case 1a: If $C_3 = 0$, then by (41), we get

$$Q = \frac{s \pm \sqrt{k + s^2}}{k}, \tag{48}$$

where $k > 0$ is a constant. Since

$$\phi = e^{\int \frac{Q(s)}{1 + sQ(s)} ds}.$$

we get

$$\phi = \frac{\sqrt{k + s^2} \pm s}{\sqrt{k}}.$$

Thus $F = \alpha\phi(\beta/\alpha)$ is an Einstein (α, β)-metric of Randers type.

Case 1b: If $C_3 \neq 0$, then by (41), we get

$$s - \frac{2Q}{C_3} - k\sqrt{2Q^2 + C_3} = 0, \tag{49}$$

where k is a non-zero constant, that is,

$$Q = \frac{\frac{2s}{C_3} \pm \sqrt{\left(\frac{2s}{C_3}\right)^2 - \left(\frac{4}{C_3^2} - 2k^2\right)(s^2 - C_3 k^2)}}{\frac{4}{C_3^2} - 2k^2}.$$

We can simplify express Q in the following form

$$Q = \frac{\lambda}{\delta} s \pm \sqrt{\delta + \lambda s^2}, \tag{50}$$

where λ and δ are constants and $\delta > 0$. In this case, $C_3 = \frac{2(\lambda - \delta^2)}{\delta}$.

If $K \neq 0$, then by (42), we get

$$\phi^2 = \frac{C_1 s^2 + 2bM_1 sQ - b^2 M_2 Q^2 - C_4(b^2 - s^2)}{(n-1)b^2 K}.$$

Plugging (50) into the above expression, one get

$$\phi^2 = a + bs^2 + cs\sqrt{\delta + \lambda s^2},$$

where a, b and c are constants. Using the above formula, we get

$$\begin{aligned} Q &= \frac{(\phi^2)'}{2\phi^2 - s(\phi^2)'} \\ &= \frac{c\delta + 2c\lambda s^2 + 2bs\sqrt{\delta + \lambda s^2}}{c\delta s + 2a\sqrt{\delta + \lambda s^2}}. \end{aligned}$$

Comparing it with (50), we get

$$b = a\frac{\lambda + \delta^2}{\delta}, \qquad c = 2a.$$

Then

$$\phi^2 = a\Big\{1 + (\frac{\lambda}{\delta} + \delta)s^2 + 2s\sqrt{\delta + \lambda s^2}\Big\} = a\Big(\sqrt{\delta}s + \sqrt{1 + \frac{\lambda}{\delta}s^2}\Big)^2.$$

Thus F is of Randers type.

If $K = 0$, we are going to express M_1, M_2 and C_3 in terms of some constants and C_4. Now (42) is reduced to

$$C_1 s^2 + 2bQM_1 s - b^2 M_2 Q^2 = C_4(b^2 - s^2). \tag{51}$$

Plugging (50) into (51) yields

$$C_1 + 2bM_1\frac{\lambda}{\delta} - b^2 M_2 \lambda - b^2 M_2\Big(\frac{\lambda}{\delta}\Big)^2 + C_4 = 0, \tag{52}$$

$$M_1 - bM_2\frac{\lambda}{\delta} = 0, \tag{53}$$

$$C_4 + M_2\delta = 0. \tag{54}$$

We get

$$C_1 = \frac{b^2\lambda(\lambda - \delta^2) - \delta^3}{\delta^3}C_4, \qquad M_1 = -\frac{b\lambda}{\delta^2}C_4, \qquad M_2 = -\frac{1}{\delta}C_4.$$

In this case,

$$\overline{\mathrm{Ric}} = C_1\alpha^2 + C_2(b^2\alpha^2 - \beta^2) - C_3 s_{0m}s^m{}_0, \tag{55}$$

$$s^i{}_m s^m{}_i = M_2 = -\frac{1}{\delta}C_4, \tag{56}$$

$$s^m{}_{0;m} = b^{-1}M_1\beta = -\frac{\lambda}{\delta^2}\beta C_4, \tag{57}$$

where, $C_3 = \frac{2(\lambda - \delta^2)}{\delta}$ and

$$C_2 = -\frac{\lambda(\lambda - \delta^2)}{\delta^3}C_4$$

by (45). From (47), we have

$$(\delta + b^2\lambda)C_4 = 0. \tag{58}$$

By the assumption, $s_{ij} \neq 0$. Thus the matrix $(s^i{}_m s^m_j)$ is semi-negative definite. Then $M_2 = s^i{}_m s^m_i < 0$, which means that $C_4 > 0$. It follows from (58) that

