Symbolic dynamics for Lozi maps

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Abstract

In this paper we study the family of the Lozi maps $L_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$, $L_{a,b} = (1+y-a|x|, bx)$, and their strange attractors $\Lambda_{a,b}$. We introduce the set of kneading sequences for the Lozi map and prove that it determines the symbolic dynamics for that map. We also introduce two other equivalent approaches.

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1 Introduction

Symbolic dynamics and the Milnor – Thurston kneading theory are very powerful tools in studying one-dimensional dynamics of unimodal maps, such as the tent maps, or the quadratic maps. One of the most important ingredients in the kneading theory for unimodal maps is the kneading sequence, which is defined as the itinerary of the critical value. This symbol sequence is a complete invariant of the topological conjugacy classes of unimodal maps with negative Schwarzian derivative (when there is no periodic attractor). A key step in proving this fact is that the set of all possible itineraries of such a map is completely characterized by its kneading sequence. For a unimodal map f with the turning point c, restricted to an invariant interval

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$I = [f^2(c), f(c)] \subseteq [0,1]$, called the core of f, this characterization is as follows. A sequence $\overrightarrow{x} = (x_i)_{i \in \mathbb{N}_0}$ is an itinerary of some point $x \in I$ if and only if $\sigma^n(\overrightarrow{x}) \preceq \overrightarrow{k}(f)$ for every $n \in \mathbb{N}_0$, where $\overrightarrow{k}(f)$ is the kneading sequence of f, σ is the shift map, $\preceq$ is the parity-lexicographical ordering, and $\mathbb{N}_0$ is the set of all non-negative integers ($\mathbb{N}$ will be the set of positive integers). Besides this, the kneading theory plays a fundamental role in various applications, for example in proving monotonicity of the topological entropy for the family of skew tent maps, see [4].

On the other hand, no theory of comparable rigor for such 'good symbolic invariants' is currently available for general once-folding mappings of the plane such as the Hénon and the Lozi mappings. The obstruction to constructing a similar theory for such mappings are that they lack critical points in the usual sense (their Jacobians never vanish) and that their dynamical space lack a natural order which one-dimensional dynamics *a priori* have.

In this paper we study the family of the Lozi maps $L_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$L_{a,b} = (1 + y - a|x|, bx)$$

and their strange attractors $\Lambda_{a,b}$. We introduce the set of kneading sequences for the Lozi map and prove that the set of all itineraries of points in $\Lambda_{a,b}$ is completely characterized by the set of kneading sequences of $L_{a,b}$. This characterization for the Lozi maps has the same flavor as the Milnor – Thurston characterization for the unimodal maps mentioned above. The difference is that the Lozi map has countably many kneading sequences and one needs criteria when to use which sequence. We also introduce a *folding pattern* and a *folding tree*, which can replace the set of kneading sequences. They carry the same information as the set of kneading sequences, coded in a different way.

The paper is organized as follows. In Section 2 we summarize basic information about Lozi maps. In Sections 3 and 4 we define various notions used later in the paper. In Section 5 we introduce orders which can partially replace the natural orders on an interval and in the set of itineraries, that work so well for unimodal interval maps. In Section 6 we present the three equivalent approaches to coding the basic information about a Lozi map: the set of kneading sequences, the folding pattern, and the folding tree. In Section 7 we show how the set of kneading sequences (or the folding pattern, or the folding tree) determine the symbolic dynamics for a Lozi map.

2 Preliminaries

The family of piecewise affine mappings $L_{a,b} = (1 + y - a|x|, bx)$ of the plane into itself was given by Lozi in 1978 [2]. The results of his numerical investigations for the values of parameters $a = 1.7$ and $b = 0.5$ suggested the existence of a strange attractor. Figure 1 shows the strange attractor for Lozi's original choice of parameters.

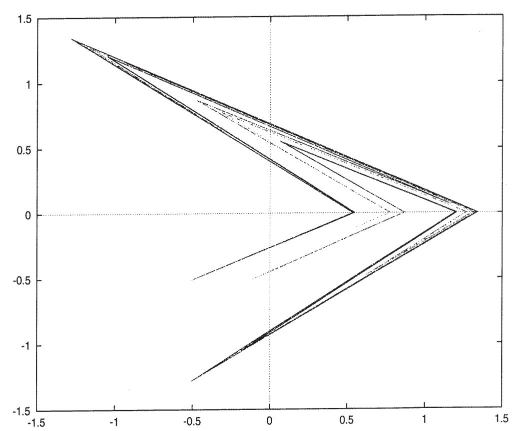

Figure 1: The Lozi attractor for $a = 1.7$ and $b = 0.5$.

A mathematical justification for the existence of strange attractors of the Lozi maps was given by the first author in 1980 [3]. It was proved there that the Lozi mappings have strange attractors for $(a, b) \in S$, where the set S is shown in Figure 2 (the figure is a copy of a figure in [3]) and is given by the

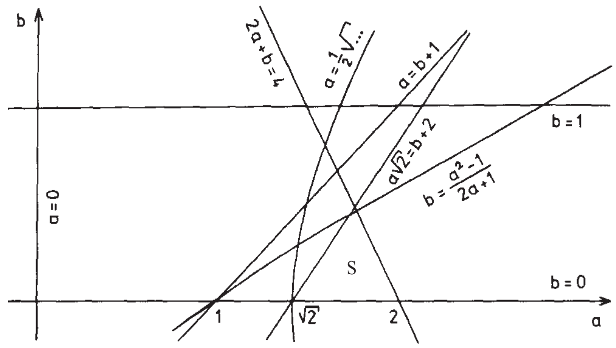

Figure 2: The set S.

following inequalities: $b > 0$, $a\sqrt{2} > b+2$, $b < \frac{a^2-1}{2a+1}$, $2a+b < 4$.[1]

Let $(a,b) \in S$ and, for simplicity, denote $L := L_{a,b}$. The map L is a homeomorphism which linearly maps the left half-plane onto the lower one and the right one onto the upper one. There are two fixed points: $X = (\frac{1}{1+a-b}, \frac{b}{1+a-b})$ in the first quadrant and $Y = (-\frac{1}{a+b-1}, -\frac{b}{a+b-1})$ in the third quadrant. They are hyperbolic. Note that the Lozi map L is not everywhere differentiable, and therefore its hyperbolic structure can be understood only as the existence of a hyperbolic splitting at those points at which it may exist (for which the derivative exists at the whole trajectory). Recall, the *stable*, respectively *unstable, manifold* of a fixed point P (or a periodic point in general), W^s_P, respectively W^u_P, is an invariant curve which emanates from P,

$$W^s_P = \{T : L^n(T) \to P \text{ as } n \to \infty\}, W^u_P = \{T : L^{-n}(T) \to P \text{ as } n \to \infty\}.$$

For the Lozi map L the stable and unstable manifolds are broken lines, and therefore not the differentiable manifolds. The half of the unstable manifold

[1]In several figures we use values of a and b that are not in this set, in order to get a better picture.

