Synthetic foundations of cevian geometry, III: The generalized orthocenter

Igor Minevich and Patrick Morton[1]

August 17, 2015

1 Introduction.

In Part II of this series of papers [mm2] we studied the conic $\mathcal{C}_P = ABCPQ$ and its center Z, where P is a point not on the extended sides of triangle ABC or its anticomplementary triangle, and $Q = K(\iota(P)) = K(P')$ is the complement of the isotomic conjugate of P (*isotomcomplement*[2], for short) with respect to ABC. When $P = Ge$ is the Gergonne point of ABC, $\mathcal{C}_P$ is the Feuerbach hyperbola. To prove this, we introduce in Section 2 a generalization of the orthocenter H which varies with the point P.

The generalized orthocenter H of P is defined to be the intersection of the lines through the vertices A, B, C, which are parallel, respectively, to the lines QD, QE, QF, where DEF is the cevian triangle of P with respect to ABC. We prove that the point H always lies on the conic $\mathcal{C}_P$, as does the corresponding generalized orthocenter H' for the point P' (Theorem 2.8). Thus, the cevian conic $\mathcal{C}_P$ lies on the nine points

$$A, B, C, P, Q, P', Q', H, H',$$

where $Q' = K(\iota(P')) = K(P)$.

In the first two parts [mm1] and [mm2] we used the affine maps $T_P, T_{P'}, \mathcal{S}_1 = T_P \circ T_{P'}$, and $\lambda = T_{P'} \circ T_P^{-1}$, where T_P is the unique affine map which takes

[1]The authors were partially supported by an Honors Research Fellowship from the IUPUI Honors Program.

[2]This is Grinberg's terminology.

ABC to DEF and $T_{P'}$ is defined similarly for the point P'. (See Theorems 2.1 and 3.4 of [mm2].) In Section 2 of this paper we prove the affine formula

$$H = K^{-1}T_{P'}^{-1}K(Q) \tag{1}$$

for the point H that we defined above and deduce that H and H' are related by $\eta(H) = H'$, where η is the affine reflection we made use of in Part II [mm2]. (See Theorem 2.4.) The point H is the anti-complement of the point

$$O = T_{P'}^{-1}K(Q), \tag{2}$$

which is a generalization of the circumcenter. Several facts from Part I [mm1], including the Quadrilateral Half-turn Theorem, allow us to give a completely synthetic proof of the affine formulas (1) and (2). We show that the circumscribed conic $\tilde{\mathcal{C}}_O$ of ABC whose center is O is the *nine-point conic* (with respect to the line at infinity l_∞) for the quadrangle formed by the point Q and the vertices of the anticevian triangle of Q (for ABC). Furthermore, the complement $K(\tilde{\mathcal{C}}_O)$ is the nine-point conic $\mathcal{N}_H$ of the quadrangle $ABCH$. When $P = Ge$ is the Gergonne point, $Q = I$ is the incenter, P' is the Nagel point, and (1) and (2) yield affine formulas for the orthocenter and circumcenter as functions of I.

In Section 3 we study the relationship between the nine-point conic $\mathcal{N}_H$, the circumconic $\tilde{\mathcal{C}}_O$, and the inconic $\mathcal{I}$, which is the conic tangent to the sides of ABC at the traces D, E, F of the point P. Its center is Q. (See [mm1, Theorem 3.9].) We also study the maps

$$\mathsf{M} = T_P \circ K^{-1} \circ T_{P'} \quad \text{and} \quad \Phi_P = T_P \circ K^{-1} \circ T_{P'} \circ K^{-1}.$$

We show that these maps are homotheties or translations whose centers are the respective points

$$S = OQ \cdot GV = OQ \cdot O'Q' \quad \text{and} \quad Z = GV \cdot T_P(GV),$$

and use these facts to prove the Generalized Feuerbach Theorem, that $\mathcal{N}_H$ and $\mathcal{I}$ are tangent to each other at the point Z. The proof boils down to the verification that Φ_P takes $\mathcal{N}_H$ to $\mathcal{I}$, leaving Z and its tangent line to $\mathcal{N}_H$ invariant. Thus, this proof continues the theme, begun in Part I [mm1], of characterizing important points as fixed points of explicitly given affine mappings.

When $\mathcal{N}_H$ is an ellipse, the fact that $\mathcal{N}_H$ and $\mathcal{I}$ are tangent could be proved by mapping $\tilde{\mathcal{C}}_O$ to the circumcircle and ABC to a triangle $A'B'C'$ inscribed in the same circumcircle; and then using the original Feuerbach theorem for $A'B'C'$. The proof we give does not assume Feuerbach's original theorem, and displays explicitly the intricate affine relationships holding between the various points, lines, and conics that arise in the proof. (See Figure 2 in Section 3.) It also applies when $\mathcal{N}_H$ is a parabola or a hyperbola, and when Z is infinite. (See Figures 3 and 4 in Section 3. Also see [mo], where a similar proof is used to prove Feuerbach's Theorem in general Hilbert planes.)

In Section 4 we determine synthetically the locus of points P for which the generalized orthocenter is a vertex of ABC. This turns out to be the union of three conics minus six points. (See Figure 5 in Section 4.) We also consider a special case in which the map M is a translation, so that the circumconic $\tilde{\mathcal{C}}_O$ and the inconic are congruent. (See Figures 6 and 7.)

The results of this paper, as well as those in Part IV, could be viewed as results relating to a generalized notion of perpendicularity. The inconic replaces the incircle, and the lines QD, QE, QF replace the perpendicular segments connecting the incenter to the points of contact of the incircle with the sides of ABC. In this way we obtain the generalized incenter Q, generalized orthocenter H, generalized circumcenter O, generalized nine-point center N, etc., all of which vary with the point P. Using the polarity induced by the inconic, we have an involution of conjugate points on the line at infinity, unless P lies on the Steiner circumellipse. Now Coxeter's well-known development (see [co1], Chapter 9 and [bach]) of Euclidean geometry from affine geometry makes use of an elliptic involution on the line at infinity (i.e., a projectivity ψ from l_∞ to itself without fixed points, such that $\psi(\psi(X)) = X$ for all X). This involution is just the involution of perpendicularity: the direction perpendicular to the direction represented by the point X at infinity is represented by $\psi(X)$.

Our development is, however, not equivalent to Coxeter's. If P lies inside the Steiner circumellipse, then the inconic is an ellipse and the involution is elliptic, but if P lies outside the Steiner circumellipse, the inconic is a hyperbola and the involution is hyperbolic. Furthermore, if P is on the Steiner circumellipse, then $P' = Q = H = O$ is at infinity, the inconic is a parabola, and there is no corresponding involution on the line at infinity. However, interesting theorems of Euclidean geometry can be proved even in the latter two settings, which cannot be derived by applying an affine map to the standard results, since an affine map will take an elliptic involution on l_∞ to another elliptic involution.

2 Affine relationships between Q, O, and H.

We continue to consider the affine situation in which Q is the isotomcomplement of P with respect to triangle ABC, and DEF is the cevian triangle for P with respect to ABC, so that $D = AP \cdot BC$, $E = BP \cdot AC$, and $F = CP \cdot AB$. As in Parts I and II, $D_0E_0F_0 = K(ABC)$ is the medial triangle of ABC.

Definition 2.1. *The point O for which $OD_0 \parallel QD, OE_0 \parallel QE$, and $OF_0 \parallel QF$ is called the* **generalized circumcenter** *of the point P with respect to ABC. The point H for which $HA \parallel QD, HB \parallel QE$, and $HC \parallel QF$ is called the* **generalized orthocenter** *of P with respect to ABC.*

We first prove the following affine relationships between Q, O, and H.

Theorem 2.2. *The generalized circumcenter O and generalized orthocenter H exist for any point P not on the extended sides or the anticomplementary triangle of ABC, and are given by*

$$O = T_{P'}^{-1}K(Q), \quad H = K^{-1}T_{P'}^{-1}K(Q).$$

Remark. The formula for the point H can also be written as $H = T_L^{-1}(Q)$, where $L = K^{-1}(P')$ and T_L is the map T_P defined for $P = L$ and the anticomplementary triangle of ABC.

Proof. We will show that the point $\tilde{O} = T_{P'}^{-1}K(Q)$ satisfies the definition of O, namely, that

$$\tilde{O}D_0 \parallel QD, \quad \tilde{O}E_0 \parallel QE, \quad \tilde{O}F_0 \parallel QF.$$

It suffices to prove the first relation $\tilde{O}D_0 \parallel QD$. We have that

$$T_{P'}(\tilde{O}D_0) = K(Q)T_{P'}(D_0) = K(Q)A_0'$$

and

$$T_{P'}(QD) = P'A_3',$$

by I, Theorem 3.7. Thus, we just need to prove that $K(Q)A_0' \parallel P'A_3'$. We use the map $\mathcal{S}_2 = T_{P'}T_P$ from I, Theorem 3.8, which takes ABC to $A_3'B_3'C_3'$. We have $\mathcal{S}_2(Q) = T_{P'}T_P(Q) = T_{P'}(Q) = P'$. Since $\mathcal{S}_2$ is a homothety or translation, this gives that $AQ \parallel \mathcal{S}_2(AQ) = A_3'P'$. Now note that $M' = K(Q)$ in I, Corollary 2.6, so

$$K(Q)A_0' = M'A_0' = D_0A_0',$$

by that corollary. Now the Quadrilateral Half-turn Theorem (I, Theorem 2.5) implies that $AQ \parallel D_0A_0'$, and therefore $P'A_3' \parallel K(Q)A_0'$. This proves the formula for O. To get the formula for H, just note that $K^{-1}(OD_0) = K^{-1}(O)A, K^{-1}(OE_0) = K^{-1}(O)B, K^{-1}(OF_0) = K^{-1}(O)C$ are parallel, respectively, to QD, QE, QF, since K is a dilatation. This shows that $K^{-1}(O)$ satisfies the definition of the point H. □

Corollary 2.3. *The points O and H are ordinary whenever P is ordinary and does not lie on the Steiner circumellipse $\iota(l_\infty)$. If P does lie on $\iota(l_\infty)$, then $O = H = Q$.*

Proof. If P lies on $\iota(l_\infty)$, then $P' = Q$ is infinite, and since K is a dilatation, we have that $O = H = T_{P'}^{-1}(Q) = T_{P'}^{-1}T_P^{-1}(Q) = \mathcal{S}_1^{-1}(Q) = Q$ by I, Theorems 3.2 and 3.8. □

To better understand the connection between the point P and the points O and H of Theorem 2.2 we consider the circumconic $\tilde{\mathcal{C}}_O$ on ABC with center O. We will show that this circumconic is related to the nine-point conic $\mathcal{N}_{P'}$ (with respect to l_∞) on the quadrangle $ABCP'$. Recall that this is the conic through the diagonal points and the midpoints of the sides of the quadrangle [co1, p. 84]; alternatively, it is the locus of the centers of conics which lie on the vertices of the quadrangle $ABCP'$. Three of these centers are the points

$$D_3 = AP' \cdot BC, \quad E_3 = BP' \cdot AC, \quad F_3 = CP' \cdot AB,$$

which are centers of the degenerate conics

$$AP' \cup BC, \quad BP' \cup AC, \quad CP' \cup AB.$$

Since the quadrangle is inscribed in $\mathcal{C}_P$, its center Z lies on $\mathcal{N}_{P'}$.

