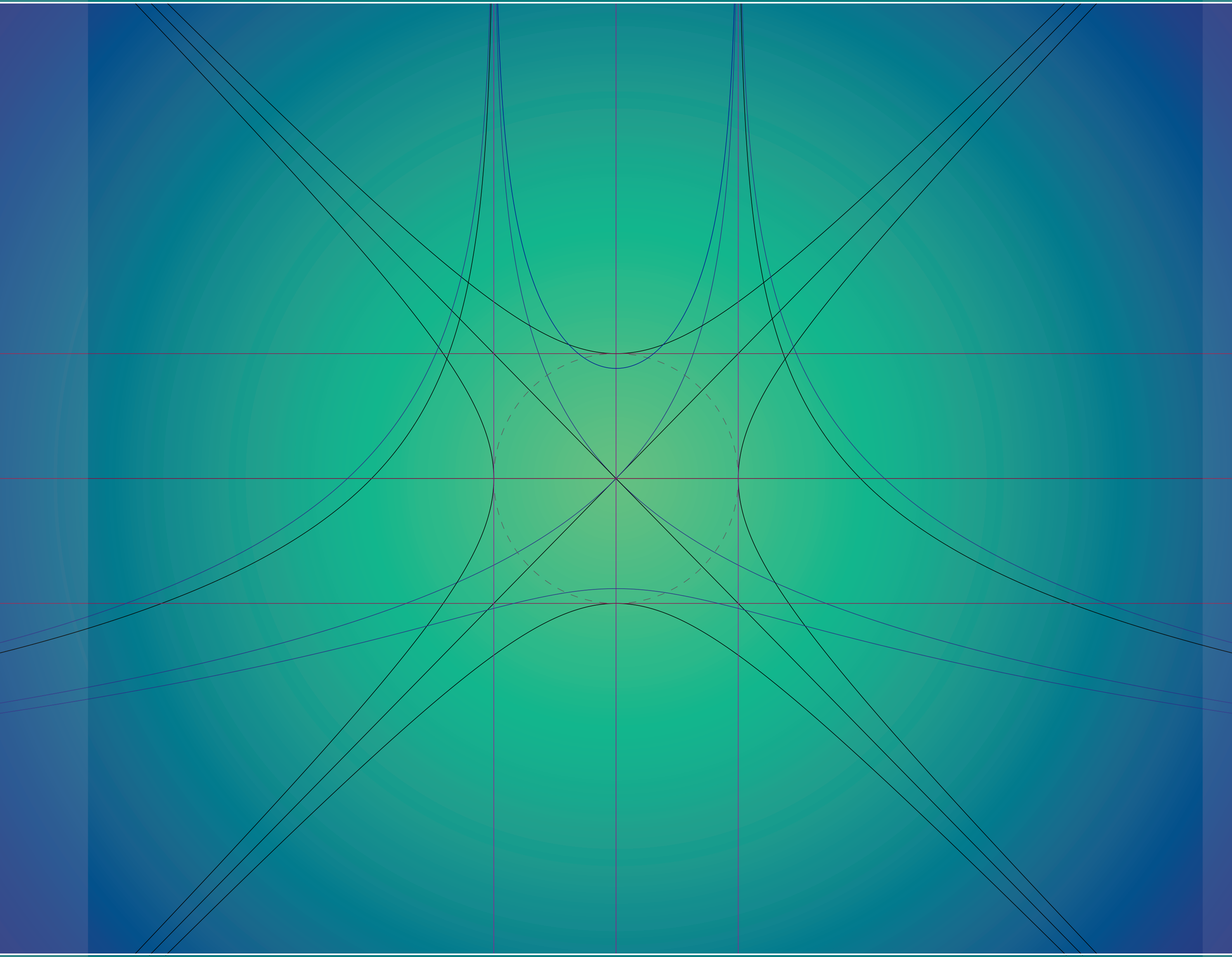
Deformation and Contraction of Symmetries in Special Relativity



A Dissertation by

Lukas Konstantin Brunkhorst

Deformation and Contraction of Symmetries in Special Relativity

Am 28. Februar 2017
dem Fachbereich 1 der Universität Bremen
zur Erlangung des Grades
Doktor der Naturwissenschaften (Dr. rer. nat)
vorgelegte Dissertation

von
Lukas K. Brunkhorst,
geboren am 19.05.1989 in Bremen.

Gutachter:
Prof. Dr. Domenico Giulini
Prof. Dr. Peter Schupp

Datum des Dissertationskolloquiums: Montag, der 8. Mai 2017

ABSTRACT

This dissertation gives an account of the fundamental principles underlying two conceptionally different ways of embedding Special Relativity into a wider context. Both of them root in the attempt to explore the full scope of the Relativity Postulate.

The first approach uses Lie algebraic analysis alone, but already yields a whole range of alternative kinematics that are all in a quantifiable sense near to those in Special Relativity, while being rather far away in a qualitative way. The corresponding models for spacetime are seen to be four-dimensional versions of the prototypical planar geometries associated with the work of Cayley and Klein. The close relationship between algebraic and geometric methods displayed by these considerations is being substantialized in terms of light-like spacetime extensions.

The second direction of departures from Special Relativity stresses and develops the algebraic view on spacetime by considering Hopf instead of Lie algebras as candidates for the description of kinematical transformations and hence spacetime symmetry. This approach is motivated by the belief in the existence of a quantum theory of gravity, and the assumption that such manifests itself in nonlinear modifications of the laws of Special Relativity at length scales comparable to the Planck length.

The twofold character of this work, and the presentation of an example for the fully geometric character of a specific Hopf algebraic deformation of the Poincaré algebra, enable a conclusion that speculates on a possible relationship between the two developed viewpoints via the technique of nonlinear realizations. A non-perturbative approach to the latter is given which generalizes to all the considered geometries.

ABSTRACT (GERMAN)

Die vorliegende Arbeit stellt grundlegende Prinzipien zweier konzeptionell verschiedener Wege dar, die Poincaré-Symmetrie der Spezielle Relativitätstheorie durch eine allgemeinere Interpretation des Relativitätspostulats in einen breiteren Rahmen zu rücken.

Der erste dieser Ansätze benutzt allein Lie-algebraische Methoden, führt jedoch bereits auf eine Reihe alternativer kinematischer Modelle. Diese sind alle in quantifizierbarer Weise jenem der Speziellen Relativitätstheorie nahe, weichen in qualitativem Sinne jedoch drastisch ab. Die entsprechenden Raumzeitmodelle werden vorgestellt und im Sinne der Klassifizerung möglicher Geometrien nach Cayley und Klein eingeordnet. Es wird erläutert, wie sich in speziellen lichtartigen Raumzeiterweiterungen der enge Zusammenhang zwischen algebraischen und geometrischen Methoden manifestiert.

Die zweite Klasse möglicher Abweichungen von der Speziellen Relativitätstheorie erweitert auf Kosten der unmittelbaren geometrischen Anschauung den Symmetriebegriff hin zu Hopfalgebren. Dies wird motiviert durch die Annahme der Existenz einer Quantentheorie der Gravitation, die sich in nichtlinearen Modifikationen speziell-relativistischer Transformationsgesetze niederschlägt.

Der zweiteilige Aufbau der vorliegenden Arbeit ermöglicht die Rückführung einer besonders viel beachteten Hopf-algebraischen Verallgemeinerung der Poincaré-Algebra auf rein geometrische Begriffe. Dies führt zur Formulierung einer Annahme über die Anwendbarkeit ähnlicher Methoden auf eine ganze Klasse von Hopf-Algebren. Als Schlüssel dazu wird die Theorie der nichtlinearen Realisierungen von Lie-Gruppen vorgestellt und über die störungstheoretische Beschreibung hinaus erweitert.

CONTENTS

1. Introduction

Nature presents itself to us in uncountable different shapes and guises. Underlying, however, there are certain invariable structures which to determine is the subject of natural sciences, and it belongs to its assumptions that the basic rules by which Nature abides exist independently of those who seek to discover them. In physics, this becomes manifest in the existence of symmetries, and the application of relativity theory.

Fundamental physics occupies a singular status as a subdiscipline of natural sciences, in that its aim is to understand only the most elementary constituents of nature, and ask only the most fundamental questions. For a long time, and tightly connected with the names of Galilei and Newton, its great achievement was to provide a unifying framework for the description of moving bodies. In the 19th and 20th century then, with the discovery of electromagnetism and the nuclear forces, new phenomena came into focus which were less accessible to the human eye alone, and whose observation necessitated the use of ever more ingenious setups in the laboratory. Along with this evolution a change in what was understood as an observing system took place. While Galilei was still concerned with the physical experiences made by actual sailors under deck of a ship [GD53], the more general conception of an observer includes his usage of measurement devices capable of assigning to all recorded measurement values definitive coordinates in space and time, thereby defining a frame of reference.

In order to infer physical laws from experiments, it almost appears as a necessity that all reference frames are in principle suitable for this task. This means that one should expect the existence of well-defined operations that translate physical quantities as observed in one laboratory system into their form as observed in any other system—and with them the laws describing their mutual dependence.

The (Galilean) Relativity Postulate goes further. It assumes the existence of a privileged class of reference frames, which are distinguished from all others by the property that observed from these, the fundamental physical laws assume their simplest form, which, in addition, stays the same in all such frames. Since they are operationally distinguished as being free of forces, so that their motion is determined by their property of inertia alone, these reference frames have been called 'inertial frames'. The fact that the fundamental laws are found to share the same form in all such frames does not mean that the quantities determined by these laws have the same value. On the contrary, since inertial frames are a subset of all possible references frames, there exist nontrivial transformations between them that relate all physical quantities. It appears adequate to call this subset of all transformations between reference frames *inertial transformations*. It is distinguished by the property of leaving the fundamental laws invariant, which means nothing but that inertial transformations constitute a symmetry.

To Galilei and Newton it would appear that the invariance under inertial transformations is a property of the fundamental physical laws. The conceptual leap that is primarily attributed to Einstein is to instead understand this invariance property as belonging to space and time itself. This becomes plausible when realizing how inertial frames are always characterized with respect to one another in terms of spatio-temporal notions, namely by their location (in space and time), their orientation, and their velocity. Adopting this view, the basic physical laws are seen to merely inherit the symmetry properties that spacetime dictates.

Given the Relativity Postulate, an actual theory of relativity, and hence of spacetime as we argued, consists in specifying the set of inertial transformations. This is equivalent with fixing the allowable kinematics for a system, which could be a moving point particle or a field in spacetime. For his construction of the Special Theory of Relativity, Einstein added the assumption (well-motivated by the famous thought experiments as well as Maxwell's formulation of electrodynamics) that the speed of light be measured to have the same numerical value in all inertial frames. This allowed him to deduce the Poincaré group as the proper structure implementing inertial transformations, and at the same time, suggest Minkowski spacetime M as the corresponding model for space and time.

It is important to be aware of how closely considerations about inertial symmetry go hand in hand with the conception of spacetime. For Special Relativity, this is mathematically achieved by the unique characterization of the Poincaré group as the automorphism group of Minkowski space [Giu10]. While it might be more customary to think of symmetries as capturing the indifference of a certain object against a given set of operations, implementations of the Relativity Postulate justify turning this around. We will see how, for a given group of inertial transformations, one can infer a corresponding spacetime geometry. This puts relativity theory in close contact with the work of Klein and his Erlangen programme, in which the very notion of geometry is reduced to that of group actions [Kle72b; Kle72a].

Triggered by an unease felt in postulating the existence of a fundamental constant of nature, one might raise the question, and indeed numerous authors have done so in the past, whether the invariance of the speed of light is strictly needed as an input in order to arrive at the Poincaré group. Investigations of this kind show however that when dropping this as a requirement, other assumptions about the nature of inertial transformations or the properties of spacetime step into place, so that it becomes a matter of choice which basic postulates to start from.

From a practical standpoint, one could object and wonder whether such abstract consideration should not better be regarded as a subject of formal axiomatization programmes rather than expecting from them actual physical insights. But this would depreciate the impact that spacetime concepts have had on the evolution of physics. In a sense even, the complete history of classical physics since the time of Galilei can be understood in such terms, namely as the abandoning of Galilei's conception of absolute time in favour of the space-and-time unifying structure of Minkowski space, and finally the trade of the latter for a general semi-Riemannian manifold in order to include gravity. It is true however, that there is a different spirit to inquiring the nature of relativity than there is, say, to developing a theory of the atom. The former has a more universal character and should, in principle, serve as a foundation not only for the analysis of atomic processes, but also that of macroscopic phenomena, and indeed that of physical systems of all kind. This is why Special Relativity is sometimes called a 'meta-theory'. From a theory-building perspective, it then seems promising to take seriously any attempts of putting Einstein's theory into a wider context. In this manner one can then try to build a 'meta-meta-theory' that is able to uncover hidden and possibly useful structures in the set of theories implementing the Relativity Postulate. It is this conceptual background into which this dissertation should be seen embedded.

An attitude of the described kind puts much emphasis on the role of symmetry in physics. Apart from their impact on spacetime models, this can further be justified by the observation of how symmetry concepts have largely dictated the developments in modern fundamental physics over the past century. In particular the swift progression from the introduction of quantum mechanics to the formulation of the standard model of particle physics as it is known today would be hard to imagine otherwise. In this case actually, invariance properties of a second kind enter. Their recognition was sparked once more by Maxwell's Electrodynamics—this time however not by its behaviour in space and time, but by the indifference of the field strength against redefinitions of the potential. Transformations belonging to this second kind are in general not concerned with spacetime (except in the sense that they can depend on it) but act on the internal configuration space of fields, or, wave-functions. They go by the name of gauge transformations, and indeed much of a quantum field theory is already fixed by requiring it to be Poincaré-invariant and specifying the gauge group. In the standard model, the latter is famously given by $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, which leaves as one remnant at low energies an isomorphic copy of the last factor, responsible for the phase invariance observed in Electromagnetism. Arguably the trademark for the group-theoretical basis of modern particle physics is the Higgs boson, whose Nobel-awarded prediction preceded its 2012 discovery at the LHC by almost half a century.

One of the principal lessons from the success of the standard model is that symmetries can even be useful in the construction of theories when they are not actually present under the conditions at which the theories can be tested. This idea lies at the origin of a larger theme that goes by the name of spontaneous symmetry breaking.

The way in which the Poincaré group enters quantum theory is by determining the possible particle content, namely via its irreducible unitary representations [Wig39]. In this sense it captures not only a notion of spacetime, but is also responsible for the very language that we employ for the description of the basic building blocks of matter. It underlines the universal character of Special Relativity that it is applicable both to macroscopic and subatomic systems alike—the transformations that an observer at the train station platform has to perform on his clocks and rods in order to compare with someone on the passing train are the same transformations as those that govern what we consider as an elementary physical system.

Much could be speculated about why it is that symmetries have proven so useful in physics. In the case of continuous, global symmetries like those of Special Relativity, this can partly be understood from Noether's great insight that every symmetry of this kind entails a conservation law for an associated quantity. These conserved quantities are typically of chief physical interest, like energy in the case of time translation invariance, and their existence simplifies the formulation of a theory. Also, symmetries of this type can be used to find new solutions to a theory from known ones. Although for gauge symmetries these reasons do not apply as easily, they too are able to manifest desirable features like locality in a theory. Both types of symmetry hence provide a rigid structure, a point of orientation in the otherwise daunting landscape of theories. It should then perhaps be taken as a gracious feature of nature that at a fundamental level, it allows itself to be described in such simplistic terms.

While it might well be that eventually gauge symmetries turn out to share a common origin, or at least a common mathematical representation with spacetime symmetries, this thesis keeps its focus on the latter, with only a few extended remarks towards the end. It does so in full awareness of Einstein's more complete heritage, the General Theory of Relativity, which denies the existence of any rigid structure to spacetime while incorporating gravity as its curvature, and recovers Special Relativity only in sufficiently small regions. The semi-Riemannian metric structure of the theory demands in fact that at each point in spacetime, the homogeneous Lorentz part of the Poincaré group survives strictly.

Apart from conceptual reflections upon alternative realizations of the Relativity Postulate, there is a further reason to question the seemingly firm and well-tested place the symmetries of Special Relativity occupy in modern fundamental physics. This is the search for a complete quantum theory of gravity that should supposedly take over at length scales comparable to the Planck length, which is however so tiny ($\sim 10^{-35}\,\mathrm{m}$, roughly a hundredth of a billionth of a billionth of a proton's radius) that for all purposes of current physics its finiteness has played no role. Endeavours with such motivation are based on the belief that a fundamentally new theory will also manifest itself in a different conception of spacetime, or even abandon it completely, and that this will simultaneously demand an emancipation from the Poincaré, or, (local) Lorentz group. If this is so, one expects an intermediate regime in which remnant effects of the fundamental theory already surface but can still be mathematically represented in terms of structures that are in some sense close to the conventional ones [Smo06; Mav07].

For instance, these could be parametrized by the magnitude of vacuum expectation values for fields which become dynamical only at much larger energies than have been experimentally probed so far. At low energies, they could be regarded as fixed, and would therefore define preferred reference frames. Many of the experimental efforts to test Lorentz invariance have been directed into searches for such effects, which is reflected by tight constraints on the associated parameters. As perhaps the most impressive one, we only mention here the upper bound on the anisotropy of the speed of light, which has been determined to lie at $\Delta c/c \sim 10^{-18}$ in an earthbound laboratory [Nag+15]. We will only briefly comment on scenarios for Lorentz violation of this kind, which are formally treated by considering proper subgroups of the Lorentz group. Instead we stress that the absence of preferred frames would not directly imply that Lorentz invariance holds exactly in the way in which it appears in Special Relativity.

For example, one might consider scenarios in which Lorentz symmetry is not strictly broken, but rather slightly 'deformed', hence realizing the idea of 'closeness' in a more subtle sense. Suitable mathematical structures capturing this idea were found in the class of Hopf algebras, and models of this kind continue to be under investigation today under the general rubric of non-commutative geometry. The ansatz is in spirit quite similar to the one of Supersymmetry, where the Poincaré Lie algebra is also extended to a more general algebraic structure. Here however, one does not postulate any new symmetries, but instead a modification of the known ones.

To a classical relativist, who is comfortable with the smooth geometric pictures the theory of Lie groups creates, these more algebraically-oriented approaches to departures from Special Relativity can seem like opening Pandora's box, due to the sudden appearance of a whole new variety of possible relations between inertial transformations and spacetime coordinates. Over time, however, consensus emerged in the phenomenology of quantum gravity over how to think about such modifications, and what is more, how to extract from them predictions that can be tested. The primary suggestion for this goal is to use the modified dispersion relations for fundamental particles that result from alternative spacetime symmetry algebras, and compare them with astrophysical observations of phenomena like gamma-ray bursts, which involve energies that are high enough [Ame+98].

An overview over the multitude of further tests of Lorentz invariance and its present status as a fundamental symmetry of nature is provided in [Mat05; EL06]. While recognizing the importance of testing the familiar physical mechanisms for possible Lorentz violations, it is at the same time necessary to gain structural clarity about them, and search for possibly yet unknown ones. In this line of thinking, the particularly exposing proposal of Very Special Relativity [CG06] showed that a null result in experiments like the above-mentioned one of Michelson and Morley are in fact not speaking against a preservation of the full Lorentz group itself—a result that motivates research in similar directions. It is of course not be expected that any algebraic structure that can formally be located in the vicinity of the Poincaré group will be of interest to Planck scale phenomenology. But as we will see, physical relevance is suggested surprisingly often, and offering a more whole view on spacetime symmetries in physics.

The structure of this dissertation is as follows. Chapter 2 displays the neighbourhood of the Poincaré algebra as a Lie algebra and thereby summarizes the three main scenarios that formalize possible departures from Special Relativity on the classical level. In Chapter 3, new light is cast onto a particular attempt to derive all possible, physically meaningful kinematics that follow from the Relativity Postulate. The corresponding model geometries are presented, and found to fit into an early classification of possible geometries due to Cayley and Klein. In relaxing the concept of a metric to include degenerate cases, the Lie symmetry groups of the kinematic study are shown to generate the (accordingly generalized) isometries of the model spacetimes, and the notion of spacetime curvature is extended. Contact is also made with proposals to understand relativistic limits by the help of spacetime extensions. While Chapter 2 and 3 are constrained to Lie-theoretic investigations, Chapter 4 motivates the transition to Hopf algebra in order to discuss the κ-Poincaré algebra as a specific example of interest in detail. Its surprisingly geometric nature is taken to suggest investigating if there is a more general connection between Hopf algebras of similar type and nonlinear realizations of Lie groups. Finally, some remarks are made about related streams of research as well as a side project that has been pursued by the author during his time working on the topics of this dissertation.

In the main text, some notational conventions as well as particular mathematical terms are being used without immediate explanation. They are clarified in the Appendix, Sec. A.1 and A.2.

2. LIE ALGEBRAIC NEIGHBOURHOOD

Once the assumption is made that the kinematical group should be of Lie type, one can ask: What are the possibilities? This is to say, if one were to forget all experimental hints, what are the structural properties that one would impose on the set of transformations between inertial observers? In particular, given the observational evidence for Lorentz invariance, what are the possibilities for departures from *it* to appear? Answering this question does not only help in constructing suitable test theories but can also serve as a means to categorize models for fundamental interactions that go beyond the standard one.

This chapter locates the Poincaré algebra within the space of abstract Lie algebras. In order to do so, each of the three subsections discusses a different notion of 'vicinity' for Lie algebras. Intuitively, *deformations* are modifications of a Lie algebra that render it 'less abelian', i.e. with less commutators vanishing. In contrast, *contractions* are limits of a Lie algebra that make it 'more abelian', by turning certain commutators to zero. We will later see that the 'non-relativistic' limit, which is often referred to in physics, has a natural interpretation in these terms.

Both of the aforementioned procedures maintain the dimension of a Lie algebra. This is a feature that distinguishes them from a third, and more conventional relation two Lie algebras can have, namely when one is a subalgebra of the other. This is the notion of Lie algebraic vicinity that is explored in standard testing schemes for violations of Lorentz invariance.

2.1. LIE ALGEBRA DEFORMATIONS

Deformations of Lie algebras are characterized in terms Lie algebra cohomology, a mathematical subject which is originally due to Chevalley and Eilenberg [Cla48], and was developed in [Ger64], but also [Lév67]. Its basic object of study is a particular nilpotent operator δ, and its action on the family of n-cochains C_n on the Lie algebra $\mathfrak{g}$. These are linear maps

$$C_n : \Lambda^n(\mathfrak{g}) \to V\,, \tag{2.1}$$

where Λ^n denotes the totally antisymmetric part of the n-fold tensor product, and V provides a representation,

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)\,, \tag{2.2}$$

i.e.

$$\rho([e_a, e_b]) = \rho(e_a)\rho(e_b) - \rho(e_b)\rho(e_a) \qquad \forall\, e_a, e_b \in \mathfrak{g}\,, \tag{2.3}$$

and $\rho(e_a)(\rho(e_b)v) = (\rho(e_a)\rho(e_b))v$ on vectors $v \in V$. Then, the operator δ, which raises the degree of cochains,

$$\delta : C_n \to C_{n+1}\,, \tag{2.4}$$

is defined as

$$\begin{aligned}
&\delta f(e_1, e_2, \ldots, e_{n+1}) = \\
&= \quad \rho(e_1)(f(e_2, e_3, \ldots, e_{n+1})) - \rho(e_2)(f(e_1, e_3, ..., e_{n+1})) \pm \ldots \\
&\quad - f([e_1, e_2], e_3, \ldots, e_{n+1}) + f([e_1, e_3], e_2, \ldots, e_{n+1}) \pm \ldots \\
&= \quad \sum_{i=1}^{n+1} (-1)^{i+1} \rho(e_i)(f(e_1, e_2, \ldots, \hat{e}_i, \ldots, e_{n+1})) \\
&\quad - \sum_{i<j=1}^{n+1} (-1)^{i+j+1} f([e_i, e_j], e_1, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1})\,.
\end{aligned} \tag{2.5}$$

for $f \in C_n$ and $e_i \in \mathfrak{g}$, and with hats above missing entries. δ is called the coboundary operator, and it satisfies

$$\delta^2 = 0. \tag{2.6}$$

Proof. We introduce some notation to prepare for the calculation. Let $n \in \mathbb{N}^+$, $f \in C_n$, and denote by $\{e_i\}_{i=1..n+2} \in \mathfrak{g}$ a set of arbitrary elements of the Lie algebra $\mathfrak{g}$. Furthermore, we call $\{m|k_1, k_2, \ldots\}$ the set of integers between (and including) 1 and m, with the exception of $k_1, k_2, \ldots$, and $\langle m|k_1, k_2, \ldots\rangle$ the same set, but numerically ordered. Also, we denote by $e_{\langle m|k_1,k_2,\ldots\rangle}$ the numerically ordered list of Lie algebra elements $\{e_i\}$, with $i \in \{m|k_1, k_2, \ldots\}$ (taking $e_{\langle m\rangle} := e_{\langle m|\{\}\rangle}$). Lastly, $\#_i\langle m|k_1, k_2, \ldots\rangle$ will be the position of i in the list $\langle m|k_1, k_2, \ldots\rangle$.
Now we can write

$$\begin{aligned}
&\delta^2 f(\langle e\rangle_{\{n+2\}}) = \\
&= \sum_{i=1}^{n+2} (-1)^{i+1} \rho(e_i)(\delta f(e_{\langle n+2|i\rangle})) - \sum_{i<j=1}^{n+2} (-1)^{i+j+1} \delta f([e_i, e_j], e_{\langle n+2|i,j\rangle}) \\
&= \sum_{i=1}^{n+2} (-1)^{i+1} \Bigg(\sum_{k\in\{n+2|i\}} (-1)^{\#_k\langle n+2|i\rangle+1} \rho(e_i)(\rho(e_k)(f(e_{\langle n+2|i,k\rangle}))) \\
&\quad - \sum_{k<l\in\{n+2|i\}} (-1)^{\#_k\langle n+2|i\rangle+\#_l\langle n+2|i\rangle+1} \rho(e_i)(f([e_k, e_l], e_{\langle n+2|i,k,l\rangle}))\Bigg)
\end{aligned} \tag{2.7a}$$

[continues on next page]

$$
\begin{aligned}
&- \sum_{i<j=1}^{n+2} (-1)^{i+j+1} \Bigg(\rho([e_i, e_j])(f(e_{\langle n+2|i,j\rangle})) \\
&\quad + \sum_{k \in \{n+2|i,j\}} (-1)^{\#_k \langle n+2|i,j\rangle} \rho(e_k)(f([e_i, e_j], e_{\langle n+2|i,j,k\rangle})) \\
&\quad - \sum_{k \in \{n+2|i,j\}} (-1)^{\#_k \langle n+2|i,j\rangle + 1} f([[e_i, e_j], e_k], e_{\langle n+2|i,j,k\rangle}) \\
&\quad - \sum_{k<l \in \{n+2|i,j\}} (-1)^{\#_k \langle n+2|i,j\rangle + \#_l \langle n+2|i,j\rangle + 1} f([e_k, e_l], [e_i, e_j], e_{\langle n+2|i,j,k,l\rangle}) \Bigg).
\end{aligned}
\tag{2.7b}
$$

There are four different kinds of terms, differing in where and how the Lie bracket is taken (inside or outside f etc.). In order for the whole expression to vanish in the generic case, terms of each kind need to annihilate separately, as they are independent. That is to say, we will never be able to say anything about how, e.g., $\rho(e_1)(\rho(e_2)(f(\langle n+2|i,k\rangle)))$ and $\rho(e_1)(f([e_2, e_3], \langle n+2|i,k,3\rangle))$ relate, without actually knowing the bracket relations for a specific Lie algebra A. Because $\delta^2 f$ as a whole must be totally antisymmetric in all $\{e_i\}$, we then also know that those sets of terms must be completely skew in and for themselves when summed. We now analyse the four sets successively.

- Terms involving $f(e_{\langle n+2|i,j\rangle})$ for some i, j:

$$
\begin{aligned}
&\big(+ \rho(e_1)(\rho(e_2)(f(e_{\langle n+2|1,2\rangle}))) - \rho(e_2)(\rho(e_1)(f(e_{\langle n+2|1,2\rangle}))) \\
&- \rho([e_1, e_2])(f(e_{\langle n+2|1,2\rangle}))\big) \pm \text{perm.}
\end{aligned}
\tag{2.8}
$$

  The first three terms alone sum up to zero due to (2.3), hence their permutations do so as well.

- Terms involving $f([e_i, e_j], e_{\langle n+2|i,j,k\rangle})$ for some i, j, k:

$$
\begin{aligned}
&\big(- \rho(e_1)(f([e_2, e_3], e_{\langle n+2|1,2,3\rangle})) + \rho(e_2)(f([e_1, e_3], e_{\langle n+2|1,2,3\rangle})) \\
&- \rho(e_3)(f([e_1, e_2], e_{\langle n+2|1,2,3\rangle})) - \rho(e_2)(f([e_1, e_3], e_{\langle n+2|1,2,3\rangle})) \\
&+ \rho(e_3)(f([e_1, e_2], e_{\langle n+2|1,2,3\rangle})) + \rho(e_1)(f([e_2, e_3], e_{\langle n+2|1,2,3\rangle}))\big) \pm \text{perm.} \\
&= 0.
\end{aligned}
\tag{2.9}
$$

- Terms involving $f([[e_i, e_j], e_k], e_{\langle n+2|i,j,k\rangle})$ for some i, j, k:

$$
\begin{aligned}
&+ f([[e_1, e_2], e_3], e_{\langle n+2|1,2,3\rangle}) + f([[e_2, e_3], e_1], e_{\langle n+2|1,2,3\rangle}) \\
&- f([[e_1, e_3], e_2], e_{\langle n+2|1,2,3\rangle}) \pm \text{perm.} \\
&= 0\,,
\end{aligned}
\tag{2.10}
$$

  due to the Jacobi identity for $\mathfrak{g}$.

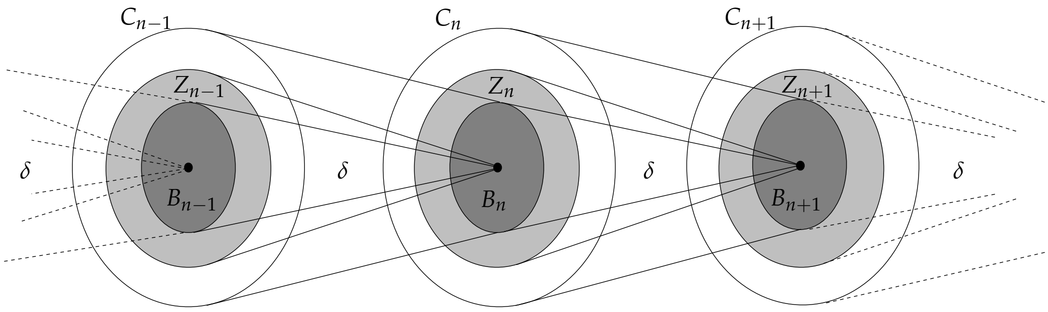


Figure 2.1.: A schematic illustration of general cohomologies induced by an operator like δ, adapted to our notation for Lie algebra cohomology. Darker shading means inclusion.

- Terms involving $f([e_i, e_j], [e_k, e_l], e_{\langle n+2|i,j,k,l\rangle})$ for some i, j, k, l:

$$\begin{aligned}&\big(+f([e_3,e_4],[e_1,e_2],e_{\langle n+2|1,2,3,4\rangle})\\&\ -f([e_2,e_4],[e_1,e_3],e_{\langle n+2|1,2,3,4\rangle})\\&\ +f([e_2,e_3],[e_1,e_4],e_{\langle n+2|1,2,3,4\rangle})\\&\ +f([e_1,e_4],[e_2,e_3],e_{\langle n+2|1,2,3,4\rangle})\\&\ -f([e_1,e_3],[e_2,e_4],e_{\langle n+2|1,2,3,4\rangle})\\&\ +f([e_1,e_2],[e_3,e_4],e_{\langle n+2|1,2,3,4\rangle})\big)\pm\text{perm.}\\&\ =0,\end{aligned} \tag{2.11}$$

  due to the antisymmetry of f.

Consequently, $\delta^2 f(e_{\langle n+2\rangle})$ vanishes all together, and since we assumed nothing for f or the $\{e_i\}$, $\delta^2 f$ does so in general. □

This means that there is an infinite sequence

$$C_0 \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} \dots \tag{2.12}$$

which has the property that the images

$$C_n \cap \operatorname{im} \delta =: B_n\,, \tag{2.13}$$

called the *coboundaries* of degree n, all lie within the subspaces

$$C_n \cap \ker \delta =: Z_n\,, \tag{2.14}$$

called *cocycles* of degree n. The situation is illustrated in Fig. 2.1. The n-th cohomology class is then given by the quotient of these vector spaces,

$$H_n := Z_n / B_n\,. \tag{2.15}$$

The set of cohomology classes hence measures the lack in exactness[1] of the series (2.12). In order to study deformations of Lie algebras, one has to pick as the representation vector space V the vector space that underlies the Lie algebra $\mathfrak{g}$ itself, and for the representation ρ simply the adjoint representation,

$$(\rho, V) \to (\mathrm{ad}, \mathfrak{g}) \tag{2.16}$$

so that one can write

$$\rho(e_i)(e_j) = \mathrm{ad}_{e_i}(e_j) = [e_i, e_j] \qquad e_i, e_j \in \mathfrak{g}\,. \tag{2.17}$$

It is this particular case which is assumed for the remainder of this section.