W_X^u of the fixed point X that starts to the right intersects the horizontal axis for the first time at the point $Z = (\frac{2+a+\sqrt{a^2+4b}}{2(1+a-b)}, 0)$. Let us consider the triangle Δ with vertices Z, $L(Z)$ and $L^2(Z)$, see Figure 3. In the mentioned

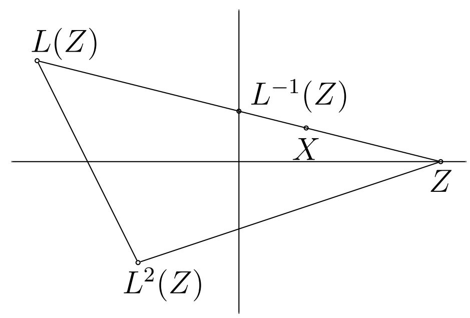

Figure 3: The triangle Δ.

paper [3] it was proved that $L(\Delta) \subset \Delta$ and moreover, that

$$\Lambda = \bigcap_{n=0}^{\infty} L^n(\Delta)$$

is the strange attractor. Moreover, $L|_\Lambda$ is topologically mixing, and Λ is the closure of W_X^u. Recall that an *attractor* is the set that is equal to the intersection of images of some its neighborhood, and such that the mapping restricted to this set is topologically transitive. An attractor is called *strange* if it has a fractal structure.

We code the points of Λ in the following standard way. To a point $P = (P_x, P_y) \in \Lambda$ we assign a bi-infinite sequence $\overline{p} = \ldots p_{-2}\, p_{-1} \raisebox{0.3ex}{\tiny$\blacksquare$}\, p_0\, p_1\, p_2 \ldots$ such that

$p_n = -1$ if $P_x^n \leq 0$ and

$p_n = +1$ if $P_x^n \geq 0$, where $L^n(P) = (P_x^n, P_y^n)$.

The dot shows where the 0th coordinate is. Moreover, to simplify notation, we use just symbols $+$ and $-$ instead of $+1$ and -1.

A bi-infinite symbol sequence $\overline{q} = \ldots q_{-2}\, q_{-1} \raisebox{0.3ex}{\tiny$\blacksquare$}\, q_0\, q_1\, q_2 \ldots$ is called *admissible* if there is a point $Q \in \Lambda$ such that $\overline{q}$ is assigned to Q. We will call this sequence an *itinerary* of Q. Obviously, some points of Λ have more than one itinerary. We denote the set of all admissible sequences by Σ_Λ. It is a metrizable topological space with the usual product topology. Since the half-planes that we use for coding, intersected with Λ, are compact, the space Σ_Λ is compact. From the hyperbolicity of L it follows that for every admissible sequence $\overline{q}$ there exists only one point $Q \in \Lambda$ with this itinerary. The detailed proof can be extracted from the paper of Ishii [1]. Because of this uniqueness, we have a map $\pi : \Sigma_\Lambda \to \Lambda$, such that $\overline{q}$ is an itinerary of $\pi(\overline{q})$. Clearly, $\pi \circ L = \sigma \circ \pi$.

In fact, Ishii was identifying the itineraries of the same point and he proved that the shift map in the quotient space is conjugate (via the map induced by π) with L on Λ. In our setup, this just means that π is continuous, and is a semiconjugacy between (Σ_Λ, σ) and (Λ, L).

3 Definitions

Let us consider the unstable manifold of the fixed point X, W^u_X. It is an image of the real line under a map which is continuous and one-to-one. For simplicity, we denote it by $R := W^u_X$. We denote by R^+ the half of R that starts at the fixed point X and goes to the right and intersects the horizontal axis for the first time at the point Z. By R^- we denote the other half of R that also starts at the fixed point X and goes to the left and intersects the vertical axis for the first time at the point $L^{-1}(Z)$. Let $[A, B] \subset R$ denote an arc of R with boundary points A and B, and let $(A, B) = [A, B] \setminus \{A, B\}$. For a point $P \in R$ and $\epsilon > 0$ let

$$(P - \epsilon, P + \epsilon) = \{Q \in R : d(P, Q) < \epsilon\} \subset R,$$

where $d(P, Q)$ denotes length of the arc $[P, Q]$.

We introduce an ordering $\lhd$ on R in the following natural way: For $P, P' \in R^+$ we say that

$$P \lhd P' \quad \text{if} \quad [X, P] \subset [X, P'].$$

For $P, P' \in R^-$ we say that

$$P \triangleleft P' \text{ if } [X, P] \supset [X, P'].$$

Also, if $P \in R^-$ and $P' \in R^+$, we say that $P \triangleleft P'$. Note that $L(R^+) = R^-$ and $L(R^-) = R^+$.

Gluing points. We call a point $G = (G_x, G_y) \in R$ a *gluing point* if $G_x = 0$ and there is no $k \in \mathbb{N}$ such that $G_x^{-k} = 0$. Let us denote the set of all gluing points by $\mathcal{G}$.

Turning points. We call a point $T = (T_x, T_y) \in R$ a *turning point* if $T = L(G)$ for some $G \in \mathcal{G}$. In this case $T_y = 0$. Let us denote the set of all turning points by $\mathcal{T}$. Let us denote by $\widehat{\mathcal{T}} := \{L^j(T) : T \in \mathcal{T}, j \in \mathbb{N}\}$ the set of *postturning points*.

Basic points. Let

$$\mathcal{E} = \mathcal{G} \cup \mathcal{T} \cup \widehat{\mathcal{T}} = \{L^j(G) : G \in \mathcal{G}, j \in \mathbb{N}_0\},$$

We call the points of $\mathcal{E}$ the *basic points*.

We can think of R in two ways. The first one is R as a subset of the plane. The second way is to "straighten" it and consider it the real line. Our order on R is the natural order when we use the second way. Note that the topology in R is different in both cases.

Lemma 1. *The set $\mathcal{E}$ is discrete, and therefore countable.*

Proof. Note that $L(Z) \triangleleft L^{-1}(Z) \triangleleft Z$ are three consecutive basic points, where $L^{-1}(Z)$ is a gluing point and Z is a turning point. Note also that $L|_R$ is expanding and $R = \bigcup_{i=0}^{\infty} L^i([L(Z), Z])$. If P and Q are two consecutive points of $\mathcal{E}$, then between $L(P)$ and $L(Q)$ there is at most one point of $\mathcal{E}$. Therefore by induction we see that in $L^i([L(Z), Z])$ there are only finitely many points of $\mathcal{E}$. This proves that $\mathcal{E}$ is discrete. $\square$

The set of all gluing points is discrete. Let

$$\mathcal{G} = \{G_n, n \in \mathbb{Z}\},$$

where $G_0 = L^{-1}(Z)$ and $G_n \lhd G_{n+1}$ for every $n \in \mathbb{Z}$. The set of all turning points is also discrete. Let

$$\mathcal{T} = \{T_n, n \in \mathbb{Z}\},$$

where $T_0 = Z$ and $T_n \lhd T_{n+1}$ for every $n \in \mathbb{Z}$. We have $L(G_n) = T_{-n}$ for every $n \in \mathbb{Z}$. Let

$$\mathcal{E} = \{E_n, n \in \mathbb{Z}\},$$

where $E_0 = G_0$ and $E_i \lhd E_{i+1}$ for every $i \in \mathbb{Z}$. Note that $E_1 = T_0$, $E_2 = G_1$, $E_{-1} = L(T_0)$, and $E_{-2} = G_{-1}$.