Theorem 2.4. *a) The point $K(Q)$ is the center of the nine-point conic $\mathcal{N}_{P'}$ (with respect to l_∞) for the quadrangle $ABCP'$.*

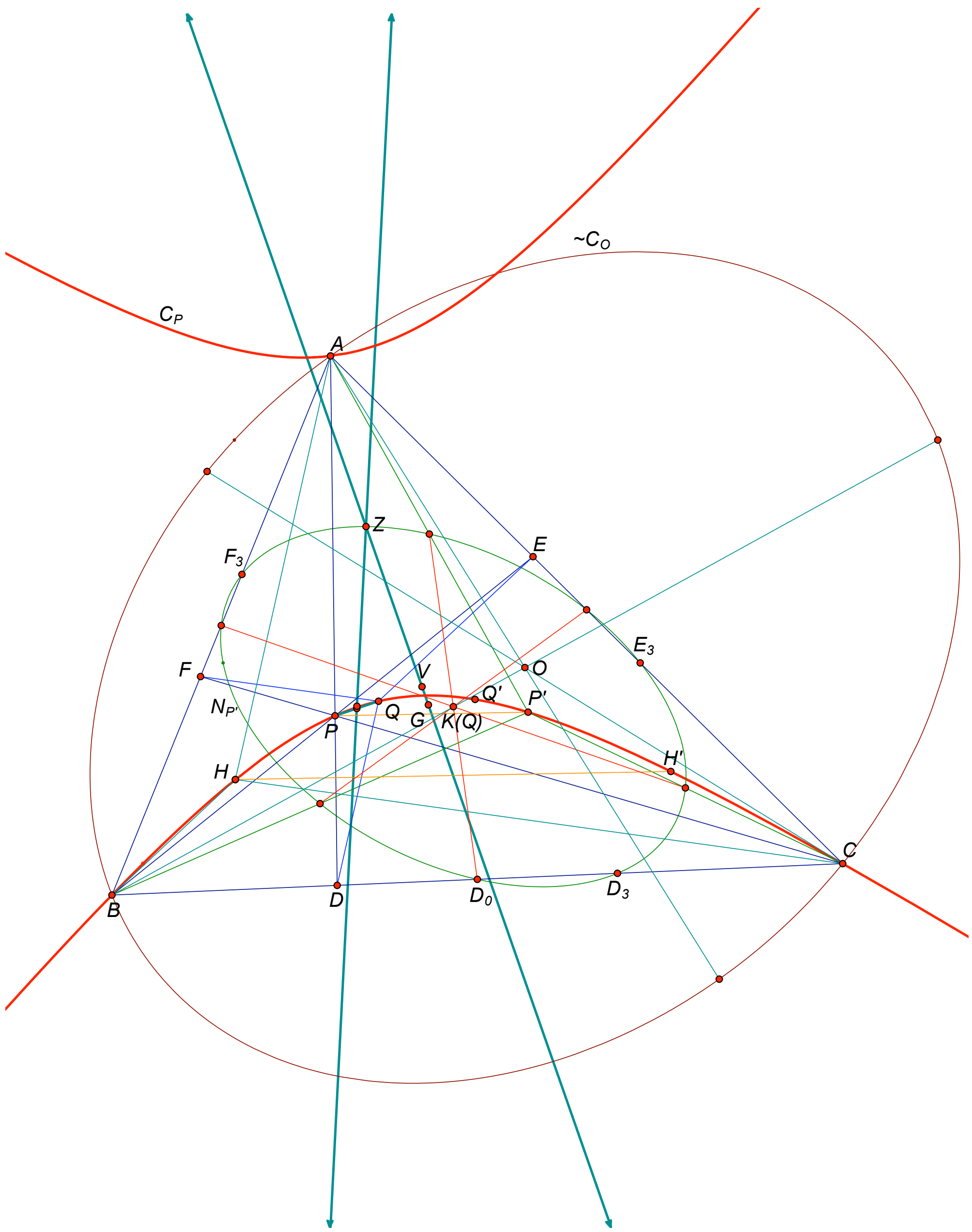

Figure 1: Circumconic $\tilde{\mathcal{C}}_O$ and Nine-point Conic $\mathcal{N}_{P'}$

b) The circumconic $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$ is the nine-point conic for the quadrangle formed by the point Q and the vertices of the anticevian triangle of Q. Its center is $O = T_{P'}^{-1}K(Q)$.

c) If P does not lie on a median or on $\iota(l_\infty)$ and O', H' are the points of Theorem 2.2 corresponding to the point P', then $O' = \eta(O)$ and $H' = \eta(H)$. Thus OO' and HH' are parallel to PP'.

Proof. a) If P' is ordinary, the conic $\mathcal{N}_{P'}$ lies on the midpoints D_0, E_0, F_0, the vertices D_3, E_3, F_3, and the midpoints R_a', R_b', R_c' of AP', BP', and CP'. By I, Corollary 2.6, the point $K(Q)$ is the midpoint of segments $D_0R_a', E_0R_b', F_0R_c'$, which are chords for the conic $\mathcal{N}_{P'}$. As the midpoint of two intersecting chords, $K(Q)$ is conjugate to two different points at infinity (with respect to the polarity associated with $\mathcal{N}_{P'}$), and so must be the pole of the line at infinity. If P' is infinite, the nine-point conic $\mathcal{N}_{P'}$ lies on the midpoints D_0, E_0, F_0, the vertices D_3, E_3, F_3, and the harmonic conjugate of $AQ \cdot l_\infty = Q$ with respect to A and Q, which is Q itself. (See [co1, 6.83, p. 84].) In this case we claim that $Q = K(Q)$ is the pole of l_∞. The line l_∞ intersects $\mathcal{N}_{P'}$ at least once at Q; it is enough to show that l_∞ intersects $\mathcal{N}_{P'}$ *only* at Q, because that will imply it is the tangent at Q, hence its pole is Q. Suppose l_∞ also intersects $\mathcal{N}_{P'}$ at a point X. The nine-point conic $\mathcal{N}_{P'}$ is the locus of centers (that is, poles of l_∞) of conics that lie on A, B, C, and $P' = Q$, which means X is the pole of l_∞ with respect to one of these conics $\mathcal{C}$. Now X cannot be D_3, E_3, or F_3, which are the centers of the degenerate conics through $ABCQ$, because none of these points lies on l_∞. Thus $\mathcal{C}$ is nondegenerate. By assumption, X lies on its polar l_∞ with respect to $\mathcal{C}$, so X lies on $\mathcal{C}$ and l_∞ is the tangent line at X. But we assumed that Q and X are distinct points on l_∞, so l_∞ is also a secant of $\mathcal{C}$, a contradiction.

b) By I, Corollary 3.11, the anticevian triangle $A'B'C'$ of Q is $T_{P'}^{-1}(ABC)$. By I, Theorem 3.7, $Q = T_{P'}^{-1}(P')$, so the quadrangle $ABCP'$ is mapped to quadrangle $A'B'C'Q$ by the map $T_{P'}^{-1}$. The diagonal points D_3, E_3, F_3 of quadrangle $ABCP'$ map to A, B, C so $T_{P'}^{-1}(\mathcal{N}_{P'})$ is certainly a circumconic for triangle ABC with center $T_{P'}^{-1}K(Q) = O$, by Theorem 2.2.

c) By Theorem 2.2, II, Theorem 2.4, and the fact that the map η (see the discussion following Prop. 2.3 in Part II) commutes with the complement map, we have that

$$\eta(O) = \eta T_{P'}^{-1}K(Q) = T_P^{-1}K(\eta(Q)) = T_P^{-1}K(Q') = O',$$

and similarly $\eta(H) = H'$. $\square$

We show now that there are 4 points P which give rise to the same generalized circumcenter O and generalized orthocenter H. These points arise in the following way. Let the vertices of the anticevian triangle for Q with respect to ABC be denoted by

$$Q_a = T_{P'}^{-1}(A), Q_b = T_{P'}^{-1}(B), Q_c = T_{P'}^{-1}(C).$$

Then we have

$$A = Q_bQ_c \cdot QQ_a, B = Q_aQ_c \cdot QQ_b, C = Q_aQ_b \cdot QQ_c.$$

This clearly implies that QQ_aQ_b is the anticevian triangle of Q_c with respect to ABC. Similarly, the anticevian triangle of any one of these four points is the triangle formed by the other three (analogous to the corresponding property of the excentral triangle). We let P_a, P_b, P_c be the anti-isotomcomplements of the points Q_a, Q_b, Q_c with respect to ABC, so that

$$P_a = \iota^{-1}K^{-1}(Q_a),\ P_b = \iota^{-1}K^{-1}(Q_b),\ P_c = \iota^{-1}K^{-1}(Q_c).$$

Theorem 2.5. *The points P, P_a, P_b, P_c all give rise to the same generalized circumcenter O and generalized orthocenter H.*

Proof. We use the characterization of the circumconic $\tilde{\mathcal{C}}_O$ from Theorem 2.4(b). It is the nine-point conic $\mathcal{N}_Q$ for the quadrangle $Q_aQ_bQ_cQ$. But this quadrangle is the same for all of the points P, P_a, P_b, P_c, by the above discussion, so each of these points gives rise to the same circumconic $\tilde{\mathcal{C}}_O$. This implies the theorem, since O is the center of $\tilde{\mathcal{C}}_O$ and H is the anti-complement of O. $\square$

Corollary 2.6. *The point-set $\{A, B, C, H\}$ is the common intersection of the four conics $\mathcal{C}_Y$, for $Y = P, P_a, P_b, P_c$.*

Proof. If P does not lie on a median of ABC, the points Q, Q_a, Q_b, Q_c are all distinct, since $Q_aQ_bQ_c$ is the anticevian triangle of Q with respect to ABC. It follows that the points P, P_a, P_b, P_c are distinct, as well. If two of the conics $\mathcal{C}_Y$ were equal, say $\mathcal{C}_{P_a} = \mathcal{C}_{P_b}$, then this conic lies on the points Q_a, Q_b and $QQ_c \cdot Q_aQ_b = C$, which is impossible. This shows that the conics $\mathcal{C}_Y$ are distinct. $\square$

The points Q_a, Q_b, Q_c are the analogues of the excenters of a triangle, and the points P_a, P_b, P_c are the analogues of the external Gergonne points. The traces of the points P_a, P_b, P_c can be determined directly from the definition of H; for example, Q_cD_c, Q_cE_c, Q_cF_c are parallel to AH, BH, CH, which are in turn parallel to QD, QE, QF.

The next theorem shows that the points H and H' are a natural part of the family of points that includes P, Q, P', Q'.

Theorem 2.7. *If the ordinary point P does not lie on a median of ABC or on $\iota(l_\infty)$, we have:*

a) $\lambda(P) = Q'$ *and* $\lambda^{-1}(P') = Q$.

b) $\lambda(H) = Q$ *and* $\lambda^{-1}(H') = Q'$.

Remarks. This gives an alternate formula for the point $H = \lambda^{-1}(Q) = \eta\lambda^2(P)$. Part b) gives an alternate proof that $\eta(H) = H'$.

Proof. Part a) was already proved as part of the proof of II, Theorem 3.2. Consider part b), which we will prove by showing that

$$T_P^{-1}(H) = T_{P'}^{-1}(Q). \tag{3}$$

The point $T_P^{-1}(H)$ is the generalized orthocenter for the point $T_P^{-1}(P) = Q'$ and the triangle $T_P^{-1}(ABC) = \tilde{A}\tilde{B}\tilde{C}$, which is the anticevian triangle for Q' with respect to ABC (I, Corollary 3.11(a)). It follows that the lines $T_P^{-1}(H)\tilde{A}$ and QA are parallel, since Q is the isotomcomplement of Q' with respect to $\tilde{A}\tilde{B}\tilde{C}$ (I, Theorem 3.13), and A is the trace of the cevian $\tilde{A}Q'$ on $\tilde{B}\tilde{C}$. We will show that the line $T_{P'}^{-1}(Q)\tilde{A}$ is also parallel to QA. For this, first recall from I, Theorem 2.5 that QD_0 and AP' are parallel. Part II, Theorem 3.4(a) implies that Q, D_0, and $\lambda(A)$ are collinear, so it follows that $Q\lambda(A)$ and $D_3P' = AP'$ are parallel. Now apply the map $T_{P'}^{-1}$. This shows that $T_{P'}^{-1}(Q\lambda(A)) = T_{P'}^{-1}(Q)T_{P'}^{-1}\lambda(A) = T_{P'}^{-1}(Q)\tilde{A}$ is parallel to $T_{P'}^{-1}(D_3P') = AQ$, as claimed. Applying the same argument to the other vertices of $T_P^{-1}(ABC) = \tilde{A}\tilde{B}\tilde{C}$ gives (3), which is equivalent to $\lambda(H) = Q$. Now apply the map η to give $\lambda^{-1}(H') = Q'$. □

Theorem 2.8. *If P does not lie on a median of triangle ABC or on $\iota(l_\infty)$, the generalized orthocenter H of P and the generalized orthocenter H' of P' both lie on the cevian conic $\mathcal{C}_P = ABCPQ$.*

Proof. Theorem 2.7(b) and the fact that λ maps the conic $\mathcal{C}_P$ to itself (II, Theorem 3.2) imply that H lies on $\mathcal{C}_P$, and since Q' lies on $\mathcal{C}_P$, so does H'. □

Corollary 2.9. *When P is the Gergonne point of ABC, the conic $\mathcal{C}_P = ABCPQ$ is the Feuerbach hyperbola $ABCHI$ on the orthocenter H and the incenter $Q = I$. The Feuerbach hyperbola also passes through the Nagel point (P'), the Mittenpunkt (Q'), and the generalized orthocenter H' of the Nagel point, which is the point of intersection of the lines through A, B, C which are parallel to the respective lines joining the Mittenpunkt to the opposite points of external tangency of ABC with its excircles.*

Part II, Theorem 3.4 gives six other points lying on the Feuerbach hyperbola. Among these is the point $A_0P \cdot D_0Q'$, where D_0Q' is the line joining the Mittenpunkt to the midpoint of side BC, and A_0P is the line joining the Gergonne point to the opposite midpoint of its cevian triangle. Using I, Theorem 2.4 and the fact that the isotomcomplement of the incenter $Q = I$ with respect to the excentral triangle $I_aI_bI_c$ is the Mittenpunkt Q' (see I, Theorem 3.13), it can be shown that the line D_0Q' also passes through the excenter I_a of triangle ABC which lies on the angle bisector of the angle at A. Thus, $A_0P \cdot D_0Q' = A_0P \cdot I_aQ'$. By II, Theorem 3.4(b), the lines DQ and $A_3'P'$ also lie on this point.