For low degrees $n \leq 2$, the spaces C_n, B_n and Z_n then have clear interpretations. In particular, elements of Z_1 may be regarded as basis changes in the vector space underlying $\mathfrak{g}$ which do not change the bracket structure. This is obvious when writing an infinitesimal ($\epsilon^2 = 0$) change of basis

$$e_i' = \Phi(e_i) = (\mathrm{id} + \epsilon f)(e_i) = e_i + \epsilon f(e_i), \tag{2.18}$$

and demanding compatibility with the Lie brackets,

$$\begin{aligned}
& & [e_i, e_j]' &= [e_i', e_j'] \\
&\Leftrightarrow & [e_i, e_j] + \epsilon f([e_i, e_j]) &= [e_i, e_j] + \epsilon[f(e_i), e_j] + \epsilon[e_i, f(e_j)] \\
&\Leftrightarrow & f([e_i, e_j]) &= [e_i, f(e_j)] - [e_j, f(e_i)] \\
&\Leftrightarrow & \delta f(e_1, e_2) &= 0\,.
\end{aligned} \tag{2.19}$$

Hence 1-cocycles define the infinitesimal Lie algebra automorphisms, or, derivations,

$$Z_1 = \mathrm{Der}(\mathfrak{g})\,. \tag{2.20}$$

Conversely, there may as well be elements $f \in C_1$ for which $\delta f \neq 0$. Those are maps from $\mathfrak{g}$ to itself which *do* change the Lie bracket, namely exactly by δf, which is an element of B_2. Note that the Lie bracket itself is a 2-cochain. An infinitesimal change of it,

$$[e_i, e_j]' = [e_i, e_j] + \epsilon F(e_i, e_j)\,, \tag{2.21}$$

will in general lead to a different Lie algebra. The need for the new Lie algebra to also satisfy the Jacobi identity is expressed by demanding that $F \in Z_2$, which is seen when spelling out

$$\begin{aligned}
0 &= [e_1, [e_2, e_3]']' + \text{cycle} \\
&= [e_1, [e_2, e_3]] + \epsilon\big([e_1, F(e_2, e_3)] + F(e_1, [e_2, e_3])\big) + \text{cycle} \\
&= 0 + \epsilon \delta F(e_1, e_2, e_3).
\end{aligned} \tag{2.22}$$

[1] In an exact sequence, the image of one map *equals* the kernel of the next one.

Elements of Z_2 may be regarded as infinitesimal instances of what are called *deformations* of the Lie algebra $\mathfrak{g}$, which can be viewed as mappings

$$[\cdot,\cdot] \to [\cdot,\cdot] + \tau F(\cdot,\cdot) + \tau^2 F^{(2)}(\cdot,\cdot) + \tau^3 F^{(3)}(\cdot,\cdot) + \dots \tag{2.23}$$

that are convergent as power series in the real parameter τ near the origin. To be precise, in order for a 2-cocyle to be the linear term in such an expansion, it has to fulfil integrability conditions captured by H_3 [Lév67]. Note that deformations only change the bracket structure of a Lie algebra, while leaving the underlying vector space, and in particular its dimension fixed.

Finally, we recognize H_2 as the set of deformations which cannot be affected by mere (non-singular) redefinitions of generators. In physics, one is often inclined to call such deformations 'non-trivial'.

Example. *We illustrate the above considerations with the rotation algebra $\mathfrak{so}(3)$, which is spanned by $\{J_i\}_{i=1,2,3}$, and has the Lie bracket*

$$[J_i, J_j] = \epsilon_{ijk} J_k \,. \tag{2.24}$$

Consider the following change in generators, viewed as defining a 1-cochain on $\mathfrak{so}(3)$:

$$\begin{pmatrix} J_1' \\ J_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \qquad J_3' = J_3. \tag{2.25}$$

For these new generators, the Lie brackets are different in general, namely:

$$\begin{aligned} [J_1', J_2'] &= (ad - bc) J_3 \\ [J_2', J_3'] &= d J_1 - c J_2 \\ [J_1', J_2'] &= a J_2 - b J_1 \,. \end{aligned} \tag{2.26}$$

For the choice $a = \cos\phi = d,\ b = -\sin\phi = -c$ however, they return to their original form,

$$[J_i', J_j'] = \epsilon_{ijk} J_k' \,, \tag{2.27}$$

meaning that in this case the basis change (2.25) *becomes a 1-cocycle. On the other hand, if we understand the RHS of Eq.* (2.25) *as the RHS of a deformation* (2.23)*, that is, as a 2-cochain, then we immediately conclude by definition that it actually is a 2-coboundary. In fact, $\mathfrak{so}(3)$ has no non-trivial deformations at all, $H_2 = 0$. This follows from the simple observation that the space of possible structure constants for three-dimensional real Lie algebras is of dimension $3 \times \frac{3\times 2}{2} - 1 = 8$ (the generic number of independent components of the structure constants are reduced by the condition imposed on them by the Jacobi identity), while $GL(\mathbb{R}^3)$ is parametrised by nine real numbers, so that one will always be able to find a redefinition of generators that cancels a given change in the Lie bracket.*

Lie algebras like $\mathfrak{so}(3)$, which admit no non-trival deformations at all are called *rigid*. It is a prominent result in Lie algebra cohomology [Ger64] that a sufficient condition for rigidity is $H_2 = 0$, which is true in particular for all semisimple Lie algebras [Jac62]. However, prevailing in physics are non-simple, semidirect sum Lie algebras, as they will also play a major role in this work. Such Lie algebras have non-vanishing second cohomology and generically (although not always [Ric67]) lack rigidity.

By interpreting Lie algebra deformations as paths on a manifold that is coordinatized by the structure constants of d-dimensional Lie algebras (here taken real), one obtains a geometrical understanding for equivalent ('trivial') deformations in terms of the quotient

$$\mathcal{O} = \mathrm{GL}(\mathbb{R}^d)/\mathrm{Aut}(\mathfrak{g}) . \tag{2.28}$$

From here we turn to the application which is of particular interests to us: we ask for deformations of the Poincaré Lie algebra $\mathbf{P}$.

A concise argument can be made when realizing from the start that there are two obvious deformations of $\mathbf{P}$, and then showing that these are unique. We begin by a definition of $\mathbf{P}$.

Definition 1. *The Poincaré Lie algebra $\mathbf{P} = (V, [\cdot,\cdot])$ consists of a ten-dimensional real vector space V that admits a basis $\{M_{mn} = -M_{nm}, P_m\}$ in which the Lie brackets take the form*

$$\begin{aligned} [M_{mn}, M_{m'n'}] &= 4\eta_{[m'[n} M_{m]n']} , \\ [M_{mn}, P_l] &= 2\eta_{l[n} P_{m]} , \\ [P_m, P_n] &= 0 , \end{aligned} \tag{2.29}$$

with $\eta_{mn} = \mathrm{diag}(-1,1,1,1)$. The subalgebra that is closed by the $\{M_{mn}\}$ is that of the Lorentz Lie algebra, and the $\{P_m\}$ are called translations.

This basis-dependence in this definition of $\mathbf{P}$ make it appear rather ad hoc, but we will uncover the abstract structure of $\mathbf{P}$ in the the next section. We do so just as we will describe its geometric interpretation in the course of this, namely mostly in a contextual manner, i.e. by embedding it in its algebraic and geometric neighbourhood. To do so, the above definition serves a practical starting point. For an invariant definition of $\mathbf{P}$ as the Lie algebra of the automorphism group of Minkowski space, see [Giu10].

The immediate observation that can be made, and which starts our argument, is that changing

$$[P_m, P_n] \to \pm\epsilon M_{mn} \tag{2.30}$$

results, depending on the sign, in the de Sitter Lie algebras $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$, which we abbreviate by $\mathbf{dS}_+$ and $\mathbf{dS}_-$, respectively. While they will play a central role later on, for now it suffices to recognize that these Lie algebras are simple, and hence allow no further deformations. The remaining question is then if there are yet more deformations of $\mathbf{P}$. In order to give an answer, consider the orbits of $\mathbf{P}$ and $\mathbf{dS}_\pm$. We denote their dimension by N and N', respectively. In general,

$$\begin{aligned} \dim B_2 &= \dim \mathcal{O} \\ &= d^2 - \dim(\mathrm{Der}(\mathfrak{g})) \\ &= d^2 - (\dim(\mathrm{IDer}(\mathfrak{g})) - \dim(Z(\mathfrak{g})) + \dim(\mathrm{ODer}(\mathfrak{g}))) , \end{aligned} \tag{2.31}$$

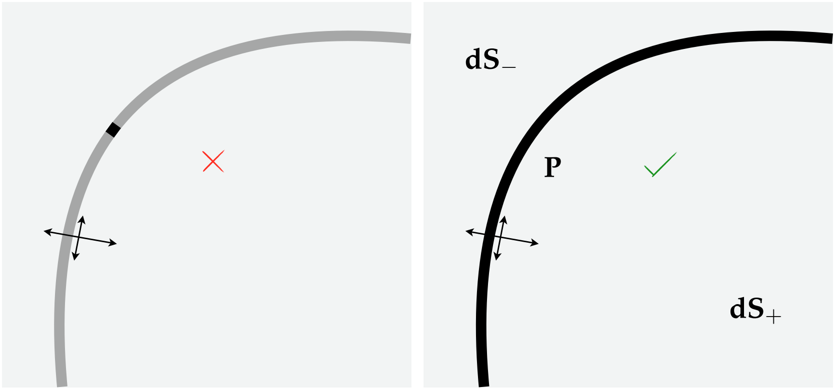


Figure 2.2.: Depiction of two possibilities for the orbits of equivalence (cf. (2.28)) for the Poincaré **P** and de Sitter $\mathbf{dS}_\pm$ Lie algebras. In the left case, their dimensions differ by two instead of one, which is ruled out by the one-dimensionality of ODer **P**.

where Z means centre, IDer denotes inner derivations (i.e. bracketing with elements from $\mathfrak{g}$), ODer the outer ones and in the first line it was used that discrete automorphisms do not contribute. Since ODer is zero in the semisimple case [Jac62], which is in addition centerless, one can infer $\dim B_2 = d^2 - d$, in particular so for $\mathbf{dS}_\pm$, where $d = 10$. For **P**, which is still centerless, it is found that the outer derivations are the dilatations [Mic64], whose action on **P** in the above basis is

$$[D, M_{mn}] = 0, \qquad [D, P_m] = -P_m \,. \tag{2.32}$$

We can hence conclude that $N' - N = 90 - 89 = 1$, which implies that on the manifold of structure constants, there cannot be any extra directions besides the $\mathbf{dS}_\pm$ ones in which **P** could be deformed—the orbit of **P** is a hypersurface separating the orbits of $\mathbf{dS}_+$ and $\mathbf{dS}_-$. The situation is illustrated in Fig. 2.2.

To arrive at this result, it is of course also possible to explicitly compute the second cohomology class of the Poincaré Lie algebra. This is done in [Lév67].

The discussion of this chapter revolved around a very particular notion of deformation. Most importantly, we stayed in the category of Lie algebra. When Lie algebraic concepts are employed in physics however, very often use is made not exclusively of their properties as Lie algebras. Rather, they are employed in their guise as universal enveloping algebras, or more precisely, their generalization as Hopf algebras. All these concepts will be explained in detail in later parts of this thesis. The point to note here is that, if we define our notions of symmetry to be realized via a different mathematical structure, equivalent to Lie algebras for some applications but perhaps richer under other circumstances, then we will be forced to also consider deformations to take place in a new realm.

2.2. Wigner-Inönü contractions

Can we think of another kind of procedure that lets us explore the space of d-dimensional Lie algebras? If we wish to change some of the structure of a Lie algebra, we will first need to distinguish some subspace of it with respect to which the change is going to take place. We do this by relabelling the generators as follows:

$$e_i = \begin{cases} e_I & I = 1 \dots \dim \mathfrak{g} - r, \\ e_{\hat{I}} & \hat{I} = \dim \mathfrak{g} - r + 1 \dots \dim \mathfrak{g}, \end{cases} \tag{2.33}$$

so that the Lie algebra divides into

$$\begin{aligned} [e_I, e_J] &= c^K{}_{IJ} e_K + c^{\hat{K}}{}_{IJ} e_{\hat{K}} \,, \\ [e_I, e_{\hat{J}}] &= c^K{}_{I\hat{J}} e_K + c^{\hat{K}}{}_{I\hat{J}} e_{\hat{K}} \,, \\ [e_{\hat{I}}, e_{\hat{J}}] &= c^K{}_{\hat{I}\hat{J}} e_K + c^{\hat{K}}{}_{\hat{I}\hat{J}} e_{\hat{K}} \,. \end{aligned} \tag{2.34}$$

Clearly, simply changing the values of some of the structure constants will violate the Jacobi identity and hence transport us out of the category of Lie algebras. But there is something else that we are allowed to do which does not spoil the Lie algebra structure, and that is rescaling some of the generators by a real number t, like

$$\hat{e}_{\hat{I}} := t e_{\hat{I}} \,, \tag{2.35}$$

while the unhatted generators are left unchanged. Actually, a truly different Lie algebra will only emerge in the singular limit, where

$$t \to 0 \,. \tag{2.36}$$

This limiting procedure, initially suggested in [IW88], corresponds to a situation where the transformations associated with the hatted generators are considered to be negligibly small. The central point is that, provided the unhatted generators close a Lie subalgebra, i.e.,

$$c^{\hat{K}}{}_{IJ} = 0 \,, \tag{2.37}$$

then the limit takes the original Lie algebra (2.34) into a new one, given in terms of the new generators by

$$\begin{aligned} [e_I, e_J] &= c^K{}_{IJ} e_K \,, \\ [e_I, \hat{e}_{\hat{J}}] &= c^{\hat{K}}{}_{I\hat{J}} \hat{e}_{\hat{K}} \,, \\ [\hat{e}_{\hat{I}}, \hat{e}_{\hat{J}}] &= 0 \,. \end{aligned} \tag{2.38}$$

This now is exactly the structure of a *semidirect sum*. Such are abstractly characterized by a splitting of the short exact sequence

$$0 \to \mathfrak{n} \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{h} \to 0 \tag{2.39}$$

via a homomorphism

$$j : \mathfrak{h} \hookrightarrow \mathfrak{g} \tag{2.40}$$

where $\mathfrak{n}$ maps (necessarily) to an ideal in $\mathfrak{g}$, the projection

$$p : \mathfrak{g} \twoheadrightarrow \mathfrak{h} \simeq \mathfrak{g}/\mathfrak{n}, \tag{2.41}$$

consists of setting that ideal to zero, and j is such that

$$p \circ j = \mathrm{id}_{\mathfrak{h}}\,. \tag{2.42}$$

If such a j cannot be found, one is back to the generic case of an extension, i.e. only the short exact sequence (2.39) without (2.40). One either speaks of an upward extension of $\mathfrak{n}$ by $\mathfrak{h}$ or a downward extension of $\mathfrak{h}$ by $\mathfrak{n}$. This terminology is due to [Con+85] and the most precise, as highlighted in [Giu15]. Often however, the attributes 'downward' and 'upward' are omitted.

We can see now that Lie algebras resulting from Wigner-Inönü contractions are special Lie algebras in a two-fold sense: Firstly they are semidirect sums. But secondly, $\mathfrak{n}$ is *abelian* for them. In general, one writes for semidirect sums

$$\mathfrak{g} \simeq \mathfrak{h} \ltimes_{\varphi} \mathfrak{n} \tag{2.43}$$

(or alternatively $\mathfrak{g} \simeq \mathfrak{n}\,{}_{\varphi}\!\rtimes \mathfrak{h}$), where

$$\varphi : \mathfrak{h} \hookrightarrow \mathrm{Aut}(\mathfrak{n}) \tag{2.44}$$

is determined by the way the ideal in $\mathfrak{g}$ is acted upon,

$$\varphi_j(h) : n \mapsto i^{-1}([j(h), i(n)]) \qquad \forall\, h \in \mathfrak{h},\ n \in \mathfrak{n}\,. \tag{2.45}$$

In our example (2.38), φ is given via the structure constants $c^{\hat{K}}{}_{IJ}$. However, once two initially unrelated Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$ are chosen, generically a freedom in the choice of φ and therefore the mixed structure constants arises. In the case of a semidirect sum for which φ can be taken to coincide with the vector representation of $\mathfrak{h}$ (i.e. $\mathfrak{n}$ abelian and in addition of the right dimension) the declaration of φ is often omitted. For example, this is the case for the Lie algebra **P** of the Poincaré group, which is built up from the four-dimensional Lie algebra of ordinary translations, $\mathbb{R}^4$, and the Lorentz algebra $\mathfrak{so}(1,3)$, in such a way that

$$0 \to \mathbb{R}^4 \overset{i}{\hookrightarrow} \mathbf{P} \overset{p}{\twoheadrightarrow} \mathfrak{so}(1,3) \to 0 \tag{2.46}$$

splits. This is all there is to say to characterize **P** invariantly, since for extensions in general the choice of splitting homomorphism j is not unique. In fact there is one for each member in the ideal,

$$j_m(h) = j_0(h) + m\,, \qquad \forall\, h \in \mathfrak{h},\ m \in \mathrm{im}(i)\,. \tag{2.47}$$

For each of them, however, the effected automorphism is identical and equals the defining (i.e. 4-vector) representation π_0,

$$\varphi_{j_m} \equiv \varphi_0(h) : n \mapsto \pi_0(h)(n)\,, \qquad \forall\, h \in \mathfrak{h},\; n \in \mathfrak{n}\,. \tag{2.48}$$

The freedom in splitting homomorphism is not unique to **P**, but characteristic for all semidirect sum Lie algebras. To add further language that we will also employ at later stages, extensions by vector representation are frequently referred to as 'inhomogeneous' versions of their unextended part, so that the inhomogeneity is given by the translations. For example, in the case of **P**, the term 'inhomogeneous Lorentz algebra' is used interchangeably, and abbreviated $\mathfrak{iso}(1,3)$.

All these consideration suggest a compelling geometric picture of Wigner-Inönü contractions. In the limit where the contraction parameter becomes zero, the Lie group manifold to which the Lie algebra integrates is being stretched in some directions (the hatted ones in our example) and eventually adopts the same vector space structure as exhibited by the ideal part of its Lie algebra in those directions. We will later observe how this is being reflected in the quotient manifolds of the Lie group.

The impact of group theory in physics is in fact to a large extent mediated via representation theory. This becomes particularly apparent in field theory, where one may wonder in which sense, for example, the Schrödinger equation is a limiting case of the Klein-Gordon equation. Asking how the representation theory of a Lie group changes under contraction was of primary concern in the original work of Wigner and Inönü. The general lesson is that while the group becomes more abelian by contraction, the existence of central extensions tends to become more favorable, and with it, the possibilities for non-trivial projective representations. The mentioned 'non-relativistic' limit from a Klein-Gordon field to the quantum-mechanical wave function is a prime example here. While group-theoretically, the Galilei group is a strict contraction of the Poincaré with respect to the subgroup of rotations, wave functions solving the Schrödinger equations transform properly only under the Bargmann group, a central extension of the Galilei group which will play a role later on in this thesis. Since they appear and gain relevance in the context of group contractions, a short characterization of central extensions shall complete this section. It should be stressed however that apart from the Bargmann case, their role in the subject will remain largely omitted for the rest of this work.

Staying on the level of Lie algebras once more, we use again a short exact sequence to describe the nature of central extensions. In this case it is

$$0 \to \mathrm{c} \overset{k}{\hookrightarrow} \bar{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} \to 0, \tag{2.49}$$

and $\mathrm{im}(k)$ is now not only ideal, but in the centre of the extended Lie algebra $\bar{\mathfrak{g}}$,

$$k(\mathrm{c}) \subseteq Z(\bar{\mathfrak{g}})\,, \tag{2.50}$$

meaning it commutes with the whole of $\mathfrak{g}$. In this case one would say that $\bar{\mathfrak{g}}$ is a central extenstion of $\mathfrak{g}$ by c.

Central extension involving discrete groups play an important role for the global structure of Lie groups. In fact the group that is usually considered as the 'full' Lorentz group is $\mathbf{L} = \mathrm{O}(1,3)$, and also includes transformations that change the orientation in time and/or space. These are commonly denoted by

$$V = \{e, P, T, PT\}\,, \tag{2.51}$$

where P stands for parity and T for time reversal. Subjected to the relations $P^2 = e = T^2, TP = PT$, V inherits the group structure

$$\mathrm{V} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\,. \tag{2.52}$$

What is known as the restricted Lorentz group $\mathbf{L_0} = \mathrm{SO}_0(1,3)$, and which stands in one-to-one correspondence with the Lie algebra $\mathfrak{so}(1,3)$ (via the exponential map (3.109)), is then only one of four disconnected components of $\mathbf{L}$. The situation is captured again by a short exact sequence, although now on group level:

$$1 \to \mathbf{L_0} \hookrightarrow \mathbf{L} \twoheadrightarrow \mathrm{V} \to 1\,, \tag{2.53}$$

while the same language as before applies: $\mathbf{L}$ is an upward extension of $\mathbf{L}_0$ by V (or a downward extension of V by $\mathbf{L}_0$).

Things are complicated further when considering double covers, as is demanded for applications to the physics of spinors. In most applications only the double cover of the restricted Lorentz group is used, which is isomorphic to the group of complex matrices with unit determinant. It is, in contrast to the previous case, a *down*ward extension of $\mathbf{L}_0$,

$$1 \to \mathbb{Z}_2 \hookrightarrow \mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathbf{L}_0 \to 1\,. \tag{2.54}$$

If, on the other hand, one were to regard the full Lorentz group $\mathbf{L}$ to be the correct starting point for the double cover, one faces also other candidates than $\mathrm{SL}(2,\mathbb{C})$ as spin groups, which is due to the fact that V itself has more than one double cover [Tra05].

Note finally that all these consideration about global properties of the different variations of the Lorentz group directly translate to their inhomogeneous, Poincaré versions, since the additional translationary part introduces no new topological features.

2.3. Spontaneous breaking

Symmetry concepts have been central for the successful developments in quantum field theory during the 20th century. Apart from the properties inherited from Minkowski spacetime, it is the idea of internal symmetry that lies at the heart of all the current models in particle physics. That is, instead of considering single-valued fields (real or complex) on spacetime, the fields (here generically denoted Ψ) are allowed to take values in more general vector spaces which are assumed to be endowed with the action of a Lie group G. The model, characterized by an action functional S, is then constructed to be invariant. Schematically,

$$\Psi(x) \to (g\Psi)(x), \;\; g \in G, \qquad S[\Psi] = S[g\Psi]. \tag{2.55}$$

Most prominently, this group action may be permitted a spacetime dependence, which then necessitates the introduction of additional vector ('gauge') fields. It is remarkable that this construction permits to describe three of the four known fundamental interactions.

There is a sense in which the transition from spacetime independent ('global', or, 'rigid') internal transformations towards gauge transformation entails quite a dramatic shift in attitude. The notion of symmetry suddenly becomes entangled with that of redundancy, which occasionally prompts arguments in favour of the position that in this case symmetry degenerates to a mere descriptional tool, only remotely connected with physical invariance properties of nature.

In a similar spirit, i.e. from a more practical standpoint that utilizes symmetries as structure-inducing tools, quantum field theories regularly address scenarios in which some gauge group is assumed fundamentally, but only so in order to consider its remnants (i.e. one of its subgroups) in situations of very particular interest. This is the concept of spontaneous symmetry breaking.

A spontaneous breakdown of symmetry occurs whenever a G-invariant action has unstable stationary points, i.e., field configurations Ψ_0 for which $\delta S[\Psi_0] = 0$, that however do not minimize S. The true set of ground states, which do minimize the action, must then still be left invariant by the action of G, only so as a set however. Each particular representative of it is only stabilized by some subgroup H of G. Phrased differently, while the model still behaves covariantly under transformations from G, its solutions only exhibit the remnant H-symmetry. As a result, one can identify the set of ground states with the quotient G/H.

A large stream of research asks whether in fact Lorentz symmetry might be spontaneously broken, due to some yet unknown effects with possible quantum-gravitational origin [KS89]. Here one distinguishes between Lorentz violation stemming from the matter section, and violations that are due to the geometry of spacetime [Kos04]. The approach is often presented in a way that omits a peculiarity: In applications of the mechanism of spontaneous symmetry breaking it is usually assumed that the original symmetry is present at high energies and broken at low energies. Given however the experimental bounds on violations of Lorentz invariance at the energy scales presently accessible, this reasoning must be turned around for a for Lorentz symmetry. Nevertheless, while perhaps counter-intuitive, a loss of symmetry at higher energies is in principle possible [Wei74].

In reaction to this remark one might actually feel compelled to investigate a further direction: Instead of a spontaneous breakdown from the Lorentz group to some of its subgroups, consider in turn the Lorentz group itself as the low-energy remnant of some larger group governing at high energies. Obvious candidates discussed in the literature are the de Sitter groups and the conformal group [SW80; CM10; CM13; ISS70]. In String or M-Theory, where spacetime itself is fundamentally higher-dimensional, vastly more possibilities for isometry groups open up. In that case, when what is actually referred to is the higher-dimensional Lorentz group, one should not see the scenario of its spontaneous breaking too critical from a phenomenological standpoint. For after all, this must naturally happen at some point, since we experience only a four-dimensional world, in everyday live as well as in all experiments performed so far.

Among the subgroups to which the Lorentz group might be broken there are some obvious ones, for which tests are constantly being performed. Others are not so obvious, and more difficult to test for. A complete account of subgroups of the Lorentz group can be found in [Hal]. In fact, as shown in [CG06], there are some which would actually qualify as a complete substitute for the Lorentz group in its role in describing fundamental spacetime symmetries. The largest of them, $\mathrm{SIM}(2)$ will now briefly be described.

Subgroups of the Lorentz group can be characterized by providing the substructure of momentum space (or, Minkowski vector space) which they leave invariant. For example, the subgroup of three-dimensional rotations is given by those elements of the Lorentz group which stabilize timelike vectors[2].

Consider a lightlike four-momentum with Cartesian components

$$l_m = (1, 0, 0, 1). \tag{2.56}$$

Clearly, it is left invariant by rotations around the z-axis, which form a one-parametric subgroup generated by J_3. There are two more directions in the Lorentz Lie algebra which generate transformations leaving the above momentum vector invariant. Those are that of the combinations $J_1 - K_2$ and $J_2 + K_1$ of boost and rotation generators in the x-y-plane. Lastly, boosts in z-direction merely scale the light momentum given above. In effect, we can say that the Lorentz subgroup generated by the subset

$$\{J_3, J_1 - K_2, J_2 + K_1, K_3\} \tag{2.57}$$

of its Lie algebra stabilizes null lines. It is called the Similitude Group, $\mathrm{SIM}(2)$. Cohen and Glashow pointed out that in order to construct $\mathrm{SIM}(2)$-invariant algebraic expressions in the four-momenta p and q of two scattering particles, one needs to take quotients. For example, the simplest possible one would behave as

$$\frac{p.l}{q.l} \to \frac{p.l'}{q.l'} = \frac{p.l}{q.l}, \tag{2.58}$$

where a dot denotes the scalar product in terms of the (inverse) Minkowski metric. The central observation was that such terms are not captured by conventional, perturbative approaches to quantum field theory, and hence in particular not so in the framework of the Standard Model Extension (SME), a major test theory for Lorentz invariance violation in the Standard Model [Mat05; KR11].

[2] Note the implicit precision of this formulation: The fact that there are infinitely many timelike vectors exactly corresponds to the status of the rotation group in the Lorentz group. It is not an invariant subgroup, but conjugation with boosts leads to isomorphic copies.

3. Cayley-Klein model spacetimes

The previous chapter located Special Relativity from an algebraic perspective by characterizing its kinematical transformations—given by the Poincaré Lie algebra $\mathbf{P}$ on the infinitesimal level—and distinguishing three types of close relatives. Alternatively, one may adopt a geometric attitude, and try to embed Minkowski space itself, rather than its symmetries, into a generalizing scheme. Note that the Relativity Postulate restricts the possibilities: one ought to aim at preserving the interpretation in terms of kinematical transformations, i.e. one should consider only those geometric structures that are able to describe relations between inertial systems. This, of course, immediately raises the question how one should represent inertial systems geometrically. Two complementary answers seem conceivable at first sight: Since it is the inertial observers that are to be regarded as elementary objects it would be legitimate to demand that they be modelled as the *points* of the geometry. On the other hand, trajectories of observers are conventionally identified with *lines* in spacetime. In fact, both concepts become intertwined in the subject of projective geometry, and this suggests its applicability to the problem.

Early after the advent of Special Relativity in 1905, its natural interpretation in terms of Minkowskian geometry was emphasized. Not as widely recognized is the fact that this was not the only non-Euclidean geometry known to that day. Building on earlier work by Cayley [Cay59], Klein had arrived at a classification of projective planar geometries by distinguishing three possible ways both of measuring *distances* (on lines) and *angles* (between lines) [Kle71; Kle73]. The different combinations of what he referred to as the 'elliptic', 'parabolic' and 'hyperbolic' length measures, and the equally-named angle measures, yield nine distinct geometries [Som09]. Two of them soon received a relativistic interpretation: the geometries governing the transformations between Galilean and Lorentzian observers were understood as distinct from Euclidean geometry only in their angular measures (parabolic respectively hyperbolic instead of elliptic) [Yag21]. 'Angles' hence received an interpretation as attributes of an inertial system determining its state of motion ('velocities').

This chapter should be read as a prolongation of the Cayley-Klein classification in the following senses. Most importantly, we let its geometries ascend from the plane to four dimensions in order to arrive at a spacetime interpretation. Physically, this means that we generalize to the case of more than two, and non-collinearly moving inertial systems. By a common argument which will be repeated in Sec. 3.2, some geometries (those with elliptic angular measure) do not lead to valid spacetime models and, given our intention, are consequently left out from the start. With this restriction, we are able to obtain in Sec. 3.1, through physically motivated limiting procedures, all *Cayley-Klein spacetime models* from only two of them. These are de Sitter and Anti-de Sitter space, i.e. the four-dimensional quotient spaces of the Lie groups corresponding to $\mathbf{dS}_+$ and $\mathbf{dS}_-$, the unique deformations of **P**. In Sec. 3.2, the limiting geometries are then, in turn, found to possess infinitesimal automorphisms described by the Wigner- Inönü contractions of $\mathbf{dS}_{\pm}$. In this way we work out in detail, and extend, what has been proposed in [Fer84].

In Sec. 3.3, the theory of Lie groups is used to compare the meaning of curvature in Cayley-Klein spacetimes to the Riemannian one. This could be understood as the appreciation of a pre-Riemannian notion of curvature, which, had it been interpreted in the spirit of Special Relativity, would have served to predict the curvature of spacetime in advance of the formulation of General Relativity. In this way we pay late justice to a 'missed opportunity', as it has been called by Dyson [Dys72].

Lastly, Sec. 3.4 puts the considerations of this chapter into the context of selected topics within General Relativity.

3.1. Contractions of geometry

In order to lift the planar Cayley-Klein geometries to higher dimensions, we make the assumption of rotational invariance, i.e we think of one of the coordinates in the plane (the one that is interpreted as parametrizing space) as the radial spacetime coordinate. Stated differently, the planar geometries arise as planar slices of spacetime which include the time axis. The assumption will find its representation in the first of the algebraic requirements of the next section.

Minkowski space as a zero-curvature limit

We start from $\mathbb{R}^5$, considered as differentiable manifold with Cartesian coordinates X^M, and equip it with one of the following three bilinear forms:

$$\begin{aligned} \eta_+ &:= -(\mathrm{d}X^0)^2 + \mathrm{d}\vec{X}^2 + (\mathrm{d}X^4)^2 \,, \\ \eta_0 &:= (\mathrm{d}X^4)^2 \,, \\ \eta_- &:= -(\mathrm{d}X^0)^2 + \mathrm{d}\vec{X}^2 - (\mathrm{d}X^4)^2 \,. \end{aligned} \tag{3.1}$$

This triple could in fact be seen to arise from an underlying form

$$\begin{aligned} \tilde{\eta} &:= -(\mathrm{d}\Xi^0)^2 + \mathrm{d}\vec{\Xi}^2 + (\mathrm{d}X^4)^2 \\ &= \begin{cases} \eta_+ & \text{with} \quad \Xi^m = X^m, \\ \eta_0 & \text{with} \quad \Xi^m = \iota X^m, \\ -\eta_- & \text{with} \quad \Xi^m = i X^m, \end{cases} \end{aligned} \tag{3.2}$$

where $m = 0 \dots 3$ and ι is a so-called dual number, for which $\iota^2 = 0$. This threefold number system is heavily made use of in [Yag21].

Let us call the structures resulting from the above elements accordingly,

$$\begin{aligned} \mathbb{R}^5_+ &:= (\mathbb{R}^5, \eta_+), \\ \mathbb{R}^5_0 &:= (\mathbb{R}^5, \eta_0, \eta_0^\#), \\ \mathbb{R}^5_- &:= (\mathbb{R}^5, \eta_-), \end{aligned} \tag{3.3}$$

where in addition we define

$$\eta_0^\# := -\frac{\partial}{\partial X^0} \otimes \frac{\partial}{\partial X^0} + \frac{\partial}{\partial \vec{X}} \dot{\otimes} \frac{\partial}{\partial \vec{X}}, \tag{3.4}$$

which can be viewed as a generalized dual of η_0, characterizable by

$$\eta_0^\# = (\mathrm{id} - \eta_0 \otimes \eta_0)(\eta_+^\sharp). \tag{3.5}$$

Here, $\sharp$ is one of the two, mutually inverse, 'musical' isomorphisms. The second, denoted $\flat$, is the map from tangent to cotangent space defined on any manifold with metric g as

$$\flat : v \mapsto v^\flat = g(v) \tag{3.6}$$

for a vector v. $\flat$ and $\sharp$ are extended to tensors in the straightforward way, so that when, as above, applied to the metric itself, $\sharp$ simply yields the dual metric, with coefficient matrix inverse to that of g. We will use the musical isomorphisms only if it is clear which metric g has been used in their definition. In the present case, it is η_+. Note that for degenerate metrics, $\sharp$ and $\flat$ cease to be isomorphisms, which is why we used the symbol # as a generalisation in Eq. (3.4).

Denoting

$$X := X^M \frac{\partial}{\partial X^M}, \tag{3.7}$$

(sometimes called Euler vector field,) we find families, parametrised by a real number L, of two types of hyperboloids, namely

$$\begin{aligned} H_{+,L} &:= \left\{ p \in \mathbb{R}^5_+ \,\middle|\, \eta_+(X,X)|_p = L^2 \right\} , \\ H_{-,L} &:= \left\{ p \in \mathbb{R}^5_- \,\middle|\, \eta_-(X,X)|_p = -L^2 \right\} , \end{aligned} \tag{3.8}$$

and in addition the set

$$E := \left\{ p \in \mathbb{R}^5_+ \,\middle|\, \eta_0(X,X)|_p = 1 \right\} / \mathbb{Z}_2 \simeq \mathbb{R}^4 \tag{3.9}$$

as submanifolds of our $\mathbb{R}^5$. Note that in Eq. (3.9), we identified the two disconnected components of the set satisfying $(X^4)^2 = 1$ by the $\mathbb{Z}_2$-action $X^4 \mapsto -X^4$. The reason why L disappears in the definition of E is understood when regarding E as the limiting set of either $H_{+,L}$ or $H_{-,L}$ for $L \to \infty$. This is done by renormalizing $X^4 \to LX^4$ in the fourth direction before taking the limit, which is equivalent to taking the remaining coordinates X^m to be small. Hence the effect is equivalent to introducing the dual numbers in Eq. (3.2).