We call the arcs between two consecutive basic points $[E_i, E_{i+1}]$, $i \in \mathbb{Z}$, the *basic arcs*.

We denote the jth image of the kth turning point by $T_k^{j+1} := L^j(T_k)$, $j \in \mathbb{N}_0$ (note that $T_k^1 = T_k$). Also, we denote the x- and y-coordinates of the basic points as follows: $G_k = (G_{k,x}, G_{k,y})$, $T_k = (T_{k,x}, T_{k,y})$, $T_k^j = (T_{k,x}^j, T_{k,y}^j)$.

Let us now consider itineraries of points of R. Let Σ_R denote the set of all itineraries of all points of R. For a sequence $\overline{p} = (p_i)_{i\in\mathbb{Z}} \in \Sigma_R$ and $n \in \mathbb{Z}$, we will call the left-infinite sequence $\overleftarrow{p_n} = \dots p_{n-2}\, p_{n-1}\, p_n$ a *left tail* of $\overline{p}$ and the right-infinite sequence $\overrightarrow{p_n} = p_n\, p_{n+1}\, p_{n+2} \dots$ a *right tail* of $\overline{p}$. We will call a finite sequence $W = w_1 \dots w_k$ a *word* and denote its length by $|W|$, $|W| = k$. We will denote an infinite to the right (respectively, left) sequence of $+$s by $+^\infty$ (respectively, ${}^\infty{+}$).

The itinerary of the fixed point X (which is in the first quadrant) is $\overline{x} = {}^\infty{+}\cdot{+}^\infty$. Since R is the unstable manifold of X, for every point $P \in R$ and its itinerary $\overline{p}$, there is $n \in \mathbb{Z}$ such that $\overleftarrow{p_n} = {}^\infty{+}$. Therefore, there exists the smallest integer $k > n$ such that $P_x^k = 0$ and R crosses the y-axis at $L^k(P)$. Also $P_y^{k+1} = 0$, $P_x^{k+1} > 0$ and there exists $\delta > 0$ such that for all points $Q \in (P - \delta, P + \delta)$ we have $0 < Q_x^{k+1} \le P_x^{k+1}$. In other words, R makes a turn at the point $L^{k+1}(P)$. Moreover, if there exists also $l > k$ such that $P_x^l = 0$, then R does not cross y-axis at $L^l(P)$, but also makes a turn at $L^l(P)$. In other words, there exists $\epsilon > 0$ such that for every point

$Q \in (P-\epsilon, P+\epsilon)$, $Q \neq P$, and its itinerary $\overline{q}$, we have either $Q_x^l < P_x^l$ and hence $q_l = -$, or $Q_x^l > P_x^l$ and hence $q_l = +$. Therefore, instead of considering two itineraries of P, we consider only one, with $p_l = -$ in the first case and $p_l = +$ in the second case. This amounts to removing from Σ_R isolated points. The remaining part of Σ_R will be denoted Σ_R^e. In such a way every point $P \in R$ has at most two itineraries, and if there are two of them, then they differ at one coordinate k, and then $L^k(P)$ is a gluing point. In such a case we will sometimes write $p_k = \pm$.

The set Σ_R^e has a very useful property.

Lemma 2. *Assume that $\overline{p} \in \Sigma_R^e$ and $n \in \mathbb{N}$. Then there is $\overline{q} \in \Sigma_R^e$ such that $p_{-n}, \dots p_n = q_{-n} \dots q_n$ and $\overline{q}$ is the only itinerary of $\pi(\overline{q})$.*

Proof. Since the map L is linear on the left and right half-planes, the set of points of Δ that have an itinerary $\overline{q}$ such that $p_{-n}, \dots p_n = q_{-n} \dots q_n$ is a closed convex polygon, perhaps degenerate. Therefore, in a neighborhood of $P = \pi(\overline{p})$ in R (in the topology of the real line), its intersection with R is a closed interval J, perhaps degenerate to a point. However, if J is only one point, then the itinerary $\overline{p}$ is isolated and hence one of the removed ones, that is, it does not belong to Σ_R^e. Therefore, J is a nondegenerate closed interval.

The set R intersects the y-axis only at countable number of points. Therefore the set of points $Q \in R$ such that $L^k(Q)$ belongs to the y-axis for some $k \in \mathbb{Z}$, is countable. Thus, there are points $Q \in J$ such that for every $k \in \mathbb{Z}$ the point $L^k(Q)$ does not belong to the y-axis. Such Q has only one itinerary. This completes the proof. $\square$

4 Basic arcs and coding

Recall that we call the arcs between two consecutive basic points $[E_i, E_{i+1}]$, $i \in \mathbb{Z}$, the *basic arcs*.

Let $P \in R$ be a point and let $\overline{p}$ be its itinerary. Note that

$$P \in (G_0, T_0) = (E_0, E_1) \;\Rightarrow\; \overleftarrow{p}_0 = {}^{\infty}+.$$

Since always $T_{0,x} > 0$ and $T^2_{0,x} < 0$ (that follows easily from the assumptions on a and b), we have $L([E_0, E_1]) \ni G_0$. This implies that

$$L([E_0, E_1]) = [T^2_0, G_0] \cup [G_0, T_0] = [E_{-1}, E_0] \cup [E_0, E_1]$$

and

$$P \in (E_{-1}, E_0) \;\Rightarrow\; \overleftarrow{p}_0 = {}^{\infty}+ - .$$

Consider now $L([T^2_0, G_0]) = [T_0, T^3_0]$. If $T^3_{0,x} \geq 0$ then $[T_0, T^3_0]$ does not contain any gluing point (both boundary points are in the right half-plane) and hence $[T_0, T^3_0]$ is a basic arc

$$L([E_{-1}, E_0]) = [E_1, E_2] \subset R^+$$

and

$$P \in (E_1, E_2) \;\Rightarrow\; \overleftarrow{p}_0 = {}^{\infty}+ - +.$$

If $T^3_{0,x} < 0$ then $G_1 \in [T_0, T^3_0]$ and

$$L([E_{-1}, E_0]) = [T_0, G_1] \cup [G_1, T^3_0] = [E_1, E_2] \cup [E_2, E_3] \subset R^+.$$

Hence

$$P \in (E_1, E_2) \;\Rightarrow\; \overleftarrow{p}_0 = {}^{\infty}+ - +,$$

$$P \in (E_2, E_3) \;\Rightarrow\; \overleftarrow{p}_0 = {}^{\infty}+ - -$$

(see Figure 4).