From (3) we deduce

Theorem 2.10. *Assume that the ordinary point P does not lie on a median of ABC or on $\iota(l_\infty)$.*

a) The point Q is the perspector of triangles $D_0E_0F_0$ and $\lambda(ABC)$.

b) The point $\tilde{H} = T_P^{-1}(H) = T_{P'}^{-1}(Q)$ is the perspector of the anticevian triangle for Q' (with respect to ABC) and the medial triangle of the anticevian triangle of Q. Thus, the points $\tilde{A} = T_P^{-1}(A), \tilde{H}$, and $T_{P'}^{-1}(D_0)$ are collinear, with similar statements for the other vertices.

c) The point H is the perspector of triangle ABC and the medial triangle of $\lambda^{-1}(ABC)$. Thus, A, H, and $\lambda^{-1}(D_0)$ are collinear, with similar statements for the other vertices.

d) The point $\tilde{H}$ is also the perspector of the anticevian triangle of Q and the cevian triangle of P'. Thus, $\tilde{H}$ is the P'-ceva conjugate of Q, with respect to triangle ABC.

e) The point H is also the perspector of $\lambda^{-1}(ABC)$ and the triangle $A_3B_3C_3$.

Proof. a) As in the previous proof, the points Q, D_0, and $\lambda(A)$ are collinear, with similar statements for $Q, E_0, \lambda(B)$ and for $Q, F_0, \lambda(C)$. Therefore, Q is the perspector of triangles $D_0E_0F_0$ and $\lambda(ABC)$.

b) Now apply the map $T_{P'}^{-1}$, giving that $T_{P'}^{-1}(Q)$ is the perspector of $T_{P'}^{-1}(D_0E_0F_0)$, which is the medial triangle of $T_{P'}^{-1}(ABC)$, and the triangle $T_{P'}^{-1}\lambda(ABC) = T_P^{-1}(ABC)$. The result follows from (3) and I, Corollary 3.11.

c) This part follows immediately by applying the map T_P to part (b) or λ^{-1} to part (a).

d) I, Theorem 3.5 gives that A, A_1' (on E_3F_3), and Q are collinear, from which we get that $T_{P'}^{-1}(A) = A', T_{P'}^{-1}(A_1') = D_1' = D_3$, and $T_{P'}^{-1}(Q) = \tilde{H}$ are collinear. This and the corresponding statements for the other vertices imply (d).

e) Applying the map T_P to (d) show that $\lambda^{-1}(A), A_3$, and H are collinear, with similar statements for the other vertices. □

3 The generalized Feuerbach Theorem.

Recall that the 9-point conic $\mathcal{N}_H$, with respect to the line l_∞ is the conic which passes through the following nine points:

the midpoints D_0, E_0, F_0 of the sides BC, AC, AB;
the midpoints R_1, R_2, R_3 of the segments AH, BH, CH;
the traces (diagonal points) H_1, H_2, H_3 of H on the sides BC, AC, AB.

In this section we give a synthetic proof, based on the results of this paper, that $\mathcal{N}_H$ is tangent to the inconic of ABC whose center is Q, at the generalized Feuerbach point Z. We start by showing that the conics $\mathcal{N}_H$ and $\tilde{\mathcal{C}}_O$ are in the same relationship to each other as the classical 9-point circle and circumcircle.

Theorem 3.1. *Assume that P is a point for which the generalized circumcenter O is not the midpoint of any side of ABC, so that H does not coincide with any of the vertices A, B, or C. As in Theorem 2.4, $\tilde{\mathcal{C}}_O$ is the unique circumconic of ABC with center O. Then the 9-point conic $\mathcal{N}_H$ is the complement of $\tilde{\mathcal{C}}_O$ with respect to ABC. It is also the image of $\tilde{\mathcal{C}}_O$ under a homothety H centered at H with factor $1/2$. Its center is the midpoint N of segment HO.*

Proof. Let T_1, T_2, T_3 denote the reflections of A, B, C through the point O, and denote by S_1, S_2, S_3 the intersections of AH, BH, CH with the conic $\mathcal{C} = \tilde{\mathcal{C}}_O$. Let R_O the mapping which is reflection in O. By Theorem 2.2, $K(H) = O$, so $K(O) = N$. Hence $K(\mathcal{C})$ is a conic through the midpoints of the sides of ABC with center N. Reflecting the line AH in the point O, we obtain the parallel line T_1S_1' with $S_1' = \mathsf{R}_O(S_1)$ on $\mathcal{C}$. This line contains the point $\bar{H} = \mathsf{R}_O(H) = K^{-1}(H)$, since the centroid G lies on OH. If we define $R_1 = T_1G \cdot AH$, then triangle HGR_1 is similar to $\bar{H}GT_1$. Hence, $R_1 = K(T_1)$ lies on $K(\mathcal{C})$ and $R_1H = K\mathsf{R}_O(AH)$, so R_1 is the midpoint of AH. In the same way, $R_2 = K(T_2)$ and $R_3 = K(T_3)$ lie on the conic $K(\mathcal{C})$. This shows that $K(\mathcal{C})$ lies on the midpoints of the sides of the quadrangle $ABCH$, and so is identical with the conic $\mathcal{N}_H$. Since the affine map $\mathsf{H} = K\mathsf{R}_O$ takes A, B, and C to the respective midpoints of AH, BH, CH and fixes H, it is clear that H is the homothety centered at H with factor $1/2$, and $\mathsf{H}(\mathcal{C}) = K\mathsf{R}_O(\mathcal{C}) = K(\mathcal{C}) = \mathcal{N}_H$. □

Proposition 3.2. *Assume the hypothesis of Theorem 3.1. If O is not on the line AH, and the intersections of the cevians AH and AO with the circumconic $\tilde{\mathcal{C}}_O$ are S_1 and T_1, respectively, then S_1T_1 is parallel to BC.*

Proof. (See Figure 2.) First note that P does not lie on the median AG, since H and O lie on this median whenever P does, by Theorem 2.4 and the arguments in the proof of II, Theorem 2.7, according to which T_P and $T_{P'}$ fix the line AG when it lies on P. Thus, H cannot lie on any of the secant lines AB, AC, or BC of the conic $\mathcal{C}_P = ABCPQ$, because H is also on this conic and does not coincide with a vertex. It is also easy to see that H does not lie on a side of the anticomplementary triangle $K^{-1}(ABC)$, since Q does not lie on a side of ABC. By the proof of I, Theorem 2.5 (Quadrilateral Half-turn Theorem), with H as the point P and $O = K(H)$ as the point Q', we know that R_1 is the reflection of O in the midpoint N_1 of E_0F_0. Since R_1 is the midpoint of AH, the line R_1O is a midline of triangle AHT_1; and since N_1 is the midpoint of segment R_1O, we know that AN_1 intersects the opposite side HT_1 in its midpoint M, with N_1 halfway between A and M. However, $AN_1 \cong N_1D_0$, because N_1 is the midpoint of the midline E_0F_0 of triangle ABC. Therefore, $D_0 = M$ is the midpoint of HT_1. Now, considering triangle HS_1T_1, with $OD_0 \parallel HS_1$ (on AH), we know that OD_0 intersects S_1T_1 in its midpoint. But D_0 is the midpoint of the chord BC of the conic $\tilde{\mathcal{C}}_O$, and S_1T_1 is also a chord of $\tilde{\mathcal{C}}_O$. Since O is the center of $\tilde{\mathcal{C}}_O$, it follows that OD_0 is the

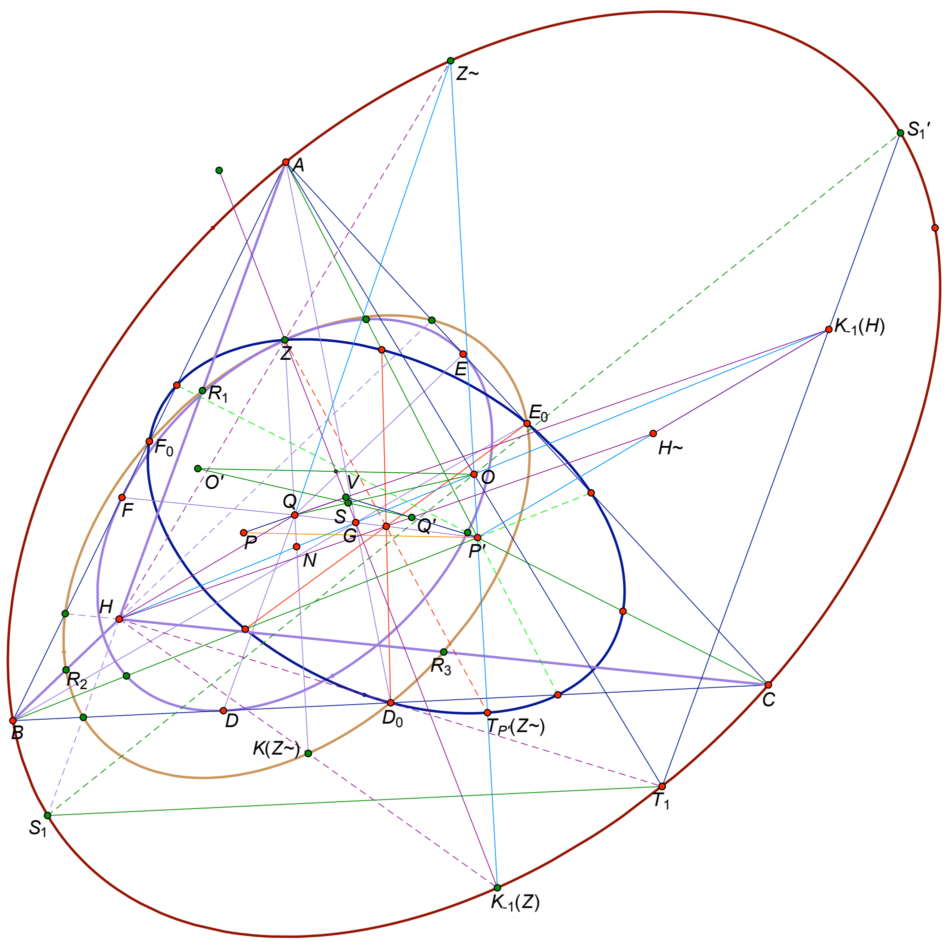

Figure 2: $\mathcal{N}_{P'}, \tilde{\mathcal{C}}_O$, $\mathcal{N}_H$, and $\mathcal{I}$ $(K_{-1} = K^{-1})$

polar of the points at infinity of both BC and S_1T_1 (with respect to $\tilde{\mathcal{C}}_O$), whence these lines must be parallel. $\square$

When the point H does coincide with a vertex, we define $\mathcal{N}_H$ by $\mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$.

Proposition 3.3. *If the generalized orthocenter $H = A$, then $\mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$ is the unique conic on the vertices of the quadrangle $AD_0E_0F_0$ which is tangent to the conic $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$ at A.*

Proof. If $H = A$, then $O = D_0$ is the center of $\tilde{C}_O$, so the reflection of A through D_0 lies on $\tilde{C}_O$, and this reflection is precisely the anticomplement $A' = K^{-1}(A) = K^{-2}(D_0)$ of A. Then clearly $A = K(A')$ lies on $\mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$. Thus $\mathcal{N}_H$ is a conic on the vertices of $AD_0E_0F_0$. The tangent line t' to $\tilde{\mathcal{C}}_O$ at A' is taken by the complement map to a parallel tangent line t_1 to $\mathcal{N}_H$ at A. But the tangent line t_2 to $\tilde{\mathcal{C}}_O$ at A is also parallel to t' because the line AA' lies on the center O (dually, $a \cdot a'$ lies on l_∞). It follows that $t_1 = t_2$, hence $\mathcal{N}_H$ is tangent to $\tilde{\mathcal{C}}_O$ at A. $\square$

In this case, we take the point S_1 in the above proposition to be the intersection of the conic $\tilde{\mathcal{C}}_O$ with the line t_A through A which is parallel to QD, and we claim that we still have $S_1T_1 \parallel BC$. To prove this we first prove the following theorem.