We can now endow each of the above three submanifolds with metrics that arise naturally from the above construction. For $H_{+,L}$ and $H_{-,L}$ these are simply given by restriction of the ambient metrics $\eta_\pm$. In the flat case of E, we can take

$$\eta := \left(\eta_0^\sharp|_E\right)^\flat . \tag{3.10}$$

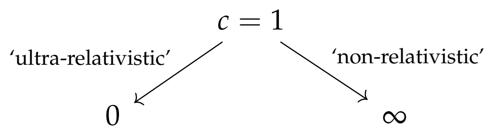
Leaving the dependence on L implicit, the resulting structures are called

$$\begin{aligned} \text{de Sitter spacetime} \quad \mathrm{dS}_+ &:= \Big(H_+, \eta_+|_{H_+}\Big) , \\ \text{Minkowski spacetime} \quad \mathrm{M} &:= (E, \eta) , \\ \text{Anti de Sitter spacetime} \quad \mathrm{dS}_- &:= \Big(H_-, \eta_-|_{H_-}\Big) , \end{aligned} \tag{3.11}$$

where dS_- is often denoted AdS.

Threefold Relativity

In Einstein's theory of Relativity, one is free to rescale coordinate axes by the speed of light c. This is made evident by a conventional choice of units to the effect that $c = 1$. We do not want to adopt this choice, but instead make use of the original scaling freedom in dicussing metrics of Eq. (3.1). It is in fact useful for the purposes of this section to treat c like a parameter, which enables us to consider the following two formal limits:



The names commonly employed for them are given in quotation marks since truly, both limits should still be regarded as realisation of one and the same principle of Relativity, which states the equivalence of inertial frames regarding the validity of the fundamental physical laws. What changes is merely the assignment of how to transform from one inertial reference frame to another.

What $c \to \infty$ really means is that one investigates physical systems whose individual degrees of freedom evolve much slower than the speed at which information is passed between them. In field theory, one might say that excitations are spread instantaneously through the system. But this limit holds also for Newtonian gravity, in which all planets of the solar system would begin to leave their orbits at the very same moment somebody took away the sun. A direct consequence is that in 'non-relativistic' physics, particles can have infinite velocity.

On the other hand, taking $c \to 0$ says that one studies phenomena for which the constituents of a system evolve so quickly compared to how information flows globally that actually it makes sense to consider them decoupled from each other. In this sense, one could also call the resulting dynamics 'ultralocal'. A free particle, however 'fast', would not propagate spatially [Duv+14].

Together, we refer to the two limits as the *causal limits* of the Lorentzian principle of Relativity. From an abstract point of view, they may be viewed as affecting the light cone structure, as later illustrated in Fig. 3.3, where the mathematical representation of these limits will also be discussed. For the order in which we chose to present the following different geometric structures, the causal limits are the secondary criterion. As the primary one we chose the sign of curvature. Since a proper discussion of curvature in our generalized setting will have to wait until Sec. 3.3, for the moment it can simply be taken to indicate whether the concerned geometry stems from dS_+ ('positive curvature') or dS_- ('negative curvature'), or whether it can equally be obtained from both of them ('vanishing curvature').

POSITIVE CURVATURE

We have, with $[T] = \text{time}$,

$$\eta_+ = -c^2\mathrm{d}T^2 + \mathrm{d}\vec{X}^2 + (\mathrm{d}X^4)^2 \tag{3.12}$$

to measure *distances* on $\mathbb{R}^5$, and

$$\tilde{\eta}_+ = -\mathrm{d}T^2 + \frac{1}{c^2}\mathrm{d}\vec{X}^2 + \mathrm{d}S^2, \tag{3.13}$$

with $S := X^4/c$, to measure *time intervals*. While the former posesses a well-defined limit only for $c \to 0$, the latter does so only for $c \to \infty$. Call

$$\begin{aligned} \eta_+^{\downarrow} &:= \lim_{c\to 0} \eta_+ = \mathrm{d}\vec{X}^2 + (\mathrm{d}X^4)^2 \\ \text{and} \quad \eta_+^{\uparrow} &:= \lim_{c\to\infty} \tilde{\eta}_+ = -\mathrm{d}T^2 + \mathrm{d}S^2, \end{aligned} \tag{3.14}$$

and similarly as done previously, define

$$\begin{aligned} C &:= \left\{ p \in \mathbb{R}^5_+ \,\middle|\, \underline{\eta}_+(X,X)\Big|_p = L^2 \right\} \\ hC^{\circlearrowleft} &:= \left\{ p \in \mathbb{R}^5_+ \,\middle|\, \underline{\tilde{\eta}}_+(X,X)\Big|_p = \tau^2 \right\} \end{aligned} \tag{3.15}$$

with τ a constant that should be viewed as the limiting value

$$\tau := \lim_{L,c\to\infty} L/c\,, \tag{3.16}$$

which is however not further constrained, generically. Regardless, one obtains

$$\begin{aligned} \text{Para-Euclidean spacetime} \quad \mathrm{pEuc} &:= \left(C, \eta_+^{\downarrow}\Big|_C\right) \\ \text{and Newton+Hooke spacetime} \quad \mathrm{NH}_+ &:= \left(hC^{\circlearrowleft}, \eta_+^{\uparrow}\Big|_{hC^{\circlearrowleft}}\right). \end{aligned} \tag{3.17}$$

We will later become more explicit, but note already that C and $hC^{\circlearrowleft}$ are higher-dimensional versions of a cylinder and its hyperbolic analogue, with time flowing in the axial and the radial direction, respectively.

Vanishing curvature

By rescaling $X^0 = cT$ as above we now extract from the Minkowski metric η a limiting *distance* and a *time* metric

$$\begin{aligned} \eta^{\downarrow} &:= \lim_{c\to 0} \eta = \mathrm{d}\vec{X}^2 \,, \\ \eta^{\uparrow} &:= \lim_{c\to\infty} \frac{1}{c^2}\eta = -\mathrm{d}T^2 \,, \end{aligned} \tag{3.18}$$

by which we define

$$\begin{aligned} \text{Carroll spacetime} \quad \mathrm{Car} &:= \left(E, \eta^{\downarrow}\Big|_E\right) \\ \text{and Galilei spacetime} \quad \mathrm{Gal} &:= \left(E, \eta^{\uparrow}\Big|_E\right) . \end{aligned} \tag{3.19}$$

Negative curvature

Proceeding in analogy to the positive curvature case, define

$$\begin{aligned} \eta_-^{\downarrow} &:= \lim_{c\to 0} \eta_- = \mathrm{d}\vec{X}^2 - (\mathrm{d}X^4)^2 \,, \\ \eta_-^{\uparrow} &:= \lim_{c\to\infty} \tilde{\eta}_- = -\mathrm{d}T^2 - \mathrm{d}S^2 \,, \end{aligned} \tag{3.20}$$

and from these,

$$\begin{aligned} hC &:= \left\{ p \in \mathbb{R}^5_- \mid \eta_-^{\downarrow}(X,X)\Big|_p = -L^2 \right\} \\ C^{\circlearrowleft} &:= \left\{ p \in \mathbb{R}^5_- \mid \eta_-^{\uparrow}(X,X)\Big|_p = -\tau^2 \right\} , \end{aligned} \tag{3.21}$$

in order to arrive at what we call

$$\begin{aligned} \text{Para-Minkowski spacetime} \quad \mathrm{pM} &:= \left(hC, \eta_-^{\downarrow}\Big|_{hC}\right) \\ \text{and Newton-Hooke spacetime} \quad \mathrm{NH}_- &:= \left(C^{\circlearrowleft}, \eta_-^{\uparrow}\Big|_{C^{\circlearrowleft}}\right) . \end{aligned} \tag{3.22}$$

Note that the notation for hC and $C^{\circlearrowleft}$ is chosen complementary to the positive curvature case. In three embedding dimensions, $C^{\circlearrowleft}$ and hC are scaled and rotated versions of C and $hC^{\circlearrowleft}$, respectively. This is visible in Fig. 3.1, in which all curved of the presented geometries are depicted.

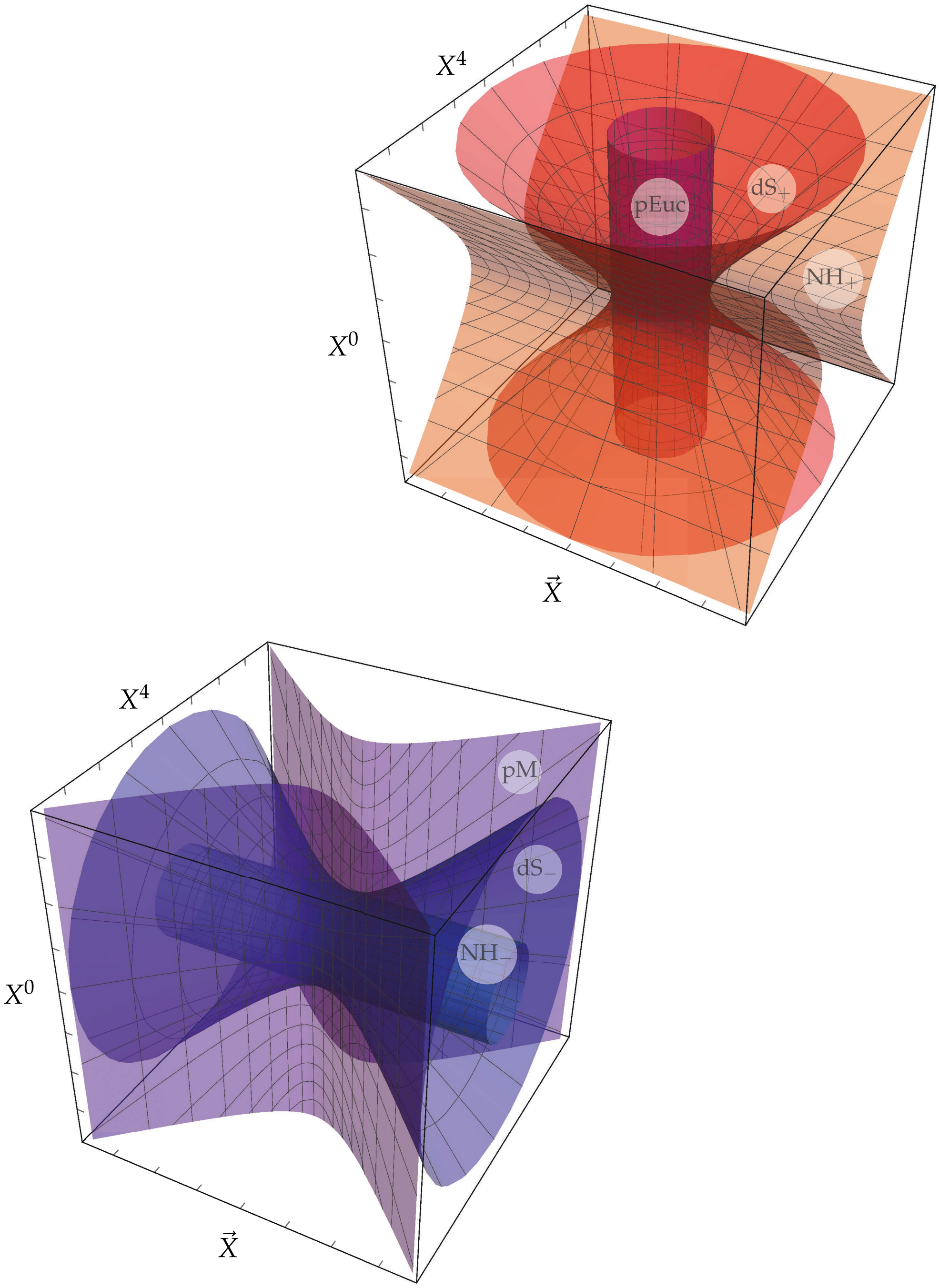


Figure 3.1.: Depicted are two-dimensional analogues of six of the model spacetimes discussed in the text. Their four-dimensional versions can be obtained by imagining each point in the planes replaced by a two-sphere spatially (in $\vec{X}$-direction). X^0 is a time coordinate, while the dimension of X^4 as well as that of the fundamental curvature constant depends on the causal limit taken—it is time for $c \to \infty$ and length for $c \to 0$. The missing model geometries would look like ordinary, flat planes in this picture.

COORDINATE SYSTEMS

For the sake of explicitness, and since the author has not been able to find a similar discussion in the literature, let us provide a unified way to coordinatize the model spacetimes introduced so far. Since the limiting procedures involved to obtain the derived spaces C, hC, $C^{\circlearrowleft}$, $hC^{\circlearrowleft}$ and E from the prototypical hyperboloids $H_\pm$ are very global in character, they become particularly clear in terms of the global coordinates which we chose for $H_\pm$. We will see that these directly translate to complete coordinates for the limiting geometries as well. Other coordinate systems are of course possible, and perhaps of practical interest. In Appendix A.3, the coordinate systems that are referred to in this work are being compared.

Let us denote the intrinsic coordinates by $x^\mu = (x^0, x^1, \theta, \phi)$. In order to keep the presentation concise it is necessary for now to forget again about the subtle but crucial meaning of dimensionality involved in the previous discussion. We therefore simply treat x^μ dimensionless. At the same time, we consider the normalized geometries only, i.e. formally take L and τ to equal 1. Then the embedding of the three Lorentzian spaces in $\mathbb{R}^5$ is given by

$$X^M(x^\mu) = \begin{cases} \begin{pmatrix} \sinh x^0 \\ \vec{n}\cosh x^0 \sin x^1 \\ \cosh x^0 \cos x^1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in [0,\pi) \quad \text{on } \mathrm{dS}_+, \\ \begin{pmatrix} x^0 \\ x^1\vec{n} \\ 1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in \mathbb{R} \quad \text{on M}, \\ \begin{pmatrix} \cosh x^1 \sin x^0 \\ \vec{n}\sinh x^1 \\ \cosh x^1 \cos x^0 \end{pmatrix} \text{ with } x^0 \in [0,\pi) \text{ and } x^1 \in \mathbb{R} \quad \text{on } \mathrm{dS}_-, \end{cases} \tag{3.23}$$

with $\vec{n}$ being a vector on the unit two-sphere,

$$\vec{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} . \tag{3.24}$$

This choice of coordinates in the de Sitter case is called *closed slicing*. The name stems from the fact that $x^0 =$ const. describes spherical, hence closed, hypersurfaces isomorphic with S^3. The dS_- coordinates chosen have a similar geometric interpretation, only that now 'time and space are switched', i.e., the foliation is now in the x^1-direction, with the hypersurfaces of constant x^1 being products of the type $\mathrm{S}^1 \times \mathrm{S}^2$. Spherically polar coordinates on Minkowski space arise in the intermediate, flat limit.

Coordinates on the non-relativistic contractions arise by neglecting all higher than linear terms in x^1:

$$X_\infty^M(x^\mu) = \begin{cases} \begin{pmatrix} \sinh x^0 \\ x^1 \vec{n} \cosh x^0 \\ \cosh x^0 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in \mathbb{R} \quad \text{in NH}_+, \\ \begin{pmatrix} x^0 \\ x^1 \vec{n} \\ 1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in \mathbb{R} \quad \text{in Gal}, \\ \begin{pmatrix} \sin x^0 \\ x^1 \vec{n} \\ \cos x^0 \end{pmatrix} \text{ with } x^0 \in [0, \pi) \text{ and } x^1 \in \mathbb{R} \quad \text{in NH}_-, \end{cases} \tag{3.25}$$

Conversely, the ultra-relativistic contractions arise when nonlinear terms in x^0 are neglected:

$$X_0^M(x^\mu) = \begin{cases} \begin{pmatrix} x^0 \\ \vec{n} \sin x^1 \\ \cos x^1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in [0, \pi) \quad \text{on pEuc}, \\ \begin{pmatrix} x^0 \\ x^1 \vec{n} \\ 1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in \mathbb{R} \quad \text{on Car}, \\ \begin{pmatrix} x^0 \cosh x^1 \\ \vec{n} \sinh x^1 \\ \cosh x^1 \end{pmatrix} \text{ with } x^0 \in \mathbb{R} \text{ and } x^1 \in \mathbb{R} \quad \text{on pM}. \end{cases} \tag{3.26}$$

Other coordinate choices are of course possible for these model spacetimes. A most fruitful approach to construct and classify them arises from group theory. Actually, all the presented geometries can independently be understood as cosets of Lie groups acting as symmetry transformations on them. This is the viewpoint of the next section, and it will even allow us to discover two more geometries that fit into the scheme presented so far.

3.2. Automorphisms

Instead of *defining* as above, in an admittedly to some extent arbitrary way, the different geometric structures, they can also be *derived* as the spacetime models corresponding to a given kinematical group, i.e. the group of transformations between inertial frames. Taking the conventional path and trying to extract symmetries from degenerate geometries like Gal, Car, $\text{NH}_\pm$, pEuc and pM, would necessitate adapting the concept of isometries appropriately to more general automorphisms, i.e. diffeomorphims that preserve the generalized metric. This is however achieved automatically from the group level, which also provides more physical insight as we will see.

In a 'group-first' approach, the freedom then shifts from a choice of geometrical set-up as above, towards the formulation of requirements one asks the kinematical group to meet. As mentioned in the introduction, numerous investigations of this kind have been led in the past; mostly intended to single out the Poincaré group, and to put Einstein's Relativity Postulate on firm mathematical ground. Examples are given by [BG69; Zee63; Lév76], in which also further references are given. In this section we subscribe to a study that was done on Lie algebraic level, and suggests to allow a whole neighbourhood of kinematics alternative to those of Special Relativity, in which the geometric substructures of $(\mathbb{R}^5, \eta^{\pm})$ defined in the previous section find a natural place.

Given the result from Sec. 2.1 that the de Sitter algebras are the only Lie algebraic deformations of the Poincaré algebra, this section could also be advertised as exploring all their Wigner-Inönü contractions under the assumption of spatial isotropy.

KINEMATICS, ALGEBRAICALLY

In the following we reiterate the analysis which Bacry and Lévy-Leblond (henceforth BLL) performed in their original article [BL68]. In doing so, we emphasize some technical aspects and make a number of reinterpretations.

The starting point is the formulation of the following three requirements which any 'possible kinematics' would have to satisfy:

1. Isotropy of space,
2. Parity and time reversal invariance,
3. Non-periodicity of boost[1] transformations.

The mathematical representation of these assumptions was chosen in terms of the Lie algebra of the kinematical group, which is presumed to consist of the familiar ten type of generators with the common interpretation:

$\vec{J}$	spatial rotations
$\vec{K}$	boosts
$\vec{P}$	spatial translations
H	time translations

The first of the BLL assumptions is formalized in [BL68] by requiring that the generators of spatial rotations form a Lie sub-algebra, and that all generators should decompose under the Lie bracket, into the usual vector representations, i.e.

$$[J_i, V_j] = \epsilon_{ijk} V_k \qquad \vec{V} \in \{\vec{J}, \vec{K}, \vec{P}\} \tag{3.27}$$

[1] We borrow the term 'boost' from the standard Lorentz case, but use it for *all* transformations that are taken into consideration for implementing the concept of transitioning from a frame at rest to a moving frame.

and invariants proportional to H,

$$[J_i, H] = 0\,. \tag{3.28}$$

The second requirement is taken to demand the existence of the discrete set of Lie algebra automorphisms

$$\begin{cases} P: & \{\vec{J},\vec{K},\vec{P},H\} & \mapsto \{\vec{J},-\vec{K},-\vec{P},H\}\,, \\ T: & \{\vec{J},\vec{K},\vec{P},H\} & \mapsto \{\vec{J},-\vec{K},\vec{P},-H\}\,, \\ PT: & \{\vec{J},\vec{K},\vec{P},H\} & \mapsto \{\vec{J},\vec{K},-\vec{P},-H\}\,, \end{cases} \tag{3.29}$$

interpreted as parity P and time reversal T, and the joint operation $PT = TP$. Taken together, respecting the first two requirements almost fixes the Lie algebra already. The third requirement then rules out a few remaining candidates[2]. It translates as allowing only those groups in which boosts in any given direction form non-compact subgroups. This ensures that relative velocities between inertial systems always maintain a unique value.

While the third requirement seems indisputable to the author, from a fundamental standpoint one may raise objections against the first two and their implementation.

For instance, it would appear that the discovery of CP violation in electroweak interactions disqualifies the second assumption as a legitimate starting point. However, the fact that the operations (3.29) constitute automorphisms of the kinematical Lie algebra does not imply their presence as spacetime-induced symmetries in any given theory. It merely opens up this *possibility*, namely by facilitating a discrete extension of the kinematical group in the manner of (2.53). In fact, any Lie algebraic analysis excludes the ability to draw conclusions about the topological nature of the kinematical group.

Further, the proposal of Very Special Relativity that was briefly mentioned earlier would imply a violation of spatial isotropy (by the existence of a preferred lightlike direction). The hypothesis that there is a lower-dimensional spacetime symmetry group fundamentally is of course the very starting point for test theories like the Standard Model Extension.

These last remarks were meant to put the BLL axioms into perspective, and to clarify that the possible kinematical groups meeting these requirements should not necessarily be considered to be of equally fundamental importance as the Poincaré group. Only the de Sitter cases, which satisfy the BLL requirements, have a natural interpretation as being more fundamental, since they carry an additional fundamental unit of length (which might however be zero of course). A legitimate view on the BLL classification would hence be to postulate that what we observe as the symmetry group of Special Relativity, and mistake as the Poincaré group, is truly only a (very good) approximation of one of the de Sitter groups—its unique deformations in the Lie theoretic sense. The other resulting kinematical groups could then be understood as equally-justified remnants of $\mathbf{dS}_{\pm}$, which become applicable as its 'approximation' only in specific situations. This is the position we take.

[2]The ruled-out groups correspond to a Euclidean, or, Riemannian model geometry for spacetime.

The bracket relations between ten generators $M_{MN} = M_{[MN]}$ of the (Anti) de Sitter algebra are

$$\begin{aligned}[M_{KL}, M_{MN}] &= \eta_{LM} M_{KN} + \eta_{KN} M_{LM} - \eta_{LN} M_{KM} - \eta_{KM} M_{LN} \\ &= 4\, \eta_{[M[L} M_{K]N]} \\ &= 4\, \eta_{[M[L} \delta^{K'}_{K]} \delta^{N'}_{N]} M_{K'N'}\end{aligned} \tag{3.30}$$

with $\eta_{MN} = \eta_{\pm}(\partial_M, \partial_N)$ the Cartesian components of the 5-dimensional Minkowski metric. In order to begin the BLL analysis, it is necessary to identify them with the physical generators (3.2). We do this by defining

$$\begin{aligned}J_i &:= \frac{1}{2}\epsilon_{ijk} M_{kj}\,, \\ K_i &:= M_{i0}\,, \\ P_i &:= \frac{1}{L} M_{i4}\,, \\ H &:= \frac{c}{L} M_{40}\,.\end{aligned} \tag{3.31}$$

The resulting form of the de Sitter Lie algebras is

$$\begin{array}{llll}[\vec{J},\vec{J}] = \vec{J} & [\vec{J},\vec{K}] = \vec{K} & [\vec{J},\vec{P}] = \vec{P} & [\vec{J},H] = 0 \\ & [\vec{K},\vec{K}] = -\vec{J} & [\vec{K},\vec{P}] = \frac{1}{c}H & [\vec{K},H] = c\vec{P} \\ & & [\vec{P},\vec{P}] = \pm\frac{1}{L^2}\vec{J} & [\vec{P},H] = \pm\frac{c}{L^2}\vec{K}\,.\end{array} \tag{3.32}$$

Here we have introduced a shorthand that makes use of the covariance under the subalgebra of rotations: If there is a vector on the right hand side of a relation, then in components it is summed over with ϵ_{ijk} as shown in Eq. (3.27). Otherwise no sum appears, but instead an additional δ_{ij}. For example, $[\vec{K},\vec{P}] = \frac{1}{c}H$ stands for $[K_i, P_j] = \frac{1}{c}\delta_{ij}H$.

Note that the relations (3.30) generalize to (pseudo-)orthogonal Lie algebras in any dimension D, which one customarily denotes by $\mathfrak{so}(n, D-n)$ when the metric has signature $(n, D-n)$. Accordingly, we say that

$$\begin{aligned}\text{the de Sitter Lie algebra} \qquad & \mathbf{dS}_+ := \mathfrak{so}(1,4) \\ \text{and the Anti de Sitter Lie algebra} \qquad & \mathbf{dS}_- := \mathfrak{so}(2,3).\end{aligned}$$

In order to draw a connection between the present algebraic considerations and the geometric constructions in Sec. 3.1, we make use of a common construction for implementing (pseudo-)orthogonal Lie algebras in terms of vector fields on $\mathbb{R}^D$. In our case, this can be formulated in terms of a map

$$\begin{aligned}\Omega^{\pm}: \quad & \mathbf{dS}_{\pm} \to \mathrm{Vec}(\mathbb{R}^5) \\ & M_{MN} \mapsto \Omega^{\pm}_{MN} := (\omega^{\pm}_{MN})^K{}_L X^L \partial_K\,.\end{aligned} \tag{3.33}$$

Computing the Lie bracket for the vector fields $\Omega^{\pm}_{MN}$ yields

$$\begin{aligned}
[\Omega^{\pm}_{KL}, \Omega^{\pm}_{MN}] &= \left[(\omega^{\pm}_{KL})^{K'}{}_{L'}X^{L'}\partial_{K'}, (\omega^{\pm}_{MN})^{M'}{}_{N'}X^{N'}\partial_{M'}\right] \\
&= (\omega^{\pm}_{KL})^{K'}{}_{L'}(\omega^{\pm}_{MN})^{M'}{}_{N'}\left[X^{L'}\partial_{K'}, X^{N'}\partial_{M'}\right] \\
&= (\omega^{\pm}_{KL})^{K'}{}_{L'}(\omega^{\pm}_{MN})^{M'}{}_{N'}\left(X^{L'}\delta^{N'}_{K'}\partial_{M'} - X^{N'}\delta^{L'}_{M'}\partial_{K'}\right) \\
&= \left((\omega^{\pm}_{KL})^{N'}{}_{L'}(\omega^{\pm}_{MN})^{M'}{}_{N'} - (\omega^{\pm}_{KL})^{M'}{}_{N'}(\omega^{\pm}_{MN})^{N'}{}_{L'}\right) X^{L'}\partial_{M'} \\
&= -\left[(\omega^{\pm}_{KL}), (\omega^{\pm}_{MN})\right]^{N'}{}_{L'} X^{L'}\partial_{N'} \,.
\end{aligned} \tag{3.34}$$

This means that the map $M_{MN} \mapsto \Omega^{\pm}_{MN}$ becomes a Lie algebra anti-homomorphism if we take the matrices $(\omega^{\pm}_{MN})$ to constitute the defining ('fundamental') representation of $\mathbf{dS}_{\pm}$, i.e.

$$\begin{aligned}
\omega^{\pm} : \quad \mathbf{dS}_{\pm} &\to \mathfrak{gl}(\mathbb{R}^5) \\
M_{MN} &\mapsto (\omega^{\pm}_{MN}) \,,
\end{aligned} \tag{3.35}$$

with the commutation relations between the matrices $(\omega^{\pm}_{MN})$ coinciding with those of the abstract Lie algebra elements M_{MN}. If, in reverse, one chooses $\omega^{\pm}$ to be an anti-homomorphism (e.g. by taking the transpose), then $\Omega^{\pm}$ becomes a proper one. Alternatively, one could choose vector fields

$$\tilde{\Omega}^{\pm}_{MN} = X^K (\tilde{\omega}^{\pm}_{MN})_K{}^L \partial_L \,, \tag{3.36}$$

which satisfy

$$[\tilde{\Omega}^{\pm}_{KL}, \tilde{\Omega}^{\pm}_{MN}] = X^{K'}\left[(\tilde{\omega}^{\pm}_{KL}), (\tilde{\omega}^{\pm}_{MN})\right]_{K'}{}^{N'} \partial_{N'} \,, \tag{3.37}$$

so that both $\tilde{\omega}^{\pm}_{MN}$ and $\tilde{\Omega}^{\pm}_{MN}$ fulfil the $\mathbf{dS}_{\pm}$ relations. However, we prefer the pair ($\Omega^{\pm}$, $\omega^{\pm}$) to be an anti-homomorphism and a homomorphism, respectively, for the following reason: The shift in the coordinate functions X^M that is induced along the flow of the vector fields $\Omega^{\pm}_{MN}$ is obtained by ordinary matrix-vector multiplication from the left:

$$\Omega^{\pm}_{MN} X^L = (\omega^{\pm}_{MN})^L{}_K X^K \,. \tag{3.38}$$

We remark in passing that this means on the other hand that the action on dual vectors, i.e. on momentum space, strictly is to be taken from the right. While for $\mathbf{dS}_{\pm}$ this distinction only amounts to changes of signs in some of the transformation parameters, it becomes important for the degenerate cases which are to be discussed.

Explicitly, one finds for the fundamental representation (with some antisymmetric parameter set $\sigma^{MN} = -\sigma^{NM}$)

$$\frac{1}{2}\sigma^{MN}(\omega_{MN}) = \begin{pmatrix} 0 & -\sigma^{01} & -\sigma^{02} & -\sigma^{03} & -\sigma^{04} \\ -\sigma^{01} & 0 & \sigma^{12} & \sigma^{13} & \sigma^{14} \\ -\sigma^{02} & -\sigma^{12} & 0 & \sigma^{23} & \sigma^{24} \\ -\sigma^{03} & -\sigma 13 & -\sigma^{23} & 0 & \sigma^{34} \\ \mp\sigma^{04} & \mp\sigma^{14} & \mp\sigma^{24} & \mp\sigma^{34} & 0 \end{pmatrix} \tag{3.39}$$

Note that $\omega^+_{mn} = \omega^-_{mn} =$ with $\eta_\pm = \mathrm{diag}(-1,1,1,1,\pm 1)$, so that we can also write $\omega_{mn} := \omega^\pm_{mn}$. Due to the linearity of $\omega^\pm$ (and hence $\Omega^\pm$), we may use a notation that corresponds more closely to the assignment of dimensionful generators in Eq. (3.31), and write

$$\begin{aligned} \Omega_{J_i} &:= \frac{1}{2}\epsilon_{ijk}\Omega_{kj}\,, \\ \omega_{J_i} &:= \frac{1}{2}\epsilon_{ijk}\omega_{kj} \qquad \text{etc.}, \end{aligned} \tag{3.40}$$

i.e. similarly for $\vec{K}$, $\vec{P}$ and H. A full parametrization of the de Sitter algebras then takes the following form:

$$\mathbf{dS}_\pm: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{v}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega^\pm_{\vec{P}} + b\,\omega^\pm_H = \begin{pmatrix} 0 & \vec{v}^t & \frac{c}{L}b \\ \vec{v} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \frac{1}{L}\vec{a} \\ \pm\frac{c}{L}b & \mp\frac{1}{L}\vec{a}^t & 0 \end{pmatrix}, \tag{3.41}$$

with dimensionless $\vec{\alpha}$ and $\vec{v}$, but $[b] =$ time and $[\vec{a}] =$ length, and where $(\boldsymbol{\omega}_i)^j{}_k = -\epsilon_{ijk}$. From the fact that rotations form a subalgebra it follows immediately that conjugating group elements amounts to transforming the parameters appropriately. Denoting any pure rotation by

$$R = \exp(\vec{\alpha}\cdot\omega_{\vec{J}}) = \begin{pmatrix} 1 & \vec{0}^t & 0 \\ \vec{0} & \exp(\vec{\alpha}\cdot\vec{\boldsymbol{\omega}}) & \vec{0} \\ 0 & \vec{0}^t & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{0}^t & 0 \\ \vec{0} & \mathbf{R} & \vec{0} \\ 0 & \vec{0}^t & 1 \end{pmatrix}, \tag{3.42}$$

one has

$$\begin{aligned} R(\exp(\vec{\alpha}\cdot\vec{J}))R^{-1} &= \exp((\mathbf{R}\vec{\alpha})\cdot\vec{J})\,, \\ R(\exp(\vec{v}\cdot\vec{K}))R^{-1} &= \exp((\mathbf{R}\vec{v})\cdot\vec{K})\,, \\ R(\exp(\vec{a}\cdot\vec{P}))R^{-1} &= \exp((\mathbf{R}\vec{a})\cdot\vec{P})\,, \\ R(\exp(bH))R^{-1} &= \exp(bH)\,. \end{aligned} \tag{3.43}$$

In general, (pseudo-)orthogonal transformations are defined to be those which preserve the norm. On the infinitesimal level considered here, this means geometrically that the Lie derivative of the metric with respect to the generating vector fields vanishes,

$$\mathrm{Lie}_{\Omega^\pm_{MN}}\eta_\pm = 0\,. \tag{3.44}$$

This is equivalent to the statement that the flow of all $\Omega^\pm_{MN}$ lies within the hyperboloidal hypersurfaces (3.8). Rather central for the purposes of this chapter is the observation that this means that $\Omega^\pm_{MN}$ remain to generate isometries for the induced metrics. In other words, they also satisfy the Killing equation in the de Sitter spaces, as defined in (3.11),

$$\mathrm{Lie}_{\Omega_{MN}|_{H_\pm}}\eta_\pm|_{H_\pm} = 0\,. \tag{3.45}$$

Hence, although we will see towards the end of this work when an intrinsic analysis does become necessary, there is no need for it in finding the abstract symmetry structure. For now it is equivalent, and easier, to use the embedded view.

We can now ask, with an eye on the classification of the preceding section: How do the vector fields $\Omega^{\pm}_{MN}$ change for the limiting geometries of Sec. 3.1, and what replaces $\eta_{\pm}$? Due to the homomorphism (3.33), this question can be answered by an appropriate analysis of the corresponding abstract Lie algebras. Since it will be clear from the context which case it meant, we will omit the index $\pm$ on the generators in all what follows.

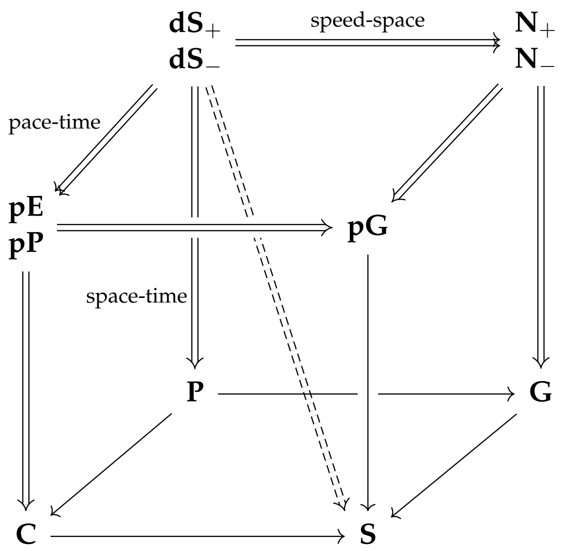


Figure 3.2.: The kinematical cube, as discovered in [BL68]. Arrows indicate Wigner-Inönü contraction. For those pointing downwards, it is physically effected by restriction to small spacetime intervals. Along arrows to the right, one considers only small velocities and small spacelike distances. Going from the back to the front means assuming high velocities (small paces) and short durations.