$$\lambda = -b^{-2}\delta.$$

Further, we may define a scalar function $\tau = \tau(x)$ such that

$$C_4 = -(n-1)\delta\tau.$$

We get

$$\begin{aligned} C_1 &= -(n-1)\tau\frac{b^2\lambda(\lambda-\delta^2)-\delta^3}{\delta^2} = -(n-1)\tau b^{-2}, \\ C_2 &= (n-1)\tau\frac{\lambda(\lambda-\delta^2)}{\delta^2} = (n-1)b^{-2}(b^{-2}+\delta)\tau, \\ C_3 &= -2(b^{-2}+\delta). \end{aligned}$$

Plugging them into (50), (55), (56) and (57) and letting $K_1 := -b^{-2}$, $K_2 := b^{-2}(b^{-2}+\delta)$ and $K_3 := C_3$ yield Theorem 1.2 (ii).

Case 2: $c'_{19}(s) \neq 0$ for some s. Then by (40), there is a number $\tau = \tau(x)$ independent of s such that

$$C'(s) = -\tau c'_{19}. \tag{59}$$

Equivalently, there is a scalar function $\rho = \rho(x)$ independent of s such that

$$C(s) + \tau c_{19} = -\rho. \tag{60}$$

By (40) and (59), we obtain

$$\bar{s}_{0m}\bar{s}^m{}_0 = \tau\bar{\alpha}^2. \tag{61}$$

Then (39) becomes

$$\bar{G}_{00} = \rho\bar{\alpha}^2. \tag{62}$$

Let us rewrite (60) as

$$G_{11}s^2 + s^m{}_{1;m}c_{24}bs + b^2 s^i{}_m s^m_i c_{26} - (n-1)Kb^2\phi^2 + (b^2-s^2)(\tau c_{19}+\rho) = 0. \tag{63}$$

In a general coordinate system, by Lemma 3.1 and (38), (62), we can get

$$\overline{\mathrm{Ric}} = C_1\alpha^2 + C_2(b^2\alpha^2 - \beta^2), \tag{64}$$

where

$$C_1 = G_{11},\ C_2 = b^{-2}(\rho - G_{11}). \tag{65}$$

By (61) and $s_{1m} = b^{-1}s_m = 0$, we have the following

$$s_{0m}s^m_{\ 0} = b^{-2}\tau(b^2\alpha^2 - \beta^2). \tag{66}$$

It follows from (66) that

$$s^i_{\ m}s^m_{\ i} = (n-1)\tau. \tag{67}$$

By (27), we get

$$s^m_{\ 1;m} = -b^{-1}s^i_{\ m}s^m_{\ i} = -(n-1)b^{-1}\tau. \tag{68}$$

Further, by (38) and (34), we have

$$s^m_{\ 0;m} = \frac{s^m_{\ 1;m}}{\sqrt{b^2-s^2}}s\bar{\alpha} + \bar{s}^m_{\ 0;m} = s^m_{\ 1;m}b^{-1}\beta = -(n-1)b^{-2}\tau\beta.$$

Then we have the following

Lemma 3.2 *Assume that an almost regular (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ is an Einstein metric,* $\mathrm{Ric} = (n-1)KF^2$*, and satisfies (5) and $c'_{19}(s) \neq 0$. Then*

$$\overline{\mathrm{Ric}} = C_1\alpha^2 + C_2(b^2\alpha^2 - \beta^2), \tag{69}$$

$$s_{0m}s^m_{\ 0} = b^{-2}\tau(b^2\alpha^2 - \beta^2), \tag{70}$$

$$s^m_{\ 0;m} = -(n-1)b^{-2}\tau\beta, \tag{71}$$

where $C_1 = G_{11},\ C_2 = b^{-2}(\rho - G_{11})$ and $\tau = \tau(x),\ \rho = \rho(x)$ are scalar functions independent of s on the manifold.