If in addition $T^4_{0,x} < 0$ (and $T_{-1,x} > 0$, which holds for all $(a, b) \in S$, implying $T_{i,x} > 0$ for all $i \in \mathbb{Z}$), we have

$$L^2([E_{-1}, E_0]) = L([T_0, G_1]) \cup L([G_1, T^3_0]) = [T^2_0, T_{-1}] \cup [T_{-1}, T^4_0] \subset R^-.$$

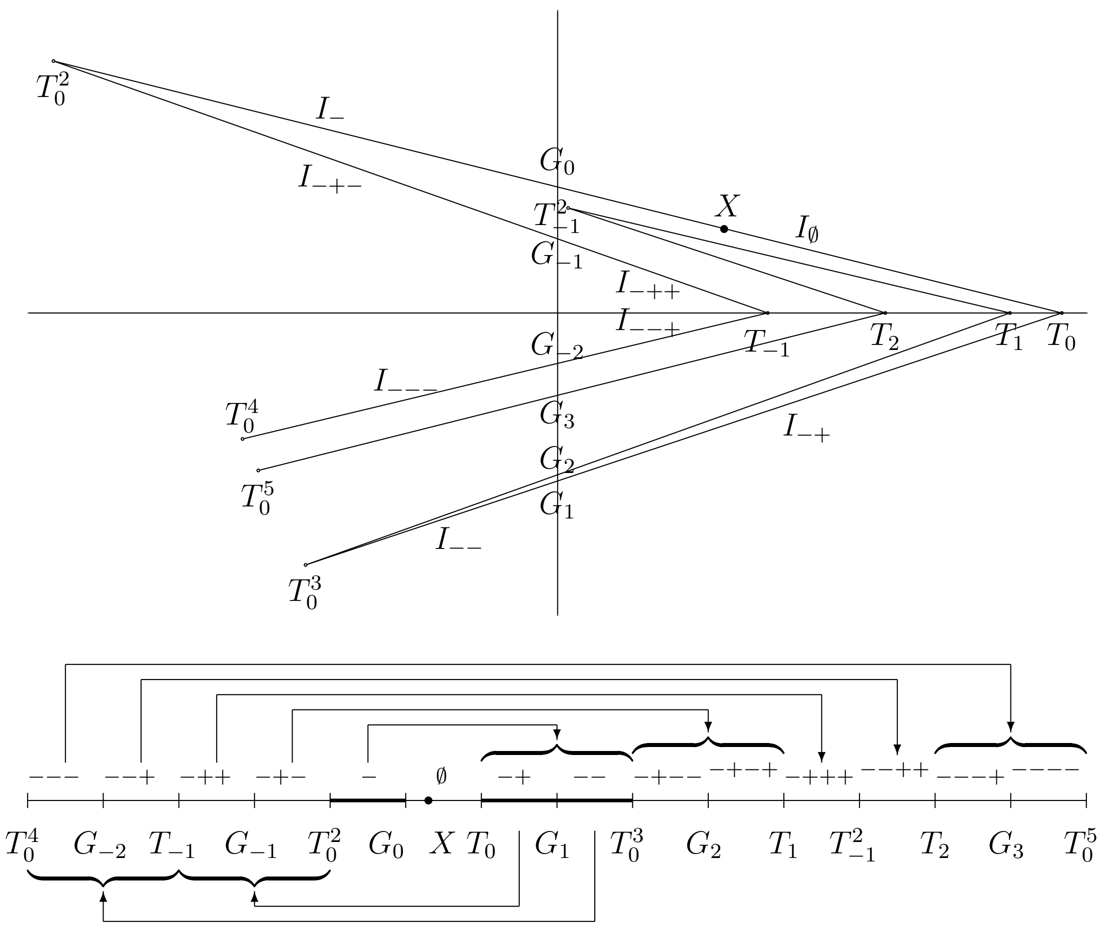

Figure 4: Basic points and basic arcs for $a = 1.75$ and $b = 0.5$.

Since T_0^2 and T_0^4 are in the left half-plane and T_{-1} is in the right half-plane, we have $[T_0^2, T_{-1}] \ni G_{-1}$ and $[T_{-1}, T_0^4] \ni G_{-2}$, implying

$$\begin{aligned} L^2([E_{-1}, E_0]) &= [T_0^2, G_{-1}] \cup [G_{-1}, T_{-1}] \cup [T_{-1}, G_{-2}] \cup [G_{-2}, T_0^4] \\ &= [E_{-1}, E_{-2}] \cup [E_{-2}, E_{-3}] \cup [E_{-3}, E_{-4}] \cup [E_{-4}, E_{-5}] \subset R^-. \end{aligned}$$

Hence

$$P \in (E_{-1}, E_{-2}) \;\Rightarrow\; \overleftarrow{p_0} = {}^{\infty}\!+ - + -,$$

$$P \in (E_{-2}, E_{-3}) \;\Rightarrow\; \overleftarrow{p_0} = {}^{\infty}\!+ - + +,$$

$$P \in (E_{-3}, E_{-4}) \;\Rightarrow\; \overleftarrow{p_0} = {}^{\infty}\!+ - - +,$$
$$P \in (E_{-4}, E_{-5}) \;\Rightarrow\; \overleftarrow{p_0} = {}^{\infty}\!+ - - -$$

(see Figure 4).

Continuing this procedure, we can code basic arcs $[E_i, E_{i+1}]$, $i \in \mathbb{Z}$, with finite words of $-$s and $+$s in the following way. Since the map L on R is expanding, for every basic arc J as above there exists the smallest n such that $L^{-n-1}(J) \subset [G_0, T_0]$. For points $P \in J$ we have then $\overleftarrow{p_0} = {}^{\infty}\!+a_{-n} \dots a_{-1}a_0 = {}^{\infty}\!+\alpha$. In this case we will use the following notation: $I_\alpha := [E_i, E_{i+1}]$. In particular, $[E_0, E_1] = I_\emptyset$.

Observe that for every $m \in \mathbb{N}, m \geq 1$, all basic arcs of $L^{m-1}(I_-)$ are coded by words of length m. Moreover, if m is even, $L^{m-1}(I_-) \subset R^+$, and if m is odd, $L^{m-1}(I_-) \subset R^-$.