Theorem 3.4. *The map $\mathsf{M} = T_PK^{-1}T_{P'}$ is a homothety or translation taking the conic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$, which is the conic with center Q tangent to the sides of ABC at the points D, E, F.*

Remark. Below we will show that the fixed point (center) of the map M is the point $S = OQ \cdot GV$, which coincides with the point $\gamma_P(P) = Q \cdot Q'$, where γ_P is the generalized isogonal map, to be defined in Part IV of this paper, and $Q \cdot Q'$ is the point whose barycentric coordinates are the products of corresponding barycentric coordinates of Q and Q'.

Proof. The proof of I, Theorem 3.8 shows that M is a homothety or translation, since K fixes all points at infinity. This map takes the triangle $T_{P'}^{-1}(D_0E_0F_0)$, which is inscribed in $\tilde{\mathcal{C}}_O$ by Theorem 2.4, to the triangle DEF, which is inscribed in the inconic. It also takes the center $O = T_{P'}^{-1}K(Q)$ of $\tilde{\mathcal{C}}_O$ to the center Q of $\mathcal{I}$. Now Q is never the midpoint of a side of triangle DEF, because, for instance, $Q = A_0$ would imply $Q = T_P^{-1}(A_0) = D_0$ and then $P' = K^{-1}(Q) = K^{-1}(D_0) = A$, contradicting the fact that P and P' do not lie on the sides of ABC. It follows that O is never the midpoint of a side of $T_{P'}^{-1}(D_0E_0F_0)$, and this implies that $\tilde{\mathcal{C}}_O$ is mapped to $\mathcal{I}$. This result holds even if the point $O = Q$ is infinite, since then M fixes the tangent at Q, i.e., the line at infinity. □

Corollary 3.5. *The tangent to the conic $\tilde{\mathcal{C}}_O$ at the point $T_{P'}^{-1}(D_0)$ is parallel to BC, with similar statements for the other vertices of the medial triangle of the anticevian triangle of Q.*

Proof. The tangent to $\tilde{\mathcal{C}}_O$ at $T_{P'}^{-1}(D_0)$ is mapped by M to and therefore parallel to the tangent to $\mathcal{I}$ at D, which is BC. □

We also need the following proposition in order to complete the proof of the above remark, when $O = D_0$.

Proposition 3.6. *Let ψ_1, ψ_2, ψ_3 be the mappings of conjugate points on l_∞ which are induced by the polarities corresponding to the inconic $\mathcal{I}$ (center Q), the circumconic $\tilde{\mathcal{C}}_O$ (center O), and the 9-point conic $\mathcal{N}_H$ (center N). If O and Q are finite, then $\psi_1 = \psi_2 = \psi_3$.*

Proof. If T is any projective collineation, and π is the polarity determining a conic $\mathcal{C}$, then the polarity determining $T(\mathcal{C})$ is $T\pi T^{-1}$. If q is any non-self-conjugate line for π, then $T(q)$ is not self-conjugate, since its pole $T(Q)$ does not lie on $T(q)$. If ψ is the mapping of conjugate points on q, then the polar of a point A on q is $\pi(A) = a = Q\psi(A)$. Hence, the polar of $T(A)$ on $T(q)$ is

$$T\pi T^{-1}(T(A)) = T\pi(A) = T(Q\psi(A)) = T(Q)T(\psi(A)) = T(Q)T\psi T^{-1}(T(A)).$$

This shows that the mapping of conjugate points on $T(q)$ is $T\psi T^{-1}$. Now apply this to $\tilde{\mathcal{C}}_O$ and $\mathcal{N}_H = K(\tilde{\mathcal{C}}_O)$, with $T = K$ and $q = l_\infty$. The complement mapping fixes the points on l_∞, so $\psi_3 = K\psi_2K^{-1} = \psi_2$. Theorem 3.4 and a similar argument shows that $\psi_1 = \psi_2$. □

Corollary 3.7. *The conclusion of Proposition 3.2 holds if $A = H$ and $O = D_0$, where S_1 is defined to be the intersection of the conic $\tilde{\mathcal{C}}_O$ with the line through A parallel to QD.*

Proof. The line $AO = AT_1$ is a diameter of $\tilde{\mathcal{C}}_O$, so the direction of AS_1, which equals the direction of QD, is conjugate to the direction of S_1T_1. But the direction of QD is conjugate to the direction of BC, since BC is tangent to $\mathcal{I}$ at D, and since $\psi_1 = \psi_2$. Therefore, S_1T_1 and BC lie on the same point at infinity. □

Proposition 3.2 and Corollary 3.7 will find application in Part IV. To determine the fixed point of the mapping M in Theorem 3.4, we need a lemma.

Lemma 3.8. *If P does not lie on $\iota(l_\infty)$, the point $\tilde{H} = T_P^{-1}(H)$ is the midpoint of the segment joining P' and $K^{-1}(H)$, and the reflection of the point Q through O.*

Proof. Let H_1 be the midpoint of $P'K^{-1}(H)$. The Euclidean quadrilateral $H_1HQK^{-1}(H)$ is a parallelogram, because $K^{-1}(QH) = P'K^{-1}(H)$ and the segment $H_1K^{-1}(H)$ is therefore congruent and parallel to QH. The intersection of the diagonals is the point O, the midpoint of $HK^{-1}(H)$, so that O is also the midpoint of H_1Q. On the other hand, $K(Q)$ is the midpoint of segment $P'Q$, so Theorem 2.4 gives that $O = T_{P'}^{-1}K(Q)$ is the midpoint of $T_{P'}^{-1}(P'Q) = Q\tilde{H}$, by (3). This implies that $H_1 = \tilde{H}$. □

Theorem 3.9. *If P is ordinary and does not lie on a median of ABC or on $\iota(l_\infty)$, the fixed point (center) of the map $\mathsf{M} = T_PK^{-1}T_{P'}$ is $S = OQ \cdot GV = OQ \cdot O'Q'$.*

Remark. The point S is a generalization of the insimilicenter, since it is the fixed point of the map taking the circumconic to the inconic. See [ki2].

Proof. Assume first that M is a homothety. The fixed point S of M lies on OQ, since the proof of Theorem 3.4 gives that $\mathsf{M}(O) = Q$. Note that $O \neq Q$, since $T_{P'}(Q) = P' \neq K(Q)$, by I, Theorem 3.7. We claim that $\mathsf{M}(O') = Q'$ also. We shall prove the symmetric statement $\mathsf{M}'(O) = Q$, where $\mathsf{M}' = T_{P'}K^{-1}T_P$. We have that

$$\mathsf{M}'(O) = T_{P'}K^{-1}T_P(T_{P'}^{-1}K(Q)) = T_{P'}K^{-1}\lambda^{-1}(K(Q)).$$

Now $K(Q)$ is the midpoint of $P'Q$, so $K^{-1}\lambda^{-1}(K(Q))$ is the midpoint of $K^{-1}\lambda^{-1}(P'Q) = K^{-1}(QH) = P'K^{-1}(H)$ (Theorem 2.7), and therefore coincides with the point $\tilde{H}$, by the lemma. Therefore, by (3), we have

$$\mathsf{M}'(O) = T_{P'}(\tilde{H}) = Q.$$

Switching P and P' gives that $\mathsf{M}(O') = Q'$, as claimed. Therefore, $S = OQ \cdot O'Q'$. Note that the lines OQ and $O'Q'$ are distinct. If they were not distinct, then O, O', Q, Q' would be collinear, and applying K^{-1} would give that H, H', P', P are collinear, which is impossible since these points all lie on the cevian conic $\mathcal{C}_P$. (Certainly, $H \neq P$, since otherwise $O = T_{P'}^{-1}K(Q) = K(P) = Q'$, forcing $K(Q) = T_{P'}(Q') = Q' = K(P)$ and $P = Q$. Similarly, $H \neq P'$, so these are four distinct points.) This shows that $\eta(S) = S$, so S lies on GV and $S = OQ \cdot GV$.

If M is a translation, then it has no ordinary fixed points, and the same arguments as before give that $OQ \parallel O'Q' \parallel GV$ and these lines are fixed by M. But then M is a translation along GV, so its center is again $S = OQ \cdot GV = OQ \cdot O'Q'$. $\square$

Proposition 3.10. *If P does not lie on a median of ABC, then $T_P K^{-1}(Z) = Z$.*

Proof. We first use II, Theorem 4.1, when P and P' are ordinary. The point Z is defined symmetrically with respect to P and P', since it is the center of the conic $\mathcal{C}_P = \mathcal{C}_{P'}$. Therefore II, Theorem 4.1 yields $Z = GV \cdot T_{P'}(GV)$, so the last theorem implies that

$$T_P K^{-1}(Z) = T_P K^{-1}(GV) \cdot T_P K^{-1} T_{P'}(GV) = T_P(GV) \cdot GV = Z,$$

since the point S lies on GV. If P lies on $\iota(l_\infty)$, then $T_P K^{-1}(Z) = Z$ follows immediately from II, Theorem 4.3 and the proof of II, Theorem 2.7 (in the case that P' is infinite), since T_P is a translation along GG_1 taking G to $G_1 = T_P(G)$. If P is infinite, then P' lies on $\iota(l_\infty)$, in which case $T_P K^{-1} = T_{P'}^{-1} K^{-2}$ by I, Theorem 3.14. This mapping fixes Z by II, Theorem 4.3. $\square$

Proposition 3.11. *If P does not lie on a median of ABC, then Z lies on the 9-point conic $\mathcal{N}_H$ of the quadrangle $ABCH$, and $K^{-1}(Z)$ lies on $\tilde{\mathcal{C}}_O$.*

Proof. As we remarked in the paragraph just before Theorem 2.4, the point Z lies on $\mathcal{N}_{P'}$. Theorem 2.4 implies that $T_{P'}^{-1}(Z)$ lies on $T_{P'}^{-1}(\mathcal{N}_{P'}) = \tilde{\mathcal{C}}_O$. By Proposition 3.10, with P' in place of P, $T_{P'}^{-1}(Z) = K^{-1}(Z)$. Since $K^{-1}(Z)$ lies on $\tilde{\mathcal{C}}_O$, the point Z lies on $K(\tilde{\mathcal{C}}_O) = \mathcal{N}_H$. $\square$

Proposition 3.12. *a) The map $\Phi_P = \mathsf{M} \circ K^{-1}$ satisfies*

$$\Phi_P(K(S)) = S, \ \Phi_P(N) = Q, \ \textit{and} \ \Phi_P(K(Q')) = T_P(P).$$

The center of the homothety or translation Φ_P is the common intersection of the lines GV, NQ, and $K(Q')T_P(P)$.

b) Also, $\Phi_P = \Phi_{P'}$, and the maps $T_P K^{-1}$ and $T_{P'} K^{-1}$ commute with each other.

c) $T_{P'}(P')$ lies on the line OQ, and $T_P(P)$ lies on $O'Q'$.