Now, by (67) and (68), (63) becomes

$$C_1s^2 - (n-1)\{\tau c_{24}s - \tau c_{26}b^2 + Kb^2\phi^2\} + (b^2 - s^2)(\tau c_{19} + \rho) = 0. \tag{72}$$

Lemma 3.3 *If (72) holds, then*

$$C_1 = (n-1)(\delta_1 K + \delta_2\tau), \tag{73}$$

$$\rho = (n-1)(K + \delta_3\tau), \tag{74}$$

where $\delta_1, \delta_2, \delta_3$ are constants.

Proof: Firstly, letting $s = 0$ in (72) yields (74), where

$$\delta_3 = -\left[c_{26}(0) + \frac{1}{n-1}c_{19}(0)\right]$$

is a constant.

Differentiate (72) with respect to s twice. Then, by (74) and letting $s = 0$ yields (73) . Q.E.D.

Now (72) become

$$\{\delta_1 K + \delta_2 \tau\} s^2 - \{\tau c_{24} s - \tau c_{26} b^2 + K b^2 \phi^2\}$$
$$+(b^2 - s^2)\{\frac{1}{n-1}\tau c_{19} + K + \delta_3 \tau\} = 0,$$

that is,

$$K\{\delta_1 s^2 - b^2\phi^2 + (b^2 - s^2)\}$$
$$+\tau \left\{\delta_2 s^2 + (b^2 - s^2)\delta_3 - (c_{24}s - c_{26}b^2) + \frac{1}{n-1}(b^2 - s^2)c_{19}\right\} = 0. \quad (75)$$

Note that $K = K(x)$ and $\tau = \tau(x)$ are scalar functions on M while their coefficients are functions of s, independent of $x \in M$.

Under the condition that β is not parallel with respect to α, we assert that $\tau \neq 0$. In fact, if $\tau = 0$ at some points, then we can conclude that β is closed, $s_{ij} = 0$, by (70). Further, we can see that β is parallel with respect to α by (5). It is a contradiction. Thus one can see that $K = k\tau$ for some constant k from (75). Then (75) is reduced to

$$k\{\delta_1 s^2 - b^2\phi^2 + (b^2 - s^2)\}$$
$$+ \left\{\delta_2 s^2 + (b^2 - s^2)\delta_3 - (c_{24}s - c_{26}b^2) + \frac{1}{n-1}(b^2 - s^2)c_{19}\right\} = 0. \quad (76)$$

It is just (14). In this case, by Lemma 3.2 and Lemma 3.3,

$$C_1 = (n-1)[\delta_1 k + \delta_2]\tau, \qquad C_2 = (n-1)b^{-2}[(1-\delta_1)k + (\delta_3 - \delta_2)]\tau.$$

This completes the proof of Theorem 1.2 (iii) by letting $K_1 := \delta_1 k + \delta_2$ and $K_2 := b^{-2}[(1-\delta_1)k + \delta_3 - \delta_2]$. In this case, we do not exclude (α, β)-metrics of Randers type.

4 Einstein metrics on S^3

We now consider a special family of Randers metrics $\bar{F} = \alpha + \beta$ of constant flag curvature $K = 1$ on S^3. This family of Randers metrics were first introduced in [3]. We shall use them to construct Einstein (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ with $\mathrm{Ric} = 2F$, $\mathrm{Ric} = 0$ and $\mathrm{Ric} = -2F$, respectively, but none of them are of constant flag curvature.

We view S^3 as a Lie group and let η^1, η^2, η^3 be the standard right invariant 1-form on S^3 such that

$$d\eta^1 = 2\eta^2 \wedge \eta^3, \qquad d\eta^2 = 2\eta^3 \wedge \eta^1, \qquad d\eta^3 = 2\eta^1 \wedge \eta^2.$$

For any number $\varepsilon \geq 0$, let $\theta^1 := (1+\varepsilon)\eta^1$, $\theta^2 := \sqrt{1+\varepsilon}\eta^2$ and $\theta^3 := \sqrt{1+\varepsilon}\eta^3$. Let

$$\alpha := \sqrt{[\theta^1]^2 + [\theta^2]^2 + [\theta^3]^2}, \qquad \beta := b\theta^1,$$

where $b = \sqrt{\varepsilon/(1+\varepsilon)} < 1$. The Levi-Civita connection forms $(\theta_j{}^i)$ are given by

$$d\theta^i = \theta^j \wedge \theta_j{}^i,$$

where $\theta_j{}^i + \theta_i{}^j = 0$ and

$$\theta_2{}^1 = \theta^3, \quad \theta_3{}^1 = -\theta^2, \quad \theta_3{}^2 = \frac{1-\epsilon}{1+\epsilon}\theta^1.$$