Flat spacetime limits

We introduced Wigner-Inönü contractions in Sec. 2.2. There we saw that in order to contract an abstract Lie algebra with respect to a given subalgebra, one had to rescale all generators of its complement by a parameter that is afterwards taken to zero. Now, by our assignment of physical generators we already introduced two parameters, c and L, which, carrying dimensions, can be used to clarify the interpretation of the contraction processes. In this way we build a bridge between the abstract mathematical apparatus and how it is commonly applied in the physics literature.

POINCARÉ

We start with the presumably familiar case of the Poincaré Lie algebra **P**. It is obtained from either $\mathbf{dS}_+$ and $\mathbf{dS}_-$ in the same way, namely, by sending the curvature radius L to infinity. When c is kept constant, this automatically also lets the time scale L/c go to infinity, and hence establishes the direct reflection of the abstract rescaling (2.35) in our choice of physical generators (3.31). This connections persists in an analogous way for all the contractions discussed in the following, and we will not emphasize it every time.

Taking $L \to \infty$ results in modifying those bracket relations involving only translation, namely in the following way:

$$L \to \infty \ : \ \begin{cases} [\vec{P}, \vec{P}] \to 0\,, \\ [\vec{P}, H] \to 0\,, \end{cases} \tag{3.46}$$

which changes the Lie algebra structure to the semidirect sum

$$\mathbf{P} \simeq \mathfrak{so}(1,3) \ltimes \mathbb{R}^4\,. \tag{3.47}$$

When using this notation now and for all the cases to come, we bear in mind that the indicated isomorphism consists in a choice of splitting homomorphism for the corresponding short exact sequence like the one in Eq. (2.46).

Geometrically, this limit can be understood as arising once only negligibly small regions (compared to L) in the curved spacetimes $(\mathrm{A})\mathrm{dS}$ are considered. The fundamental representation becomes

$$\mathbf{P}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{v}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{v}^t & cb \\ \vec{v} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{a} \\ 0 & \vec{0}^t & 0 \end{pmatrix}, \tag{3.48}$$

which translates to the corresponding vector fields. It is also possible to take the limit for the latter directly from their definition (3.33), which reminds us that the physical rescaling of generators goes along with associating a dimension to the coordinates. In particular, $X^4 = L\Xi^4$ with dimensionless Ξ^4. In this way,

$$\vec{a}\cdot\Omega_{\vec{P}} = \frac{1}{L}\left(\vec{a}\cdot\vec{X}\,\partial_4 - L\Xi^4\,\vec{a}\cdot\vec{\partial}\right) \xrightarrow{L\to\infty} -\vec{a}\cdot\vec{\partial} \tag{3.49}$$

where in the limit we redefined the parameter $\vec{a}$ to absorb Ξ^4. Similarly, for the time translations,

$$b\,\Omega_H = b\frac{c}{L}\left(X^0\,\partial_4 + X^4\,\partial_0\right) = b\frac{c}{L}\left(\frac{c}{L}T\,\partial_{\Xi^4} + \frac{L}{c}\Xi^4\,\partial_T\right) \xrightarrow{L\to\infty} b\,\partial_T\,. \tag{3.50}$$

For the remaining Lie algebras of the BLL scheme, we will not write out the vector fields any more, since they follow directly from the representative matrices.

GALILEI

In addition to considering only small *distances*, we now imagine considering only such motion involving *velocities* that are very small compared to the speed of light. However, in order to even be able to speak about velocities as parameters for kinematical transformations that determine the rate in time at which a body passes segments in space, we need to rescale all boost generators such that they obtain dimensions of inverse velocity, i.e.

$$\vec{K} \to \frac{1}{c}\vec{K} \qquad (\text{in } \mathbf{P})\,. \tag{3.51}$$

The Lie brackets are unaffected by this rescaling as long as we replace each instance of $\vec{K}$ by $c\vec{K}$.[3] Formally, the restriction to low velocities now amounts to sending the reference speed c to infinity, which pictorially says that the light cones open up completely. In addition to the change (3.46) affected by only considering small distances (both in time and space), we now receive

$$\text{for} \quad c \to \infty \;:\; \begin{cases} [\vec{K}, \vec{K}] \to 0\,, \\ [\vec{K}, \vec{P}] \to 0\,, \\ [\vec{K}, H] \to \vec{P}\,, \end{cases} \tag{3.52}$$

while the brackets involving rotations remain the same (since the factors of c cancel). Observe from here that, apart from brackets including rotations, the only non-vanishing commutators left in the kinematical algebra are the ones between the new, Galilean boosts and temporal translations. The resulting algebraic structure can be understood as a nested semidirect sum, and is called the (inhomogeneous) Galilei algebra

$$\mathbf{G} \simeq (\mathfrak{so}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4\,. \tag{3.53}$$

It is possible to rearrange the abelian parts towards

$$\mathbf{G} \simeq (\mathfrak{so}(3) \times \mathbb{R}) \ltimes \mathbb{R}^6\,, \tag{3.54}$$

which emphasizes a different aspect of **G**. While the first decomposition stresses its features as a spacetime symmetry algebra and as the low-velocity remnant of **P**, the second one suggests to interpret it in terms of invariances of a non-relativistic phase space (under rotations and time translations). The latter corresponds in fact to its usage in quantum mechanics. Also, it puts it more closely into the low-curvature vicinity of the Newton-Hooke algebras, which will be discussed.

[3] Imagine we had introduced a new symbol, say $\vec{G} = \vec{K}/c$, and aimed to express the Lie algebra relations (without changing them) in terms of $\vec{G}$ instead of $\vec{K}$. This way of phrasing the Lie algebra contractions to be discussed in this chapter would however force us to introduce different sets of generators for each limit, which the author hesitated to do.

Manifesting both of the above decompositions, the matrices ω_{MN} of the fundamental representation become for the Galilei algebra

$$\mathbf{G}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{v}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{0}^t & b \\ \vec{v} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{a} \\ 0 & \vec{0}^t & 0 \end{pmatrix} \tag{3.55}$$

Carroll

The Galilean limit might still be close to a physicist's, or in fact perhaps everyone's intuition. The opposite limit, found in [Lév65], describes more extreme situations, namely those in which the speed of light may safely be regarded to be zero, hence squeezing the light cone to a line. Although one is tempted to disqualify further investigations in this direction as unphysical, as was done in [BL68], it has, as indicated in the introductory remarks to the chapter, proven useful in the disguise of an ultra-local approximation, describing situations in which field degrees of freedom at different points in space evolve completely independently (or more precisely, on negligible time scales compared with those governing spatial interactions) [Dau98]. It is in this sense that the Carrollian limit is often thought of as the high-velocity limit of the Poincaré algebra.

Going beyond the abstract Wigner-Inönü prescription, in the literature there exist very different ways of how to dicuss the Carroll limit from the Lie algebra point of view (e.g. compare [Lév65; Duv+14; Har15]). To us, the most consistent approach, which also fits into the more general picture that is to be created here, goes via a redefinition of the units in which we measure the *intensity of motion*. Conventionally, the appropriate physical quantity considered is *velocity*, which has dimensions of distance per time, and was used to define the Galilean limit. We may however turn this around and take as a measure for an object's speed its *pace*[4], which has dimensions of time per distance. But this means that we need to rescale our boost generators from Eq. (3.31) like

$$\vec{K} \to c\vec{K} \qquad (\text{in } \mathbf{P}), \tag{3.56}$$

so that they obtain the dimension of $(\text{pace})^{-1}$ (which is equal to (length/time) and hence the dimension of c). With this choice, the modified Lie algebra relations involving boosts become, in the limit

$$\text{for} \quad c \to 0 \ : \ \begin{cases} [\vec{K},\vec{K}] \to 0\,, \\ [\vec{K},\vec{P}] \to H\,, \\ [\vec{K},H] \to 0\,, \end{cases} \tag{3.57}$$

while the rest is unchanged compared to $\mathbf{P}$. The resulting Lie algebra is called the Carroll

[4] Typical objects for which this is frequently done are runners of all distances, who like to categorize their training sessions in terms of how much time they needed for a, mostly fixed, distance. Note that 'fast' (high-velocity) translates to a *low* pace.

algebra[5]. Its abstract structure is, similarly to **G**,

$$\mathbf{C} \simeq (\mathfrak{so}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4, \tag{3.58}$$

where, however, the difference lies in the way in which the outer semidirect sum is taken, or to be precise, in φ of Eq. (2.44). One may try to find a similar rearrangement of abelian parts as in the Galilei case. This time however, it would correspond to swapping boosts with spatial instead of time translations, and hence merely lead to a different physical interpretation (now as 'phase-time' transformations) of the same abstract structure.

Again, all this becomes manifest in the limiting matrices of the five-dimensional representation ($[\vec{u}]$ = pace):

$$\mathbf{C}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{u}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{u}^t & b \\ \vec{0} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{a} \\ 0 & \vec{0}^t & 0 \end{pmatrix} \tag{3.59}$$

We would like to stress at this point how the appropriate choice of dimension for the boost parameters made the Galilean and Carrollian limit arise rather naturally. Fig. 3.4 shows the geometrical interpretation behind the different notions for the *intensity of motion*. The important role of group parametrizations in physical applications was also stressed in [LP79].

The static limit

The theory of Wigner-Inönü contractions allows for a further limiting regime of flat space relativistic physics satisfying the three BLL requirements. While the meaning of the Carroll limit is already less obvious than the Galilean one, it is probably fair to say that the following limit parts even further with one's intuition. This is because the very concept of motion becomes obsolete, and the physical dimension which we associate with its measure ('velocities') becomes ambiguous.

Starting from either the Galilei algebra **G** or the Carroll algebra **C**, the procedure that reflects this circumstance is a rescaling of the boost generators by a *dimensionless* parameter, i.e.

$$\vec{K} \to \epsilon\vec{K} \qquad (\text{in } \mathbf{G} \text{ or } \mathbf{C}), \tag{3.60}$$

followed by taking this parameter to zero, so that

$$\text{for} \quad \epsilon \to 0 \; : \; \begin{cases} [\vec{K}, H] \to 0 & (\text{in } \mathbf{G}), \\ [\vec{K}, \vec{P}] \to 0 & (\text{in } \mathbf{C}), \end{cases} \tag{3.61}$$

[5] Lévy-Leblond chose this name in [Lév65], appealing to events from Lewis Carroll's children's stories *Alice in Wonderland* and *Through the Looking-Glass* that illustrate features of the seemingly acausal Carrollian dynamics—most strikingly, the circumstance that boosting does not lead to movement, which can be seen from the rightmost commutator of Eq. (3.57) and is illustrated by the *Red Queen's Race*.

while all other relations remain the same again. In addition, the static Lie algebra is also reached by direct contraction from the (A)dS with respect to the rotations, i.e. by sending

$$(\vec{K}, \vec{P}, H) \to (\epsilon\vec{K}, \epsilon\vec{P}, \epsilon H) \qquad (\text{in } \mathbf{dS}_\pm)\,, \tag{3.62}$$

and then letting ϵ go to zero. Hence, depending on which of the three paths we take towards this Lie algebra, we naturally associate with the generators of boosts the dimension of either velocity, pace or none at all. This underlines the meaninglessness of motion that is demonstrated by the commutativity of boosts and translations both in space and time. The resulting Lie algebra is appropriately called the Static Lie algebra. Its structure is

$$\mathbf{S} \simeq \mathfrak{so}(3) \ltimes \mathbb{R}^7\,, \tag{3.63}$$

the only non-vanishing brackets being those among generators of rotations. The Static Lie algebra is hence as close to abelian as possible given the BLL assumptions (and their implementation). As a notable consequence, it is impossible to find limiting representative matrices from the ω_{MN}—there is no five-dimensional representation for **S**. An obvious alternative is six-dimensional, and suggests an interpretation in terms rotations in a (3+3+1)-dimensional phase space that has been extended in time direction[6]:

$$\mathbf{S}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{\mu}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{0}^t & b & 0 \\ \vec{0} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{a} & \vec{\mu} \\ 0 & \vec{0}^t & 0 & 0 \\ 0 & \vec{0}^t & 0 & 0 \end{pmatrix}\,, \tag{3.64}$$

with a free choice of $[\mu] \in \{\text{length/time},\ \text{time/length},\ [1]\}$.

Curved spacetime limits

The limit we started from in the preceding section was that of small distances in space and time, leaving a commutative translationary sector in the Lie algebra. Now we will see that it is just as sensible to consider geometries that are equally distinguishable regarding their causal structure as the ones just found, but imply spacetime curvature. This is achieved by finding the $0 \leftarrow c \to \infty$ limits already from the de Sitter Lie algebras $\mathbf{dS}_\pm$.

Newton±Hooke

As in the transition from **P** to **G**, we prepare for the low-velocity limit of $\mathbf{dS}_\pm$ by adapting from dimensionless rapidities to dimensionful velocites, and therefore take

$$\vec{K} \to \frac{1}{c}\vec{K} \qquad (\text{in } \mathbf{dS}_\pm), \tag{3.65}$$

[6] While the representative matrices for **P**, **G** and **C** have been put to use in applications, their form for **S** and all further (curved spacetime) cases seems to be new to the literature.

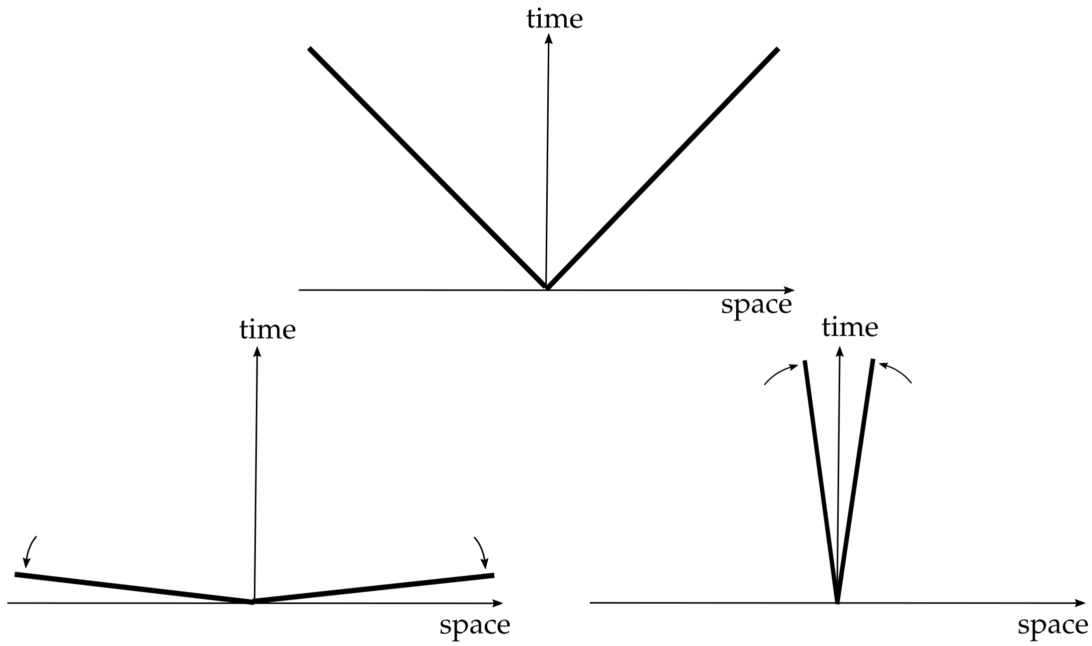


Figure 3.3.: An illustration of the causal structure associated with the different notions of relativity discussed in the text. Depicted are two-dimensional versions of the future light cones (technically, the null sets of the metric in each tangent space) for $c = 1$ (*top*), $c \to \infty$ (*bottom left*) and $c \to 0$ (*bottom right*). In 1+3 dimensions, one needs to imagine everything rotated around the time axis, so that (if one includes the origin) the topology changes from $(\mathbb{R}^+ \times S^2) \cup \{0\}$ to $\mathbb{R}^3$ and $\mathbb{R}^+ \cup \{0\}$ for $c \to \infty$ and $c \to 0$, respectively. In the cotangent (i.e. momentum spaces) the limits are interchanged.

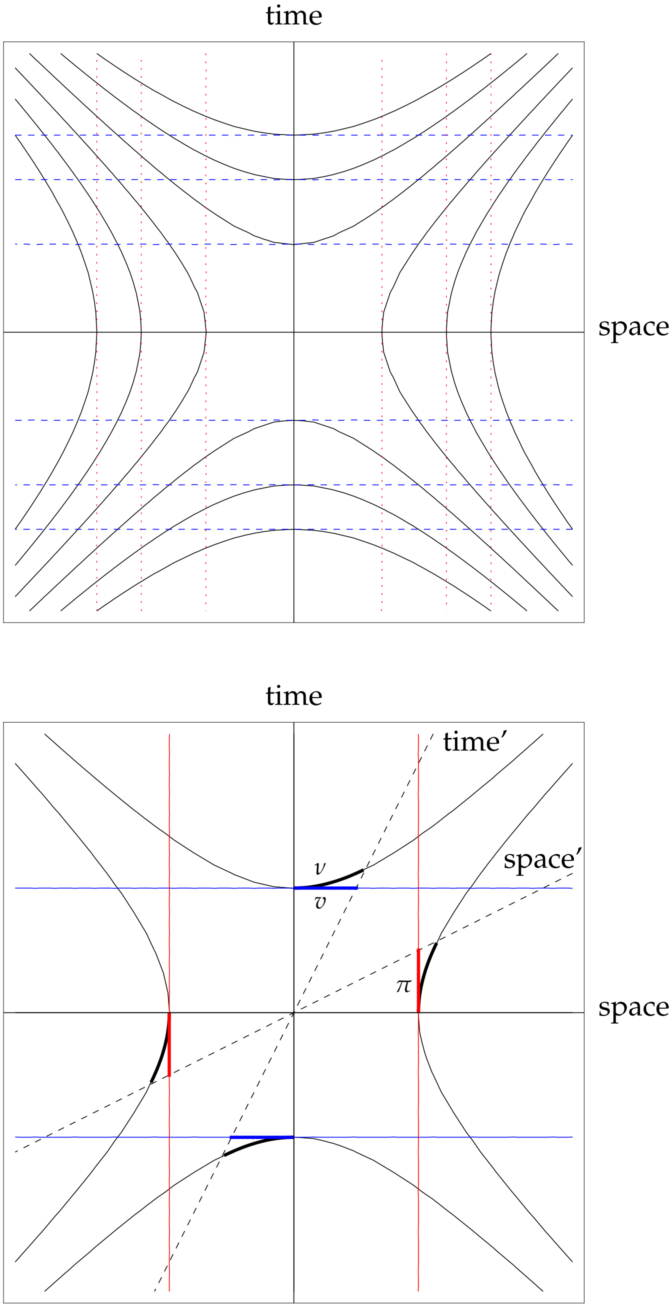


Figure 3.4.: *Top:* Orbits of the Lorentz (solid) group as well as the homogeneous Galilei (dashed) and Carroll (dotted) groups in the tangent spaces to spacetime. *Bottom:* The relative rapidity (ν), velocity (v) and pace (π) of two Lorentzian, Galilean and Carrollian inertial systems, respectively, corresponds to the length of the thick line segments on the unit orbits ('velocity' 'hyperboloids'), measured in terms of the restricted Minkowski metric. *Both* figures generalize to 1+3 dimensions via rotation about the time axis. The corresponding picture in momentum space arises by previously swapping the two axes.

so that new factors of c arise in the relations (3.32). The subtlety that arises when trying to take $c \to \infty$ lies in the commutator between spatial and temporal translations, which after the redefinition (3.65) reads $[\vec{P}, H] = \pm\frac{c^2}{L^2}\vec{K}$. It tells us that the envisaged limit must be accompanied by a restriction to small spatial distances, i.e. we need to simultaneously take $L \to \infty$. But then, the new commutation relations read

$$\begin{aligned} &[\vec{K},\vec{K}] = 0 \qquad [\vec{K},\vec{P}] = 0 \qquad [\vec{K},H] = \vec{P} \\ &\qquad\qquad\qquad\; [\vec{P},\vec{P}] = 0 \qquad [\vec{P},H] = \pm\frac{1}{\tau^2}\vec{K}\,, \end{aligned} \tag{3.66}$$

where one may define

$$\tau = \lim_{c,L\to\infty} \frac{L}{c} \tag{3.67}$$

to be the left-over curvature scale with dimension of time. The resulting Lie algebras are abstractly characterized as semidirect sums of the type

$$\mathbf{N}_{+} \simeq (\mathfrak{so}(3) \times \mathfrak{so}(1,1)) \ltimes \mathbb{R}^6\,, \tag{3.68}$$

$$\mathbf{N}_{-} \simeq (\mathfrak{so}(3) \times \mathfrak{so}(2)) \ltimes \mathbb{R}^6\,, \tag{3.69}$$

and may be represented in terms of the following matrices:

$$\mathbf{N}_{\pm}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{v}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{0}^t & \frac{1}{\tau}b \\ \vec{v} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \frac{1}{\tau}\vec{a} \\ \pm\frac{1}{\tau}b & \vec{0}^t & 0 \end{pmatrix} \tag{3.70}$$

PARA-POINCARÉ AND -EUCLID

Measuring motion in terms of paces asks us, like in the transition from **P** to **C**, to perform the substitution

$$\vec{K} \to c\vec{K} \qquad (\text{in } \mathbf{dS}_{\pm}), \tag{3.71}$$

which naturally makes factors of $1/c$ disappear from the right hand sides of the $\mathbf{dS}_{\pm}$ relations (3.32). Hence, in this case, we can directly let $c \to 0$ to obtain Lie algebras that differ from the de Sitter ones by the following commutators:

$$\begin{aligned} &[\vec{K},\vec{K}] = 0 \qquad [\vec{K},\vec{P}] = H \qquad [\vec{K},H] = 0 \\ &\qquad\qquad\qquad\; [\vec{P},\vec{P}] = \frac{1}{L^2}\vec{J} \qquad [\vec{P},H] = \pm\frac{1}{L^2}\vec{K} \end{aligned} \tag{3.72}$$

We realize the following structure, depending on the sign of the last relation:

$$\mathbf{pE} \simeq \mathfrak{so}(4) \ltimes \mathbb{R}^4 \qquad (+)\,, \tag{3.73}$$

$$\mathbf{pP} \simeq \mathfrak{so}(1,3) \ltimes \mathbb{R}^4 \qquad (-)\,. \tag{3.74}$$

Their five-dimensional representation is as follows.

$$\mathbf{pE}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{\pi}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{\pi}^t & \frac{1}{L}b \\ \vec{0} & \vec{\alpha}\cdot\vec{\omega} & \frac{1}{L}\vec{a} \\ 0 & -\frac{1}{L}\vec{a}^t & 0 \end{pmatrix} \tag{3.75}$$

$$\mathbf{pP}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{\pi}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{\pi}^t & \frac{1}{L}b \\ \vec{0} & \vec{\alpha}\cdot\vec{\omega} & \frac{1}{L}\vec{a} \\ 0 & +\frac{1}{L}\vec{a}^t & 0 \end{pmatrix} \tag{3.76}$$

These algebras are isomorphic to the Euclidean and the Poincaré Lie algebra, respectively, from where they derive their names. The physical interpretation is however rather different, which is due to the fact that the roles of boosts and spatial translations are interchanged. Bringing the boosts into the $\mathfrak{so}(4)$ part in exchange for the spatial translations is not allowed by the third of the BLL axioms, so that, different from the Poincaré group, the Euclidean group drops out as a candidate spacetime symmetry in this classification.

Our choice of terminology actually serves a mnemonic purpose here: *p*ace-time and *sp*ace-time contractions yield isomorphic Lie algebras.

Para-Galilei

As in the flat case, one may concatenate the speed-space and pace-time contractions from the de Sitter Lie algebras, leading to a spacetime geometry with completely degenerate causal structure. This time however, one can still identify a notion of curvature. Redefining

$$\begin{aligned} &\vec{K} \to \frac{1}{\tau^2}\vec{K} \qquad (\text{in } \mathbf{N}_\pm) \\ \text{or} \qquad &\vec{K} \to \frac{1}{L^2}\vec{K} \qquad (\text{in } \mathbf{pE} \text{ or } \mathbf{pP}), \end{aligned} \tag{3.77}$$

leads to new boost generators that have dimensions of $(\text{length} \times \text{time})^{-1}$. Irrespectively, in the limit where $\tau \to \infty$, respectively $L \to \infty$, the resulting Lie algebra has, apart from the unaffected ones involving rotations, the following set of commutation relations:

$$\begin{aligned} [\vec{K},\vec{K}] = 0 \qquad [\vec{K},\vec{P}] &= 0 \qquad [\vec{K},H] = 0 \\ [\vec{P},\vec{P}] &= 0 \qquad [\vec{P},H] = \pm\vec{K} \end{aligned} \tag{3.78}$$

The algebra structure is, independently of the sign in the last relation,

$$\mathbf{pG} \simeq (\mathfrak{so}(3) \times \mathbb{R}) \ltimes \mathbb{R}^6\,, \tag{3.79}$$

which should be compared with $\mathbf{N}_+$ and $\mathbf{N}_-$. Abstractly speaking, **pG** can be regarded as lying right in between the two Newton±Hooke algebras, the difference only being that the time-translationary factor in the left part of the semidirect sum is contracted to $\mathbb{R}$. On the other hand, the name 'Para-Galilean' is chosen in analogy to **pP** and **pE**.

Here, the similarity is with the Galilei algebra **G**, and interchanges the roles of *time* translations and boosts. The matrix representation that we have been using many times now becomes for

$$\mathbf{pG}: \qquad \vec{\alpha}\cdot\omega_{\vec{J}} + \vec{\lambda}\cdot\omega_{\vec{K}} + \vec{a}\cdot\omega_{\vec{P}} + b\,\omega_H = \begin{pmatrix} 0 & \vec{\lambda}^t & b \\ \vec{0} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{0} \\ 0 & \mp\vec{a}^t & 0 \end{pmatrix}, \tag{3.80}$$

with $[\vec{\lambda}] = \text{length} \times \text{time}$. This form is natural when imagining **pG** as a limit of **pP** and **pE**. Equivalently, originating from $\mathbf{N}_\pm$, one arrives (up to irrelevant signs) at the transposed matrices

$$\begin{pmatrix} 0 & \vec{0}^t & 0 \\ \vec{\lambda} & \vec{\alpha}\cdot\vec{\boldsymbol{\omega}} & \vec{a} \\ \pm b & \vec{0}^t & 0 \end{pmatrix}. \tag{3.81}$$

As in the case of **S**, separations in spacetime are not well-defined in Para-Galilean kinematics, since boosts do no longer act effectively on spacetime, and hence *any* combination of a (rotationally invariant) length and a time measure could be chosen. This statement can be formalized in terms of the definitions given in Sec. 3.3. There it will also be shown how the presence of spacetime curvature can be traced back to the non-commutativity between translations, which is still the case for **pG**. Displaying one but not the other, the Para-Galilean limit hence suggests a striking independence between the notions of spacetime curvature and that of distances, which seems to be unrecognized in the literature.

Infinite curvature limits

Contrary to the contractions discussed so far, the limit $L \to 0$ cannot be understood as rendering the embedding metric degenerate in some directions. Instead of being the *unit* norm set with respect to a *degenerate* metric, the resulting structure in the infinite curvature limits are built upon *null* sets of the *intact* metric. As we shall see, this also means that the symmetry group stays the same in the infinite curvature limit. This is why it does not appear in the original BLL discussion. While for H_+, the result fairly standard at least in one dimension lower, the corresponding analysis for H_- seems less well-known.

Infinite positive curvature

We need to analyse the limiting set

$$\lim_{L\to 0} H_{+,L}, \tag{3.82}$$

i.e., the subset of $\mathbb{R}^5$ where

$$\eta_+(X,X) = -(X^0)^2 + \vec{X}^2 + (X^4)^2 = 0\,. \tag{3.83}$$

This is nothing but the light cone of five-dimensional Minkowski space $\mathbb{R}^5_+$. But in fact, the Euler vector field X from (3.7) is only defined on $\mathbb{R}^5 \setminus \{0\}$ so that what we are truly left with is the punctured light cone

$$K := S^3 \times (\mathbb{R} \setminus \{0\}) \subset \mathbb{R}^5. \tag{3.84}$$

The coordinatization from Eq. (3.23) needs to be changed slightly towards

$$X^M(x^\mu) = \begin{pmatrix} \pm t \\ tn \end{pmatrix}, \qquad t > 0 \tag{3.85}$$

with the four-dimensional Euclidean unit vector

$$n = \begin{pmatrix} \vec{n} \sin\chi \\ \cos\chi \end{pmatrix}. \tag{3.86}$$

The induced, degenerate metric becomes a multiple of the round metric of S^3,

$$g = t^2 \delta_{|_K} = t^2 \mathrm{d}\Omega_3^2 \,. \tag{3.87}$$

Asking which vector fields Z on K satisfy

$$\mathrm{Lie}_Z g = 0 \tag{3.88}$$

will yield the isometries we are after. If we decompose

$$Z = Z^t \partial_t + Z_{\|} \tag{3.89}$$

with purely spatial $Z_{\|}$ (i.e. along the S^3 part), we can rewrite Eq. (3.88) as

$$\mathrm{Lie}_Z g = 2Z^t t \mathrm{d}\Omega_3^2 + t^2 \mathrm{Lie}_{Z_{\|}} \mathrm{d}\Omega_3^2 \,, \tag{3.90}$$

which tells us that $Z_{\|}$ may be a conformal vector field of S^3,

$$Z_{\|} \in \mathrm{Conf}(S^3) \,, \tag{3.91}$$

as long as Z^t is chosen appropriately. One finds that it must be

$$\begin{aligned} Z^t &= -\frac{t}{6} \mathrm{Tr} \left[\left(\mathrm{d}\Omega_3^2 \right)^\sharp \left(\mathrm{Lie}_{Z_{\|}} \mathrm{d}\Omega_3^2 \right) \right] \\ &= -\frac{t}{3} \mathrm{Div} Z_{\|} \end{aligned} \tag{3.92}$$

in order to counterbalance the shift in time direction evoked by $Z_{||}$. To show this in components, one can use that for the connection components that appear in the covariant divergence Div one has

$$\mathrm{d}x^i (\nabla_j \partial_i) = \left\{ {i \atop ji} \right\} = \left\{ {i \atop ij} \right\} = \frac{1}{2} g^{ik} \partial_j g_{ik} \,. \tag{3.93}$$

Now in fact,

$$\mathrm{Conf}(S^3) \simeq \mathfrak{so}(1,4), \tag{3.94}$$

which becomes clear as follows: Firstly, recall that the subalgebra of four-dimensional rotations actually generates the proper isometries,

$$\mathrm{Lie}_{Z_\parallel} g = \mathrm{Lie}_{Z_\parallel} \mathrm{d}\Omega_3^2 = 0 \qquad \text{for} \quad Z_\parallel \in \mathfrak{so}(4) \subset \mathfrak{so}(1,4)\,. \tag{3.95}$$

Boosts, on the other hand, scale vectors on the light sphere. This is seen by choosing as origin, say, $n_0 := (1, \vec{0}, 1)^t$, which is being scaled by a boost in the fourth direction,

$$(\omega_{40})^L{}_K n_0^K = n_0^L\,. \tag{3.96}$$

Due to Eq. (3.43) then, this holds for any vector on the light sphere and a boost in its direction:

$$\Rightarrow \quad n^M (\omega_{M0})^L{}_K n^K = n^L \tag{3.97}$$

Note that this behaviour makes it appear rather natural to extend the symmetry concept as a whole to that of conformal transformations. If this is done, due to the peculiarities stemming from the degeneracy of the metric structure, one is however immediately facing a whole family of symmetry groups. Each of them being infinite-dimensional by itself, they go by the name of Newman-Unti groups. One particular member of this family is the Bondi-Metzner-Sachs group, which was used to understand the pheonomenon of gravitational radiation from an asymptotic point of view [BBM62; Sac62]. The infinite-dimensional character of these groups stems from the possibility of infinitely many translations in time-direction, the 'supertranslations', which replace the boosts above. $\mathrm{Conf}(S^3)$ lies within each of the Newman-Unti groups, in that for it, the supertranslations are fixed by the requirement (3.92). For an extended discussion of these topics, see e.g. [DGH14b].