Proof. a) The map Φ_P is a homothety or translation by the same argument as in Theorem 3.4. We have $\Phi_P(K(S)) = \mathsf{M}(S) = S$, while

$$\begin{aligned}\Phi_P(N) &= T_P \circ K^{-1} \circ T_{P'} \circ K^{-1}(K(O)) = T_P K^{-1} T_{P'}(O)\\ &= T_P K^{-1}(K(Q)) = T_P(Q) = Q,\end{aligned}$$

and

$$\begin{aligned}\Phi_P(K(Q')) &= T_P \circ K^{-1} \circ T_{P'} \circ K^{-1}(K(Q')) = T_P K^{-1} T_{P'}(Q')\\ &= T_P K^{-1}(Q') = T_P(P).\end{aligned}$$

It follows that Φ_P fixes the three lines GV, NQ, and $K(Q')T_P(P)$.

b) Note that

$$\begin{aligned}\eta\Phi_P &= \eta \circ T_P \circ K^{-1} \circ T_{P'} \circ K^{-1}\\ &= T_{P'} \circ \eta \circ K^{-1} \circ T_{P'} \circ K^{-1} = T_{P'} \circ K^{-1} \circ T_P \circ K^{-1} \circ \eta\\ &= \Phi_{P'}\eta,\end{aligned}$$

since η and K^{-1} commute. On the other hand, the center of Φ_P lies on the line GV, which is the line of fixed points for the affine reflection η. It follows that $\eta\Phi_P\eta = \Phi_{P'}$ has l_∞ as its axis and $\eta(F) = F$ as its center, if F is the center of Φ_P (a homology or an elation). But both maps Φ_P and $\Phi_{P'}$ share the pair of corresponding points $\{K(S), S\}$, also lying on GV. Therefore, the two maps must be the same. (See [co2, pp. 53-54].)

c) From a) and b) we have $\Phi_P(K(Q)) = \Phi_{P'}(K(Q)) = T_{P'}(P')$. On the other hand, S on OQ implies that $K(S)$ lies on $NK(Q)$, so $K(Q)$ lies on the line $K(S)N$. Mapping by Φ_P and using a) shows that $T_{P'}(P')$ lies on $\Phi_P(K(S)N) = SQ = OQ$. □

The Generalized Feuerbach Theorem. *If P does not lie on a median of triangle ABC, the map*

$$\Phi_P = \mathsf{M} \circ K^{-1} = T_P \circ K^{-1} \circ T_{P'} \circ K^{-1}$$

takes the 9-point conic $\mathcal{N}_H$ to the inconic $\mathcal{I}$ and fixes the point Z, the center of $\mathcal{C}_P$. Thus, Z lies on $\mathcal{I}$, and the conics $\mathcal{N}_H$ and $\mathcal{I}$ are tangent to each other at Z. The same map Φ_P also takes the 9-point conic $\mathcal{N}_{H'}$ to the inconic $\mathcal{I}'$ which is tangent to the sides of ABC at D_3, E_3, F_3. The point Z is the center of the map Φ_P (a homology or elation).

Proof. The mapping Φ_P takes $\mathcal{N}_H$ to $\mathcal{I}$ by Theorems 3.1 and 3.4. Applying Proposition 3.10 to the points P' and P, we see that Φ_P fixes Z, so Z lies on $\mathcal{I}$ by Proposition 3.11. First assume Z is an ordinary point. As a homothety with center Z, Φ_P fixes any line through Z, and therefore fixes the tangent t to $\mathcal{N}_H$ at Z. Since tangents map to tangents, t is also the tangent to $\mathcal{I}$ at Z, which proves the theorem in this case. If $Z \in l_\infty$ and $\mathcal{N}_H$ is a parabola, the same argument applies, since the tangent to $\mathcal{N}_H$ at Z is just $l_\infty = \Phi_P(l_\infty)$. Assume now that $\mathcal{N}_H$ is a hyperbola. Then Z must be a point on one of the asymptotes t for $\mathcal{N}_H$, which is also the tangent at Z. But Z is on the line GV, and by Proposition 3.12 the center of Φ_P lies on GV. It follows that if Φ_P is a translation, it is a translation along GV, and therefore fixes the parallel line t. This will prove the assertion if we show that Φ_P cannot be a homothety when Z is infinite, i.e., it has no ordinary fixed point. Let X be a fixed point of Φ_P on the line GV. Writing $\Phi_P = \mathsf{M}_1\mathsf{M}_2$, with $\mathsf{M}_1 = T_P \circ K^{-1}$ and $\mathsf{M}_2 = T_{P'} \circ K^{-1}$, we have by part b) of Proposition 3.12 that

$$\Phi_P(\mathsf{M}_1(X)) = \mathsf{M}_1(\mathsf{M}_2\mathsf{M}_1(X)) = \mathsf{M}_1(X),$$

so $\mathsf{M}_1(X)$ is a fixed point of Φ_P on the line $\mathsf{M}_1(GV) = T_P(GV)$. Assuming X is ordinary, this shows that $\mathsf{M}_1(X) = X$, since a nontrivial homothety has a unique ordinary fixed point. Hence, $X \in GV \cdot T_P(GV)$. But Z is infinite and $Z = GV \cdot T_P(GV)$, so this is impossible. Thus, Φ_P has no ordinary fixed point in this case and its center is Z. □

Corollary 3.13. *a) If $N = K(O)$ is the midpoint of segment OH, the center Z of $\mathcal{C}_P$ lies on the line QN, and $Z = GV \cdot QN$.*

b) The point $K^{-1}(Z)$ lies on the line OP', so that $K^{-1}(Z) = GV \cdot OP'$ lies on $\tilde{\mathcal{C}}_O$. This point is the center of the anticevian conic $T_P^{-1}(\mathcal{C}_P)$. (See II, Theorem 3.3.)

c) If Z is infinite and the conics $\mathcal{N}_H$ and $\mathcal{I}$ are hyperbolas, the line QN is a common asymptote of $\mathcal{N}_H$ and $\mathcal{I}$.

Proof. For part a), Proposition 3.12 shows that the center Z of Φ_P lies on the line NQ. For part b), just note that $K^{-1}(NQ) = OP'$ and $K^{-1}(\mathcal{N}_H) = \tilde{\mathcal{C}}_O$. The last assertion follows from Proposition 3.10. For part c), the asymptote of $\mathcal{N}_H$ through Z must lie on the center of $\mathcal{N}_H$, which is N, and the asymptote of $\mathcal{I}$ through Z must lie on the center of $\mathcal{I}$, which is Q. Therefore, the common asymptote is QN. □

The Generalized Feuerbach Theorem applies to all four points of Theorem 2.5, and therefore generalizes the full statement of Feuerbach's theorem in the case that P is the Gergonne point. Thus, $\mathcal{N}_H$ is tangent to four distinct conics, each of which is tangent to the sides of ABC, namely, the inconics corresponding to each of the points P, P_a, P_b, P_c. Figure 3 shows the configuration in case P is outside the Steiner circumellipse, in which case $\mathcal{N}_H, \tilde{\mathcal{C}}_O$, and $\mathcal{I}$ are hyperbolas. The point marked 1 is a general point on the conic $\mathcal{N}_{P'}$, and the points marked 2 and 3 are the images of 1 on $T_{P'}^{-1}(\mathcal{N}_{P'}) = \tilde{\mathcal{C}}_O$ and on $K(\tilde{\mathcal{C}}_O) = \mathcal{N}_H$, respectively. As P varies on a line perpendicular to BC in this picture, the locus of the point Z is pictured in teal. This locus consists of three branches which are each tangent to a side of ABC at its midpoint. Figure 4 pictures a situation in which Z is infinite. The point P in this figure was found using the ratios $BD/BC = \frac{15}{16}$ and $BF/AF = \frac{6}{5}$.

Theorem 3.14. *The point $\tilde{Z} = \mathsf{R}_O K^{-1}(Z)$ is the fourth point of intersection of the conics $\mathcal{C}_P$ and $\tilde{\mathcal{C}}_O$, the other three points being the vertices A, B, C.*

Proof. Theorem 3.1 and Proposition 3.11 show that $\tilde{Z} = \mathsf{R}_O K^{-1}(Z) = \mathsf{H}^{-1}(Z)$ lies on $\tilde{\mathcal{C}}_O$. Since $T_{P'}$ maps $\tilde{\mathcal{C}}_O$ to $\mathcal{N}_{P'}$, we know that the half-turns through the points O and $K(Q) = T_{P'}(O)$ are conjugate by $T_{P'}$, namely:

$$T_{P'} \circ \mathsf{R}_O \circ T_{P'}^{-1} = \mathsf{R}_{K(Q)}.$$

Therefore, $T_{P'}(\tilde{Z}) = T_{P'}\mathsf{R}_O K^{-1}(Z) = T_{P'}\mathsf{R}_O T_{P'}^{-1}(Z) = \mathsf{R}_{K(Q)}(Z)$, the second equality following from Proposition 3.10. In other words, Z and $T_{P'}(\tilde{Z})$ are opposite points on the conic $\mathcal{N}_{P'}$. Furthermore, Z lies on QN, so $T_{P'}(\tilde{Z})$ lies on the parallel line $l = \mathsf{R}_{K(Q)}(QN)$, and since $K(Q)$ is the midpoint of QP', l is the line through P' parallel to QN, i.e. $l = OP' = K^{-1}(QN)$. Hence $T_{P'}(\tilde{Z})$ lies on OP', while Corollary 3.13b) implies that $\tilde{Z} = \mathsf{R}_O K^{-1}(Z)$ also lies on OP'. Therefore, $\tilde{Z}, P'$, and $T_{P'}(\tilde{Z})$ are collinear. Now II, Corollary 2.2b) implies that $\tilde{Z}$ lies on $\mathcal{C}_{P'} = \mathcal{C}_P$. This shows that $\tilde{Z} \in \mathcal{C}_P \cap \tilde{\mathcal{C}}_O$. □

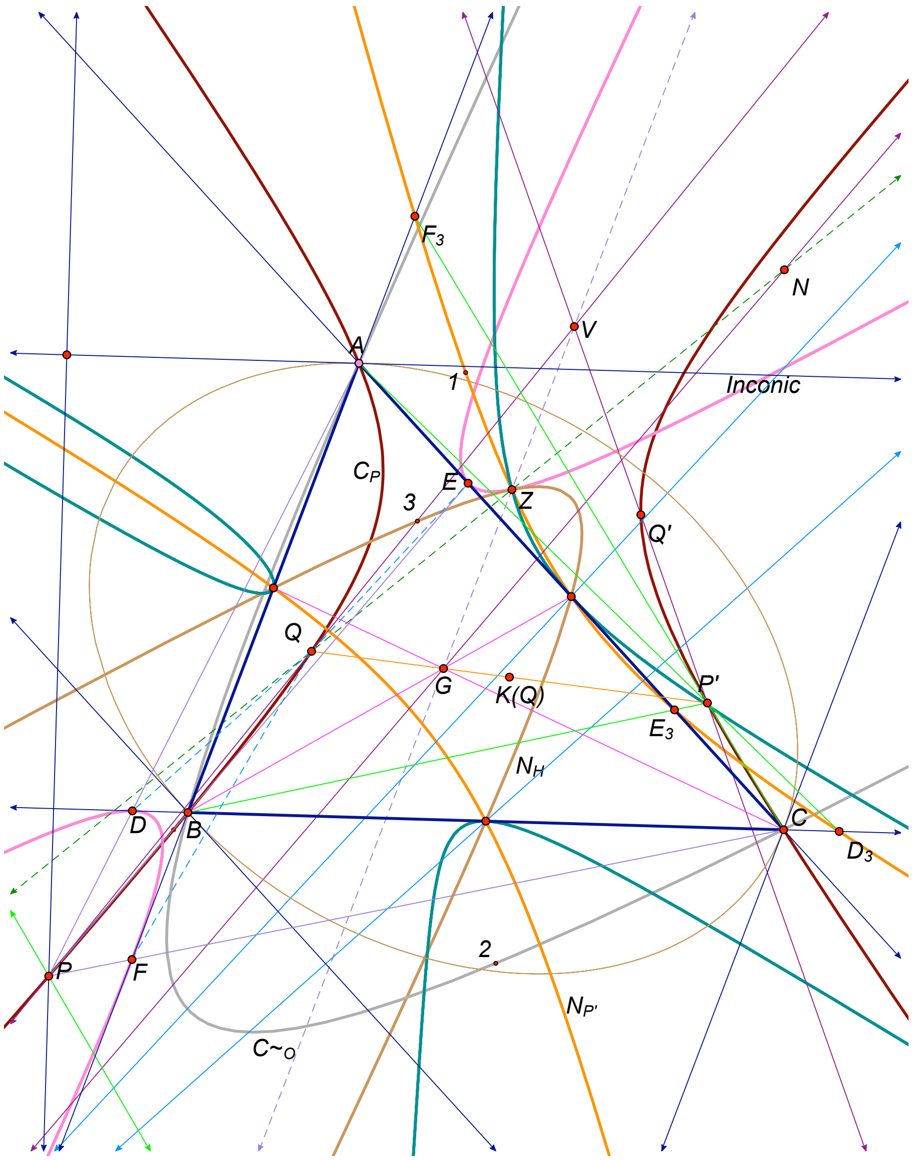

Figure 3: $\mathcal{N}_H$ (light brown) and $\mathcal{I}$ (pink) tangent at Z. (The locus of Z is pictured in teal as P varies on a line. The ellipse is the Steiner circumellipse.)