5 Orders

Let us look at the points of R and their itineraries. On R (when we think about it as the real line) we have the natural order $\triangleleft$. We want to introduce a corresponding order in the itineraries. Since the situation is similar as for unimodal interval maps, we start with the usual *parity-lexicographical order*. Let $\overrightarrow{p} = p_0p_1\dots$, $\overrightarrow{q} = q_0q_1\dots$ be two different right-infinite sequences or finite words. Let $k \in \mathbb{N}_0$ be the smallest integer such that $p_k \neq q_k$. Then $\overrightarrow{p} \prec \overrightarrow{q}$ if either $p_0 \dots p_{k-1}$ is even (contains an even number of $+$s) and $p_k < q_k$, or $p_0 \dots p_{k-1}$ is odd (contains an odd number of $+$s) and $q_k < p_k$. Here, if $k = 0$, then $p_0 \dots p_{k-1}$ is the empty word, and $-1 < +1$, that is $- \prec +$ (if $p_k = \pm$, or $q_k = \pm$, then by convention $- \prec \pm \prec +$). (We also allow that $\overrightarrow{p} = p_0p_1 \dots p_n$ is a finite word and $\overrightarrow{q} = q_0q_1\dots$ is a right-infinite sequence, or vice versa, and in this case we say that $\overrightarrow{p} \prec \overrightarrow{q}$ if $p_0p_1\dots p_n \prec q_0q_1\dots q_n$.) While this does not work if the lengths of $\overrightarrow{p}$ and $\overrightarrow{q}$ are different and one of them is the beginning of the other one, we will not encounter such situations.

When we want to define some reasonable order in Λ, we have to use similar ideas as in the relativity theory (in the space-time). Recall that we have a forward invariant unstable cone of directions. (see [3]). We will call two distinct points $P, Q \in \Lambda$ *comparable* if the direction of the straight line containing P and Q belongs to this cone.

Lemma 3. *Assume that $P, Q \in \Lambda$ are comparable. Then $\overrightarrow{p_0} \prec \overrightarrow{q_0}$ if and only if the x-coordinate of P is smaller than the x-coordinate of Q.*

Proof. The map L expands distances on straight lines whose direction is in the unstable cone. The expansion factor is at least a constant larger than 1 dependent only of a and b. Moreover, the unstable cone is mapped to itself by the derivative of L and Λ is bounded (see [3]). Therefore, there is $n \in N_0$ such that L^n is linear on the straight line segment with endpoints P, Q and one of the points $L^n(P)$, $L^n(Q)$ lies in the left half-plane, while the other one lies in the right half-plane. The smallest such n is exactly the smallest $n \geq 0$ for which $p_n \neq q_n$. For each i between 0 and $n-1$ the order in the x-direction between the points $L^{i+1}(P)$ and $L^{i+1}(Q)$ is opposite to the analogous order between $L^i(P)$ and $L^i(Q)$ if $p_i = +$, and the same if $p_i = -$. Comparing this with the definition of the parity-lexicographical order gives the result described in the lemma. $\square$

Note that the direction of the local unstable manifold of X is in the unstable cone, so by the invariance of the unstable cone we get that the direction of every straight line segment contained in R is contained in the invariant cone. In particular, we get immediately the following result.

Lemma 4. *Assume that $P, Q \in [G_0, T_0]$ and $P \neq Q$. Then $\overrightarrow{p_0} \prec \overrightarrow{q_0}$ if and only if $P \lhd Q$.*

This allows us to relate the orders $\prec$ and $\lhd$.

Observe that a point $P \in R$ belongs to $[G_0, T_0]$ if and only if $\overleftarrow{p_0} = {}^{\infty}+$. Let us define the *generalized parity-lexicographical order* on the set Σ_R in the following way. Let $\overline{p}, \overline{q} \in \Sigma_R$, $\overline{p} \neq \overline{q}$. Let $n \in \mathbb{N}$ be a positive integer such that $\overleftarrow{p}_{-n} = \overleftarrow{q}_{-n} = {}^{\infty}{+}$. Then $\overline{p} \prec \overline{q}$ if either

1. n is even and $\overrightarrow{q}_{-n+1} \prec \overrightarrow{p}_{-n+1}$, or
2. n is odd and $\overrightarrow{p}_{-n+1} \prec \overrightarrow{q}_{-n+1}$.

By the definition of the parity-lexicographical order and since L reverses orientation on $[G_0, T_0]$, this order is well defined (it does not depend on the choice of n).

From this definition and Lemma 4 we get immediately the following result.

Lemma 5. *Let $P, Q \in R$ be two different points and let $\overline{p}, \overline{q}$ be their itineraries. Then $P \triangleleft Q$ if and only if $\overline{p} \prec \overline{q}$.*

Another straightforward consequence of Lemma 3 is the following lemma. It compares right tails of itineraries of points of R to the corresponding right tails of itineraries of the turning points.

Lemma 6. *Assume that $P, Q \in R$, the arc $[P, Q]$ is a straight line segment (as a subset of Λ), and Q is a turning point. Then $\overrightarrow{p}_0 \prec \overrightarrow{q}_0$.*

Arc-codes. We call a word α an *arc-code* if there exists a basic arc $[E_j, E_{j+1}]$ such that $[E_j, E_{j+1}] = I_\alpha$. Note that, by definition, every arc-code of length ≥ 1 starts with $-$. Also, in this case, if I_α is a basic arc and $|\alpha|$ is even (respectively odd), then $I_\alpha \subset R^+$ (respectively $I_\alpha \subset R^-$).

Lemma 7. *Let α, β be two different arc-codes and let I_α, I_β be the corresponding basic arcs. If α and β have different lengths, but $|\alpha|$ and $|\beta|$ have the same parity, then $|\alpha| > |\beta|$ if and only if the basic arc I_α is farther from X then the basic arc I_β $(d(I_\alpha, X) > d(I_\beta, X))$. If α and β have the same length, then $\alpha \prec \beta$ if and only if $d(I_\alpha, X) > d(I_\beta, X)$.*

Proof. If α and β have different lengths, but $|\alpha|$ and $|\beta|$ have the same parity, then we take n of the same parity as $|\alpha|$ and $|\beta|$ and such that $L^{-n}(I_\alpha)$ and

$L^{-n}(I_\beta)$ are contained in $[G_0, T_0]$. Choose $P \in L^{-n}(I_\alpha)$ and $Q \in L^{-n}(I_\beta)$. Compare $\overrightarrow{p}_0$ with $\overrightarrow{q}_0$. By the parity assumptions, they both start with the odd number of +s. If $|\alpha| > |\beta|$ then $\overrightarrow{q}_0$ starts with more +s, so $\overrightarrow{q}_0 \prec \overrightarrow{p}_0$, and therefore $Q \lhd P$. This means that I_β is closer to X than I_α.

If α and β have the same length, we make the same construction. Then $\alpha \prec \beta$ is equivalent to $\overrightarrow{q}_0 \prec \overrightarrow{p}_0$ (remember of the odd number of +s in front), and, as before, I_β is closer to X than I_α. $\square$

6 Three approaches

In this section we will present three approaches to coding the main information about the unstable manifold R of X, foldings of R and the dynamics of L on R. Those approaches will be via kneading sequences, the folding pattern, and the folding tree.

Kneading sequences. For each $n \in \mathbb{Z}$ the itinerary $\overline{k}^n$ of the nth turning point T_n is a *kneading sequence*. Let

$$\mathcal{K} := \{\overline{k}^n : n \in \mathbb{Z}\}$$

be the set of all kneading sequences of L. Similarly as for interval maps, $\mathcal{K}$ contains the information about the basic properties of L. Sometimes we will call $\mathcal{K}$ the *kneading set*.