Infinite negative curvature

Now the limiting set to be investigated is

$$\lim_{L\to 0} H_{-,L} \tag{3.98}$$

i.e. the subset of $\mathbb{R}^5$ where

$$\eta_-(X,X) = -(X^0)^2 + \vec{X}^2 - (X^4)^2 = 0\,. \tag{3.99}$$

Excluding the origin as before, it has the form

$$J := (S^1 \times S^2) \times (\mathbb{R} \setminus \{0\}) \tag{3.100}$$

and may be coordinatized by

$$X^M(x^\mu) = \begin{pmatrix} \chi \sin t \\ \pm\chi\vec{n} \\ \chi \cos t \end{pmatrix}, \tag{3.101}$$

so that the metric induced on it becomes

$$h := \chi^2(\mathrm{d}t^2 + \mathrm{d}\Omega_2^2) = \chi^2 h_{\|} \tag{3.102}$$

which is again degenerate, but now in the direction of χ instead of t. This does however not mean that we cannot proceed in analogy. Again we can decompose the Killing vector fields Y, satisfying

$$\mathrm{Lie}_Y h = 0\,, \tag{3.103}$$

into a vertical and a horizontal part,

$$Y = Y^\chi \partial_\chi + Y_{\|}\,. \tag{3.104}$$

As before, $Y_{\|}$ can only depend on the horizontal coordinates, and consists of proper isometries for $h_{\|}$, and those that act by a rescaling, hence

$$\mathfrak{so}(2,3) \simeq \mathrm{Conf}(S^1 \times S^2)\,. \tag{3.105}$$

The division now is sightly different to the positive curvature case. This is because there is now a whole circle of 'boosts',

$$\cos(t) M_{i4} + \sin(t) M_{i0}\,, \tag{3.106}$$

so that the number of proper horizontal isometries is reduced to four,

$$\mathrm{Lie}_{Y_{\|}} h_{\|} = 0 \qquad \text{for} \qquad Y_{\|} \in \mathfrak{so}(2) \times \mathfrak{so}(3)\,. \tag{3.107}$$

The condition that fixes the analogue supertranslation has the same form as before,

$$Y^\chi = -\frac{1}{6}\chi \mathrm{Tr}\left(h_{\|}^{\sharp}(\mathrm{Lie}_{Y_{\|}} h_{\|})\right) = -\frac{1}{3}\mathrm{Div} Y_{\|}\,. \tag{3.108}$$

Quotient spacetimes: Coordinates continued

In order to see how for each of the discussed kinematical Lie algebras a natural model for spacetime arises, one needs to ascend to the level of finite transformations. This is done via the exponential map, which makes use of the fact that for Lie group manifold G, the tangent space $T_e G$ at the identity element can be identified with its Lie algebra $\mathfrak{g}$. It is defined as

$$\begin{aligned} \exp:\ \mathfrak{g} &\to G \\ M &\mapsto c(1) \end{aligned} \tag{3.109}$$

with $c : \mathbb{R} \to G$ being the unique path in the Lie group G whose image is the one-parameter subgroup with tangent vector at the identity coinciding with M,

$$\dot{c}(0) = M, \qquad c(0) = e\,. \tag{3.110}$$

Scalar multiplication in the argument of exp simply yields a corresponding point on the same path,

$$\exp(tM) = C(t), \qquad t \in \mathbb{R}, \tag{3.111}$$

for if $C_t(1) \equiv \exp(tM)$, then $\dot{C}_t(0) = tM = t\dot{C}(0)$, and hence $C_t(\lambda) = C(t\lambda)\ \forall \lambda \in \mathbb{R}$. This means that, for example,

$$\exp(\vec{v}\cdot\vec{K}) \tag{3.112}$$

parametrizes all boosts with magnitude $|\vec{v}|$ in the direction of $\vec{v}$. In the defining representation, exp becomes simply the matrix exponential function.

Similarly as a general curved manifold can only locally be identified with its tangent spaces, it is also usually not possible to identify the whole of $\mathfrak{g}$ with the whole of G, since exp can lack both injectivity and surjectivity. If, however, one restrict to a small neighbourhood in $\mathfrak{g}$ then exp will yield a good, one-to-one copy of it within a neighbourhood of the identity element $e \in G$. From there, it is then possible to obtain any further element of the identity component of the group by a finite product, i.e.

$$\forall g \in G \quad \exists \quad \{M_k\}_{k=1\ldots N<\infty} \quad \text{s.t.} \quad g = \exp(X_1)\ldots\exp(X_N)\,. \tag{3.113}$$

A further important property of the exponential map is that it maps Lie subalgebras to Lie subgroups. For proofs of these statements, and further discussion of subtleties connected with the exponential map, the reader is referred to [Hal15].

It is now a characteristic feature common to all of the BLL algebras that boosts and rotations do form a subalgebra. We will denote it by $\mathfrak{h}$, and by H the generated subgroup of the full kinematical group G. The quotient manifold

$$G/H\,, \tag{3.114}$$

i.e. the set of all left cosets gH in $G \ni g$, should be viewed as the space of translations relative to the observer who performs the distinction of boosts and rotations amongst all possible kinematical transformations. For example he could do so by asking which of all transformation fix his origin of coordinates, hence only rotate the axes. The claim about the nature of a kinematical group, namely that it comprises *all* (continuous, structure-preserving) spacetime transformations, it is then legitimate to assume that there is a one-to-one correspondence between G/H and the union of all possible positions in space and time. In this sense, G/H can be taken to model the geometry of spacetime. Note already that G acts transitively on it via isometries. This means that the Lie algebra of Killing vector fields, necessarily maximal in dimension, will at every point be isomorphic to $\mathfrak{g}$. By construction, the action of G on the quotient is however not free [7], its ambiguity being encoded exactly in the homogeneous transformations H, which fix the identity coset eH. The latter defines a preferred point in G/H—an 'origin'. From the observer point of view this seems natural, but its existence in an abstract model for spacetime is rather unsatisfactory. For this reason one might actually want to model spacetime by an 'origin-free' space Σ, which only after an arbitrary point $p \in \Sigma$ on it is singled out obtains coset structure. This can be formulated by demanding that G acts transitively on Σ and that for two points $p, q \in \Sigma$, the respective stabilizing subgroups H_p and H_q ($H_p.p = p$, $H_q.q = q$) are conjugate to each other, i.e. $\forall g \in G$ satisfying $g.p = q$ one has $H_q = gH_pg^{-1}$, and $H_p \simeq H \; \forall p \in \Sigma$. In this sense then, one can at every point realize the isomorphism $\Sigma \simeq G/H$. A concise way to express this is by defining Σ to be the equivalence class

$$\Sigma := [G/H] \tag{3.115}$$

defined by

$$[G/H] := \left\{ G/H' \;\middle|\; \exists g \in G: \; H' = gHg^{-1} \right\} . \tag{3.116}$$

Although in everything that follows this subtlety in deriving a notion of spacetime from the kinematical group will not further be discussed, it remains in the background as a major interpretative element, in particular whenever only specific representatives G/H of Σ are analysed.

The main interest in this section shall be to show that the group-theoretically obtained spacetime models exactly match those initially introduced in terms of constant curvature subsets of $\mathbb{R}^5$. This can quickly be shown from the embedded view, using the representation in terms of the matrices (3.41). If one is on the other hand interested in explicit results concerning intrinsic properties, one is forced to consider coordinate systems on G/H. This naturally goes along with different parametrizations of the group G itself. That such considerations are important is apparent, for instance, already from the special-relativistic law for velocity composition, where the well-known relation $\vec{\beta} = \frac{\vec{v}}{|v|} \tanh |\vec{v}|$ between velocity $\vec{v} = \vec{\beta}c$ and boost vector $\vec{v}$ only holds if one parametrizes Lorentz group elements as

$$h = \exp(\vec{v} \cdot \vec{K}) R \tag{3.117}$$

(R being a rotation) and identifies $\vec{v}$ with the spatial components of the 4-velocity u^μ after normalisation, $v^i = u^i / u^0$. Also, the style in parametrization of the rotation group varies between practical applications. Our standpoint will be that when studying the transformations induced by a group on its coset spaces, there are distinguished

[7] A free group action is defined as having no fixed points, i.e. no points which are left invariant by group elements other than the identity. A free and transitive group action is also called simply transitive, and is both one-to-one and onto, hence allows to identify the group (as a manifold) with the space it acts on. Spaces allowing such an action are called torsors.

parametrizations for the group, and hence distinguished coordinates on the quotient. This remark will be illustrated by Ex. (3.3) for the case of the 2-sphere, where it is found that the standard spherical polar coordinates are generated when choosing

$$R(\vec{\alpha}) = \exp(\alpha^1 J_3)\exp(\alpha^2 J_2)\exp(\alpha^3 J_3) =: r(\alpha_1, \alpha_2)\exp(\alpha^3 J_3) \tag{3.118}$$

as group parametrisation. This is to say that only then the unit vector $\vec{n}$ acquires the familiar form (3.24) by application of $R(\phi, \theta, \alpha^3)$ (or simply $r(\phi, \theta)$) to the coordinate origin at the north pole. Including also the translations, a full group element can then be written

$$g_\pm = T_\pm(b, \vec{a})h \tag{3.119}$$

where, if one aims at a complete coordinate system for G/H, the translational part must be chosen depending on the sign of curvature:

$$\begin{aligned} T_+(b, \vec{a}) &= \exp(\vec{a}\cdot\vec{P})\exp(bH) \\ T_-(b, \vec{a}) &= \exp(bH)\exp(\vec{a}\cdot\vec{P}) \end{aligned} \tag{3.120}$$

This parametrisation exhibts very clearly the relationship between Lie group and associated spacetime, for if we put

$$b \to t, \qquad \vec{a} \to \vec{x}, \tag{3.121}$$

then the interpretation of spacetime points as left cosets becomes manifest: We can simply read off the representatives by setting $\vec{v} = \vec{\alpha} = 0$, i.e., $h \to e$. Applying the so-obtained elements in the fundamental representation to $(0, \vec{0}^t, 1)$, the expression from Eq. (3.23) is recovered.

Remark. *From here on, we will no longer make a distinction between the abstract Lie algebras and their fundamental representation. We do so for notational simplicity, and because we use the Lie algebra representation only for computational purposes. The main results will be independent of it, and concern properties of the abstract Lie algebra only.*

3.3. Rethinking constant curvature

The way in which the BLL algebras were introduced already appealed to the physical intuition behind the different contractions in associating curvature to some of them. The following construction will make this picture more precise. In particular, we will extend the definition of spacetime curvature to situations of degenerate metrics. The necessity for doing so was recognized already in formal treatments of the Newtonian limit of General Relativity (see for example [Ehl97]). The discussion of this chapter has the advantage of unifying 'non-relativistic' with 'ultra-relativistic' notions.

The fundamental tool that we will need is the Maurer-Cartan form Γ on the kinematical group G. It takes values in the Lie algebra $\mathfrak{g} \simeq T_eG$ of G, and so can be written as a map

$$\Gamma : TG \to \mathfrak{g}\,. \tag{3.122}$$

It reverses the push-forward of Lie algebra elements from the identity to a group element g by the left group action on itself,

$$\Gamma(v) = (\mathrm{L}_{g^{-1}})_* v \qquad \forall v \in T_g G\,. \tag{3.123}$$

Bearing in mind all peculiarities concerning the distinction between homogeneous transformations and translations, note that at least as vector spaces one may always write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}\,, \tag{3.124}$$

where $\mathfrak{h} = \mathrm{span}(\vec{J}, \vec{K})$ and $\mathfrak{t} = \mathrm{span}(H, \vec{P})$. Note that all BLL algebras share the structure

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{t}] \subseteq \mathfrak{t}, \qquad [\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{h}\,. \tag{3.125}$$

While, as we stressed, this structure is not G-invariant, it *is* left unaffected by the adjoint action of H. This is important, for otherwise the definition of what a translation is would not only depend on the point in spacetime, but in addition on the velocity and orientation of the observer.

We decompose Γ according to the above split of $\mathfrak{g}$ as

$$\Gamma = \omega + \theta \tag{3.126}$$

where

$$\omega = \Gamma|_{\mathfrak{h}} \qquad \text{and} \qquad \theta = \Gamma|_{\mathfrak{t}}\,. \tag{3.127}$$

For computational purposes it is useful to employ the formula for Γ in a representation of g, i.e. when its Lie algebra part as well as group elements are matrices. In that case, which we will presume in everything that follows, one can write

$$\Gamma = g^{-1}\mathrm{d}g\,. \tag{3.128}$$

Γ descends to the quotient G/H as a natural connection as follows. First understand G as the total space of a principal H-bundle with base the coset manifold G/H, the projection being canonically $g \to gH$ for all $g \in G$. A section

$$s : G/H \to G, \tag{3.129}$$

defines a local representative connection one-form on G/H via pull-back,

$$\underline{\Gamma} := s^*\Gamma \tag{3.130}$$

We can now compute the curvature associated with this connection according to Cartan, but not before we make a choice of what we actually wish to call curvature. To be precise, the θ part will in fact correspond to torsional components, while it is the ω part which reflects the Riemannian notion of curvature. Cartan's structure equations tell us that their sum will always vanish. Using the generalization of the wedge product for differential forms to Lie algebra valued ones α, β,

$$(\alpha \wedge \beta)(v, w) := \frac{1}{2} \left([\alpha(v), \alpha(w)] - [\alpha(w), \alpha(v)] \right), \tag{3.131}$$

(v, w being generic vectors), we have

$$\mathrm{D}_\Gamma \Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma = 0. \tag{3.132}$$

And indeed, we would rather like to approach the situation with our intuition of Riemannian geometry, so that we define

$$R = \mathrm{d}\omega + \omega \wedge \omega, \tag{3.133}$$

i.e., the restriction of $D_\Gamma \Gamma$ to $\mathfrak{h}$ as the curvature two-form of interest—again along with its pulled-back version[8]

$$\mathcal{R} = \mathrm{d}\varpi + \varpi \wedge \varpi, \qquad \varpi = s^* \omega. \tag{3.134}$$

To see that this quantity behaves well under homogeneous transformations, let $\omega \equiv \omega_{|g}$ and denote $\omega' := \omega \equiv \omega_{|gh}$. Explicitly,

$$\begin{aligned} \omega' &= (gh)^{-1} \mathrm{d}(gh) \\ &= h^{-1} g^{-1} (\mathrm{d}gh + g\mathrm{d}h) \\ &= h^{-1} \omega h + h^{-1} \mathrm{d}h, \end{aligned} \tag{3.135}$$

where h is taken here to be the image under s. For the global right H action on the bundle, the second, inhomogeneous term vanishes, and the equivariance property that one requires of so-called Ehresmann connections is recovered. From here however, we have

$$\mathrm{d}\omega' = \mathrm{d}h^{-1} \wedge \omega h + h^{-1} \mathrm{d}\omega h - h^{-1} \omega \wedge \mathrm{d}h + \mathrm{d}h^{-1} \wedge \mathrm{d}h, \tag{3.136}$$

and furthermore

$$\begin{aligned} \omega' \wedge \omega' &= (h^{-1} \omega h + h^{-1} \mathrm{d}h) \wedge (h^{-1} \omega h + h^{-1} \mathrm{d}h) \\ &= h^{-1} \Big(\omega \wedge \omega h + \omega \wedge \mathrm{d}h + \mathrm{d}h \wedge (h^{-1} \omega h) + \mathrm{d}h \wedge (h^{-1} \mathrm{d}h) \Big). \end{aligned} \tag{3.137}$$

Using

$$\mathrm{d}h^{-1} = -h^{-1} \mathrm{d}h h^{-1} \tag{3.138}$$

we note that most terms cancel in the sum, and what remains is

$$R' = \mathrm{d}\omega' + \omega' \wedge \omega' = h^{-1} (\mathrm{d}\omega + \omega \wedge \omega) h = h^{-1} R h = \mathrm{Ad}_{h^{-1}} R. \tag{3.139}$$

[8]Pullbacks are compatible with the wedge product.

This is a consistent result, as R' is again $\mathfrak{h}$-valued. Only now, the curvature of G/H depends on the section s taken to explore the coset manifold, namely in terms of the adjoint action of H on its Lie algebra. And this does not change for $\mathcal{R}$, since s^* only affects form parts. How do we reconcile this with the Riemannian view, where there is no need to refer to any algebraic structure, the defining quantity being a metric? The answer is delivered by θ, the $\mathfrak{t}$-valued part of the Maurer-Cartan form. This is because its descendant on G/H,

$$\vartheta := s^*\theta, \tag{3.140}$$

is a pointwise isomorphism

$$\vartheta_{|x} : T_x(G/H) \to T \quad \forall x \in G/H. \tag{3.141}$$

We call it the *coframe*, and its inverse

$$e := \vartheta^{-1} \tag{3.142}$$

simply the corresponding *frame*, or *vielbein*[9]. In components,

$$\begin{aligned} \vartheta = \vartheta^a_\mu \, \mathrm{d}x^\mu \otimes t_a, \qquad & e = e^\mu_a \, t^{*a} \otimes \frac{\partial}{\partial x^\mu}, \\ \vartheta^a_\mu e^\mu_b = \delta^a_b, \qquad & \vartheta^a_\nu e^\mu_a = \delta^\mu_\nu, \end{aligned} \tag{3.143}$$

with coordinates x^μ on G/H, and (dual) basis $t^{(*)a}$ of T ($t^{*a}(t_b) = \delta^a_b$). The idea shall be now to retrieve a metric for G/H from purely Lie algebraic considerations. For this purpose, note that $\mathfrak{h}$ acts linearly on the vector space $\mathfrak{t}$ in its adjoint representation. Hence we can ask for the invariant bilinear forms on $\mathfrak{t}$ under these transformations, or even more specifically, the symmetric ones among those. Parametrizing them by

$$\tilde{g} = \tilde{g}_{ab} \, t^{*a} \otimes t^{*b}, \tag{3.144}$$

with $[h_{a'}, t_b] = f^c{}_{a'b} \, t_c$ for some basis $\{h_{a'}\}$ of $\mathfrak{h}$, the invariance requirement

$$\begin{aligned} & \mathrm{coadj}_{\mathfrak{h}} \tilde{g} = 0 \\ \Leftrightarrow \qquad & \tilde{g}(\mathrm{adj}_{\mathfrak{h}} t_a, t_b) = -\tilde{g}(t_a, \mathrm{adj}_{\mathfrak{h}} t_b) \quad \forall a, b \end{aligned} \tag{3.145}$$

translates to

$$f^c{}_{a'a} \, \tilde{g}_{cb} + f^c{}_{a'b} \, \tilde{g}_{ac} = 0 \quad \forall a', a, b. \tag{3.146}$$

[9]This terminology is more general than indicated here. In fact, it is standard wherever the geometry of a general Riemannian or Lorentzian manifold is described in terms of its frame bundle. In this sense, we employ some notions here in a rather specific situation. In another sense, namely that we do not require H to be a strictly (pseudo-)orthogonal group, *we* will stay more general. This is also why we refrain from calling ω a spin connection, as it is done in other places.

Assuming a solution to this equation, it defines a metric on G/H by

$$g := \tilde{g}(\vartheta \otimes \vartheta) \tag{3.147}$$

with components

$$g_{\mu\nu} = \vartheta^a_\mu \vartheta^b_\nu \tilde{g}_{ab} \,. \tag{3.148}$$

Note that by its definition, the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{t}$ always solves Eq. (3.146).

Crucially for us, the metric obtained in the described manner will not always be non-degenerate as it is in Lorentzian or Riemannian geometry. As seen when evaluating Eq. (3.145), or equivalently Eq. (3.146) for the BLL algebras, g degenerates exactly for the 'non-relativistic' Lie algebras.

Very much in the spirit that already allowed us to take both the $c \to \infty$ as well as the $c \to 0$ limit of the Poincaré algebra by recognizing the freedom in choosing the units in which to measure speeds, instead of attempting to find an algebraic definition of a *metric*, we could equally well try the same for a *co*metric. Simply speaking, we also need a replacement for the inverse metric if we want to stick to the ideology of our approach to kinematics. Luckily, such can be found.

In analogy to Eq. (3.144) we now start with

$$\tilde{\gamma} = \tilde{\gamma}^{ab} t_a \otimes t_b \tag{3.149}$$

which we demand again to be symmetric, $\tilde{\gamma}_{ab} = \tilde{\gamma}_{ba}$, and H-invariant,

$$\mathrm{adj}_{\mathfrak{h}} \tilde{\gamma} = 0. \tag{3.150}$$

Explicitly, the latter now reads

$$f^a{}_{a'c} \tilde{\gamma}^{cb} + f^b{}_{a'c} \tilde{\gamma}^{ac} = 0 \quad \forall a', a, b. \tag{3.151}$$

A solution to this equation will then determine a cometric on G/H, defined by

$$\gamma := (e \otimes e)(\tilde{\gamma}), \tag{3.152}$$

or,

$$\gamma^{\mu\nu} = e^\mu_a e^\nu_b \tilde{\gamma}^{ab}. \tag{3.153}$$

If $\mathfrak{h} \simeq \mathfrak{so}(1,3)$ (or more generally, if $\mathfrak{h}$ is semisimple), it will be true that $\gamma = g^{-1}$, at least up to some constant, which may then be used for normalization. Otherwise, γ and g can actually annihilate, as is the case for the BLL algebras where $\mathfrak{h}$ is not the Lorentz algebra. Saving the degenerate situation for later, we shall illustrate the Riemannian case by an

Example. *Consider the simplest possible case, namely*

$$G = \mathrm{SO}(3), \qquad H = \mathrm{SO}(2) \subset \mathrm{SO}(3). \tag{3.154}$$

If group elements are parametrized by (ϕ, θ, χ) *as*

$$R = \exp(\phi J_3)\exp(\theta J_2)\exp(\chi J_3) \tag{3.155}$$

(this peculiar choice will become clear) one can choose as canonical representatives of the left cosets elements of the form

$$r = \exp(\phi J_3)\exp(\theta J_2). \tag{3.156}$$

To see this, one might imagine, in the defining (3d vector) representation, to pick as origin o *of our coordinate system the unit vector in the third direction, which fixes* J_3 *as the generator of the stabilizing subgroup, so that*

$$p = Ro = ro \tag{3.157}$$

will be a point on the coset manifold (vector on the unit sphere)

$$\mathrm{SO}(3)/\mathrm{SO}(2) \simeq \mathrm{S}^2. \tag{3.158}$$

We can now follow the prescription given above to define a metric on it. With $\mathfrak{g}$ *being*

$$[J_i, J_j] = \epsilon_{ijk} J_k\,, \tag{3.159}$$

Eq. (3.146) *becomes*

$$\epsilon_{\hat{k}3\hat{i}}\tilde{g}_{\hat{k}\hat{j}} + \epsilon_{\hat{k}3\hat{j}}\tilde{g}_{\hat{i}\hat{k}} = 0, \qquad \hat{i}, \hat{j}, \hat{k} = 1, 2 \tag{3.160}$$

The solution here is (uniquely up to a constant)

$$\tilde{g}_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}} \tag{3.161}$$

which is in fact (proportional to) the Killing form of $\mathfrak{g}$ *restricted to the subspace spanned by* J_1 *and* J_2*. The Maurer-Cartan can be written*

$$\begin{aligned}\Gamma &= R^{-1}\mathrm{d}R \\ &= \mathrm{Ad}_{\exp(-\chi J_3)}(r^{-1}\mathrm{d}r) + \exp(-\chi J_3)\mathrm{d}\exp(\chi J_3)\end{aligned} \tag{3.162}$$

with

$$r^{-1}\mathrm{d}r = \cos\theta\, J_3 \otimes \mathrm{d}\phi - \sin\theta\, J_1 \otimes \mathrm{d}\phi + J_2 \otimes \mathrm{d}\phi. \tag{3.163}$$

Hence,

$$\vartheta = \mathrm{Ad}_{\exp(-\chi J_3)}(-\sin\theta J_1 \otimes \mathrm{d}\phi + J_2 \otimes \mathrm{d}\phi) \tag{3.164}$$

But in $\tilde{g}(\vartheta \otimes \vartheta)$ *the* $\mathrm{Ad}_{\mathrm{SO}(2)}$*-action drops out by definition, so that in result*

$$\begin{aligned} g &= \tilde{g}(\vartheta \otimes \vartheta) \\ &= \mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\phi^2, \end{aligned} \tag{3.165}$$

which is readily recognized to be the standard (round) metric of the 2-sphere. Note that this exact result is not an unavoidable consequence of the general idea about how to construct a metric on a homogeneous space from the underlying Lie algebra. Rather, in addition, the final coincidence with conventional notions relies crucially on the parametrization of group elements of G *and a choice of* $H \subset G$ *(origin for* G/H*). While this circumstance might well be the source for some inconvenience at times when relying on explicit calculation using coordinatizations, the geometrically invariant statement remains of course unshaken.*

Now that we have shown how a natural metric on G/H can be defined by using the coframe, let us move on and see how the H-dependence of the curvature as defined in Eq. (3.139) can be eliminated. For this purpose, note that although the Maurer-Cartan form on G as a whole transforms inhomogeneously under the right group action, only its ω part does so under transformations from H. In contrast,

$$\theta \to \mathrm{Ad}_{h^{-1}}\theta \qquad \text{for } h \in H. \tag{3.166}$$

This means for ϑ, in components,

$$\vartheta^a_\mu \to (h^{-1})^a{}_b \vartheta^b_\mu. \tag{3.167}$$

As a result, there is a tensor on G/H which expresses H-invariant curvature information, namely

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} := e^\mu_a \mathcal{R}^a{}_{b\rho\sigma} \vartheta^b_\nu. \tag{3.168}$$

Here, the index positioning for $\mathcal{R}$ is supposed to say that its Lie algebra part has been taken in the adjoint representation. The question is now if this curvature tensor is the one which we would construct in the well-known way from the symmetric and metric-compatible affine connection associated with $g_{\mu\nu}$:

$$\mathcal{R} \stackrel{?}{=} \mathrm{Riem}[g] \tag{3.169}$$

First of all, note that such an equality can strictly only make sense if g is a Lorentzian metric, which means in particular that it has an inverse dual. From the way we constructed it however, this can only be if H is (pseudo-)orthogonal. In the following argument this will be assumed. At the same time, and in regards of the classification of possible kinematics, we bear in mind that what we are really interested in is how to generalize the correspondence (3.169) to non-semisimple H, specifically those with semidirect product structure.

Since the RHS of Eq. (3.169) depends only on the affine connection ∇, while the LHS is an expression only in ϖ as well as ϑ and its inverse e, the answer to the question posed above will depend on how we choose to relate these quantities. A natural choice is given by what in slightly other contexts is also known as the *vielbein*, or *tetrad postulate*,

$$\mathcal{D}_\mu \vartheta^a_\nu := \partial_\mu \vartheta^a_\nu - \{^\rho{}_{\mu\nu}\} \vartheta^a_\rho + \varpi^a{}_{\mu b} \vartheta^b_\nu = 0\,, \tag{3.170}$$

stating that the complete covariant derivative of the vielbein vanishes. It may be solved for the components of ∇,

$$\mathrm{d}x^\rho(\nabla_\mu \partial_\nu) =: \{^\rho{}_{\mu\nu}\}\,, \tag{3.171}$$

to wit,

$$\{^\rho{}_{\mu\nu}\} = e^\rho_a \varpi^a{}_{\mu b} \vartheta^b_\nu + e^\rho_a \partial_\mu \vartheta^a_\nu\,. \tag{3.172}$$

If H is (pseudo-)orthogonal, it follows from the presumed form of the Lie algebra of G, and from

$$\mathrm{d}\theta + \omega \wedge \theta = 0\,, \tag{3.173}$$

that torsion vanishes on the quotient,

$$\{^\rho{}_{[\mu\nu]}\} = e^\rho_a \partial_{[\mu} \vartheta^a_{\nu]} + e^\rho_a \vartheta^b_{[\nu} \varpi^a{}_{\mu]b} = 0\,. \tag{3.174}$$

Also, ϖ will be anti-symmetric in its Lie algebra parts,

$$\varpi^{ab}{}_\mu = \varpi^{[ab]}{}_\mu\,, \tag{3.175}$$

which leaves the metricity condition satisfied automatically, when respecting the relation (3.169):

$$\begin{aligned}
\nabla_\mu g_{\nu\rho} &= \tilde{g}_{ab} \left(\vartheta^a_\nu \nabla_\mu \vartheta^b_\rho + \vartheta^b_\rho \nabla_\mu \vartheta^a_\nu \right) \\
&= -\tilde{g}_{ab} \left(\vartheta^a_\nu \varpi^b{}_{c\mu} \vartheta^c_\rho + \vartheta^b_\rho \varpi^a{}_{c\mu} \vartheta^c_\nu \right) \\
&= \varpi_{ab\mu} \left(\vartheta^a_\nu \vartheta^b_\rho + \vartheta^a_\rho \vartheta^b_\nu \right) \\
&= 0
\end{aligned} \tag{3.176}$$

This then also permits the derivation of the familiar expression for the Christoffel symbols, since it implies

$$0 = g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} - \{^\sigma{}_{\mu\nu}\} g_{\sigma\rho} - \{^\sigma{}_{\mu\rho}\} g_{\nu\sigma} \right), \tag{3.177}$$

which due to Eq. (3.174) can be rearranged as

$$\begin{aligned}
\{^\lambda{}_{\mu\nu}\} = \{^\lambda{}_{(\mu\nu)}\} &= g^{\lambda\rho} \left(\partial_{(\mu} g_{\nu)\rho} - \{^\sigma{}_{\rho(\mu}\} g_{\nu)\sigma} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right).
\end{aligned} \tag{3.178}$$

If we now compute the Riemann tensor

$$\mathrm{Riem}^{\mu}{}_{\nu\rho\sigma} = 2\partial_{[\rho}\{^{\mu}{}_{\sigma]\nu}\} + 2\{^{\mu}{}_{\lambda[\rho}\}\{^{\lambda}{}_{\sigma]\nu}\}\,, \tag{3.179}$$

we find that it precisely coincides with the curvature $\mathcal{R}$ defined in Eq. (3.168). Hence, for the special case of (pseudo-)orthogonal H, we can answer the question initially raised affirmatively. Moreover, we can say that the so-called vielbein postulate was necessary to arrive at this result, and in this way, endow it with additional meaning.

Motivated by the above observation, we may now feel legitimated to work with $\mathcal{R}$ as our definition of curvature. In fact, while it reduces to the Riemann tensor in Riemannian situations, it stays well-defined under more general circumstances.

We have now achieved the position aimed at in this chapter, namely the ability to extend the notion of constant curvature to general model spacetimes, i.e. quotient spaces of arbitrary kinematical groups. The simple reason is that we can now reproduce in a well-defined way the characteristic relation for constant curvature:

$$g_{\mu\lambda}\mathcal{R}^{\lambda}{}_{\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \tag{3.180}$$

Here, Λ is a curvature constant, proportional to the inverse square of the fundamental length (or time) scale present. Note that the defining structural feature of this equation is that it is purely algebraic in the metric, i.e., involves no derivatives.

As a further step, we can go on and develop the notion of Ricci curvature. From Eq. (3.179), the Ricci tensor is obtained by taking a trace:

$$\mathrm{Ric}_{\mu\nu} = \mathrm{Riem}^{\rho}{}_{\mu\rho\nu}\,. \tag{3.181}$$

Again however, we aim at a definition that is more closely linked to our algebraic viewpoint, and the established correspondence (3.169) suggests indeed that such a definition should exist. In order to arrive at it, we step back a bit and write

$$R = \frac{1}{(2!)^2} R^{mn}{}_{kl}\, M_{mn} \otimes \theta^k \wedge \theta^l \tag{3.182}$$

for the curvature two-form on G, where

$$R^{mn}{}_{kl} = -c^{mn}{}_{kl}, \tag{3.183}$$

are the structure constants of the mixed relations of $\mathfrak{g}$, $[P_k, P_l] = \frac{1}{2}c^{mn}{}_{kl}M_{mn}$, as is readily inferred from the original definition. Here we decided to work in the basis of left-invariant one-forms, and it needs to be understood that we can only employ the form (3.182) because all remaining components are seen to be zero on the whole of TG when the Lie algebra part of the wedge product is expanded. The labelling of generators in $\mathfrak{h}$ by double indices is inspired by the non-degenerate cases, where $M_{mn} = -M_{nm}$, and indices now refer to the Cartesian basis P_m in $\mathfrak{t}$. While in the semisimple case it does not matter in which way we proceed to define the Ricci tensor, the degenerate case forces us to first consider R in its adjoint representation and only afterwards take a trace and descend to the quotient. As a result,

$$\tilde{\mathcal{R}}_{\mu\nu} = \vartheta^b_\mu e^\rho_a \mathcal{R}^a{}_{b\rho\nu} = \vartheta^n_\mu e^\rho_m \mathcal{R}^m{}_{n\rho\nu}\,. \tag{3.184}$$

Application to Newton+Hooke spacetime

Setting $\tau = 1$, exponentiating the matrices (3.70) according to Eq. (3.119) leads to a parametrization of group elements by

$$g = \begin{pmatrix} \cosh b & 0 & \sinh b \\ \vec{v} + \vec{a}\sinh b & \mathbf{R} & \vec{a}\cosh b \\ \sinh b & 0 & \cosh b \end{pmatrix}, \tag{3.185}$$

with $\mathbf{R} = \exp(\vec{\alpha}\cdot\vec{\omega})$. Group multiplication, $g'g = g''$, is then quickly seen to amount to

$$\begin{aligned} \mathbf{R}'' &= \mathbf{R}'\mathbf{R} \\ \vec{v}'' &= \mathbf{R}'(\vec{v} + \vec{a}\sinh b) + \vec{v}'\cosh b \\ &\quad + (\mathbf{R}'\vec{a}\cosh b + \vec{v}'\sinh b)\tanh(b + b') \\ b'' &= b + b' \\ \vec{a}'' &= \vec{a}' + \frac{\mathbf{R}'\vec{a}\cosh b + \vec{v}'\sinh b}{\cosh(b + b')} . \end{aligned} \tag{3.186}$$

Note that with $(\mathbf{R}, \vec{v}, b, \vec{a}) = (\mathbb{1}, 0, t, \vec{x})$, one obtains an explicit expression for the Newton-Hooke isometries from the latter two equations. In terms of the embedding coordinates (3.25), it assumes the form

$$\begin{aligned} X^M(t,\vec{x}) = \begin{pmatrix} \sinh t \\ \vec{x}\cosh t \\ \cosh t \end{pmatrix} &\mapsto \begin{pmatrix} X^0(t+b',\vec{x}) \\ \mathbf{R}'\vec{X}(t,\vec{x}) + \vec{a}'X^4(t+b',\vec{x}) + \vec{v}'X^0(t,\vec{x}) \\ X^4(t+b',\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} \sinh(t+b') \\ \mathbf{R}'\vec{x}\cosh t + \vec{a}'\cosh(t+b') + \vec{v}'\sinh t \\ \cosh(t+b') \end{pmatrix} . \end{aligned} \tag{3.187}$$

With

$$g^{-1} = \begin{pmatrix} \cosh b & 0 & -\sinh b \\ -\mathbf{R}^{-1}\vec{v}\cosh b & \mathbf{R}^{-1} & -\mathbf{R}^{-1}(\vec{a} - \vec{v}\sinh b) \\ -\sinh b & 0 & \cosh b \end{pmatrix}, \tag{3.188}$$

the Maurer-Cartan form is, when omitting the $\otimes$ symbol,

$$\begin{aligned} \Gamma = g^{-1}\mathrm{d}g &= \begin{pmatrix} 0 & 0 & \mathrm{d}b \\ \mathbf{R}^{-1}(\mathrm{d}\vec{v} + \mathrm{d}\vec{a}\sinh b) & \mathbf{R}^{-1}\mathrm{d}\mathbf{R} & \mathbf{R}^{-1}(\mathrm{d}\vec{a} - \vec{v}\mathrm{d}b) \\ \mathrm{d}b & 0 & 0 \end{pmatrix} \\ &= H\mathrm{d}b + \vec{P}\cdot\mathbf{R}^{-1}(\mathrm{d}\vec{a} - \vec{v}\mathrm{d}b) + \vec{K}\cdot\mathbf{R}^{-1}(\mathrm{d}\vec{v} + \mathrm{d}\vec{a}\sinh b) + \mathbf{R}^{-1}\mathrm{d}\mathbf{R} \\ &= \omega + \theta \end{aligned} \tag{3.189}$$

Along the section $s : (t,\vec{x}) \mapsto (\mathbf{R} = \mathbb{1}, \vec{v} = 0, b = t, \vec{a} = \vec{x})$, i.e.

$$s(x) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ \vec{x}\sinh t & \mathbb{1} & \vec{x}\cosh t \\ \sinh t & 0 & \cosh t \end{pmatrix}, \tag{3.190}$$

we receive

$$\vartheta = H\mathrm{d}t + \vec{P}\cdot\mathrm{d}\vec{x}. \tag{3.191}$$

Now we need to find $\tilde{g}$. Using the Newton+Hooke Lie algebra structure (3.66), two of the conditions it has to satisfy are

$$0 = \tilde{g}([K_i, H], P_j) + \tilde{g}(H, [K_i, P_j]) = \tilde{g}(P_i, P_j) \tag{3.192}$$

and

$$0 = \tilde{g}([J_i, H], P_j) + \tilde{g}(H, [J_i, P_j]) = \epsilon_{ijk}\tilde{g}(H, P_k)\ . \tag{3.193}$$

All the other conditions are identically fulfilled. If we further demand that $\tilde{g}(H, H) = -1$, we can infer that

$$\tilde{g} = H^* \otimes H^*, \tag{3.194}$$

or, $\tilde{g}_{ab} = \mathrm{diag}(-1, 0, 0, 0)$, and hence,

$$g = -\mathrm{d}t^2 \tag{3.195}$$

Note that in order to discuss the zero curvature limit, it would be more instructive not to use the intrinsic coordinate t, but instead the one of a Galilean observer, namely X^0. The relation is simply $T := X^0 = \sinh(t)$, and hence the metric

$$g = -\frac{\mathrm{d}T^2}{1+T^2}\,. \tag{3.196}$$

In order to evaluate the curvature tensor (3.134) we note first that from

$$\delta^a_b = \vartheta^a(e_b) = s^*\theta(e_b) = \theta(s_* e_b) \tag{3.197}$$

it follows that

$$s_* e_m \equiv P^{\#}_m \tag{3.198}$$

are the fundamental vector fields on G stemming from the translationary part of the Lie algebra. On fundamental vector fields, which arise as the push-forward from the Lie algebra by the group action on itself, the Maurer-Cartan form evaluates in a particularly simple way, namely by reversing the push-forward,

$$\Gamma(X^{\#}) = X \qquad \forall X \in \mathfrak{g}. \tag{3.199}$$

By obeying the rules of exterior calculus for Lie algebra-valued forms and using the fact that the push-forward on the group is a Lie algebra homomorphism, the curvature tensor evaluates as

$$\begin{aligned}\mathcal{R}(e_m, e_n) &= s^* R(e_m, e_n) = R(s_* e_m, s_* e_n) \\ &= R(P_m^\#, P_n^\#) = \mathrm{d}\omega(P_m^\#, P_n^\#) + (\omega \wedge \omega)(P_m^\#, P_n^\#) \\ &= P_m^\#(\omega(P_n^\#)) - P_m^\#(\omega(P_n^\#)) - \omega([P_m^\#, P_n^\#]) \\ &\quad + \frac{1}{2}\left([\omega(P_m^\#), \omega(P_n^\#)] - [\omega(P_n^\#), \omega(P_m^\#)]\right) \\ &= P_m^\#(P_n) - P_m^\#(P_n) - \omega([P_m, P_n]^\#) + 0 \\ &= -[P_m, P_n],\end{aligned} \tag{3.200}$$

so that in Newton+Hooke spacetime the only three non-vanishing components are (restoring τ)

$$\mathcal{R}_{i0} = -\frac{1}{\tau^2} K_i\,, \tag{3.201}$$

or in the adjoint representation,

$$\mathcal{R}^i{}_{0j0} = -\frac{1}{\tau^2}\delta^i_j\,. \tag{3.202}$$

The first two indices in this last expression can be taken either of Latin or Greek type, since s was chosen in such a way that the components of ϑ as well as e are those of the identity matrix. Hence our Ricci tensor has the single non-zero component

$$\tilde{\mathcal{R}}_{00} = -\frac{3}{\tau^2} \tag{3.203}$$

so that in terms of the metric (3.195) we can write

$$\tilde{\mathcal{R}}_{\mu\nu} = \frac{3}{\tau^2}\, g_{\mu\nu}\,, \tag{3.204}$$

and by analogy to the Lorentzian case, where Eq. (3.180) holds, and hence

$$\mathrm{Ric}_{\mu\nu} = \Lambda(d-1) g_{\mu\nu} \tag{3.205}$$

in a d-dimensional space with curvature constant Λ, we can say that Newton+Hooke spacetime has positive, constant curvature $1/\tau^2$. One should realize nonetheless how sensitively this result depends on our way of progression in setting up the geometric quantities of interest. In particular, note that if we took the second equation characteristic of Lorentzian constant curvature, Eq. (3.180), and naively inserted the Newton+Hooke metric, we would obtain zero. This is a feature that distinguishes the Galilean case from the Carrollian one, where more of the metric structure is preserved. The analysis itself works analogously there, however.