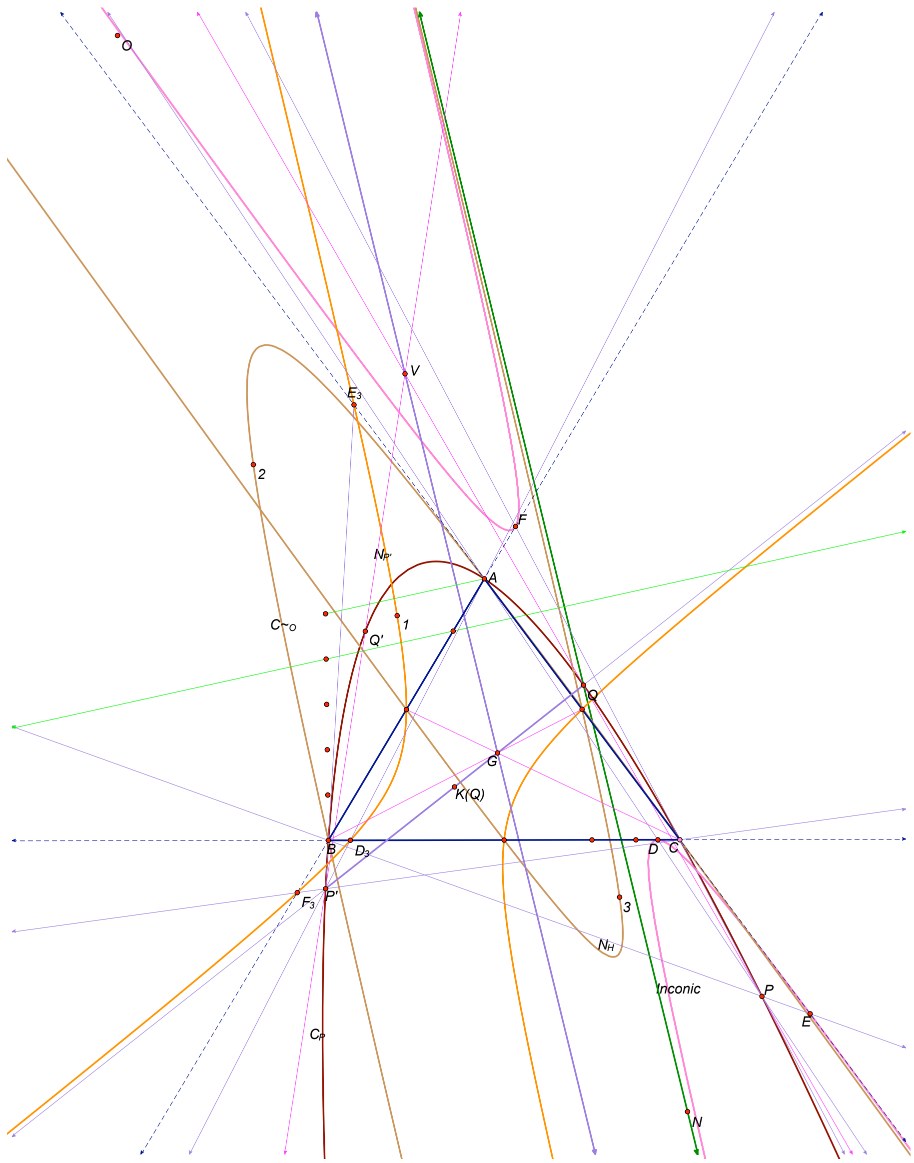

Figure 4: Z infinite, $\mathcal{N}_H$ (light brown) and $\mathcal{I}$ (pink) tangent to each other with the common asymptote QN. $\mathcal{C}_P$ is the parabola in dark brown.

4 The special case $H = A, O = D_0$.

We now consider the set of all points P such that $H = A$ and $O = D_0$. We start with a lemma.

Lemma 4.1. *Provided the generalized orthocenter H of P is defined, the following are equivalent:*

(a) $H = A$.

(b) $QE = AF$ and $QF = AE$.

(c) F_3 is collinear with Q, E_0, and $K(E_3)$.

(d) E_3 is collinear with Q, F_0, and $K(F_3)$.

Proof. We use the fact that $K(E_3)$ is the midpoint of segment BE and $K(F_3)$ is the midpoint of segment CF from I, Corollary 2.2. Statement (a) holds iff $QE \parallel AB$ and $QF \parallel AC$, i.e. iff $AFQE$ is a parallelogram, which is equivalent to (b). Suppose (b) holds. Let $X = BE \cdot QF_3$. Then triangles BXF_3 and EXQ are congruent since $QE \parallel BF_3 = AB$ and $QE = AF = BF_3$. Therefore, $BX = EX$, i.e. X is the midpoint $K(E_3)$ of BE, so Q, F_3, and $X = K(E_3)$ are collinear. Similarly, Q, E_3, and $K(F_3)$ are collinear. This shows (b) $\Rightarrow$ (c), (d).

Next, we show (c) and (d) are equivalent. Suppose (c) holds. Since P', E_3, B are collinear, $Q, K(E_3), E_0$ are collinear and the line F_3E_0 is the complement of the line BE_3, hence the two lines are parallel and

$$\frac{AF_3}{F_3B} = \frac{AE_0}{E_0E_3}. \tag{4}$$

Conversely, if the equality holds, then the lines are parallel and F_3 lies on the line through $K(E_3)$ parallel to $P'E_3$, i.e. the line $K(P'E_3) = QK(E_3)$, so (c) holds. Similarly, (d) holds if and only if

$$\frac{AE_3}{E_3C} = \frac{AF_0}{F_0F_3}. \tag{5}$$

A little algebra shows that (4) holds if and only if (5) holds. Using signed distances, and setting $AE_0/E_0E_3 = x$, we have $AE_3/E_3C = (x+1)/(x-1)$. Similarly, if $AF_0/F_0F_3 = y$, then $AF_3/F_3B = (y+1)/(y-1)$. Now (4) is equivalent to $x = (y+1)/(y-1)$, which is equivalent to $y = (x+1)/(x-1)$, hence also to (5). Thus, (c) is equivalent to (d). Note that this part of the lemma does not use that H is defined.

Now if (c) or (d) holds, then they both hold. We will show (b) holds in this case. By the reasoning in the previous paragraph, we have $F_3Q \parallel E_3P'$ and $E_3Q \parallel F_3P'$, so $F_3P'E_3Q$ is a parallelogram. Therefore, $F_3Q = P'E_3 = 2 \cdot QK(E_3)$, so $F_3K(E_3) = K(E_3)Q$. This implies the triangles $F_3K(E_3)B$ and $QK(E_3)E$ are congruent (SAS), so $AF = BF_3 = QE$. Similarly, $AE = CE_3 = QF$, so (b) holds. □

Theorem 4.2. *The locus $\mathscr{L}_A$ of points P such that $H = A$ is a subset of the conic $\overline{\mathcal{C}}_A$ through B, C, E_0, and F_0, whose tangent at B is $K^{-1}(AC)$ and whose tangent at C is $K^{-1}(AB)$. Namely, $\mathscr{L}_A = \overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$.*

Proof. Given E on AC we define F_3 as $F_3 = E_0K(E_3) \cdot AB$, and F to be the reflection of F_3 in F_0. Then we have the following chain of projectivities:

$$BE \;\bar{\wedge}\; E \;\bar{\wedge}\; E_3 \;\overset{G}{\bar{\wedge}}\; K(E_3) \;\overset{E_0}{\bar{\wedge}}\; F_3 \;\bar{\wedge}\; F \;\bar{\wedge}\; CF.$$

Then $P = BE \cdot CF$ varies on a line or a conic. We want to show: (a) for a point P thus defined, $H = A$; and (b) if $H = A$ for some P, then P arises in this way, i.e. F_3 is on $E_0K(E_3)$. Both of these facts follow from the above lemma.

Now we list four cases in which H is undefined, namely when $P = B, C, E_0, F_0$. Let $A_\infty, B_\infty, C_\infty$ represent the points at infinity on the respective lines BC, AC, AB.

1. For $E = B_\infty = E_3 = K(E_3)$, we have $E_0K(E_3) = AC$ so $F_3 = A, F = B$, and $P = BE \cdot CF = B$.

2. For $E = C$, we have $E_3 = A, K(E_3) = D_0, E_0K(E_3) = D_0E_0 \parallel AB, F = F_3 = C_\infty$, so $P = BE \cdot CF = C$.

3. For $E = E_0$, we have $E_3 = E_0$ and $K(E_0)$ is the midpoint of BE_0 by I, Corollary 2.2, so $F_3 = B, F = A$, and $P = BE \cdot CF = E_0$.

4. For $E = A$, we have $E_3 = C, K(E_3) = F_0, F_3 = F = F_0$, and $P = BE \cdot CF = F_0$.

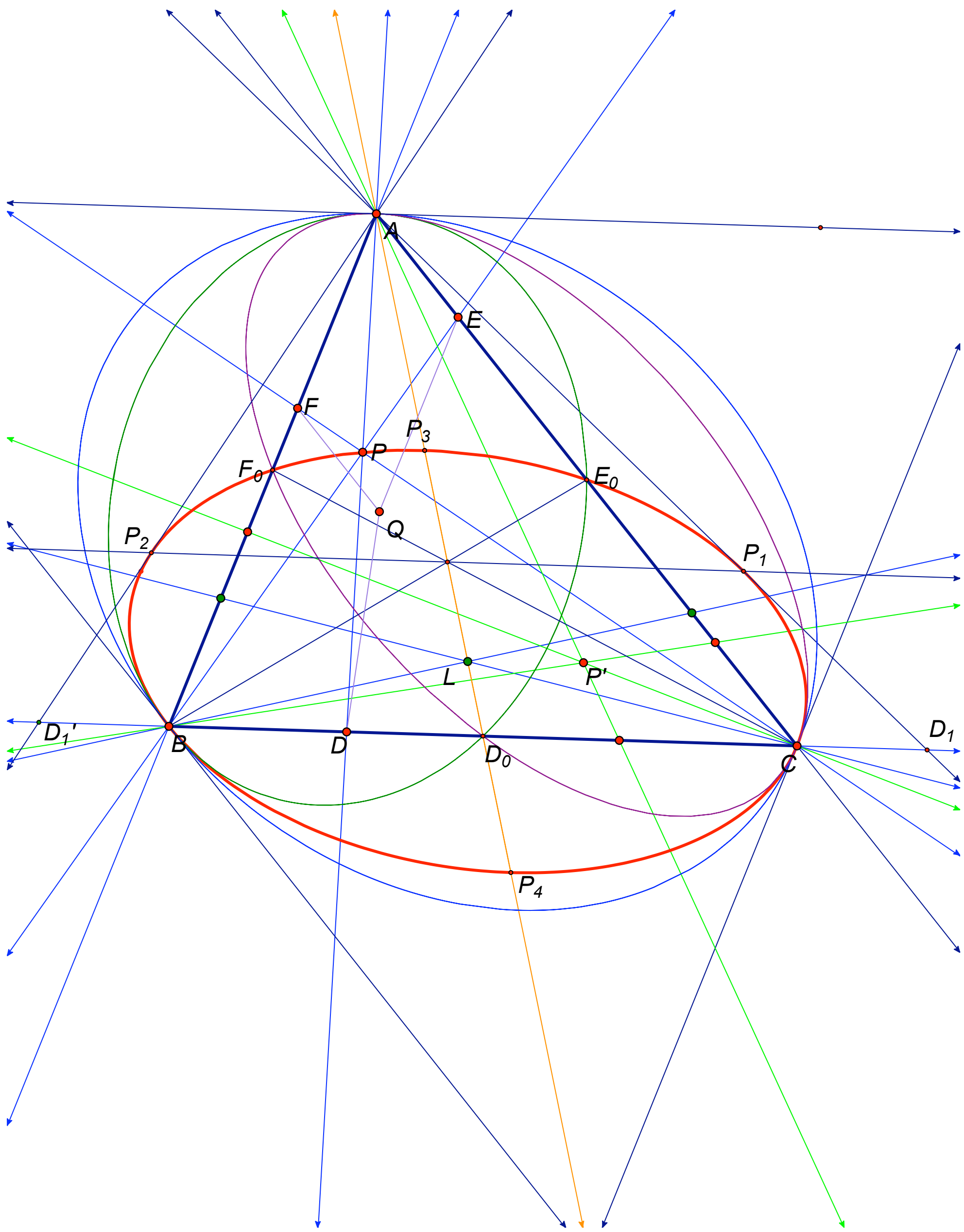

Figure 5: The conics $\overline{\mathcal{C}}_A$ (red), $\overline{\mathcal{C}}_B$ (purple), $\overline{\mathcal{C}}_C$ (green), and $\iota(l_\infty)$ (blue).

Since the four points B, C, E_0, F_0 are not collinear, this shows that the locus of points $P = BE \cdot CF$ is a conic $\overline{\mathcal{C}}_A$ through B, C, E_0, F_0. Moreover, the locus $\mathscr{L}_A$ of points P such that $H = A$ is a subset of $\overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$.