Then the Riemann curvature tensor of α is given by

$$d\theta_j{}^i - \theta_j{}^m \wedge \theta_m{}^i = \frac{1}{2}\bar{R}_j{}^i{}_{pq}\theta^p \wedge \theta^q,$$

where $\bar{R}_j{}^i{}_{pq} + \bar{R}_i{}^j{}_{pq} = 0$, $\bar{R}_j{}^i{}_{pq} + \bar{R}_j{}^i{}_{qp} = 0$ and

$$\begin{aligned}
\frac{1}{2}\bar{R}_2{}^1{}_{pq}\theta^p \wedge \theta^q &= \theta^1 \wedge \theta^2, \\
\frac{1}{2}\bar{R}_3{}^1{}_{pq}\theta^p \wedge \theta^q &= \theta^1 \wedge \theta^3, \\
\frac{1}{2}\bar{R}_1{}^2{}_{pq}\theta^p \wedge \theta^q &= -\theta^1 \wedge \theta^2, \\
\frac{1}{2}\bar{R}_3{}^2{}_{pq}\theta^p \wedge \theta^q &= \lambda\theta^2 \wedge \theta^3, \\
\frac{1}{2}\bar{R}_1{}^3{}_{pq}\theta^p \wedge \theta^q &= -\theta^1 \wedge \theta^3, \\
\frac{1}{2}\bar{R}_2{}^3{}_{pq}\theta^p \wedge \theta^q &= -\lambda\theta^2 \wedge \theta^3,
\end{aligned}$$

where $\lambda := (1-3\varepsilon)/(1+\varepsilon) = 1-4b^2$. We get an expression for $\bar{R}^i{}_j = \bar{R}_p{}^i{}_{jq}y^py^q$:

$$\begin{aligned}
\bar{R}^1{}_1 &= (y^2)^2 + (y^3)^2, \\
\bar{R}^2{}_2 &= (y^1)^2 + \lambda(y^3)^2, \\
\bar{R}^3{}_3 &= (y^1)^2 + \lambda(y^2)^2, \\
\bar{R}^1{}_2 &= -y^1y^2, \\
\bar{R}^1{}_3 &= -y^1y^3, \\
\bar{R}^2{}_3 &= -\lambda y^2y^3.
\end{aligned}$$

Then the Ricci curvature of α is given by

$$\overline{\mathrm{Ric}} = 2(y^1)^2 + (1+\lambda)[(y^2)^2 + (y^3)^2] = 2\{\alpha^2 - 2(b^2\alpha^2 - \beta^2)\}.$$

By definition, $db_i - b_j\theta_i{}^j = b_{i;j}\theta^j$. A direct computation gives

$$b_{1;1} = b_{1;2} = b_{1;3} = 0,$$

$$b_{2;1} = b_{2;2} = 0, \quad b_{2;3} = -b,$$

$$b_{3;1} = 0, \quad b_{3;2} = b, \quad b_{3;3} = 0.$$

We see that

$$r_{ij} = 0, \qquad s_j = 0.$$

We have

$$s^1{}_0 = 0, \qquad s^2{}_0 = -by^3, \qquad s^3{}_0 = by^2,$$

$$s_{01} = 0, \quad s_{02} = by^3, \quad s_{03} = -by^2.$$

We have

$$s^1{}_m s^m{}_1 = 0, \quad s^1{}_m s^m{}_2 = 0, \quad s^1{}_m s^m{}_3 = 0,$$

$$s^2{}_m s^m{}_1 = 0, \quad s^2{}_m s^m{}_2 = -b^2, \quad s^2{}_m s^m{}_3 = 0,$$

$$s^3{}_m s^m{}_1 = 0, \quad s^3{}_m s^m{}_2 = 0, \quad s^3{}_m s^m{}_3 = -b^2,$$

In short,

$$s_{0m} s^m{}_0 = -(b^2\alpha^2 - \beta^2).$$

By direction computation

$$s_{12;k} = -b\delta_{2k}, \qquad s_{13;k} = -b\delta_{3k}, \qquad s_{23;k} = 0.$$