Lastly, let us find the cometric. Boost invariance dictates

$$\begin{aligned}0 = [K_i, \tilde{\gamma}] &= \tilde{\gamma}^{mn}([K_i, P_m] \otimes P_n + P_m \otimes [K_i, P_n]) \\ &= \tilde{\gamma}^{00}(P_i \otimes H + H \otimes P_i) + \tilde{\gamma}^{0j}([P_i \otimes P_j + P_j \otimes P_i]) \qquad \forall i\end{aligned} \tag{3.206}$$

and hence $\tilde{\gamma}^{00} = \tilde{\gamma}^{0i} = 0$. The remaining spatial components must satisfy

$$\begin{aligned} 0 = [J_i, \tilde{\gamma}] &= \tilde{\gamma}^{jk}([J_i, P_j] \otimes P_k + P_j \otimes [J_i, P_k]) \\ &= \tilde{\gamma}^{jk}(\epsilon_{ijk'} P_{k'} \otimes P_k + \epsilon_{ikj'} P_j \otimes P_{j'}) \qquad \forall i \, . \end{aligned} \tag{3.207}$$

For $i = 1$ this reads

$$\begin{aligned} 0 = {} & \tilde{\gamma}^{22}(P_3 \otimes P_2 + P_2 \otimes P_3) \\ & \tilde{\gamma}^{33}(-P_2 \otimes P_3 - P_3 \otimes P_2) \\ & \tilde{\gamma}^{23}(P_3 \otimes P_3 - P_2 \otimes P_3), \end{aligned} \tag{3.208}$$

implying $\tilde{\gamma}^{22} = \tilde{\gamma}^{33}$ and $\tilde{\gamma}^{23} = 0$. Evaluating Eq. (3.207) for $i = 2, 3$ as well then yields

$$\tilde{\gamma} \propto \delta^{ij} P_i \otimes P_j \, , \tag{3.209}$$

where one may wish to take the constant of proportionality to equal one. In that case, along the chosen section s, one obtains

$$\gamma = \delta^{ij} \, \partial_i \otimes \partial_j \tag{3.210}$$

for the cometric on Newton+Hooke spacetime.

3.4. Symmetry contraction from spacetime extensions

Our aim is to demonstrate how Newton-Cartan as well as Carroll structures may be understood from a higher-dimensional Lorentzian point of view. That this is possible has been shown in the literature; see e.g. [Kue72; Duv+85; GP03; Duv+14].

Newton-Cartan and Carrol structures

A *Newton-Cartan structure* (sometimes also called *Galilei* structure) is defined by a quadruple

$$\mathrm{N} = (N, \gamma, \theta, \nabla_\infty) \, , \tag{3.211}$$

where N is a smooth d-dimensional manifold, γ a positive, symmetric, twice-*contra*variant tensor on N with a one-dimensional kernel[10] generated by θ, a non-vanishing one-*form* on N. Finally, ∇_∞ is a covariant derivative that parallel-transports both γ and θ.

A *Carroll structure* is a quadruple

$$\mathrm{C} = (C, g, \xi, \nabla_0) \, , \tag{3.212}$$

[10] Here, by kernel we mean the one under the non-standard action of vectors v on forms ω, mapping $\omega \mapsto v[\omega] := \omega(v)$, extended to the vector bundle. The mentioned property is then $(\theta \otimes 1)(\gamma) \equiv (1 \otimes \theta)(\gamma) = 0$, which in coordinates simply says $\gamma^{\mu\nu}\theta_\nu = 0 = \gamma^{\nu\mu}\theta_\nu$.

with C being again a smooth d-dimensional manifold, but now g is a positive, symmetric, twice-*co*variant tensor on C with one-dimensional kernel generated by a nowhere-vanishing *vector* field ξ, and ∇_0 parallel-transports g and ξ.

The observation made in [Duv+85] and [Duv+14] is that these definitions obtain a natural geometric interpretation in terms of *Bargmann structures* in one dimension higher. These are characterised by a quadruple

$$\mathrm{B} = (B, G, \Xi, \nabla), \tag{3.213}$$

where B is a smooth $(d+1)$-dimensional manifold with Lorentzian metric G, while Ξ is a nowhere vanishing vector field that is null with respect to G and parallel transported by ∇, the Levi-Civita covariant derivative belonging to G. These properties imply the existence of coordinates (U, V, X^A) for which, globally, $\Xi = \partial/\partial V$ and the only term in G involving V is (proportional to) $\mathrm{d}U\mathrm{d}V$.

A subclass of Bargmann structures, which first appeared in [Bri25], is given by those for which the hypersurfaces of constant coordinate U are planes, which is to say that the metric assumes the form

$$G = -2\mathrm{d}U\mathrm{d}V + H(U, \vec{X})\mathrm{d}U^2 + 2\vec{K}(U, \vec{X}) \cdot \mathrm{d}\vec{X}\mathrm{d}U + \mathrm{d}\vec{X}^2\,. \tag{3.214}$$

Such spacetimes are called *plane-fronted waves with parallel rays*, but are customarily referred to simply as *pp-waves*, abbreviating their characteristics: a *p*arallel(-transported) vector that is null, hence defining the propagation of a wave (at the speed of light) for which the wave fronts are *p*lanar.

Specifying yet further the form of the metric, one obtains so-called *plane waves*, the name being admittedly prone to some confusion. These spacetimes are of some interest in the study of gravitational waves [Per04], but moreover, due to their exquisite features, have provided a rich playground in String Theory; see [Pan03] for an overview. For plane waves metrics, one has that $\vec{K} = 0$ and $H = H_{ab}(U)X^aX^b$ quadratic in $\vec{X}$. They play a particular role in the present chapter as we are about see. Their generic relevance derives from the fact that they arise in a limiting sense from any Lorentzian metric, in a manner that is to be explained further below.

Duality from Bargmann structures

We mainly envisage two cases of application for the following. The original interest in Newton-Cartan geometry was motivated by the wish to obtain a rigorous geometrical setting for non-relativistic physics and thus $d = 4$ was chosen. We will however see that there is a second viewpoint validated that relies on the notion of Penrose limits. It is this limiting procedure that introduces a way of rigorously thinking about local non-relativistic approximations to any spacetime, as will be argued.

Given a Bargmann structure as defined in the preceding subsection, there are two possibilities for the integral curves of Ξ. They could either be closed or not, corresponding to orbits of a $U(1)$ or $\mathbb{R}$ action on B. We can however always constrain ourselves to a region of B in which no orbits close and bearing this in mind, consider the second case only for now. Imagine now being only interested into the space of flow lines of Ξ,

$$B/\mathbb{R}, \tag{3.215}$$

where the action of $\mathbb{R}$ as additive group is given by the flow of Ξ. Denoting by π the canonical projection from B to this quotient, there is the following inheritance of structures, which justifies the identification of $B/\mathbb{R}$ as a Newton-Cartan manifold:

$$\pi_* G^{-1} = \gamma \tag{3.216}$$

$$G(\Xi) = \pi^* \theta \tag{3.217}$$

As was shown in [Kue72], the Killing property of Ξ guarantees that the Levi-Civita connection of B descends to the quotient parallel-transporting both γ and θ, so that indeed one arrives at a Newton-Cartan structure.

There is a construction dual to the one above which gives rise to a Carroll structure. The duality is that of pull-back and push-forward, and has its fixed point in the musical isomorphisms on B. Instead of projecting out the flow of Ξ, one now considers the hypersurfaces orthogonal to it, that is, the distribution

$$\ker G(\Xi), \tag{3.218}$$

which is necessarily integrable by Frobenius's theorem and our definition of Bargmann structures. (In the coordinates introduced earlier, one has simply $G(\Xi) = -\mathrm{d}U$). Denoting by

$$\iota : C \hookrightarrow B \tag{3.219}$$

the embedding of one of its leaves, there are natural analogues of Eqs. (3.216) and (3.217), now built from Ξ and G themselves:

$$\iota^* G = g, \tag{3.220}$$

$$\iota_* \xi = \Xi. \tag{3.221}$$

Furthermore, the Levi-Civita connection ∇ of the ambient Bargmann structure restricts, on vectors in the leaf $\iota(C)$, to a Carroll connection ∇_∞ [Duv+14].

Application

Six of the Cayley-Klein spaces introduced earlier exhibit either Carroll or Newton-Cartan structure. Given the geometric relation presented above, one should be able to find, for each of them, at least one Bargmann structure that gives rise to it in the above sense. It seems tempting to suspect further that the Carrollian and the Galilean limit of any d-dimensional Lorentzian spacetime are obtained from the very same $(d+1)$-dimensional Bargmann structure. Although, as we will show now, this is still true in the flat case, the curved Cayley-Klein geometries provide counter-examples. This section may be viewed as the natural extension of the insight gained in [Duv+14], [GP03] and [DGH91], to Para-Euclidean and Para-Minkowski spacetime, pEuc and pM.

Let $\mathbb{R}^5$ be coordinatized by $Y^M = (U, X^1, X^2, X^3, V)$, related to the standard Cartesian coordinates by

$$U = \frac{1}{\sqrt{2}}(X^0 - X^4), \qquad V = \frac{1}{\sqrt{2}}(X^0 + X^4). \tag{3.222}$$

A flat Bargmann structure is given by $(\mathbb{R}^5, \eta, \Xi, \nabla)$, with

$$\Xi = \frac{\partial}{\partial V} \qquad \text{and} \qquad \eta = -2\mathrm{d}U\mathrm{d}V + \mathrm{d}\vec{X}^2, \tag{3.223}$$

which is nothing but Minkowski spacetime in light cone coordinates, and with one of the lightlike directions selected as wave vector. The corresponding Carroll structure, Car, is embeddable as any of the planes $U = \text{const}$. Its spatial metric is then (the pullback of) $\delta = \mathrm{d}\vec{X}^2$, and (the restriction of) Ξ is the vector field in the Carrollian time direction V. Conversely, the flat Galilei structure Gal is represented by any hypersurface with Ξ nowhere tangent to it. The simplest choice is the plane, $U = \text{const}$. It inherits $-\mathrm{d}V$ as time-measure and $(\partial/\partial\vec{X})^{\otimes 2}$ as the spatial co-metric. Really, what we consider in this way is a section, and the induced structure on it, of a principal $\mathbb{R}$-bundle, with total space being five-dimensional Minkowski spacetime and the global right action on it given by the flow of Ξ. It is obvious how this generalizes to higher dimensions, so that as a result, we can state that both the ultra-relativistic as well as the non-relativistic limit of d-dimensional Minkowski spacetime arise from its (d+1)-dimensional version, understood as a Bargmann structure, or in fact a plane wave.

We now turn on non-relativistic curvature, and ask which Bargmann structures yield the two Newton±Hooke spacetimes. While the answer was already given from a slightly extended viewpoint in [GP03], we are here in a position to show that it is unique. Essentially, the fact that $\mathrm{NH}_\pm$ are spatially planar brings the case quite close to the flat one, and suggests to still try a pp-wave ansatz,

$$G_\pm = -2\mathrm{d}t\mathrm{d}v + H_\pm(t, \vec{X})\mathrm{d}t^2 + \mathrm{d}\vec{X}^2\,. \tag{3.224}$$

Starting with the case of NH_+, we need to demand that the corresponding transformations (3.187) be isometries. Because the v-direction is going to be projected out, we have the freedom to assume for it any transformation behaviour in order to counterbalance the shifts induced from the base. Thus, we consider the diffeomorphisms

$$\begin{aligned} t &\mapsto t + b\,, \\ \vec{X} &\mapsto \mathbf{R}\vec{X} + \vec{a}\cosh(t+b) + \vec{v}\sinh t\,, \\ v &\mapsto v + c(t) + \vec{X}\cdot\vec{C}(t) + D(t,\vec{X})\,, \end{aligned} \tag{3.225}$$

where we put $\tau = 1$. Under these, the metric changes by[11]

$$\begin{aligned}\delta G_+ &= -2\mathrm{d}t(\dot{c}\mathrm{d}t + \mathrm{d}\vec{X}\cdot\vec{C} + \vec{X}\cdot\dot{\vec{C}}\mathrm{d}t + \dot{D}\mathrm{d}t + \vec{\nabla}D\cdot\mathrm{d}\vec{X}) + \delta H\mathrm{d}t^2 \\ &\quad + 2(\vec{a}\sinh(t+b) + \vec{v}\cosh t)\cdot\mathbf{R}\mathrm{d}\vec{X}\mathrm{d}t + (\vec{a}\sinh(t+b) + \vec{v}\cosh t)^2\mathrm{d}t^2 \\ &= (-2\dot{c} - 2\vec{X}\cdot\dot{\vec{C}} - 2\dot{D} + \delta H + (\vec{a}\sinh(t+b) + \vec{v}\cosh t)^2)\mathrm{d}t^2 \\ &\quad + (-2\vec{C} - 2\vec{\nabla}D)\cdot\mathrm{d}X\mathrm{d}t + 2(\vec{a}\sinh(t+b) + \vec{v}\cosh t)\cdot\mathbf{R}\mathrm{d}\vec{X}\mathrm{d}t\end{aligned} \tag{3.226}$$

where $\delta H_+ = H_+(t+b, \boldsymbol{R}\vec{X} + \vec{a}\cosh(t+b) + \vec{v}\sinh t) - H_+(t,\vec{X})$. For $\delta G_+ = 0$ one needs that

$$\vec{C} + \vec{\nabla}D = \mathbf{R}^{-1}(\vec{a}\sinh(t+b) + \vec{v}\cosh t)\,. \tag{3.227}$$

Since neither the right hand side nor $\vec{C}$ depend on spatial coordinates, D can only be an at most linear function of them. But then, in view of our parametrization for the assumed transformations in v, we can assume $D = 0$ without loss of generality. Equivalently, we could redefine c and $\vec{C}$ appropriately to eliminate D from Eq. (3.225). In effect, the last equation defines $\vec{C}$ explicitly, and we are left to infer, with a suitable choice for c, an expression for H_+ from

$$\begin{aligned}\frac{1}{2}\delta H_+ &= \dot{c} + \vec{X}\cdot\dot{\vec{C}} - \frac{1}{2}\vec{C}^2 \\ &= \dot{c} + \vec{X}\cdot\mathbf{R}^{-1}(\vec{a}\cosh(t+b) + \vec{v}\sinh t) - \frac{1}{2}(\vec{a}\sinh(t+b) + \vec{v}\cosh t)^2\,.\end{aligned} \tag{3.228}$$

A similar argument as above can be made here. The form of the equation tells us that, since c is only a function of time, H_+ can only depend quadratically on spatial coordinates. Furthermore, due to the ordinary action of rotations, we must in fact have

$$H_+ = \vec{X}^2, \tag{3.229}$$

which lets us conclude that the unique pp-wave projecting to Newton+Hooke spacetime is actually a plane wave, and in fact the simplest possible one. The transformations in the fibre direction (i.e. in v) are fixed up to a constant c_0 by

$$c_+(t) = c_{+,0} + \frac{1}{2}(\vec{a}\sinh(t+b) + \vec{v}\cosh t)(\vec{a}\cosh(t+b) + \vec{v}\sinh t). \tag{3.230}$$

In order to arrive at the negative curvature case, an analogous calculation with sinh and cosh replaced by sin and cos, respectively, reveals a change of sign in the wave profile function,

$$H_-(t,\vec{X}) = -\vec{X}^2, \tag{3.231}$$

[11] In an active interpretation one would take the Lie derivative along the vector fields generating the transformations (3.225).

and

$$c_-(t) = c_{-,0} + \frac{1}{2}(-\vec{a}\sin(t+b) + \vec{v}\cos t)(\vec{a}\cos(t+b) + \vec{v}\sin t). \tag{3.232}$$

Time, in NH_-, runs along a circle, making all dynamics acquire a periodicity corresponding to its radius τ.

What we have actually constructed now by allowing transformations in fibre direction is a non-trivial central extension of the Newton$\pm$Hooke algebra, as it is known in the Galilei case as the Bargmann algebra. Indeed, considering infinitesimal times only, we recover the formulae for the flat case, with

$$\begin{aligned} t &\to U \\ v &\to V \\ H_\pm &\to 0 \\ c(t) + \vec{X}\cdot\vec{C}(t) &\to \frac{1}{2}\vec{v}^2 t + \mathbf{R}\vec{X}\cdot\vec{v}. \end{aligned} \tag{3.233}$$

The two missing, Carrollian spaces are pEuc and pM. A naive question one might raise is whether the algebraic duality between their automorphisms and those of the Newton$\pm$Hooke spacetimes translates to the geometric duality which is the theme of the current section. That is to say, one might ask if pEuc and pM stem from exactly those plane waves that project to $NH_\pm$. In the following we will see that this is not the case, and construct the Bargmann structures in which they *can* be embedded. To the author's knowledge, this is a new result.

In contrast to the Galilean setting, the Carrollian case gives us no reason to assume a pp-wave ansatz. In principle, we would therefore have to assume as ansatz that of a generic Bargmannian metric and demand invariance under pEuc and pM transformations, respectively. Since we are now to embed instead of project, we do not need to worry about additional transformations along the Ξ-lines. Also, due to the large homogeneous subgroups SO(4) and SO(1,3), respectively, which act as rotations and translations in space, we should be able to find a form of the metric which exhibits the four-dimensional Euclidean (Minkowski) metric, that is

$$G^\pm = -\mathrm{d}f^\pm(U,\vec{X})\mathrm{d}V \pm \mathrm{d}U^2 + \mathrm{d}\vec{X}^2, \tag{3.234}$$

which, using

$$\begin{aligned} f^+(U,\vec{X}) &= \sqrt{U^2 + \vec{X}^2} =: r \\ \text{and} \quad f^-(U,\vec{X}) &= \sqrt{U^2 - \vec{X}^2} =: \rho \end{aligned} \tag{3.235}$$

becomes in adapted coordinates

$$\begin{aligned} G^+ &= -\mathrm{d}r\mathrm{d}V + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2 \\ \text{and} \quad G^- &= -\mathrm{d}\rho\mathrm{d}V - \mathrm{d}\rho^2 + \rho^2\mathrm{d}\Sigma_3^2. \end{aligned} \tag{3.236}$$

Here, shorthand was used for the standard metrics on the unit three-sphere and (one sheet of) the unit two-sheeted hyperboloid ('mass-shell', if we were in momentum space) in four dimensions,

$$\begin{aligned} \mathrm{d}\Omega_3^2 &= \mathrm{d}\psi^2 + \sin^2\psi\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \\ \mathrm{d}\Sigma_3^2 &= \mathrm{d}\chi^2 + \sinh^2\chi\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \end{aligned} \tag{3.237}$$

Now we can embed pEuc and pM as the leaves at, say, $r = 1$ and $\rho = 1$, respectively, which endows them, apart from (the restriction of) $\Xi = \partial/\partial V$, with exactly the above two expressions as spatial metrics $g^{\pm}$.

It might have occurred to the reader from their definition already that Carroll structures should be of some interest also in the study of General Relativity. Indeed, the proper understanding of event horizons, cosmological horizons, or in fact the conformal boundary of an asymptotically flat spacetime could actually be developed from a Carrollian point of view. This also puts into new light the starting point of a recent proposal to reconsider the so-called information loss paradox [HPS16; DGH14b].

As a concrete instance of the idea, we are now able to identify the event horizon $r = R_s$ of a Schwarzschild black hole as an instance of Para-Euclidean space, though in three instead of four dimensions, topologically $S^2 \times \mathbb{R}$. As immediately seen in Eddington-Finkelstein coordinates, the metric induced on it is a multiple of the round metric, $(R_s)^2\mathrm{d}\Omega_2^2$, whose kernel is given by the null generators of the horizon.

A similar situation is encountered at $\mathcal{I}^{\pm}$, the lightlike boundaries of Penrose diagrams. In this case however, the rigid pEuc is somewhat hidden, since one is there only interested in the conformal properties, which are the same as in the flat, Carroll case [DGH14a].

PERVADING LORENTZIAN GEOMETRY

We have seen now how geometries with Galilean or Carrollian structure can be understood in terms of particular Lorentzian spacetime extensions. The goal of the remainder of the section will be to show that this connection persists in fact for *arbitrary* Lorentzian spacetimes, when these are considered in a particular limit suggested by Penrose [Pen76].

NULL COORDINATES

The following sentences paraphrase a construction which was originally presented in [Pen72].

Given a spacelike hypersurface Σ of a spacetime manifold M and a point p on Σ with neighbourhood Q in M, then in this neighbourhood one can use coordinates such that the metric assumes the form

$$g = -\mathrm{d}t^2 + h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \tag{3.238}$$

where x^i $(i, j = 1, 2, 3)$ are coordinates on Σ and t is the affine parameter of the timelike geodesic with unit tangent vector coinciding with the unit normal vector of Σ at the piercing point. Such a coordinate system is called synchronous. The aim of this section is to introduce Penrose's analogous construction for lightlike geodesics.

Given a null geodesic γ_0 one can find a spacelike 2-surface Λ_0 which it pierces orthogonally. It is further possible in a neighbourhood $Q \subset M$ around γ_0 to find a smooth family of neighbouring null geodesics (γ), containing γ_0, and sharing the property of being orthogonal to Λ_0, such that their union becomes a hypersurface Ω_0 of M. As the tangent vectors to the generators of Ω_0 are all null and orthogonal to the connecting vectors[12] (which lie in Λ_0 on $\Omega_0 \cup \Lambda_0$, the inner product with the tangent vectors being preserved along the geodesics), Ω_0 must have lightlike normal vectors everywhere, i.e. it must be a null hypersurface. (Given any three linearly independent vectors, one of them null and the other two spacelike, a fourth vector can only be orthogonal to each of them if it is proportional to the original null one.) In order to extend Ω_0 to a spacetime-filling set, i.e. a null congruence within Q, one needs to consider variations of Λ_0 in a direction away from Ω_0. Introducing such a family of spacelike 2-surfaces (Λ), smoothly parametrized by a function u which becomes zero at Λ_0 (and constant on all other family members), one could then repeat the above construction for each value of u to obtain a corresponding family of null hypersufaces (Ω), which is in the aimed-at null geodesic congruence. It is possible to choose u in such a way that $\mathrm{d}u$ is dual to the vector field consisting of the normal vectors of (Ω) (which are tangent to the geodesics of (γ)). Call this vector field T. Apart from being parallel-transported along the null geodesics of the congruence, $\nabla_T T = 0$, as a gradient field it is also curlfree, which means we have obtained a twistless null geodesic congruence, as long as we restrict ourselves to a small enough region Q, so that the geodesics of (γ) do not intersect within Q.[13]

In addition to the function u, Penrose coordinates include the affine parameter v of γ_0 ($v = 0$ on Λ_0) and two spatial coordinates $x^{\hat{\imath}}$, $\hat{\imath} = 2, 3$ which are chosen such that together with u they are constant along the generators of Ω. In this system of coordinates, the most general form of the metric is

$$g = -2\mathrm{d}u\mathrm{d}v + a\,\mathrm{d}u^2 + 2b_{\hat{\imath}}\,\mathrm{d}u\mathrm{d}x^{\hat{\imath}} + c_{\hat{\imath}\hat{\jmath}}\,\mathrm{d}x^{\hat{\imath}}\mathrm{d}x^{\hat{\jmath}}, \tag{3.239}$$

with generic functions $a, b_{\hat{\imath}}, c_{\hat{\imath}\hat{\jmath}}$, except that $c_{\hat{\imath}\hat{\jmath}}$ has to be positive-definite as a matrix.

Penrose Limits

As done in the original work [Pen76], one can consider the following rescaling of coordinates in the null system introduced above:

$$X^a = (u, v, x^2, x^3) \mapsto \tilde{X}^a = (\tilde{u}, \tilde{v}, \tilde{x}^2, \tilde{x}^3) = (\lambda^{-2}u, v, \lambda^{-1}x^2, \lambda^{-1}x^3) \tag{3.240}$$

[12] By connecting vectors we mean the values of any vector field satisfying the Jacobi equation for the geodesics γ in Q. We refer to [Pen72] for an account of the Jacobi equation.

[13] Again, refer to [Pen72] for an introduction into geodesic congruences.

The metric

$$\tilde{g} = -2\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \tilde{a}\lambda^2\,\mathrm{d}\tilde{u}^2 + 2\tilde{b}_i\lambda\,\mathrm{d}\tilde{u}\mathrm{d}\tilde{x}^{\hat{i}} + \tilde{c}_{\hat{i}\hat{j}}\,\mathrm{d}\tilde{x}^{\hat{i}}\mathrm{d}\tilde{x}^{\hat{j}} \tag{3.241}$$

with $a(X^a), b_i(X^a), c_{ij}(X^a)$ equal to $\tilde{a}(\tilde{X}^a), \tilde{b}_i(\tilde{X}^a), \tilde{c}_{ij}(\tilde{X}^a)$ respectively, is conformal to the orginal one:

$$\tilde{g} = \lambda^{-2}g \tag{3.242}$$

In order to examine the limit of vanishing λ, we must constrain ourselves to regions smaller and smaller in u and $x^{\hat{i}}$. Remembering this, we obtain

$$\bar{g} := \lim_{\lambda\to 0}\tilde{g} = -2\mathrm{d}\tilde{u}\mathrm{d}v + \bar{c}_{\hat{i}\hat{j}}\,\mathrm{d}\tilde{x}^{\hat{i}}\mathrm{d}\tilde{x}^{\hat{j}}, \tag{3.243}$$

which one recognizes as the metric of a plane wave in Rosen coordinates when realising that $\bar{c}_{ij}$ must, in the limit, have lost its dependence on all coordinates but $\tilde{v} = v$, so that $\bar{c}_{\hat{i}\hat{j}}$ really becomes the restriction of $c_{\hat{i}\hat{j}}$ to the null geodesic γ_0. A more detailed discussion of Penrose limits can be found in [Bla11]. We only mention here that one can tranform to Brinkmann coordinates (U, V, X^i), $i = 1, 2$ in order to obtain $\bar{g}$ in the plane wave form of Eq. (3.214). This works via

$$\begin{aligned} \tilde{u} &= V + \frac{1}{2}c_{\hat{i}\hat{j}}\dot{e}^{\hat{i}}{}_i e^{\hat{j}}{}_j X^i X^j \\ v &= U \\ \tilde{x}^{\hat{i}} &= e^{\hat{i}}{}_i X^i \end{aligned} \tag{3.244}$$

where $e = e(v) = e(U)$ is an orthonormal frame for the metric $\bar{c}$ on Λ_0 along γ_0, i.e. its inverse $e^{-1} = \theta$ is such that

$$\bar{c}_{\hat{i}\hat{j}} = \theta^i{}_{\hat{i}}\theta^i{}_{\hat{i}}\delta_{ij}\,. \tag{3.245}$$

Furthermore it is assumed to satisfy the symmetry condition

$$c_{\hat{i}\hat{j}}\dot{e}^{\hat{i}}{}_i e^{\hat{j}}{}_j = c_{\hat{i}\hat{j}}\dot{e}^{\hat{i}}{}_{(i}e^{\hat{j}}{}_{j)}\,, \tag{3.246}$$

which can always be arranged for [BFP02]. Like this,

$$\begin{aligned} \bar{g} = &-2\mathrm{d}U\left(\mathrm{d}V + \frac{1}{2}\left(\dot{c}_{\hat{i}\hat{j}}e^{\hat{i}}{}_i e^{\hat{j}}{}_j + c_{\hat{i}\hat{j}}\ddot{e}^{\hat{i}}{}_i e^{\hat{j}}{}_j + c_{\hat{i}\hat{j}}\dot{e}^{\hat{i}}{}_i\dot{e}^{\hat{j}}{}_j\right)X^iX^j\mathrm{d}U + c_{\hat{i}\hat{j}}\dot{e}^{\hat{i}}{}_i e^{\hat{j}}{}_j X^i\mathrm{d}X^j\right) \\ &+ c_{\hat{i}\hat{j}}\left(\dot{e}^{\hat{i}}{}_i X^i\mathrm{d}U + e^{\hat{i}}{}_i\mathrm{d}X^i\right)\left(\dot{e}^{\hat{j}}{}_j X^j\mathrm{d}U + e^{\hat{j}}{}_j\mathrm{d}X^j\right) \\ = &-2\mathrm{d}U\mathrm{d}V - \left(\dot{c}_{\hat{i}\hat{j}}e^{\hat{i}}{}_i + c_{\hat{i}\hat{j}}\ddot{e}^{\hat{i}}{}_i\right)e^{\hat{j}}{}_j X^iX^j\mathrm{d}U^2 + \delta_{ij}X^iX^j, \end{aligned} \tag{3.247}$$

and one extracts the wave profile

$$H(U, X) = -\left(\dot{c}_{\hat{i}\hat{j}}(U)e^{\hat{i}}{}_i(U) + c_{\hat{i}\hat{j}}(U)\ddot{e}^{\hat{i}}{}_i(U)\right)e^{\hat{j}}{}_j(U)X^iX^j \tag{3.248}$$

The physical picture behind Penrose limits is that of a family of observers, each approaching the speed of light a bit more closely, and the world line of the limiting member eventually coinciding with the chosen null geodesic γ_0. The rescaling (3.240) should be viewed as the concatenation of a boost and an accordingly executed overall 'clock recalibration' [Pen76] by the observer, i.e.

$$(u,v) \to (\lambda^{-1}u, \lambda v, x^{\hat{i}}) \to (\lambda^{-1}\lambda^{-1}u, \lambda^{-1}\lambda v, \lambda x^{\hat{i}}) = (\lambda^{-2}u, v, \lambda x^{\hat{i}})\,. \tag{3.249}$$

As a side remark, note that the covariantly constant null vector field of the limiting plane wave is not the generator of γ_0. Instead, it is the coordinate vector field $\partial/\partial\tilde{u}$ in Rosen (or $\partial/\partial V$ in Brinkmann) coordinates stemming from the labelling of the different geodesics of the null congruence.