Strictly speaking, a turning point T_n has two itineraries. They are of the form ${}^\infty{+}\alpha^n \pm \cdot \overrightarrow{k}_0^n$, where α^n is the arc-code of the basic arc containing $L^{-2}(T_n)$. Here for $\pm$ you can substitute any of $+$ and $-$. Therefore we can think of this kneading sequence as a pair $(\alpha^n, \overrightarrow{k}_0^n)$.

While $\mathcal{K}$ is only a set, we can recover the order in it by looking at the arc-codes parts of the kneading sequences. Moreover, $\overline{k}^0$ is the only kneading sequence with the arc-code part empty. Thus, given an element $\overline{k}$ of $\mathcal{K}$ we can determine n such that $\overline{k} = \overline{k}^n$.

Folding pattern. Write the sequence

$$(\dots, E_{-3}, E_{-2}, E_{-1}, E_0, E_1, E_2, E_3, \dots),$$

replacing each E_i by the symbol G if E_i is a gluing point and T if E_i is a turning or postturning point. Add additionally the symbol X between E_0 and E_1 (that is, where it belongs). We get a sequence like

$$(\dots, T, G, T, G, T, G, X, T, G, T, G, T, T, T, G, T, \dots).$$

This is the *folding pattern* of L.

The folding pattern carries some additional information, that we can make visible. We know that L restricted to R is an orientation reversing homeomorphism that fixes X. Moreover, it maps the set of basic points bijectively onto the set of turning and postturning points. Thus, we know which symbol of the folding pattern is mapped to which one (see Figure 5).

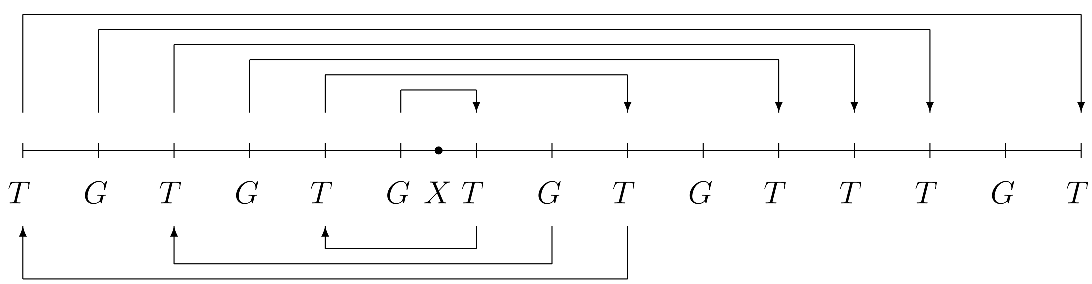

Figure 5: The action of the map on the folding pattern.

We know how to number the gluing points (the first to the left of X is G_0). This, plus the information about the action of the map, tells us which turning or postturning point is corresponding to a given symbol T. Thus, we get Gs and Ts with subscripts and (some of them) superscripts, like in Figure 4.

Another piece of information that we can read off the folding pattern is which turning and postturning points and which basic arcs are in the left or right half-plane. Namely, we know that the sign (which we use for the itineraries) changes at every symbol G. Thus, we can append our folding pattern with those signs and get a sequence like this:

$$\dots - T - G + T + G - T - G + X + T + G - T - G + T + T + T + G - T - \dots .$$

For each symbol T the signs adjacent to it from the left and right are the same, so we can say that this is the sign of this T. Note that it may happen that the corresponding postturning point is actually on the y-axis, but still it has a definite sign.

Of course, we can put some of the additional information together, for instance we can add to the folding pattern both the map and the signs (see Figure 6).

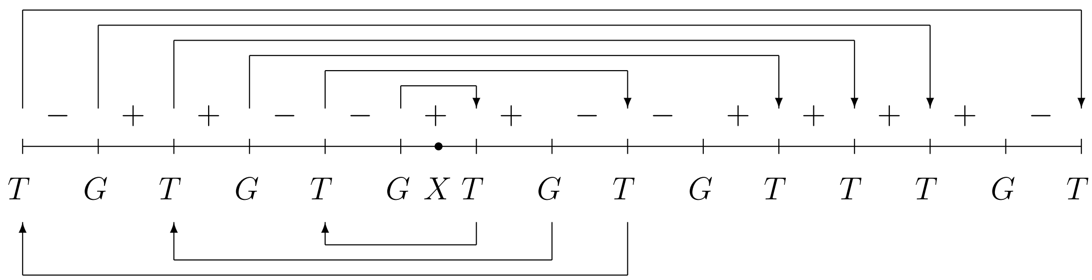

Figure 6: The action of the map on the folding pattern with signs.

Folding tree. We can think of the folding pattern as a countable Markov partition for the map L on R. Thus, we can consider the corresponding Markov graph (the graph of transitions). The vertices of this graph are the intervals $[E_i, E_{i+1}]$ (we write just the corresponding number i for them) and there is an arrow from i to j if and only if $L([E_i, E_{i+1}]) \supset [E_j, E_{j+1}]$. From the folding pattern shown in Figure 5 we get the graph shown in Figure 7 (of course this tree goes down and is infinite; we are showing only a part of it). This graph is almost a tree, so we will call it the *folding tree* of L. Except of 0 and the arrows beginning at 0, it is a subtree of the full binary tree.

This tree is in a natural way divided into *levels*. The number 0 is at level 0, the number -1 is at level 1, and in general, if the path from -1 to i has n arrows then i is at level $n+1$. It is easy to see how the levels are arranged. Starting with level 1, negative numbers are at odd levels, ordered with their moduli increasing from the left to the right. If level n ends with $-i$ then level $n+2$ starts with $-(i+1)$. Positive numbers are at even levels, ordered in a

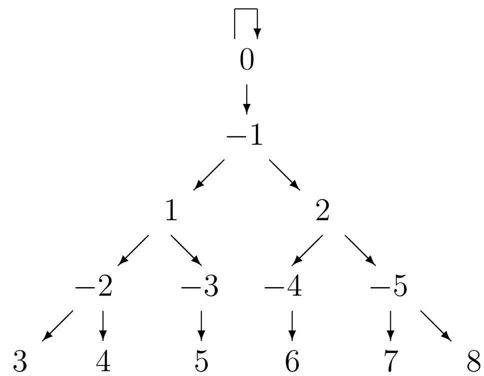

Figure 7: A folding tree with numbers of basic arcs.

similar way. Therefore, if we have the same tree without the numbers, like in Figure 8, we know where to put which number. Of course, we are talking about the tree embedded in the plane, so the order of the vertices at each level is given.

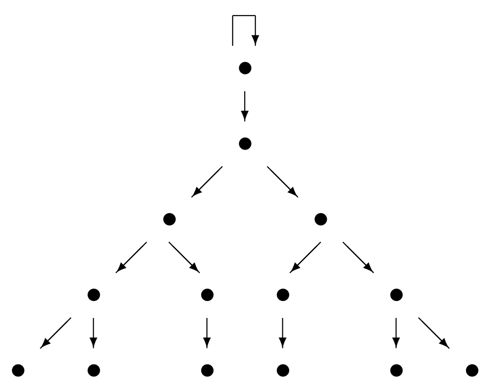

Figure 8: A "naked" folding tree.