We have

$$s^1{}_{0;0} = -b(y^2)^2 - b(y^3)^2, \qquad s^2{}_{0;0} = by^1y^2, \qquad s^3{}_{0;0} = by^1y^3.$$

$$s^1{}_{1;0} = 0, \qquad s^1{}_{2;0} = -by^2, \qquad s^1{}_{3;0} = -by^3$$

$$s^2{}_{1;0} = by^2, \qquad s^2{}_{2;0} = 0, \qquad s^2{}_{3;0} = 0$$

$$s^3{}_{1;0} = by^3, \qquad s^3{}_{2;0} = 0, \qquad s^3{}_{3;0} = 0$$

$$s^1{}_{0;1} = 0, \qquad s^1{}_{0;2} = -by^2, \qquad s^1{}_{0;3} = -by^3,$$

$$s^2{}_{0;1} = 0, \qquad s^2{}_{0;2} = by^1, \quad s^2{}_{0;3} = 0,$$

$$s^3{}_{0;1} = 0, \qquad s^3{}_{0;2} = 0, \quad s^3{}_{0;3} = by^1$$

We get

$$s^m{}_{0;m} = 2by^1 = 2\beta.$$

Therefore α and β satisfy (5), (11), (12) and (13) with

$$\tau = -b^2, \qquad K_1 = -b^{-2}, \qquad K_2 = 2b^{-2}.$$

(1) Let $\delta_2 = -b^{-2}(1-\delta_1)$, $\delta_3 = 2$ and $k = -b^{-2}$. Then (14) is simplified to

$$(b^2 - s^2)[(1+sQ)Q_s - Q^2 + 2] - [(2sQ + b^2Q^2) + 1] + \phi^2 = 0, \tag{77}$$

where $Q := \phi'/(\phi - s\phi')$. Note that $\phi = 1 + s$ is a special solution of (77). For any $\phi = \phi(s)$ satisfying (77), the metric $F = \alpha\phi(\beta/\alpha)$ is an Einstein metric with $\mathrm{Ric} = 2F^2$ on S^3.

(2) Let $\delta_2 = -b^{-2}$ and $\delta_3 = 2 - b^{-2}$ and $k = 0$. Then (14) is simplified to

$$(b^2 - s^2)[(1 + sQ)Q_s - Q^2 + 2] - [(2sQ + b^2Q^2) + 1] = 0. \tag{78}$$

For any $\phi = \phi(s)$ satisfying (78), the metric $F = \alpha\phi(\beta/\alpha)$ is an Einstein metric with $\mathrm{Ric} = 0$ on S^3.

(3) Let $\delta_2 = -b^{-2}(1-\delta_1)$, $\delta_3 = 2-2b^{-2}$ and $k = b^{-2}$. Then (14) is simplified to

$$(b^2 - s^2)[(1 + sQ)Q_s - Q^2 + 2] - [(2sQ + b^2Q^2) + 1] - \phi^2 = 0. \tag{79}$$

For any $\phi = \phi(s)$ satisfying (79), the metric $F = \alpha\phi(\beta/\alpha)$ is an Einstein metric with $\mathrm{Ric} = -2F^2$ on S^3.

To show that the above Finsler metrics on S^3 are not of constant flag curvature, we need to compute $RT^i{}_j$ in (30). We have