The existence of Penrose limits has quite an impact on the viewpoint of the present chapter, since it builds a bridge between the presented non-relativistic geometries and the local structure of any Lorentzian manifold, hence also any solution to Einstein's equations. The observation is simply that, being a plane wave, the result of a Penrose limit exhibits in particular a Bargmann structure. Consequently, it should be possible to extract elements both of Carrollian and Galilean geometry via the construction of the previous section. In this case of course, they would be three-dimensional.

4. HOPF ALGEBRA BEFORE SPACETIME GEOMETRY

So far, the discussion revolved around what might be called Lie-type geometry. This is to say, except in the last section, where contact with general Lorentzian geometry was made, the conceptions of spacetime that were employed relied on symmetry notions derived from Lie algebras and Lie groups. The types of Lie algebras discussed in detail were of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ with a six-dimensional part $\mathfrak{h}$ describing rotations in space and (generalized) boosts, and a four-dimensional complement $\mathfrak{t}$ associated with (generalized) translations. These two parts obeyed the relations

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{t}] \subseteq \mathfrak{t}\,, \qquad [\mathfrak{t},\mathfrak{t}] \subseteq \mathfrak{h}\,, \tag{3.125}$$

which maintained this abstract split form under transformations generated by $\mathfrak{h}$. We saw how this allows an observer to construct a spacetime model as the quotient space G/H of the groups generated by $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$, respectively, and how the translationary part of the Maurer-Cartan form on G can be used to identify the tangent spaces to G/H with $\mathfrak{t}$.

25 years after the classifaction of possible kinematics by Bacry and Lévy-Leblond, Bacry dressed the call for an alternative to the Poincaré Lie group into a new guise [Bac93a]. The essential feature of the more 'modern' approach, as he calls it, compared to (3.125), is to allow nonlinear expression on the right hand sides. In particular, one would like to contemplate modifications of the mixed relations,

$$[\mathfrak{h},\mathfrak{t}] \subseteq \mathfrak{t} \otimes \mathfrak{t} \otimes \ldots\,, \tag{4.1}$$

that is the manner in which rotations and boosts act in the tangent spaces of spacetime. The mathematical apparatus needed to formalize such relations is going to be introduced.

Only recently, hence not quite another 25 years later, an intimate relation has surfaced between the two works previously mentioned. This relation will be presented and refined in this chapter. Ironically, it questions if Bacry was actually right in calling the later, algebraic viewpoint more modern than the earlier Lie group-type one.

When Lie groups appear as symmetry structures in physical theories, they typically do so only in terms of particular representations, that is to say one does not actually rely on their properties as abstract objects but rather in terms of their images via homomorphisms

$$\rho : G \to \mathrm{End}(V) \simeq \mathrm{Gl}(V) \tag{4.2}$$

into a vector space V of interest. Of course, one has to make sure that the defining feature—the group multiplication law—is preserved,

$$\rho(gh) = \rho(g)\rho(h), \qquad \forall g, h \in G, \tag{4.3}$$

qualifying ρ as a homomorphism. (Here, we sloppily denoted multiplication in G and $\mathrm{GL}(V)$ in the same manner, namely by omission, i.e. concatenation of elements.)

Going even further, many times physicists content themselves with their symmetry considerations to an infinitesimal level. Indeed, for lots of applications the use of Lie algebras is sufficient. (For others of course, the global properties of the Lie group are essential, a paradigm example being the physics of particles with spin.) The author remembers a statement made by the lecturer of one of his introductory graduate courses which to his uninitiated ears sounded much like the following:

"In physics, we do not make much of a distinction between
Lie algebras and Lie groups."

How is this possible? After all, the two notions are inarguably rather different. To explain this, note that the representation of the group induces one for the Lie algebra as well, namely by push-forward

$$\rho_* : \mathfrak{g} \to \mathfrak{gl}(V) \simeq \mathcal{M}_{\dim V}\,, \tag{4.4}$$

where we identify $\mathfrak{gl}(V)$ with the set $\mathcal{M}_{\dim V}$ of $(\dim V \times \dim V)$-matrices. The homomorphism property becomes (as above, and in the following, we omit the symbol for multiplication)

$$\rho_*([a,b]) = \rho_*(a)\rho_*(b) - \rho_*(a)\rho_*(b)\,. \tag{4.5}$$

From here, one easily passes to the image of ρ itself again, i.e. recovers non-zero determinant and the group law. This goes via the matrix exponential:

$$\exp(\rho_*(a)) = \sum_{n=0}^{\infty} \frac{1}{n!}\rho_*(a)^n \quad \in \rho(G),\ \ \forall a \in \mathfrak{g}\,. \tag{4.6}$$

Since

$$\mathrm{GL}(V) \subset \mathcal{M}_{\dim V} \tag{4.7}$$

we can even make sense of mixed products like

$$\rho(g)^2\rho_*(a)^3\rho(g)\rho_*(a)^7\ldots \tag{4.8}$$

When concerned with the fundamental significance of symmetry notions one may however be hesitant in drawing conclusions from here, since they might well depend on the chosen representation. Hence, the foregoing observations raise the following question:

Is there a mathematical structure unifying Lie algebras and Lie groups
without referring to particular representations?

There is, and we gave away its name with the title of this chapter.

4.1. SURPASSING LIE

This section will differ in style from the rest of this work, in that we are going to be much more pedantic mathematically. When studying Hopf algebras initially for himself, the author found an increased degree in formal explicitness necessary in order to understand the general concepts; much more so than this was the case for Lie theory, which presumably most relativists are familiar with at least to some degree.

We start by the following

Definition 4.1.1. *A* ***unital, associative algebra*** *over a field k is a triple* (A, m, η)*, where* A *is a vector space over* k *and* $m : A \otimes A \to A$ *and* $\eta : k \to A$ *are linear maps obeying the associativity axiom*

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \tag{4.9}$$

and the unit axiom

$$m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta) = \mathrm{id} \tag{4.10}$$

after identification of the isomorphic spaces $(A \otimes A) \otimes A \simeq A \otimes A \otimes A \simeq A \otimes (A \otimes A)$ *and* $k \otimes A \simeq A \simeq A \otimes k$ *and where* id *is the identity map on* A.

In the majority of applications to physics, the relevant field k is either $\mathbb{R}$ or $\mathbb{C}$, and for our purposes it would in fact be enough to restrict everything that comes to the former case. Nevertheless, we stick to the general case in this section as far as possible. As a side remark, we also mention that it is common to denote the 'unit element' $\eta(1)$ of A simply by 1 as well.

Note at this point already that neither Lie algebras nor Lie groups satisfy the conditions of Def. 4.1.1. There are however ways, as we will see, how their basic characteristics can be fit into them.

The two composition properties (4.9) and (4.10) translate into commutativity of the following diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}\otimes m} & A \otimes A \\ \downarrow{\scriptstyle m\otimes\mathrm{id}} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{\eta\otimes\mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id}\otimes\eta} & A \otimes k \\ & \searrow\!\!\!= & \downarrow{\scriptstyle m} & =\!\!\!\swarrow & \\ & & A & & \end{array}$$

The equality signs in the right diagram mean the aforementioned identification by scalar multiplication. If we reverse all arrows in these diagrams and still demand commutativity, the so-obtained new maps give rise to a notion which is in this sense dual to that of a unital associative algebra:

Definition 4.1.2. *A* ***counital, coassociative coalgebra*** *over a field k is a triple* (C, Δ, ϵ)*, where* C *is a vector space over* k *and* $\Delta : C \to C \otimes C$ *and* $\epsilon : C \to k$ *are linear maps satisfying the coassociativity axiom*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{4.11}$$

and the counit axiom

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}\,, \tag{4.12}$$

where, as in Def. 4.1.1, identification according to isomorphic tensor product spaces is understood and id *is the identity map on C.*

The accordingly dualized diagrams look as follows:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} & & C & & \\ & /\!\!/ & \downarrow{\scriptstyle \Delta} & \backslash\!\!\backslash & \\ k \otimes C & \xleftarrow[\epsilon\otimes\mathrm{id}]{} & C \otimes C & \xrightarrow[\mathrm{id}\otimes\epsilon]{} & C \otimes k \end{array}$$

General computations involving coproducts would become quite messy rather quickly if one always wrote out $\Delta(c) = \sum_i c_{(1),i} \otimes c_{(2),i}$, which is why one usually uses the so-called Sweedler notation, in which coproducts are abbreviated as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, or just $\Delta(c) = c_{(1)} \otimes c_{(2)}$.

Whenever the context allows in the following, the attributes 'unital', 'counital', 'associative', 'coassociative' as well as the specification of k will be omitted when referring to algebras that have the respective features.

The procedure that lets us define a coalgebra from an algebra is sometimes described as 'dualizing'. In this sense, a coalgebra could be understood as the mathematical structure dual to an algebra. The essence here is, as with any duality, that one could just as well say that algebras should actually be understood as dualized coalgebras. One really puts the two notions on an equal footing.

Dualities tend to have fixed points, i.e. points, for which the duality transformation becomes equal to the identity. In situations where one wishes to learn something about one side of the duality from the other, more well-known side, fixed points are consequently considered complicated. At the same time, what happens at fixed points is typically of particular interest. The informed reader might be reminded of various dualities in String Theory. Otherwise one might take as an example the relation between Carrollian an Galilean physics discussed in the previous chapter.

Fixed points of the duality mediating between algebras and coalgebras are called *Bialgebras*. Before we give their definition, the following one is needed.

Definition 4.1.3. *A homomorphism between two algebras (A, m, η) and (A', m', η') is a linear map $f : A \to A'$ that satisfies*

$$m' \circ (f \otimes f) = f \circ m \quad \textit{and} \quad \eta' = f \circ \eta. \tag{4.13}$$

A homomorphism between two coalgebras (C, Δ, ϵ) and (C', Δ', ϵ') is a linear map $f : C \to C'$ that satisfies

$$\Delta' \circ f = (f \otimes f) \circ \Delta \quad \textit{and} \quad \epsilon' \circ f = \epsilon. \tag{4.14}$$

*Homomorphisms between algebras and coalgebras are often referred to as **algebra maps** and **coalgebra maps**, respectively.*

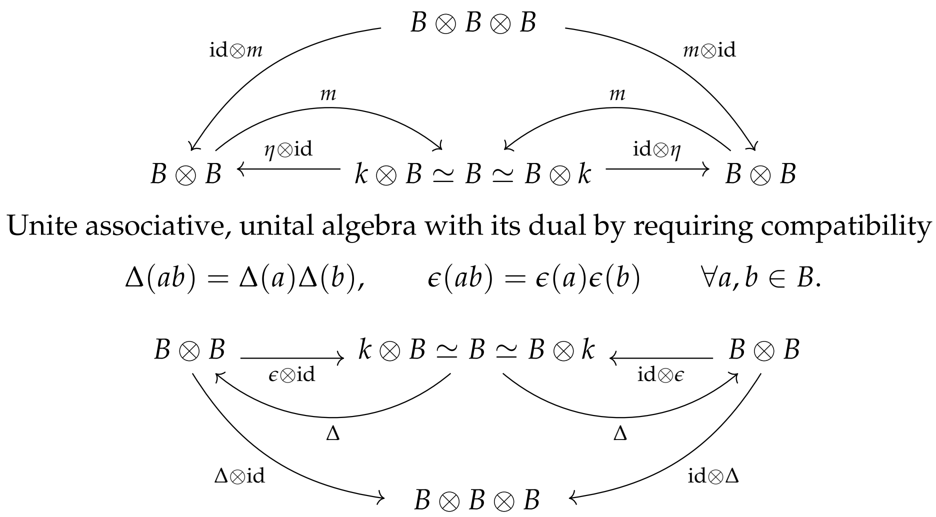


Figure 4.1.: A bialgebra ties together the structure of an algebra and a coalgebra.

Definition 4.1.4. *A* ***bialgebra*** *over a field k is a tupel $(B, m, \eta, \Delta, \epsilon)$ such that (B, m, η) is an associative algebra, (B, Δ, ϵ) is a coassociative coalgebra, and if $B \otimes B$ is endowed with the tensor product algebra and coalgebra structure then m and η are coalgebra maps.*

By tensor product algebra and coalgebra structure the following is meant. From the algebra (A, m, η), the multiplication is extended to $A \otimes A$ as

$$m(a \otimes b, c \otimes d) = m(a, c) \otimes m(b, d) \quad \forall a, b, c, d \in A, \tag{4.15}$$

from where the extended η derives. From the coalgebra (C, Δ, ϵ) the coproduct extends to $C \otimes C$ as

$$\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \tag{4.16}$$

if $\Delta(a) = a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = b_{(1)} \otimes b_{(2)}$, which then defines the extended counit ϵ. A linear map between bialgebras that is a homomorphism for both the algebra as well as the coalgebra structures is called a **bialgebra map**.

Remark 4.1.5. *The above definition remains invariant if, instead of demanding m and η to be coalgebra maps, we require Δ and ϵ to be algebra maps.*

Proof. See [Rad12]. □

The situation is summarized by the diagram in Fig. 4.1.

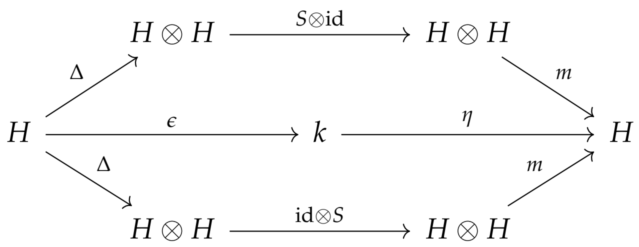
Note that bialgebras really consist of two distinct, though compatible parts, meaning that while the upper and lower half of the above diagram are commutative, they cannot be joined to form one large commutative diagram, since for example, $m \circ \Delta \neq \mathrm{id}$ in general.

There is one more structural element that needs to be added to those of a bialgebra in order to obtain a Hopf algebra.

Definition 4.1.6. *A **Hopf algebra** $(H, m, \eta, \Delta, \epsilon, S)$ is a bialgebra $(H, m, \eta, \Delta, \epsilon)$ endowed with a linear map*

$$S : H \to H\,, \tag{4.17}$$

called antipode, and rendering the following diagram commutative:



The antipode is usually being extended to act on tensor products of elements as $S \otimes S \otimes \ldots$.

As might be guessed from their definition, Hopf algebras are a rich class of mathematical objects. While they extend far beyond, it is an elementary fact that they can be seen to unify Lie algebras and Lie groups. This shall be clarified now.

Note that from an associative algebra (A, m, η) we can always extract a Lie algebra by constructing the Lie bracket as $[a, b] = m(a, b) - m(b, a)\ \forall a, b \in A$. The question of how to find an associative structure naturally related to a Lie algebra is being addressed now. First we need

Definition 4.1.7. *The **tensor algebra** (sometimes called free algebra) over a vector space V is a pair $(\iota, T(V))$ satisfying the following two universal mapping properties:*

1. *$T(V)$ is an associative algebra and $\iota : V \to T(V)$ is a linear map.*
2. *For any associative algebra A and linear map $f : V \to A$, there exists an algebra map $F : T(V) \to A$ satisfying $F \circ \iota = f$.*

We could paraphrase the second property in the definition above by saying that the tensor algebra is the most general algebra incorporating the features of its base vector space. Usually, the multiplication symbol used on the tensor algebra is $\otimes$. In the context of Hopf algebras however, it is common to reserve this for the 'exterior' tensor product between algebras, as used in the definition of the coproduct. This habit is in accord with how one usually avoids explicitly writing out multiplication in terms of the map m.

Furthermore we need

Definition 4.1.8. *An **ideal** of an associative algebra (A, m, η) over the field k is a subset $I \subseteq A$ with the following two properties:*

1. *$(I, m|_I, \eta|_I)$ is a subalgebra of (A, m, η).*
2. *$m(I, A) \subseteq I$ and $m(A, I) \subseteq I$*

A coideal *of a coassociative coalgebra* (C, Δ, ϵ) *is a subset* $J \subseteq C$ *with the following two properties:*

1. $J \subseteq ker(\epsilon)$
2. $\Delta(J) \subseteq C \otimes J + J \otimes C$

Now we are well prepared to understand how the structure of Lie algebras can be viewed in terms of particular bialgebras.

Definition 4.1.9. *Given a Lie algebra L over k, its* ***universal enveloping algebra*** *is a pair* $(\iota, U(L))$ *satisfying the following two universal mapping properties:*

1. $U(L)$ *is an associative algebra and* $\iota : L \to U(L)$ *is a Lie algebra map from L to the Lie algebra associated to* $U(L)$*.*
2. *For any associative algebra A and Lie algebra map* $f : L \to A$ *in the above sense, there exists a map of associative algebras* $F : U(L) \to A$ *satisfying* $F \circ \iota = f$*.*

A useful manner to think about universal enveloping algebras is in terms of how they arise from tensor algebras. This is now briefly going to be presented.

In the tensor algebra $(\iota, T(L))$, consider the elements of the form $d(l, l') := \iota(l)\iota(l') - \iota(l')\iota(l) - \iota([l, l'])$ where $l, l' \in L$, and call $I = (\{d(l, l')|l, l \in L\})$ the ideal in $T(L)$ generated by these differences. Informally, I measures the lack of ι in being a Lie algebra map. By forming the quotient $T(L)/I$ however, we do obtain the universal enveloping algebra $(\jmath, T(L)/I)$ with $\jmath : L \to T(L)/I$ given by $\jmath(l) = \iota(l) + I \; \forall l \in L$. This is easily seen:

$$\begin{aligned} \jmath(l)\jmath(l') - \jmath(l')\jmath(l) &= \iota(l)\iota(l') - \iota(l')\iota(l) + I \\ &= d(l, l') + \iota([l, l']) + I \\ &= \iota([l, l']) + I = \jmath([l, l']) \end{aligned}$$

Very often then, one does not introduce a new bracket symbol for this, but writes $[a, b]$ also for elements a, b of the enveloping algebra. In fact, it is possible to endow $U(L)$ obtained like this with Hopf structure, i.e. a Hopf algebra structure on $T(L)$ compatible with the relations imposed on it by the Lie bracket, hence in particular satisfying $\Delta([a, b]) = [\Delta(a), \Delta(b)]$. This 'primitive' structure is given by

$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a \\ \epsilon(a) &= 0 \\ S(a) &= -a\,, \end{aligned} \tag{4.18}$$

for all $a \in U(L),\; a \neq 1$, and

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 \\ \epsilon(1) &= 1 \\ S(1) &= 1\,. \end{aligned} \tag{4.19}$$

Taking the exponential function, one arrives at relations that constitute the analogue of Lie groups on the level of Hopf algebras. For the coproduct, one has

$$\begin{aligned}\Delta(\exp(a)) &= \sum_{n=0}^{\infty}\frac{(\Delta(a^n))}{n!} = \sum_{0}^{\infty}\frac{(\Delta(a))^n}{n!} = \sum_{0}^{\infty}\frac{1}{n!}(a\otimes 1+1\otimes a)^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}a^m\otimes a^{n-m} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{a^m}{m!}\otimes\frac{a^{n-m}}{(n-m)!} \\ &= \sum_{n=0}^{\infty}\sum_{l+m=n}\frac{a^m}{m!}\otimes\frac{a^l}{l!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^m}{m!}\otimes\frac{a^n}{n!} \\ &= \exp(a)\otimes\exp(a)\,.\end{aligned} \tag{4.20}$$

Coproducts of the form $\Delta(g) = g\otimes g$ are called 'grouplike'. In the same way one concludes that while the counit now maps all elements to the unit,

$$\epsilon(\exp(a)) = \exp(\epsilon(a)) = \exp(0) = 1\,, \tag{4.21}$$

the antipode becomes the inversion map,

$$S(\exp(a)) = \exp(S(a)) = \exp(-a) = (\exp a)^{-1}\,. \tag{4.22}$$

Hence one has gathered from $U(L)$ all elements of the Lie group belonging to the Lie algebra L. But since the resulting structure in addition inherits a vector space structure, it is in fact truly more than a Lie group, and called *group algebra*.

We have collected here only the most basic definitions underlying Hopf algebras. Yet this puts us already in the position to motivate a particularly simple type. This kind of Hopf algebra is of special interest to us, since it can be seen as a possible Hopf analogue of semidirect sum Lie algebras, hence in particular of all the considered contractions of $\mathbf{dS}_{\pm}$. In fact we will see how their Hopf algebra character lets it appear rather natural to actually consider them Hopf *deformations* of semidirect sums.

The abstract problem is that of general Hopf algebra extensions and was largely solved in [Sin72] and a brief review is contained in [MR94]. For our purposes it suffices to ask how we can adopt the characteristic features of semidirect sum Lie algebras into the Hopf algebraic setting. The leading idea here is that of dualization, which is inherent to Hopf algebras as we saw.

As a starting point recall that what defines a semidirect sum Lie algebra is the sequence

$$0 \to \mathfrak{n} \overset{i}{\hookrightarrow} \mathfrak{g} \underset{j}{\overset{p}{\rightleftarrows}} \mathfrak{h} \to 0 \tag{4.23}$$

with $p\circ j = \mathrm{id}_{\mathfrak{h}}$. In order to find the Hopf analogue we replace $\mathfrak{n}$ and $\mathfrak{h}$ by their universal envelopes $N := U(\mathfrak{n})$ and $H := U(\mathfrak{h})$, and extend (without notational distinction here) the Lie algebra homomorphisms i and p to Hopf algebra ones. In the simplest case (avoiding the Hopf version of cocycles), one also demands that j becomes an algebra homomorphism. We then have

$$1 \to N \overset{i}{\hookrightarrow} E \underset{j}{\overset{p}{\underset{\hookleftarrow}{\twoheadrightarrow}}} H \to 1\,, \tag{4.24}$$

with $p \circ j = \mathrm{id}_H$, and where it was recognized that the universal enveloping algebra of the trivial Lie algebra 0 must still feature a unit element, and is hence given here the same symbol as the trivial group 1.

The action (2.45) of $\mathfrak{h}$ onto $\mathfrak{n}$ that was induced by j in the Lie algebra translates to the present case as well. Now however, it can be dualized by a coaction of $U(\mathfrak{t})$ on $U(\mathfrak{h})$. In order to do so, we need a second projection, say q, from E, but now onto N, and inverse to i. For the same reason for which j was chosen to be an algebra homomorphism, q can be chosen to be a coalgebra one. The final sequence, of which the upper part is of the split exact type, becomes

$$1 \to N \underset{q}{\overset{i}{\underset{\twoheadleftarrow}{\hookrightarrow}}} E \underset{j}{\overset{p}{\underset{\hookleftarrow}{\twoheadrightarrow}}} H \to 1\,, \tag{4.25}$$

with $q \circ i = \mathrm{id}_N$ in addition to $p \circ j = \mathrm{id}_H$. In order to make E a Hopf algebra, one then demands commutativity from the following two diagrams:

$$\begin{array}{ccccc} H & \xrightarrow{\;j\;} & E & \xrightarrow{\;\Delta_E\;} & E \otimes E \\ {\scriptstyle \Delta_H}\downarrow & & & & \downarrow{\scriptstyle p\otimes \mathrm{id}_E} \\ H \otimes H & & \xrightarrow[\mathrm{id}_H \otimes j]{} & & H \otimes E \end{array} \qquad \begin{array}{ccccc} T & \xleftarrow{\;q\;} & E & \xleftarrow{\;m_E\;} & E \otimes E \\ {\scriptstyle m_T}\uparrow & & & & \uparrow{\scriptstyle \mathrm{id}_E\otimes i} \\ T \otimes T & & \xleftarrow[q \otimes \mathrm{id}_T]{} & & E \otimes T \end{array}$$

We can now write the action of H on T in terms of the antipode S_H of H and its coproduct Δ_H as well as the product in E (which is determined through j from the one of H) as

$$\begin{aligned} \alpha_j : N \times H &\to N \\ (n, h) &\mapsto q(j(S(h_{(1)}))i(n)j(h_{(2)}))\,, \end{aligned} \tag{4.26}$$

so that j determines the deformation away from the standard Lie type action. Meanwhile, the antipode S_T of T and its product, as well as the coproduct Δ_E in E determine a coaction

$$\begin{aligned} \beta_i : H &\to T \otimes H \\ h &\mapsto q(i(h)_{(1)})S_T(q(i(h)_{(3)})) \otimes p(i(h)_{(2)}) \end{aligned} \tag{4.27}$$

where $a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = ((\Delta_E \otimes \mathrm{id}_E) \circ \Delta_E)(a) = ((\mathrm{id}_E \otimes \Delta_E) \circ \Delta_E)(a)$. Any deformation away from the trivial action $h \mapsto 1 \otimes h$ is now implicity given by i (or, the coproduct it defines in E).

Hopf algebras E of the described kind are particular instances of so-called *bicrossproduct* algebras, which are discussed in detail in [Maj95], and denoted $E = H \triangleright\!\blacktriangleleft T$. We will use the same notation for our specific case. As in the Lie algebra case, the complexity of bicrossproducts is reduced noticeably when $\mathfrak{n}$ is abelian.

One essential conclusion that should be drawn at this point is that if we reconsider the two constituent Lie algebras $\mathbb{R}^4$ and $\mathfrak{so}(1,3)$ of the Poincaré Lie algebra $\mathbf{P}$ from the Hopf point of view, then we are immediately facing the possibility of new deformations other than $\mathbf{dS}_{\pm}$. For then the task is to find all extensions of $U(\mathfrak{so}(1,3))$ by $U(\mathbb{R}^4)$ in the above sense. A classification of such was in fact achieved in [Pod96] on the group level. Sec. 4.3 will see a particular example in the infinitesimal case.

4.2. Spacetime, curvature and momentum space

When concerned with the fundamental nature of spacetime, one necessarily needs to analyse its role in the known physical theories. That this is a nontrivial task roots in the way in which spacetime is actually addressed when writing down Lagrangians, calculating scattering amplitudes or solving a geodesic equation of motion. For all practical purposes this is rather implicit, because points in spacetime will often be addressed only by their *coordinate*, i.e., their image in a certain chart from the chosen d-dimensional spacetime manifold to $\mathbb{R}^d$. Quite frequently in fact, and most prominently so in quantum field theory in flat Minkowski space, it makes an even subtler appearance. There, it can be entirely replaced by momentum space. Often in this context, spacetime and momentum space are even geometrically identified, or at least regarded to be dual. That this should be possible in quantum physics is an old idea, initially raised by Born and today still known under the name of 'Born reciprocity' [Bor38]. Our viewpoint will be a different one, and closer to the one of General Relativity. But developments in the study of quantum groups prompt us to reconsider Born reciprocity from this geometric viewpoint, too. This will be very much in the spirit of an 80-year-old construction [Sny47], which in the author's view can be regarded as paving the way to much of current research in quantum gravity phenomenology. But let us first briefly highlight some of the conceptual differences between spacetime and momentum space, both in Special and in General Relativity. They are being opposed in Table 4.1. Note that our considerations deal alone with the geometric structure which is employed in physical theories, hence in this sense only with the scenery against which physics plays out. This is to say that the dynamical laws of a specific theory will eventually dictate how this scenery will actually be occupied by all degrees of freedom. For the moment we are also not referring to the notion of conjugate variables for fields, although it should be expected that the algebraic relations they satisfy become subject to change, too, when the underlying definition of position and momentum is refined. It is in this sense that spacetime symmetry deformations would have an impact on second quantization as well.

Ironically, Snyder's motivation in parting from the geometric structure of Special Relativity was to show that in principle, Lorentz invariance can be maintained even in a

Spacetime	Momentum space
Special Relativity	
Affine space	Vector space
No distinguished points	Zero momentum (=origin) is special
Cannot add points	Linear addition of momenta
Poincaré invariant	Lorentz invariant
General Relativity	
Differentiable manifold	Cotangent (vector) space
Points relatable by diffeomorphism	Isomorphic at different points
Equipped with metric tensor field	Inherits bilinear form
Admits local orthonormal frames	Special Relativity holds

Table 4.1.: A similar overview may be found in [Bac93b].

discrete spacetime. In hindsight, his trick consisted in reversing the roles of spacetime and momentum space, much in the sense of Born. Postulating momentum space to be a de Sitter hyperboloid, the Lorentz group is recovered as that subgroup of the de Sitter group which stabilizes points. Viewing the translationary symmetry generators as differential operators, their spectrum is then identified with spacetime. As de Sitter space is spatially compact, one obtains a notion of discrete space, while time remains continuous.

One purpose of the previous chapter was to show how tight the relationship between spacetime and its symmetries is at the level of Special Relativity. In attempts to explore the possible nature of quantum spacetime, this has often served as a basic starting point. Given a Lie algebra that describes the spacetime symmetries in a known theory, one then tries to find quantum deformations from which it is possible to recover a definition of spacetime. Most notably of course, the interest would be in the Poincaré algebra. While however a recipe exists for obtaining quantum deformations of semisimple Lie algebras [Dri87; Jim85], the Hopf algebras that are considered today as possible quantum deformations of the Poincaré algebra are of different type. Although many interpretative issues remain with the possible candidates, consensus seems to condense to the point that all of them should be understood as endowing momentum space with curvature. While the details of the full picture still seem to be far from understood, this is where the similarities with Snyder's construction begin to surface. One of the most-studied example, the κ-Poincaré algebra, almost exactly coincides with Snyder's construction. The developments that led to its discovery as well as its geometric interpretation will be presented, and slightly extended, in the following section.

4.3. κ-Minkowski spacetime

In lack of a general definition for quantum deformations of non-semisimple Lie algebras, in [Luk+91] there was presented a procedure to obtain, nevertheless, a Hopf algebra that is close to the Poincaré Lie algebra **P** in a precise sense. Actually, this procedure appears quite natural in view of the relationship of **P** towards the de Sitter Lie algebra $\mathbf{dS}_+$. It can be represented schematically by the following diagram:

$$\begin{array}{ccc} \text{de Sitter} & \xrightarrow{\;q\;} & \text{q-de-Sitter} \\ {\scriptstyle L^{-1}}\big\uparrow & & \big\downarrow{\scriptstyle L^{-1},\,\arg q} \\ \text{Poincaré} & \xleftarrow[\kappa^{-1}]{} & \kappa\text{-Poincaré} \end{array}$$

The left vertical arrow is nothing but the classical deformation of **P** to $\mathbf{dS}_+$, governed by a length scale L. Being semisimple, $\mathbf{dS}_+$ admits a quantum deformations in the sense of Drinfeld and Jimbo. As indicated by the upper horizontal arrow, such are governed by a complex parameter q, which in this case however needs to be taken to have unit modulus, so that the deformation can be undone by taking its phase $\arg q$ to zero. What however, if this limit is taken simultaneously with the one of vanishing curvature, $L \to \infty$? It turns out that the limit can be defined by taking it such that the product $L \arg q$ stays constant. Its inverse, named κ, is then a new parameter governing the new algebraic structure, which was found to still be of Hopf type. In particular, it has the property of reducing to the Poincaré Hopf algebra upon taking $\kappa \to \infty$, which is symbolized by the lower horizontal arrow. The question remains until today if there is a rigorous deformation procedure for the Poincaré Hopf algebra that coincides with the κ-Poincaré algebra, but is more generally defined for the Hopf algebras belonging to Lie algebras of semidirect sum type.

The result of the original procedure just described—we will refer to it by $\mathbf{P}_\kappa$— was a Hopf algebra with algebra relations of the form

$$\begin{aligned} &[L, L] \subseteq L \oplus T \\ &[L, T] \subseteq T \\ &[T, T] = 0, \end{aligned} \tag{4.28}$$

where L consists of the Lorentz generators and T is made up of spatial and temporal translations. Due to the T-terms in the $[L, L]$-commutators, Lorentz transformation could no longer be regarded as a subalgebra, which must have seemed like an unpleasant feature to the authors of a further piece of work that appeared two years later [MR94]. It was them who first wrote down the κ-Poincaré Hopf algebra in the form that is most commonly employed today in phenomenological discussions. The central achievement was to find a bialgebra homomorphism that brought $\mathbf{P}_\kappa$ into bicrossproduct form. This means in particular that its algebra part now has relations as those above, but with

$$[L, L] \subseteq L \oplus T \longrightarrow [L, L] \subseteq L. \tag{4.29}$$

Even more, the brackets $[L, L]$ between Lorentz generators are exactly those of the Lie algebra (which is true for the commutative translations, too). This means that in the bicrossproduct basis, obtained via a (nonlinear) redefinition of generators compatible with the the original Hopf structure, the deforming character of $\mathbf{P}_\kappa$ can be reduced to the adjoint action of the Lorentz generators on the translations. Actually, it is only the boosts that act in a fashion different from the classical, Lie algebraic one, and they do so only on spatial momenta. Relying on the general definitions underlying Hopf algebras that where provided in Sec. 4.1, the following list of properties defines $\mathbf{P}_\kappa$ as a bicrossproduct:

$$\begin{gathered}[\vec{J},\vec{J}] = \vec{J}, \qquad [\vec{J},\vec{K}] = \vec{K}, \qquad [\vec{K},\vec{K}] = -\vec{J}, \\ [\vec{P},\vec{P}] = 0, \qquad [\vec{P},H] = 0,\end{gathered} \tag{4.30}$$

$$\begin{gathered}[\vec{J},\vec{P}] = \vec{P}, \qquad [\vec{J},H] = 0, \qquad [\vec{K},H] = \vec{P}, \\ [\vec{K},\vec{P}] = \frac{\kappa}{2}\left(1 - \mathrm{e}^{-2H/\kappa}\right) + \frac{1}{2\kappa}\vec{P}^2 - \frac{1}{\kappa}\vec{P}\vec{P},\end{gathered} \tag{4.31}$$

$$\begin{gathered}\Delta(\vec{J}) = I \otimes \vec{J} + \vec{J} \otimes I, \\ \Delta(H) = I \otimes H + H \otimes I,\end{gathered} \tag{4.32}$$

$$\begin{gathered}\Delta(\vec{K}) = \vec{K} \otimes I + \mathrm{e}^{-H/\kappa} \otimes \vec{K} + \frac{1}{\kappa}\vec{P} \overset{\times}{\otimes} \vec{J}, \\ \Delta(\vec{P}) = \vec{P} \otimes I + \mathrm{e}^{-H/\kappa} \otimes \vec{P},\end{gathered} \tag{4.33}$$

$$\begin{gathered}\eta(c) = I \quad \forall c \neq 0 \in k, \\ \epsilon(H) = \epsilon(\vec{P}) = \epsilon(\vec{K}) = \epsilon(\vec{J}) = 0, \quad \epsilon(I) = 1,\end{gathered} \tag{4.34}$$

$$\begin{gathered}S(\vec{P}) = -\mathrm{e}^{H/\kappa}\vec{P}, \qquad S(H) = -H, \\ S(\vec{K}) = -\vec{K}, \qquad S(\vec{J}) = -\vec{J}\,.\end{gathered} \tag{4.35}$$

To the uninitiated reader, the definition of a Hopf algebra like this, together with the claim that it is supposed to serve as the symmetry algebra of some physical system, might seem somewhat arbitrary. For Lie algebras, this is often different due to their geometrical interpretation. What calls for a deeper understanding is the circumstance that $\mathbf{P}_\kappa$ can be derived from a purely geometric view as well. This was first noticed in [Kow02] and further understood in [KN04]. The pressing question connected to this finding is if there is a more general truth to this. In order to pose this question more precisely, we now review the geometric nature of $\mathbf{P}_\kappa$. The general viewpoint of this dissertation helps to extend it slightly, although perhaps only in sharpening the interpretation, by calling known things by their name.