We claim that if E is any point on line AC other than A, C, E_0, or B_∞, then P is a point for which H is well-defined. First, E_3 is an ordinary point because $E \neq B_\infty$. Second, because $E \neq B_\infty$, the line $E_0K(E_3)$ is not a sideline of ABC. The line $E_0K(E_3)$ intersects AB in A if and only if $K(E_3)$ lies on AC, which is true only if $E_3 = B_\infty$. The line $E_0K(E_3)$ intersects AB in B iff $K(E_3)$ is on BE_0, which holds iff E_3 is on $K^{-1}(B)B = BE_0$, and this is the case exactly when $E = E_3 = E_0$. Furthermore, the line $E_0K(E_3)$ is parallel to AB iff $K(E_3) = D_0$ and $E_3 = A$, or $K(E_3) = E_3 = E = C_\infty$, which is not on AC. Thus, the line $E_0K(E_3)$ intersects AB in an ordinary point which is not a vertex, so F_3 and F are not vertices and $P = BE \cdot CF$ is a point not on the sides of ABC.

It remains to show that P does not lie on the sides of the anticomplementary triangle of ABC. If P is on $K^{-1}(AB)$ then $F = F_3 = C_\infty$, which only happens in the excluded case $E = C$ (see Case 2 above). If P is on $K^{-1}(AC)$ then $E = B_\infty$, which is also excluded. If P is on $K^{-1}(BC)$ then P' is also on $K^{-1}(BC)$ so $Q = K(P')$ is on BC.

Suppose Q is on the same side of D_0 as C. Then P' is on the opposite side of line AD_0 from C, so it is clear that CP' intersects AB in the point F_3 between A and B. If Q is between D_0 and C, then F_3 is between A and F_0 (since F_0, C and $K^{-1}(C)$ are collinear), and it is clear that F_3E_0 can only intersect BC in a point outside of the segment D_0C, on the opposite side of C from Q. But this is a contradiction, since by construction F_3, E_0, and $K(E_3)$ are collinear, and $Q = K(P')$ lies on $K(BE_3) = E_0K(E_3)$. On the other hand, if the betweenness relation $D_0 * C * Q$ holds, then F_3 is between B and F_0, and it is clear that F_3E_0 can only intersect BC on the opposite side of B from C. This contradiction also holds when $P' = Q$ is a point on the line at infinity, since then $F_3 = B$, and B, E_0 and $Q = A_\infty$ (the point at infinity on BC) are not collinear. A symmetric argument applies if Q is on the same side of D_0 as B, using the fact that parts (c) and (d) of the lemma are equivalent. Thus, no point P in $\overline{\mathcal{C}}_A \setminus \{B, C, E_0, F_0\}$ lies on a side of ABC or its anticomplementary triangle, and the point H is well-defined; further, $H = A$ for all of these points.

Finally, by the above argument, there is only one point P on $\overline{\mathcal{C}}_A$ that is on the line $K^{-1}(AB)$, namely C, and there is only one point P on $\overline{\mathcal{C}}_A$ that is on the line $K^{-1}(AC)$, namely B, so these two lines are tangents to $\overline{\mathcal{C}}_A$. □

This theorem shows that the locus of points P, for which the generalized orthocenter H is a vertex of ABC, is the union of the conics $\overline{\mathcal{C}}_A \cup \overline{\mathcal{C}}_B \cup \overline{\mathcal{C}}_C$ minus the vertices and midpoints of the sides.

In the next proposition and its corollary, we consider the special case in which $H = A$ and D_3 is the midpoint of AP'. We will show that, in this case, the map M is a translation. (See Figure 7.) We first show that this situation occurs.

Lemma 4.3. *If the equilateral triangle ABC has sides of length 2, then there is a point P with $AP \cdot BC = D$ and $d(D_0, D) = \sqrt{2}$, such that D_3 is the midpoint of the segment AP' and $H = A$.*

Proof. (See Figure 6.) We will construct P' such that D_3 is the midpoint of AP' and $H = A$, and then show that P satisfies the hypothesis of the lemma. The midpoint D_0 of BC satisfies $D_0B = D_0C = 1$ and $AD_0 = \sqrt{3}$. Let the triangle be positioned as in Figure 3. Let $\tilde{A}$ be the reflection of A in D_0, and let D be a point on BC to the right of C such that $D_0D = \sqrt{2}$. In order to insure that the reflection D_3 of D in D_0 is the midpoint of AP', take P' on $l = K^{-2}(BC)$ with $P'\tilde{A} = 2\sqrt{2}$ and P' to the left of $\tilde{A}$. Then $Q = K(P')$ is on $K^{-1}(BC)$, to the right of A, and $AQ = \sqrt{2}$. Let E_3 and F_3 be the traces of P' on AC and BC, respectively.

We claim $BF_3 = \sqrt{2}$. Let M be the intersection of BC and the line through F_3 parallel to AD_0. Then triangles BMF_3 and BD_0A are similar, so $F_3M = \sqrt{3} \cdot MB$. Let N_1 be the intersection of BC and the line through P' parallel to AD_0. Triangles $P'N_1C$ and F_3MC are similar, so

$$\frac{F_3M}{MC} = \frac{P'N_1}{N_1C} = \frac{AD_0}{P'\tilde{A} + 1} = \frac{\sqrt{3}}{2\sqrt{2} + 1}.$$

Therefore,

$$\frac{\sqrt{3}}{2\sqrt{2} + 1} = \frac{F_3M}{MC} = \frac{\sqrt{3} \cdot MB}{MB + 2}$$

which yields that $MB = 1/\sqrt{2}$. Then $BF_3 = \sqrt{2}$ is clear from similar triangles.

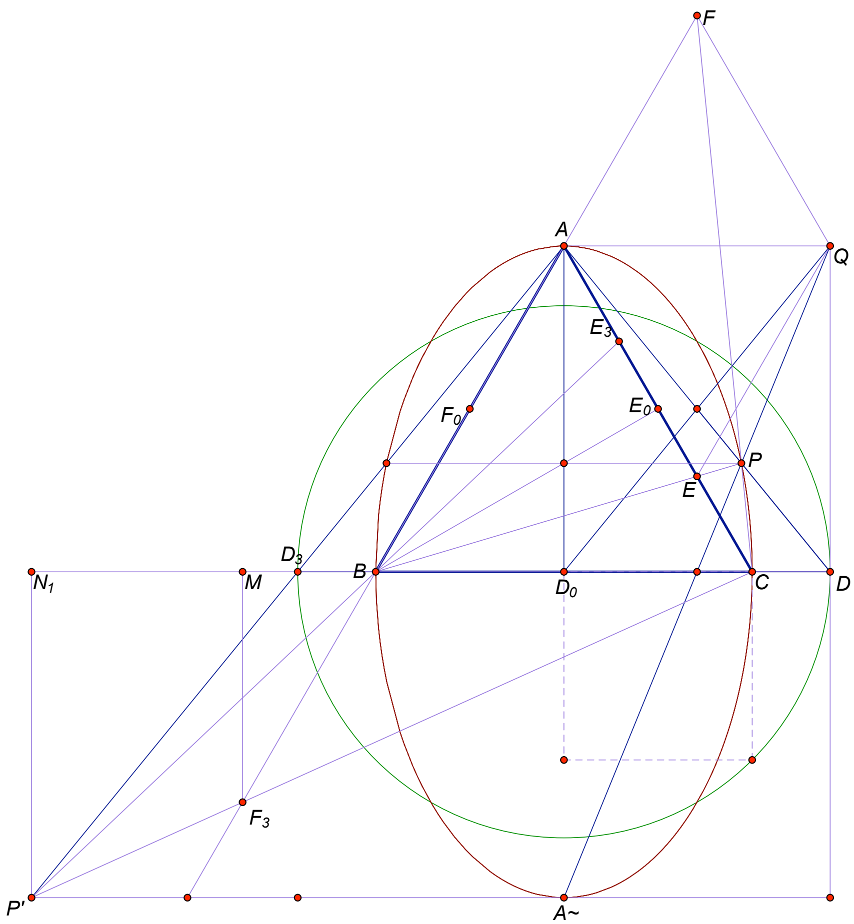

Figure 6: Proof of Lemma 4.3.

Now, let F be the reflection of F_3 in F_0 (the midpoint of AB). Then AQF is an equilateral triangle because $m(\angle FAQ) = 60^\circ$ and $AQ \cong BF_3 \cong AF$, so $\angle AQF \cong \angle AFQ$. Therefore, $QF \parallel AC$. It follows that the line through F_0 parallel to QF is parallel to AC, hence is a midline of triangle ABC and goes through D_0. (A similar proof shows that $QE \parallel AB$ so the line through E_0 parallel to QE goes through D_0.) This implies $O = D_0$. Clearly, $P = AD \cdot CF$ is a point outside the triangle ABC, not lying on an extended side of ABC or its anticomplementary triangle, which satisfies the conditions of the lemma. □

The next proposition deals with the general case, and shows that the point P we constructed in the lemma lies on a line through the centroid G parallel to BC.

Proposition 4.4. *Assume that $H = A, O = D_0$, and D_3 is the midpoint of AP'. Then the conic $\tilde{\mathcal{C}}_O = \iota(l)$, where $l = K^{-1}(AQ) = K^{-2}(BC)$ is the line through the reflection $\tilde{A}$ of A in O parallel to the side BC. The points O, O', P, P' are collinear, with $d(O, P') = 3d(O, P)$, and the map M taking $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$ is a translation. In this situation, the point P is one of the two points in the intersection $l_G \cap \tilde{\mathcal{C}}_O$, where l_G is the line through the centroid G which is parallel to BC.*

Proof. (See Figure 7.) Since the midpoint R_1' of segment AP' is D_3, lying on BC, P' lies on the line l which is the reflection of $K^{-1}(BC)$ (lying on A) in the line BC. It is easy to see that this line is $l = K^{-2}(BC)$, and hence $Q = K(P')$ lies on $K^{-1}(BC)$. From I, Corollary 2.6 we know that the points $D_0, R_1' = D_3$, and $K(Q)$ are collinear. Since $K(Q)$ is the center of the conic $\mathcal{N}_{P'}$, lying on D_0 and D_3, $K(Q)$ is the midpoint of segment D_0D_3 on BC. Applying the map $T_{P'}^{-1}$ gives that $O = T_{P'}^{-1}(K(Q))$ is the midpoint of $T_{P'}^{-1}(D_3D_0) = AT_{P'}^{-1}(D_0)$. It follows that $T_{P'}^{-1}(D_0) = \tilde{A}$ is the reflection of A in O, so that $\tilde{A} \in \tilde{\mathcal{C}}_O$. Moreover, $K(A) = O$, so $\tilde{A} = K^{-1}(A)$ lies on $l = K^{-1}(AQ) \parallel BC$.

Next we show that $\tilde{\mathcal{C}}_O = \iota(l)$, where the image $\iota(l)$ of l under the isotomic map is a circumconic of ABC (see Lemma 3.4 in Part IV). It is easy to see that $\iota(\tilde{A}) = \tilde{A}$, since $\tilde{A} \in AG$ and $AB\tilde{A}C$ is a parallelogram. Therefore, both conics $\tilde{\mathcal{C}}_O$ and $\iota(l)$ lie on the 4 points $A, B, C, \tilde{A}$. To show they are the same conic, we show they are both tangent to the line l at the point $\tilde{A}$. From Corollary 3.5 the tangent to $\tilde{\mathcal{C}}_O$ at $\tilde{A} = T_{P'}^{-1}(D_0)$ is parallel to BC, and must therefore be the line l. To show that l is tangent to $\iota(l)$, let L be a point on $l \cap \iota(l)$. Then $\iota(L) \in l \cap \iota(l)$. If $\iota(L) \neq L$, this would give three distinct points, $L, \iota(L)$, and $\tilde{A}$, lying on the intersection $l \cap \iota(l)$, which is impossible. Hence, $\iota(L) = L$, giving that L lies on AG and therefore $L = \tilde{A}$. Hence, $\tilde{A}$ is the only point on $l \cap \iota(l)$, and l is the tangent line. This shows that $\tilde{\mathcal{C}}_O$ and $\iota(l)$ share 4 points and the tangent line at $\tilde{A}$, proving that they are indeed the same conic.