$$\begin{aligned}
RT^1_{\ 1} &= -b\alpha^{-1}[(y^2)^2 + (y^3)^2]\Big\{C_{311}y^1 - Q_s b\alpha\Big\},\\
RT^1_{\ 2} &= -b\alpha^{-1}y^2\Big\{Q\alpha^2 + [(y^2)^2 + (y^3)^2]C_{311}\Big\},\\
RT^1_{\ 3} &= -b\alpha^{-1}y^3\Big\{Q\alpha^2 + [(y^2)^2 + (y^3)^2]C_{311}\Big\},\\
RT^2_{\ 1} &= -b\alpha y^2[C_{333}b^2 + Q] + b\alpha^{-1}y^1y^2\Big\{C_{311}y^1 - Q_s\alpha b - bC_{332}\alpha\Big\},\\
RT^2_{\ 2} &= -b^2[C_{331}(y^3)^2 + C_{332}(y^2)^2] + b^2Q^2\alpha^2 + 2bQ\alpha y^1 + C_{311}b\alpha^{-1}y^1(y^2)^2,\\
RT^2_{\ 3} &= b^2[C_{331} - C_{332}]y^2y^3 + b\alpha^{-1}C_{311}y^1y^2y^3,\\
RT^3_{\ 1} &= -b^2C_{332}y^1y^3 - \alpha b^3C_{333}y^3 - b\alpha Qy^3 + bC_{311}\alpha^{-1}(y^1)^2y^3,\\
RT^3_{\ 2} &= b^2[C_{331} - C_{332}]y^2y^3 + b\alpha^{-1}C_{311}y^1y^2y^3,\\
RT^3_{\ 3} &= -b^2[C_{331}(y^2)^2 + C_{332}(y^3)^2] + b^2Q^2\alpha^2 + 2Qb\alpha y^1 + b\alpha^{-1}C_{311}y^1(y^3)^2.
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
R^1{}_1 &= [1 + sQ + (b^2 - s^2)Q_s][(y^2)^2 + (y^3)^2],\\
R^2{}_2 &= (y^1)^2 + \lambda(y^3)^2 + [b^2Q^2 + 2sQ]\alpha^2 - [b^2C_{332} - sC_{311}](y^2)^2 - b^2C_{331}(y^3)^2\\
&= \frac{b^2(1 + b^2Q^2 + 2sQ)}{b^2 - s^2}[(y^2)^2 + (y^3)^2] - [b^2C_{332} - sC_{311}](y^2)^2\\
&\quad -[b^2C_{331} - \lambda](y^3)^2,\\
R^3{}_3 &= (y^1)^2 + \lambda(y^2)^2 + [b^2Q^2 + 2sQ]\alpha^2 - [b^2C_{332} - sC_{311}](y^3)^2 - b^2C_{331}(y^2)^2\\
&= \frac{b^2(1 + b^2Q^2 + 2sQ)}{b^2 - s^2}[(y^2)^2 + (y^3)^2] - [b^2C_{332} - sC_{311}](y^3)^2\\
&\quad -[b^2C_{331} - \lambda](y^2)^2.
\end{aligned}$$

Clearly, $R^1{}_1 \neq 0$. Thus when $\mathrm{Ric} = 0$, we find a Ricci-flat Finsler metric on S^3 which is not of zero flag curvature. Also, we can check that $R^1{}_1 \neq F^2(1 - l^1l_1)$ and $R^1{}_1 \neq -F^2(1 - l^1l_1)$, where $l_i := F_{y^i}$ and $l^i = g^{ij}l_j$. Thus we have found

Einstein Finsler metrics on S^3 with $K = 1$, $K = 0$ and $K = -1$, respectively, which are not of constant flag curvature.

It is surprised that there are singular Einstein (α, β)-metrics with $\mathrm{Ric} = 0$ or $\mathrm{Ric} = -(n-1)F^2$ on S^3. This is impossible if the metric is regular by the classical comparison theorems in Finsler geometry.

5 The conclusion and discussion

The research of this paper is driven by two motivitions. The first motivition is from S. S. Chern's question, that is, whether or not every smooth manifold admits an Einstein Finsler metric with Ricci-constant. We use a Killing form with non-zero constant length on a Riemannian manifold to construct a family of (α, β)-metrics and we find equations that characterize Einstein metrics among this family of (α, β)-metrics (see Theorem 1.2). By Theorem 1.2, we know that these constructed Einstein (α, β)-metrics may be Einstein Randers metrics or Ricci-flat (α, β)-metrics of non-Randers type, or Einstein almost regular (α, β)-metrics. The second motivition is from the following question: is there any Einstein-Finsler metric which is not of isotropic flag curvature on three dimensional manifolds? We have found Einstein (α, β)-metrics on S^3 with $Ric = 2F, Ric = 0$ and $Ric = -2F$, respectively, but none of them are of constant flag curvature (see Theorem 1.1).

Recently, we have learned that L. Huang had constructed a two-parameter family of almost regular Finsler metrics on S^3 . His metrics are of constant Ricci curvature $+1$ with $Ric = 2F$, but the flag curvature is not constant (see [10]). However, the technique and method used in [10] is quite different from ours in this paper.

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