In a similar way as for the folding pattern, we can add some information to the picture. The symbols G and T can be placed between the vertices of the tree. The ones that are between the last vertex of level n and the first vertex of level $n+2$, will be placed to the right of the last vertex of level n. The only exception is G_0, which has to be placed to the left of the unique vertex of level 1, in order to avoid a collision with other symbols.

We know which of the symbols are Gs. By our construction, Gs are those elements of $\mathcal{E}$ that are in the interior of some $L([E_i, E_{i+1}])$. This means that they are exactly the ones which are between the siblings (vertices where the arrows from the common vertex end). And once we have Gs and Ts marked, we can recover the signs of the vertices, because we know that the signs change exactly at Gs. Then we get the folding tree marked as in Figure 9.

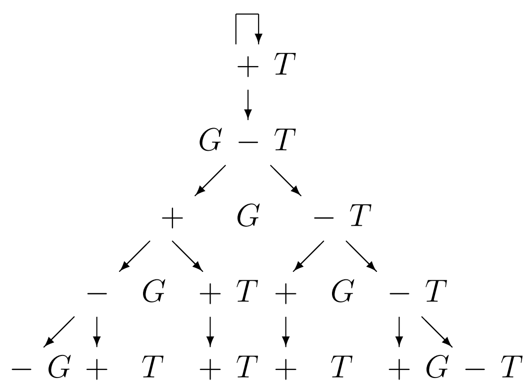

Figure 9: A folding tree with Gs, Ts and signs.

Now that we have our three objects, the kneading sequences, the folding pattern, and the folding tree, we can prove that they carry the same information.

Theorem 8. *The set of kneading sequences, the folding pattern, and the folding tree are equivalent. That is, given one of them, we can recover the other two.*

Proof. From the kneading set to the folding pattern. Suppose we know the set of the kneading sequences and we want to recover the folding pattern. As we noticed when we defined the kneading set, we know which kneading sequence is the itinerary of which point T_n. We proceed by induction. First, we know that in $[E_{-1}, E_1]$ there are three points of $\mathcal{E}$ (and X), and that they should be marked from the left to the right T, G, X, T. We also know how they are mapped by L. Now suppose that we know the points of $\mathcal{E}$ in $L^n([E_{-1}, E_1])$, how they are marked, and how they are mapped by L. Some of those points (on the left or on the right, depending on the parity of n) are not mapped to the points of this set. Then we map them to new points, remembering that $L(X) = X$ and that L is a homeomorphism of R reversing orientation. Those new points have to be marked as T, because L maps $\mathcal{E}$ onto $\mathcal{T} \cup \widehat{\mathcal{T}}$. Now we use our information about the kneading sequences. They are the itineraries of the first images of the points marked G, and since we know the action of L on $\mathcal{E} \cap L^n([E_{-1}, E_1]$, this determines the signs of all points marked T in the picture that we have at this moment. We know that the signs change at each point marked G, so we insert such a point between every pair of Ts with opposite signs (clearly, there cannot be two consecutive Gs). In such a way we get the points of $\mathcal{E}$ in $L^n([E_{-1}, E_1])$, and the information how they are marked, and how they are mapped by L. This completes the induction step.

From the folding pattern to the folding tree. This we described when we were defining the folding tree.

From the folding tree to the kneading set. As we observed, given a folding tree, we can add to it the information about the signs and the positions of the G and T symbols. The turning points are the symbols T placed directly below Gs, and additionally T_0 is the only symbol in the zeroth row. Now for every T which is a turning point, we go down along the tree, reading the signs immediately to the left of the symbols (see Figure 9). In such a way, we get the corresponding right tail of the kneading sequence (the signs do not change at Ts, so the sign of the vertex immediately to the left of a given T is the same as the sign of the x-coordinate of the corresponding turning or postturning point). Going back (up) is even simpler, since in two steps we get to a vertex and just go up the tree along the edges. □

7 Symbolic dynamics

When we want to investigate the symbolic system obtained from a Lozi map by taking the space of all itineraries and the shift on this space, the basic thing is to produce a tool for checking whether a given bi-infinite sequence is an itinerary of a point. Remember that we called such a sequence *admissible*. If a sequence is an itinerary of a point of R, we call it *R-admissible*.

Recall that when we considered the itineraries of points of R, we removed some of them, as non-essential, and we were left with the space Σ_R^e. We will call the elements of this space *essential R-admissible* sequences. From the definition it follows that this space is σ-invariant.

In the case of all admissible sequences the situation is more complicated. We do not know whether in order to get rid of the unnecessary, non-essential, sequences it is enough to remove isolated ones. Thus, we define the space Σ_Λ^e as the closure of Σ_R^e, and call the elements of Σ_Λ^e *essential admissible* sequences. As the closure of a σ-invariant space, the space Σ_Λ^e is also σ-invariant.

We have to show that the essential admissible sequences suffice for the symbolic description. We know this for R, that is, we know that $\pi(\Sigma_R^e) = R$, but the analogous property for Λ requires some simple topological consideration. We also want to show that the essential admissible sequences are really essential, that is, we cannot remove any of them from our symbolic system. Note that by the definition, Σ_Λ^e is compact.

Theorem 9. *We have $\pi(\Sigma_\Lambda^e) = \Lambda$, that is, every point of Λ has an itinerary that is an essential admissible sequence. Moreover, it is the minimal set with this property. That is, each compact subset Ξ of Σ_Λ such that $\pi(\Xi) = \Lambda$, contains Σ_Λ^e.*

Proof. Since the set Σ^e_Λ is compact, so is $\pi(\Sigma^e_\Lambda)$. It contains $\pi(\Sigma^e_R) = R$, which is dense in Λ, so it is equal to Λ.

Now suppose that $\Xi \subset \Sigma_\Lambda$ is a compact set such that $\pi(\Xi) = \Lambda$. The itineraries of all points of R which have only one itinerary have to belong to Ξ. By Lemma 2, the set of those itineraries is dense in Σ^e_R. Since Ξ is closed, we get $\Sigma^e_R \subset \Xi$, and then $\Sigma^e_\Lambda \subset \Xi$. $\square$

Now we go back to essential R-admissible sequences.

For a bi-infinite path in the folding tree (with signs of vertices), we will call the corresponding bi-infinite sequence of signs the *sign-path*.

Theorem 10. *Let $\overline{p}$ be a bi-infinite sequence of $+s$ and $-s$. Then $\overline{p}$ is essential R-admissible if and only if it is a sign-path in the folding tree.*

Proof. Assume that $\overline{p}$ is essential R-admissible. This means that there exists a point $P \in R$ with itinerary $\overline{p}$, and $\overline{p}$ is not isolated. For every $n \in \mathbb{Z}$ there is a basic arc B_n to which $L^n(P)$ belongs. Then there is an arrow in the folding tree from the vertex representing B_n to the vertex representing B_{n+1}, so $\overline{p}$ is a sign-path in the folding tree.