From this we conclude that $P = \iota(P')$ lies on $\tilde{\mathcal{C}}_O$. Hence, P is the fourth point of intersection of the conics $\tilde{\mathcal{C}}_O$ and $\mathcal{C}_P = ABCPQ$. From Theorem 3.14 we deduce that $P = \tilde{Z} = R_O K^{-1}(Z)$, which we showed in the proof of that theorem to be a point on the line OP'. Hence, P, O, P' are collinear, and applying the affine reflection η gives that O' lies on the line PP', as well. Now, Z is the midpoint of $HP = AP$, since $\mathsf{H} = K \circ \mathsf{R}_O$ is a homothety with center $H = A$ and similarity factor $1/2$. Since Z lies on GV, where $V = PQ \cdot P'Q'$, it is clear that P and Q are on the opposite side of the line GV from P', Q', and A. The relation $K(\tilde{A}) = A$ means that $\tilde{A}$ and also O are on the opposite side of GV from A and O'. Also, $J = K^{-1}(Z) = \mathsf{R}_O(\tilde{Z}) = \mathsf{R}_O(P)$ lies on the line GV and on the conic $\tilde{\mathcal{C}}_O$. This implies that O lies between J and P, and applying η shows that O' lies between J and P'. Hence, OO' is a subsegment of PP', whose midpoint is exactly $J = K^{-1}(Z)$, since this is the point on GV collinear with O and O'. Now the map η preserves distances along lines parallel to PP' (see Part II), so $JO' \cong JO \cong OP \cong O'P'$, implying that OO' is half the length of PP'. Furthermore, segment $QQ' = K(PP')$ is parallel to PP' and half as long. Hence, $OO' \cong QQ'$, which implies that $OQQ'O'$ is a parallelogram. Consequently, $OQ \parallel O'Q'$, and Theorem 3.9 shows that M is a translation. Thus, the circumconic $\tilde{\mathcal{C}}_O$ and the inconic $\mathcal{I}$ are congruent in this situation.

This argument implies the distance relation $d(O, P') = 3d(O, P)$.

The relation $O'Q' \parallel OQ$ implies, finally, that $T_P(O'Q') \parallel T_P(OQ)$, or $K(Q')P \parallel A_0Q = AQ$, since $O' = T_P^{-1}K(Q')$ from Theorem 2.2 and A_0 is collinear with A and the fixed point Q of T_P by I, Theorem 2.4. Hence $PG = PQ' = PK(Q')$ is parallel to AQ and BC. □

There are many interesting relationships in the diagram of Figure 7. We point out several of these relationships in the following corollary.

Corollary 4.5. *Assume the hypotheses of Proposition 4.4.*

a) If Q_a is the vertex of the anticevian triangle of Q (with respect to ABC)

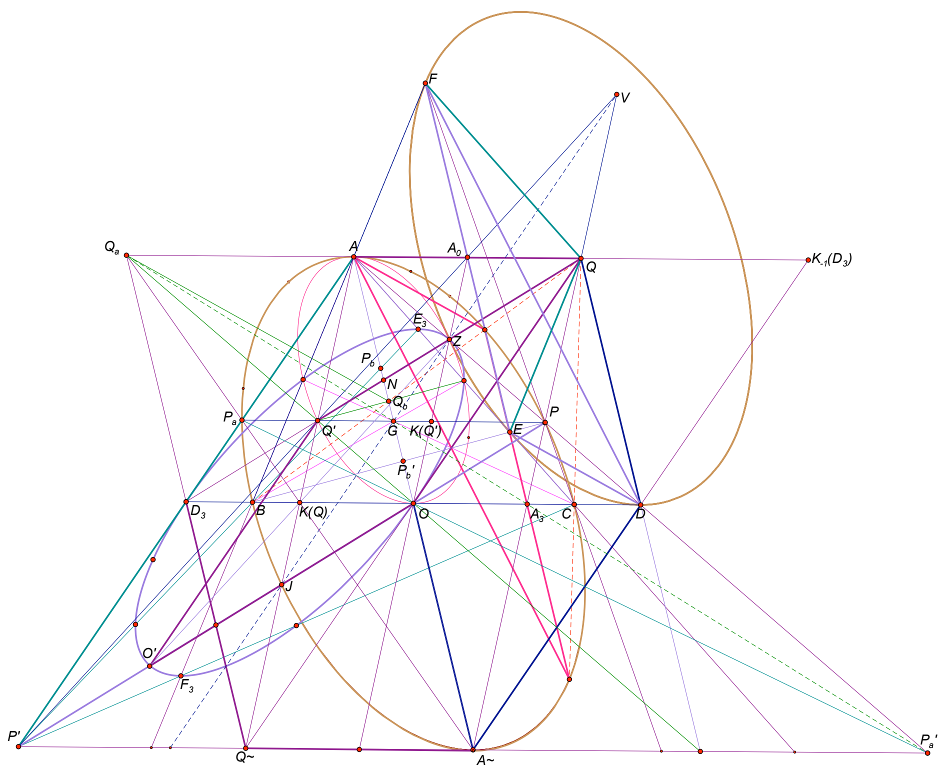

Figure 7: The case $H = A, O = D_0$.

opposite the point A, then the corresponding point P_a is the second point of intersection of the line PG with $\tilde{\mathcal{C}}_O$.

b) *The point $A_3 = T_P(D_3)$ is the midpoint of segment OD and P is the centroid of triangle ODQ.*

c) *The ratio $\frac{OD}{OC} = \sqrt{2}$.*

Proof. The anticevian triangle of Q with respect to ABC is the triangle $T_{P'}^{-1}(ABC) = Q_aQ_bQ_c$. (See I, Cor. 3.11 and Section 2 above.) Since D_3 is the midpoint of AP', this gives that $T_{P'}^{-1}(D_3) = A$ is the midpoint of $T_{P'}^{-1}(AP') = Q_aQ$. Therefore, Q_a lies on the line $AQ = K^{-1}(BC)$, so $P'_a = K^{-1}(Q_a)$ lies on the line l and is the reflection of P' in the point $\tilde{A}$. Thus, the picture for the point P_a is obtained from the picture for P by performing an affine reflection about the line $AG = A\tilde{A}$ in the direction of the line BC. This shows that P_a also lies on the line $PG \parallel BC$. The conic $\tilde{\mathcal{C}}_O$ only depends on O, so this reflection takes $\tilde{\mathcal{C}}_O$ to itself. This proves a).

To prove b) we first show that P lies on the line $Q\tilde{A}$. Note that the segment $K(P'\tilde{A}) = AQ$ is half the length of $P'\tilde{A}$, so $P'\tilde{A} \cong Q_aQ$. Hence, $Q_aQ\tilde{A}P'$ is a parallelogram, so $Q\tilde{A} \cong Q_aP'$. Suppose that $Q\tilde{A}$ intersects line PP' in a point X. From the fact that $K(Q)$ is the midpoint of D_3D_0 we know that Q is the midpoint of $K^{-1}(D_3)A$. Also, D_3Q' lies on the point $\lambda(A) = \lambda(H) = Q$, by II, Theorem 3.4(b) and Theorem 2.7 of this paper. It follows that $K^{-1}(D_3), P, P'$ are collinear and $K^{-1}(D_3)QX \sim P'\tilde{A}X$, with similarity ratio $1/2$, since $K^{-1}(D_3)Q$ has half the length of $P'\tilde{A}$. Hence $d(X, K^{-1}(D_3)) = \frac{1}{2}d(X, P')$. On the other hand, $d(O, P) = \frac{1}{3}d(O, P')$, whence it follows, since O is halfway between P' and $K^{-1}(D_3)$ on line BC, that $d(P, K^{-1}(D_3)) = \frac{1}{2}d(P, P')$. Therefore, $X = P$ and P lies on $Q\tilde{A}$.

Now, $\mathsf{P} = AD_3OQ$ is a parallelogram, since $K(AP') = OQ$, so opposite sides in AD_3OQ are parallel. Hence, $T_P(\mathsf{P}) = DA_3A_0Q$ is a parallelogram, whose side A_3A_0 lies on the line EF. Applying the dilatation $\mathsf{H} = K\mathsf{R}_O$ (with center $H = A$) to the collinear points $Q, P, \tilde{A}$ shows that $\mathsf{H}(Q), Z$, and O are collinear. On the other hand, $O = D_0, Z$, and A_0 are collinear by I, Corollary 2.6 (since $Z = R$ is the midpoint of AP), and A_0 lies on AQ by I, Theorem 2.4. This implies that $A_0 = \mathsf{H}(Q) = AQ \cdot OZ$ is the midpoint of segment AQ, and therefore A_3 is the midpoint of segment OD. Since P lies on the line PG, $2/3$ of the way from the vertex Q of ODQ to the opposite side OD, and lies on the median QA_3, it must be the centroid of ODQ. This proves b).

To prove c), we apply an affine map taking ABC to an equilateral triangle. It is clear that such a map preserves all the relationships in Figure 4. Thus we may assume ABC is an equilateral triangle whose sidelengths are 2. By Lemma 4.3 there is a point P for which $AP \cdot BC = D$ with $D_0D = \sqrt{2}, O = D_0$, and D_3 the midpoint of AP'. Now Proposition 4.4 implies the result, since the equilateral diagram has to map back to one of the two possible diagrams (Figure 4) for the original triangle. □

By Proposition 4.4 and Theorem 2.5 we know that the conic $\overline{\mathcal{C}}_A$ lies on the points P_1, P_2, P_3, P_4, where P_1 and $P_2 = (P_1)_a$ are the points in the intersection $\tilde{\mathcal{C}}_O \cap l_G$ described in Proposition 4.4 and Corollary 4.5, and $P_3 = (P_1)_b, P_4 = (P_1)_c$. (See Figure 5.) It can be shown that the equation of the conic $\overline{\mathcal{C}}_A$ in terms of the barycentric coordinates of the point $P = (x, y, z)$ is $xy + xz + yz = x^2$. Furthermore, the center of $\bar{\mathcal{C}}_A$ lies on the median AG, 6/7-ths of the way from A to D_0.

Remarks. 1. The polar of A with respect to the conic $\bar{\mathcal{C}}_A$ is the line l_G through G parallel to BC. This because the quadrangle BCE_0F_0 is inscribed in $\bar{\mathcal{C}}_A$, so its diagonal triangle, whose vertices are A, G, and $BC \cdot l_\infty$, is self-polar. Thus, the polar of A is the line l_G.

2. The two points P in the intersection $\bar{\mathcal{C}}_A \cap l_G$ have tangents which go through A. This follows from the first remark, since these points lie on the polar $a = l_G$ of A with respect to $\bar{\mathcal{C}}_A$. As a result, the points D on BC, for which there is a point P on AD satisfying $H = A$, have the property that the ratio of unsigned lengths $DD_0/D_0C \le \sqrt{2}$. This follows from the fact that $\bar{\mathcal{C}}_A$ is an ellipse: since it is an ellipse for the equilateral triangle, it must be an ellipse for any triangle. Then the maximal ratio DD_0/D_0C occurs at the tangents to $\bar{\mathcal{C}}_A$ from A; and we showed above that for these two points P, $D = AP \cdot BC$ satisfies $DD_0/D_0C = \sqrt{2}$.

References

[ac] N. Altshiller-Court, *College Geometry, An Introduction to the Modern Geometry of the Triangle and the Circle*, Barnes and Noble, New York, 1952. Reprinted by Dover Publications.

[bach] P. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer Verlag, Berlin, 1959.

[co1] H.S.M. Coxeter, *The Real Projective Plane*, McGraw-Hill Book Co., New York, 1949.

[co2] H.S.M. Coxeter, *Projective Geometry*, 2^{nd} edition, Springer, 1987.

[co3] H.S.M. Coxeter, *Introduction to Geometry*, John Wiley & Sons, Inc., New York, 1969.

[ki1] C. Kimberling, Central points and central lines in the plane of a triangle, Mathematics Magazine 67 (1994), 163-187.

[ki2] C. Kimberling, *Encyclopedia of Triangle Centers*, at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[mm1] I. Minevich and P. Morton, Synthetic foundations of cevian geometry, I: Fixed points of affine maps in triangle geometry, http://arXiv.org/abs/1504.00210, 2015.

[mm2] I. Minevich and P. Morton, Synthetic foundations of cevian geometry, II: The center of the cevian conic, http://arXiv.org/abs/1505.05381, 2015.

[mo] P. Morton, Affine maps and Feuerbach's Theorem, IUPUI Math. Dept. Preprint Series pr09-05, 2009, http://math.iupui.edu/research/research-preprints.

Dept. of Mathematics, Carney Hall
Boston College
140 Commonwealth Ave., Chestnut Hill, Massachusetts, 02467-3806
e-mail: igor.minevich@bc.edu

Dept. of Mathematical Sciences
Indiana University - Purdue University at Indianapolis (IUPUI)
402 N. Blackford St., Indianapolis, Indiana, 46202
e-mail: pmorton@math.iupui.edu