Now assume that $\overline{p}$ is a sign-path in the folding tree. If the corresponding bi-infinite sequence of vertices is $(B_n)_{n=-\infty}^{\infty}$, then the sequence

$$\left(\bigcap_{i=-n}^{n} L^{-i}(B_i) \right)_{n=0}^{\infty} = \left(\bigcap_{i=0}^{n} L^{-i}(B_i) \right)_{n=0}^{\infty}$$

of intervals of R is a nested sequence of compact sets, so there exists a point $P \in R$ such that $L^n(P) \in B_n$ for every $n \in \mathbb{Z}$. Thus, $\overline{p}$ is the itinerary of P and clearly it is not isolated, and this proves that $\overline{p}$ is essential R-admissible. $\square$

The above theorem shows that the folding tree of L determines the set of all essential R-admissible sequences. By Theorem 8, the same can be said if we replace the folding tree by the set of kneading sequences or by the folding pattern. However, in order to mimic the kneading theory for unimodal interval maps, we would like to have a more straightforward characterization of all essential R-admissible sequences by the kneading set.

First we have to remember that itineraries of all points of R start with ${}^{\infty}+$. Next thing that simplifies our task is that R is invariant for L, so the set of all essential R-admissible sequences is invariant for σ. This means that apart of the sequence ${}^{\infty}+\bullet+^{\infty}$ (which is R-admissible, because it is the itinerary of X), we only need a tool of checking essential R-admissibility of sequences of the form ${}^{\infty}+\bullet p_0p_1p_2\cdots = {}^{\infty}+\bullet\overrightarrow{p_0}$, such that $p_0 = -$.

Theorem 11. *A sequence ${}^{\infty}+\bullet\overrightarrow{p_0}$, such that $p_0 = -$, is essential R-admissible if and only if for every kneading sequence ${}^{\infty}+\alpha\pm\bullet\overrightarrow{k_0}$, such that $\alpha = p_0p_1\dots p_m$ for some m, we have $\sigma^{m+2}(\overrightarrow{p_0}) \preceq \overrightarrow{k_0}$.*

Proof. Assume first that a point $P \in R$ has the itinerary ${}^{\infty}+\bullet\overrightarrow{p_0}$, such that $p_0 = -$, a turning point Q has the itinerary ${}^{\infty}+\alpha\pm\bullet\overrightarrow{k_0}$, and $\alpha = p_0p_1\dots p_m$ for some m. If $L^{m+2}(P) = Q$, then $\sigma^{m+2}(\overrightarrow{p_0}) = \overrightarrow{k_0}$. If $L^{m+2}(P) \neq Q$, then the arc $[L^{m+2}(P), Q]$ is a straight line segment (as a subset of Λ), so by Lemma 6, $\sigma^{m+2}(\overrightarrow{p_0}) \prec \overrightarrow{k_0}$.

Assume now that a sequence ${}^{\infty}+\bullet\overrightarrow{p_0}$, such that $p_0 = -$, is given, and that for every kneading sequence ${}^{\infty}+\alpha\pm\bullet\overrightarrow{k_0}$, such that $\alpha = p_0p_1\dots p_m$ for some m, we have $\sigma^{m+2}(\overrightarrow{p_0}) \preceq \overrightarrow{k_0}$. Suppose that ${}^{\infty}+\bullet\overrightarrow{p_0}$ is not essential R-admissible. Then, by Theorem 10, it is not a sign-path in the folding tree. This means that when we go down the tree trying to find the corresponding sign-path, at a certain moment we get to a vertex from which we cannot move down to a vertex with the correct sign. That is, we found the part ${}^{\infty}+\bullet p_0p_1\dots p_n$ of a sign-path, but we cannot append it with p_{n+1}.

Denote by $\widehat{p}_{n+1}$ the sign opposite to p_{n+1}. Look at the basic arc $J := I_{p_0p_1\dots p_n\widehat{p}_{n+1}}$ as a subset of Λ. It is a straight line segment with endpoints that are turning or postturning points. Consider the endpoint which is closer to the y-axis. It is of the form $L^i(Q)$, where Q is a turning point and $i \in \mathbb{N}_0$.

The kneading sequence of Q is ${}^{\infty}\!+\alpha \pm \cdot \overrightarrow{k_0}$, with $\alpha = p_0 p_1 \dots p_m$, where $m = n - i - 1$. Thus, by the assumption, $\sigma^{n-i+1}(\overrightarrow{p_0}) \preceq \overrightarrow{k_0}$.

The point Q is a turning point, so it is the right endpoint of $L^{-i}(J)$. If the number of $+$s among $p_{n-i+1}, p_{n-i+2}, \dots, p_n$ is even, then $L^i(Q)$ is the right endpoint of J, so J is in the left half-plane. This means that $\widehat{p}_{n+1} = -$, so $p_{n+1} = +$. Both sequences $\sigma^{n-i+1}(\overrightarrow{p_0})$ and $\overrightarrow{k_0}$ start with $p_{n-i+1}, p_{n-i+2}, \dots, p_n$. Then in $\sigma^{n-i+1}(\overrightarrow{p_0})$ we have $p_{n+1} = +$, while in $\overrightarrow{k_0}$ we have $\widehat{p}_{n+1} = -$. But this means that $\overrightarrow{k_0} \prec \sigma^{n-i+1}(\overrightarrow{p_0})$, a contradiction.

Similarly, if the number of $+$s among $p_{n-i+1}, p_{n-i+2}, \dots, p_n$ is odd, then $L^i(Q)$ is the left endpoint of J, so J is in the right half-plane. This means that $\widehat{p}_{n+1} = +$, so $p_{n+1} = -$. Both sequences $\sigma^{n-i+1}(\overrightarrow{p_0})$ and $\overrightarrow{k_0}$ start with $p_{n-i+1}, p_{n-i+2}, \dots, p_n$. Then in $\sigma^{n-i+1}(\overrightarrow{p_0})$ we have $p_{n+1} = -$, while in $\overrightarrow{k_0}$ we have $\widehat{p}_{n+1} = +$. But this means that $\overrightarrow{k_0} \prec \sigma^{n-i+1}(\overrightarrow{p_0})$, a contradiction.

In both cases we got a contradiction, so ${}^{\infty}\!+\cdot\overrightarrow{p_0}$ has to be essential R-admissible. $\square$

Now, that we know which sequences are essential R-admissible, we know how to check whether a sequence $\overline{p}$ is essential admissible. Namely, since the topology in the symbolic space is the product topology, for each $n \in \mathbb{N}$ we check whether there is an essential R-admissible sequence $\overline{q}$ such that $p_{-n} \dots p_n = q_{-n} \dots q_n$. By Theorem 10 this means that for every n we have to check whether the finite sequence $p_{-n} \dots p_n$ is a finite sign-path in the folding tree.

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