The basic setting is the same as in Snyder's construction, namely de Sitter space embedded in five-dimensional Minkowski space. Both are physically viewed as spaces of momentum, so that the de Sitter curvature radius dictating the embedding has dimensions of mass, when c is set to 1. It will coincide with the (inverse) deformation parameter for the κ-Poincaré algebra. In Cartesian coordinates Π_M we have

$$H_+ \subset \mathbb{R}^5 : \quad \eta^{MN}\Pi_M\Pi_N = -\Pi_0^2 + \vec{\Pi}^2 + \Pi_4^2 = \kappa^2 \,. \tag{4.36}$$

Although we are now physically in momentum space, we would like to apply the same formalism in dealing with the isometries as was developed for coordinate space in Sec. 3.2. There we already stated that in the case of $\mathbf{dS}_\pm$ the left and right actions are equivalent. Formally, we can therefore choose to represent the Π_M also as column vectors, and take as the isometry-generating vector fields

$$\Omega_+^{MN} = (\tilde{\omega}_+^{MN})_K{}^L \Pi_L \frac{\partial}{\partial \Pi_K} \,, \tag{4.37}$$

where the matrix $(\tilde{\omega}_+^{MN})$ is numerically the same as (ω_{MN}^+), and hence the same goes for the derived, physically distinguished generators. As was announced earlier, in the following we will however omit ω for the sake of readability, and instead use the symbols M^{MN} of the abstract generators for the representing matrices.

In order to arrive at our goal of geometrically implementing $\mathbf{P}_\kappa$, we need to choose coordinates on H_+ that arise from a slicing of it by hyperplanes that are lightlike with respect to the ambient metric. If we orient them such that

$$\Pi_0 + \Pi_4 = \text{const.} \,, \tag{4.38}$$

we can choose a coordinate p_0 on H_+ that is constant on and hence labels the 3-surfaces of intersection, which satisfy

$$-(\text{const.})^2 + 2(\text{const.})\Pi_4 + \vec{\Pi}^2 = \kappa^2. \tag{4.39}$$

Note that while these have the geometry of a 3-dimensional paraboloid if the constant is different from zero, the two branches separate into two copies of a cylinder $S^2 \times \mathbb{R}$ when the intersecting hyperplanes go through the origin. Coordinates p_i within the 3-surfaces of intersection will, if good in the first case, go bad in the degenerate case. In effect, we need two coordinate patches to cover H_+ in this way, corresponding to $\Pi_0 + \Pi_4$ being positive or negative. Alternatively, we could say that the chosen coordinates cover completely the manifold given by

$$\mathbb{Z}_2 \backslash H_+, \tag{4.40}$$

i.e., exactly half of H_+, the (left) $\mathbb{Z}_2$-action being the inversion at the origin of the embedding $\mathbb{R}^5$ (it inverts the sign of all Π_M).

A convenient choice of coordinates $p_\mu = (\varepsilon, \vec{p})$ on the upper half is such that a point on the hyperboloid has Cartesian components

$$\Pi_M(p) = \begin{pmatrix} \kappa \sinh \varepsilon/\kappa + \frac{\vec{p}^2}{2\kappa} e^{\varepsilon/\kappa} \\ \vec{p} e^{\varepsilon/\kappa} \\ \kappa \cosh \varepsilon/\kappa - \frac{\vec{p}^2}{2\kappa} e^{\varepsilon/\kappa} \end{pmatrix} \,, \tag{4.41}$$

so that $\Pi_0 + \Pi_4 = \kappa e^{\varepsilon/\kappa}$. The inverse relations are

$$\begin{aligned}\varepsilon &= \kappa \ln \frac{\Pi_0 + \Pi_4}{\kappa}, \\ \vec{p} &= \frac{\kappa \vec{\Pi}}{\Pi_0 + \Pi_4}.\end{aligned} \tag{4.42}$$

Fig. A.1 gives a graphical impression of this coordinate choice.

As a homogeneous space of the de Sitter group, we should be able to reach any point on H_+ by an $SO_0(1,4)$ transformation[1] from the origin

$$o_M = \Pi_M(0) = (0, \vec{0}^t, \kappa)^t\,. \tag{4.43}$$

This is not unique, of course, the redundancy being exactly captured by the stabilizer subgroup $SO_0(1,3)$. In order to achieve the tightest correspondence between the isometry group and the space it acts upon, we would like to choose a parametrization of the group that coincides with the chosen coordinates. As it turns out, in this case it is not the 'standard' translations that are needed, i.e. the generators M^{m4} alone. Instead we use (taking $c = 1$)

$$X^i = \frac{1}{\kappa}(M^{i4} + M^{i0})\,, \tag{4.44}$$

$$T = \frac{1}{\kappa} M^{40}\,, \tag{4.45}$$

where what would usually be called the generators of spatial translations have been linearly combined with what is conventionally associated with boosts. With these we have (suppressing indices)

$$\Pi(p) = \exp(\vec{p}\cdot\vec{X}) \exp(\varepsilon T) o \tag{4.46}$$

where

$$X^i = \frac{1}{\kappa}\begin{pmatrix} 0 & \vec{n}_i^t & 0 \\ \vec{n}_i & \mathbf{0} & \vec{n}_i \\ 0 & -\vec{n}_i^t & 0 \end{pmatrix}, \qquad T = \frac{1}{\kappa}\begin{pmatrix} 0 & \vec{0}^t & 1 \\ \vec{0} & \mathbf{0} & \vec{0} \\ 1 & \vec{0}^t & 0 \end{pmatrix}. \tag{4.47}$$

For the computations to come in this section, we will make use of this matrix representation throughout. Recall that its defining feature is that it inherits the commutation relations of the abstract Lie algebra. In this case, they are

$$\begin{aligned}[\vec{X}, T] &= \frac{1}{\kappa}\vec{X}\,, \\ [\vec{X}, \vec{X}] &= 0\,.\end{aligned} \tag{4.48}$$

[1] We restrict here to the identity component of the group since what we will really be using is its Lie algebra. Geometrically, this means that we are considering H_+ endowed with a fixed orientation and a distinguished time direction. These will however not enter the discussion.

In other words, $\{\vec{X}, T\}$ form an almost commutative subalgebra of $\mathfrak{so}(1,4)$. Almost, because κ is assumed to be large for phenomenological reasons, so that we are close to the limit of vanishing curvature, $\kappa \to \infty$, which makes everything commute. In fact, we briefly encountered a lower-dimensional version of this Lie algebra already early on, when we discussed the proposal of Very Special Relativty. Removing J_3 from the set (2.57) yields the Lie algebra of so-called homotheties in 1+2 dimensions, $\mathfrak{hom}(2)$. What we have now is exactly the same, but in 1+3 dimensions, i.e. $\mathfrak{hom}(3)$. This is plausible from our embedding view, which relies on employing $\mathfrak{so}(1,4)$ as the *five*-dimensional Lorentz group.

Via the exponential map, $\mathfrak{hom}(3) \subset \mathfrak{so}(1,4)$ ascends to a Lie subgroup

$$\mathrm{HOM}(3) \subset \mathrm{SO}_0(1,4)\,, \tag{4.49}$$

which, geometrically, is nothing but the $\Pi_0 + \Pi_4 > 0$ half of de Sitter space, henceforth denoted by Ξ. With the earlier-mentioned inversion $\mathbb{Z}_2 : \Pi_M \mapsto -\Pi_M$, we may abstractly identify as manifolds

$$\mathrm{HOM}(3) \simeq \mathbb{Z}_2 \backslash (\mathrm{SO}_0(1,4)/\mathrm{SO}_0(1,3)) =: \Xi, \qquad \Xi \subset \mathbb{R}^5\,, \tag{4.50}$$

and the map from o to $\Pi(p)$, and in fact between any two points on it, becomes unique, for it now coincides with group multiplication in $\mathrm{HOM}(3)$.

Let us stand back for a second and ask where we have got. Apparently, $\mathbb{R}^4$ as the model geometry of momentum space has been replaced by a different manifold, which is at the same time a Lie group, and has curvature scale κ. It can be viewed as a deformation of $\mathbb{R}^4$ in terms of κ^{-1} much in the sense of the limiting procedures between the different Cayley-Klein geometries introduced earlier. That is to say, if κ is large, then we recover conventional models. This surely is reassuring in accepting the present model as an alternative candidate. But what is the consequence of curvature in momentum space? One might argue quite generally, that it should leave an imprint on the definition of spacetime. First of all, there is a general conceptual shift connected with starting off by considering momentum space first, instead of defining it as the cotangent space to spacetime. But if we do so, the picture is somewhat reversed, and spacetime can only be defined in relation to momentum space. That such a view should be legitimate, at least in certain situation in physics, is the claim of Born reciprocity. Table 4.1 shows why this goes beyond standard classical notions. But if we are willing to take that step the question is how big and in which direction it is. In Special Relativity, the main difference between spacetime and momentum space is that the latter lacks the affine structure of the former—with good reasons, as we discussed. Simply exchanging the structures is surely not desirable, since zero energy-momentum is undeniably a special case. On the other hand, from the perspective of quantum theory it might seem acceptable to consider Minkowski spacetime only in a particular coordinate system, hence with fixed origin, corresponding to the location of the measurement device in the laboratory. Translations away from zero to a point with Cartesian coordinates $(t, \vec{x})$ are then given by [2]

$$\exp(\vec{x}\cdot\vec{P})\exp(tH)\,. \tag{4.51}$$

[2]Eqns. (4.51)–(4.53) do not specify a particular representation of the symmetry generators.

Even if operationally perhaps ill-defined, a displacement of a given system in spacetime seems not hard to interpret. Finding a meaning for the analogous transformation in momentum space,

$$\exp(\vec{p}\cdot\vec{X})\exp(\varepsilon T)\,, \tag{4.52}$$

is not as easy, even in the Minkowski case. Certainly, it will in some sense concern the addition of momentum. For example, applying a translation in time direction to the zero momentum should correspond to the creation of a particle. Physical processes however, which realize the addition of momentum, are compositions of classical or quantum systems and scattering processes in quantum field theory. Both are rather complex and usually described under the assumption of a linear structure in momentum space.

From a technical viewpoint, in quantum mechanics one might actually be tempted to look at the two transformations (4.51) and (4.52) as one and the same, only expressed differently in configuration and in momentum space—in the same way, as Heisenberg's commutation relations

$$[\hat{X},\hat{P}] = i\hbar \tag{4.53}$$

are realized on wavefunctions $\psi(x)$ by multiplication $\hat{X}\to x$ and differentiation $\hat{P}\to -i\hbar\partial/\partial x$, and on their Fourier counterparts $\tilde{\psi}(p)$ in reverse, $\hat{X}\to i\hbar\partial/\partial p$, $\hat{P}\to p$.

All these considerations serve to motivate the interpretation of the generators in $\mathfrak{hom}(3)$ as the components of an operator corresponding to the location of a system in space and time. This is essentially Snyder's viewpoint, although due to our peculiar choice of slicing de Sitter momentum space, and the resulting choice of coordinates, we do not obtain a discretization for spacetime. We do retain however a degree of noncommutativity and in fact the so-obtained model of spacetime is called the κ-deformation of Minkowski space, or simply, κ-Minkowski space. The author hopes to have made clear, that this should be regarded a misnomer, since it omits crucial interpretative leaps. If at all, one should view it as a deformation of only the vector space structure underlying Minkowski space, seen as abelian Lie algebra.

Accepting this shift in attitude, we may continue to analyse the implications from a HOM(3) momentum space. Most importantly, it is straightforward to derive the deformed behaviour of momenta under Lorentz transformations. Infinitesimally, this is done most easily by computing the constrained vector fields associated to the generators $K^i = M^{i0}$ and $J^i = \epsilon^{ijk}M^{kj}/2$ via (4.42). The result is

$$\begin{aligned}\Omega_{\vec{J}}\Big|_{\Xi} &= \vec{p}\times\frac{\partial}{\partial\vec{p}}\\ \Omega_{\vec{K}}\big|_{\Xi} &= \vec{p}\frac{\partial}{\partial\varepsilon} + \frac{\kappa}{2}(1-\mathrm{e}^{-2\varepsilon/\kappa})\frac{\partial}{\partial\vec{p}} + \frac{1}{2\kappa}\vec{p}^2\frac{\partial}{\partial\vec{p}} - \frac{1}{\kappa}\vec{p}\,\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\,.\end{aligned} \tag{4.54}$$

Again, according to our general theme, the same result is obtained from considering how the vector (4.41) transforms under multiplication with the matrices corresponding to rotations and boosts and afterwards extracting the corresponding expression for the coordinates p_μ.

Now we are ready for the central observation of this section: When comparing (4.54) with the algebra relations (4.30) and (4.31) of the κ-Poincaré algebra, we cannot but realize that the geometric construction above provides us with a realization of the algebra part of $\mathbf{P}_\kappa$. It does so in terms of the particular coordinate functions p_μ on Ξ, which play the role of the translationary abelian ideal T, and the vector fields (4.54), which reflect the undeformed Lorentz subalgebra L (recall that the commutation relations are not changed by restriction). The bracket relations $[L, T]$ can then be replaced simply by applying these vector fields on the coordinates p_μ, i.e. reading off their components in these coordinates. To complete the algebra structure, its unit element becomes the constant function with value equal to 1. Formally, we express this realization as

$$\varrho : \mathbf{P}_\kappa \hookrightarrow (\mathrm{Vec} \times \mathcal{F})(\Xi)\,, \tag{4.55}$$

where $\mathrm{Vec}(\Xi)$ are the vector fields and $\mathcal{F}(\Xi)$ the functions on Ξ, and we take both to be smooth.

The image of the translations lies in the right factor,

$$\varrho(T) \subset \mathcal{F}(\Xi)\,, \tag{4.56}$$

and inverts the embedding map (4.41),

$$\varrho(P_\mu)(\Pi(p)) = \varrho(P_\mu)(\exp(\vec{p}\cdot\vec{X})\exp(\varepsilon T)o) = p_\mu\,. \tag{4.57}$$

An associative (and commutative) product on $\mathcal{F}(\Xi)$ is given by pointwise multiplication.

The image of the Lorentz subalgebra L lies in $\mathrm{Vec}(\Xi)$,

$$\varrho(L) \subset \mathrm{Vec}(\Xi)\,, \tag{4.58}$$

and is given on basis elements by

$$\varrho(M_{mn}) = \Omega_{mn}|_\Xi\ , \tag{4.59}$$

hence exactly by the expressions (4.54) when choosing the flat slicing coordinates. Letting the action of elements $v \in \mathrm{Vec}(\Xi)$ on a function $f \in \mathcal{F}(\Xi)$ be given by

$$v.f = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f \circ \exp(sv)\,, \tag{4.60}$$

the mixed bracket relations are realized on basis elements as

$$\varrho([M_{mn}, P_l]) = \Omega_{mn}.\varrho(P_l)\,. \tag{4.61}$$

Note how the ability to do so relies on the splitting property $[L, T] \subseteq T$ of $\mathbf{P}_\kappa$.

The extension of ϱ to the whole of L, i.e. to powers of $\vec{J}$ and $\vec{K}$, goes via composition,

$$(v'v).f = v'.(v.f)\,, \tag{4.62}$$

which also defines an associative product on Vec(Ξ). This completes to show that ϱ is an algebra homomorphism. (Note that in $\mathbf{P}_\kappa$ there is no associative product defined between the two factors L and T).

Eq. (4.60) indicates that instead of computing the restriction $\Omega|_\Xi$, the Lorentz action on Ξ may also be computed from

$$\Omega_{mn}.f(\Pi(p)) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f(\exp(s\omega_{mn})\Pi(p))\,, \tag{4.63}$$

where exp now means the matrix exponential.

In fact we can adjust the whole realization ϱ according to the isomorphism HOM(3) $\simeq$ Ξ, which makes it computationally more useful. Without notational distinction, we let

$$\varrho(P_\mu)(g(p)) = p_\mu\,, \tag{4.64}$$

where $g(p)$ are now the HOM(3) group elements themselves,

$$g(p) = \exp(\vec{p}\cdot\vec{X})\exp(\varepsilon T)\,, \tag{4.65}$$

which we may imagine in the fundamental representation ρ, so that it makes sense to consider products with Lorentz generators within the associative algebra of 5×5 matrices—cf. Eq. (4.8). We will however omit ρ and ρ_* in the following calculations for the sake of readibility.

Importantly, ϱ needs to be extended to tensor products of $\mathbf{P}_\kappa$, so that we are able to consider the images of coproducts. Without further notational distinction, we do this by defining

$$\begin{aligned}\varrho(P_\mu \otimes P_\nu)(g(p), g(q)) &:= (\varrho(P_\mu) \otimes \varrho(P_\nu))(g(p), g(q)) \\ &\equiv (\varrho(P_\mu)(g(p)))\,(\varrho(P_\nu)(g(q))) = p_\mu q_\nu\,,\end{aligned} \tag{4.66}$$

where the conventional identification by scalar multiplication

$$(\mathcal{F} \otimes \mathcal{F})(M \times M) \simeq \mathcal{F}(M \times M) \tag{4.67}$$

for a generic manifold M was used. Now it is meaningful to demand the following covariance property from the coproduct Δ_ϱ in the chosen realization:

$$\varrho \circ \Delta = \Delta_\varrho \circ \varrho\,. \tag{4.68}$$

If we restrict to the translation sector, the left hand side gives

$$\begin{aligned}&\varrho(\Delta(P_\mu))(g(p), g(q)) \\ &= \begin{pmatrix} \varrho(H)(g(p)) + \varrho(H)(g(q)) \\ \varrho(\vec{P})(g(p)) + (\mathrm{e}^{-\varrho(H)/\kappa}(g(p)))(\varrho(\vec{P})(g(q))) \end{pmatrix} \\ &= \begin{pmatrix} p_0 + q_0 \\ \vec{p} + \mathrm{e}^{-p_0/\kappa}\vec{q} \end{pmatrix},\end{aligned} \tag{4.69}$$

while the right hand side of Eq. (4.68) may be written out as

$$\Delta_\varrho(\varrho(P_\mu))(g(p),g(q)) = \Big((\varrho(P_\mu))^{(1)} \otimes (\varrho(P_\mu))^{(2)}\Big)\,(g(p),g(q)), \tag{4.70}$$

and by comparison with the last line of (4.69) we conclude the expected,

$$\begin{aligned}\Delta_\varrho(\varrho(H)) &= \varrho(H)\otimes 1 + 1\otimes\varrho(H)\\ \Delta_\varrho(\varrho(\vec{P})) &= \varrho(\vec{P})\otimes 1 + \mathrm{e}^{-\varrho(H)/\kappa}\otimes\varrho(\vec{P})\,.\end{aligned} \tag{4.71}$$

This alone is not very satisfying, since we have not gained anything compared to the abstract algebra. In order to justify the suggested geometric picture we should make profit from it here. Additional meaning would be achieved if instead of defining Δ_ϱ in terms of Δ, we could find an intrinsic counterpart in $\Xi \simeq \mathrm{HOM}(3)$. Actually, there is quite a natural candidate, and that is the group product μ. This is because, composed with $\varrho(P_\mu)$, it provides exactly what we need according to Eq. (4.67), namely a map

$$\varrho(P_\mu)\circ\mu : \mathrm{HOM}(3)\times\mathrm{HOM}(3)\to\mathbb{R} \tag{4.72}$$

Using Hadamard's lemma in the third equality (see Eqns. (A.1) and (A.2) in the Appendix), we calculate

$$\begin{aligned} g(q)g(p) &= \exp(\vec{q}\cdot\vec{X})\exp(q_0T)\exp(\vec{p}\cdot\vec{X})\exp(p_0T)\\ &= \exp(\vec{q}\cdot\vec{X})(\mathrm{Ad}_{\exp(q_0T)}\exp(\vec{p}\cdot\vec{X}))\exp(q_0T)\exp(p_0T)\\ &= \exp(\vec{q}\cdot\vec{X})\exp(\exp(\mathrm{ad}_{q_0T})\vec{p}\cdot\vec{X})\exp(q_0T)\exp(p^0T)\\ &= \exp(\vec{q}\cdot\vec{X})\exp(\mathrm{e}^{-q_0}\vec{p}\cdot\vec{X})\exp((q_0+p_0)T)\\ &= \exp((\vec{q}+\mathrm{e}^{-q_0}\vec{p})\cdot\vec{X})\exp((q_0+p_0)T)\\ &= g(q_0+p_0,\,\vec{q}+\mathrm{e}^{-q_0}\vec{p})\,,\end{aligned} \tag{4.73}$$

and find indeed the sought-for correspondence: The coproduct on the translation sector in the κ-Poincaré algebra amounts, in the realization as the particularly chosen coordinate functions on HOM(3), to group multiplication.

This last finding is reassuring in the attempt to also find a geometric procedure corresponding to the coproduct on the Lorentz part L of $\mathbf{P}_\kappa$. We mention in passing that algebraically, it is fixed by knowing $\Delta(P_\mu)$ and requiring

$$\begin{aligned}[\Delta(\vec{J}),\Delta(P_\mu)] &= \Delta([\vec{J},P_\mu])\\ [\Delta(\vec{K}),\Delta(P_\mu)] &= \Delta([\vec{K},P_\mu])\,.\end{aligned} \tag{4.74}$$

But again, we are more interested in its geometric content. And this, as it turns out, consists in generalizing the Leibniz rule for the action of $\varrho(L)$ on $\Xi = \mathrm{HOM}(3)$. We present the two necessary calculations here, which will serve as a motivation for the last section of this chapter.

As primitive elements of $\mathbf{P}_\kappa$, the generators $\vec{J}$ of rotations are a trivial example. An infinitesimal rotation around the axis given by the direction of the vector $\vec{\alpha}$, through an angle $\alpha = \sqrt{\vec{\alpha}^2}$, acts on the product of two group elements $g(p), g(q) \in \mathrm{HOM}(3)$ as

$$\begin{aligned}(1{+}\vec{\alpha}\cdot\vec{J})g(p)g(q) \\ &= [\vec{\alpha}\cdot\vec{J}, g(p)g(q)] + g(p)g(q)(1+\vec{\alpha}\cdot\vec{J}) \\ &= [\vec{\alpha}\cdot\vec{J}, g(p)]g(q) + g(p)[\vec{\alpha}\cdot\vec{J}, g(q)] + g(p)g(q)(1+\vec{\alpha}\cdot\vec{J})\,.\end{aligned} \tag{4.75}$$

On Ξ (i.e. by applying this to the origin o), the ordinary Leibniz rule follows for the variation induced by the commutator,

$$\delta(g(p)g(q)) = (\delta g(p))g(q) + g(p)(\delta g(q)), \qquad \delta = [\vec{\alpha}\cdot\vec{J}, \cdot], \tag{4.76}$$

which we may also express as

$$\mu \circ (\mathrm{ad}_{\vec{J}} \otimes 1 + 1 \otimes \mathrm{ad}_{\vec{J}})(g(p), g(q)) \tag{4.77}$$

Actually, this can even be evaluated from

$$\begin{aligned}&[\vec{\alpha}\cdot\vec{J}, \exp(\vec{p}\cdot\vec{X})] \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathrm{Ad}_{\exp(s\vec{\alpha}\vec{J})} \exp(\vec{p}\cdot\vec{X}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \exp(\exp(s\,\mathrm{ad}_{\vec{\alpha}\vec{J}})\vec{p}\cdot\vec{X}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \exp((\vec{p} + t\vec{\alpha} \times \vec{p}) \cdot \vec{X} + \mathcal{O}(s^2)) \\ &= (\vec{\alpha} \times \vec{p})\cdot\vec{X} \exp(\vec{p}\cdot\vec{X})\,.\end{aligned} \tag{4.78}$$

Note that intrinsically, this corresponds exactly to the infinitesimal flow of $\Omega_{\vec{J}}\big|_{\Xi}$:

$$\begin{aligned}&\varrho(\vec{P})(\exp((\vec{p} + \vec{\alpha} \times \vec{p}) \cdot \vec{X} + \mathcal{O}(\vec{\alpha}^2))) - \varrho(\vec{P})(\exp(\vec{p}\cdot\vec{X})) \\ &= \vec{\alpha} \times \vec{p} + \mathcal{O}(\vec{\alpha}^2) = \vec{\alpha} \cdot \Omega_{\vec{J}}\big|_{\Xi} \vec{p} + \mathcal{O}(\vec{\alpha}^2)\,.\end{aligned} \tag{4.79}$$

This simple behaviour relied on the transformation properties of $\vec{X}$ and T under rotations, which is that of a vector and a scalar:

$$[\vec{J}, \vec{X}] = \vec{X}\,, \qquad [\vec{J}, T] = 0\,. \tag{4.80}$$

For the boosts, however, the commutators are

$$[\vec{K}, \vec{X}] = T - \vec{J}/\kappa, \qquad [\vec{K}, T] = \vec{X} - \vec{K}/\kappa, \tag{4.81}$$

resulting in the need for a more careful analysis of how they act on products of HOM(3). For the purpose of deriving the generalized Leibniz rule it is sufficient to consider infinitesimal transformations, in which case we can represent boosts as

$$h = 1 + \vec{v}\cdot\vec{K}, \tag{4.82}$$

which is equivalent to assuming that the components of $\vec{v}$ are sufficiently small to neglect all terms in which they appear at least quadratically. Proceeding as done with the rotations does not work anymore, since

$$\mathrm{Ad}_h g(p) \neq g(p'). \tag{4.83}$$

Rather, we would need to disentangle the result of $\mathrm{Ad}_h g(p)$, which due to the commutation relations (4.81) contains boost and rotation parts, into L and Ξ. We therefore take another ansatz from the start, and this is

$$hg(p) = g(p')h', \tag{4.84}$$

for some p' and an element $h' \in L$, where our interest lies on the latter, since that will determine how q in the product $g(p)g(q)$ transforms. Now, we already know p' from Eq. (4.54) to be

$$\begin{aligned} p'_\mu &= p_\mu + \Omega_{\vec{v}\vec{K}}\big|_\Xi\, p_\mu \\ &= p_\mu + (\Omega_{\vec{v}\vec{K}}\big|_\Xi)_\mu \\ &= \begin{pmatrix} p_0 + \vec{v}\cdot\vec{p} \\ \vec{p} + \frac{\kappa}{2}(1 - \mathrm{e}^{-2p_0/\kappa}) + \frac{\vec{p}^2}{2\kappa}\vec{v} - \frac{1}{\kappa}\vec{v}\cdot\vec{p}\,\vec{p} \end{pmatrix} \\ &=: \begin{pmatrix} p_0 + \delta p_0 \\ \vec{p} + \delta\vec{p} \end{pmatrix} \end{aligned} \tag{4.85}$$

hence

$$\begin{aligned} h' &= g(p')^{-1} h g(p) \\ &= g(p+\delta p)^{-1} h g(p) \\ &= \left(g(p)^{-1} - \delta p_\mu g(p)^{-1} \left(\frac{\partial}{\partial p_\mu} g(p) \right) g(p)^{-1} \right) (1 + \vec{v}\cdot\vec{K}) g(p) \\ &= 1 + g(p)^{-1}(\vec{v}\cdot\vec{K}) g(p) - \delta p_\mu g(p)^{-1} \frac{\partial}{\partial p_\mu} g(p) + \mathcal{O}(\vec{v}^2), \end{aligned} \tag{4.86}$$

where the formula for the derivative of the inverse of a matrix was used. Note how in this last line the adjoint action appears, as well as the Maurer-Cartan form evaluated along the flow of $\Omega_{\vec{K}}$. We compute the matrix representation of $g(p)$, i.e the expression

$$g(p) = \exp \begin{pmatrix} 0 & \vec{p}^t/\kappa & 0 \\ \vec{p}/\kappa & \mathbf{0} & \vec{p}/\kappa \\ 0 & -\vec{p}^t/\kappa & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & \vec{0}^t & \varepsilon/\kappa \\ \vec{0} & \mathbf{0} & \vec{0} \\ \varepsilon/\kappa & \vec{0}^t & 0 \end{pmatrix}. \tag{4.87}$$

The second factor is simply

$$\exp\begin{pmatrix} 0 & \vec{0}^t & \varepsilon/\kappa \\ \vec{0} & \mathbf{0} & \vec{0} \\ \varepsilon/\kappa & \vec{0}^t & 0 \end{pmatrix} = \begin{pmatrix} \cosh\frac{\varepsilon}{\kappa} & \vec{0}^t & \sinh\frac{\varepsilon}{\kappa} \\ \vec{0} & \mathbb{1} & \vec{0} \\ \sinh\frac{\varepsilon}{\kappa} & \vec{0}^t & \cosh\frac{\varepsilon}{\kappa} \end{pmatrix}, \tag{4.88}$$

while in the first factor we benefit from the nilpotency of $\vec{X}$:

$$\begin{aligned} (\vec{p}\cdot\vec{X})^2 &= \frac{\vec{p}^2}{\kappa^2}\begin{pmatrix} 1 & \vec{0}^t & 1 \\ \vec{0} & \mathbf{0} & \vec{0} \\ 1 & \vec{0}^t & 1 \end{pmatrix} \\ (\vec{p}\cdot\vec{X})^n &= 0 \quad \text{for} \quad n \geq 3, \end{aligned} \tag{4.89}$$

so that

$$\begin{aligned} g(p) &= \begin{pmatrix} 1+\frac{\vec{p}^2}{2\kappa^2} & \frac{\vec{p}^t}{\kappa} & \frac{\vec{p}^2}{2\kappa^2} \\ \frac{\vec{p}}{\kappa} & \mathbb{1} & \frac{\vec{p}}{\kappa} \\ -\frac{\vec{p}^2}{2\kappa^2} & -\frac{\vec{p}^t}{\kappa} & 1-\frac{\vec{p}^2}{2\kappa^2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\varepsilon}{\kappa} & \vec{0}^t & \sinh\frac{\varepsilon}{\kappa} \\ \vec{0} & \mathbb{1} & \vec{0} \\ \sinh\frac{\varepsilon}{\kappa} & \vec{0}^t & \cosh\frac{\varepsilon}{\kappa} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\frac{\varepsilon}{\kappa}+\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} & \frac{\vec{p}^t}{\kappa} & \sinh\frac{\varepsilon}{\kappa}+\frac{\vec{p}^2}{2\kappa^2} \\ \frac{\vec{p}}{\kappa}\mathrm{e}^{\varepsilon/\kappa} & \mathbb{1} & \frac{\vec{p}}{\kappa}\mathrm{e}^{\varepsilon/\kappa} \\ \sinh\frac{\varepsilon}{\kappa}-\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} & -\frac{\vec{p}^t}{\kappa} & \cosh\frac{\varepsilon}{\kappa}-\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} \end{pmatrix}, \end{aligned} \tag{4.90}$$

of which the inverse is

$$g(p)^{-1} = \begin{pmatrix} \cosh\frac{\varepsilon}{\kappa}+\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} & -\frac{\vec{p}^t}{\kappa}\mathrm{e}^{\varepsilon/\kappa} & -\sinh\frac{\varepsilon}{\kappa}+\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} \\ -\frac{\vec{p}}{\kappa} & \mathbb{1} & -\frac{\vec{p}}{\kappa} \\ -\sinh\frac{\varepsilon}{\kappa}-\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} & \frac{\vec{p}^t}{\kappa}\mathrm{e}^{\varepsilon/\kappa} & \cosh\frac{\varepsilon}{\kappa}-\frac{\vec{p}^2}{2\kappa^2}\mathrm{e}^{\varepsilon/\kappa} \end{pmatrix}. \tag{4.91}$$

This facilitates the evaluation of the adjoint action on boosts:

$$\begin{aligned} &g(p)^{-1}(\vec{v}\cdot\vec{K})g(p) = \\ &\left(\left(\kappa\cosh\frac{\varepsilon}{\kappa}+\frac{\vec{p}^2}{2\kappa}\mathrm{e}^{\varepsilon/\kappa}\right)\vec{v}-\frac{\vec{v}\cdot\vec{p}}{\kappa}\mathrm{e}^{\varepsilon/\kappa}\vec{p}\right)\cdot\vec{X}+\vec{v}\cdot\vec{p}\,T+\frac{1}{\kappa}(\vec{p}\times\vec{v})\,\vec{J}. \end{aligned} \tag{4.92}$$

Left over is the Maurer-Cartan form part, i.e. the tangent vector to an integral curve of the boost action on the group, pulled-back to its Lie algebra:

$$\delta p_\mu g(p)^{-1}\frac{\partial}{\partial p_\mu}g(p) = \mathrm{e}^{\varepsilon/\kappa}\delta\vec{p}\cdot\vec{X}+\delta\varepsilon T, \tag{4.93}$$

with δp_μ from Eq. (4.85). Collecting things together, Eq. (4.86) yields to linear order in $\vec{v}$,

$$\begin{aligned} h' &= 1+\mathrm{e}^{-\varepsilon/\kappa}\vec{v}\cdot\vec{K}+\frac{1}{\kappa}(\vec{p}\times\vec{v})\cdot\vec{J} \\ &=: 1+\vec{v}'\cdot\vec{K}'. \end{aligned} \tag{4.94}$$

Comparing $h' - 1 = \vec{v}'\cdot\vec{K}'$ to the right part of $\Delta(\vec{K})$ in $\mathbf{P}_\kappa$, we find exact coincidence. This completes our geometric understanding of the coproduct in the Lorentz sector, too. We can now formulate the generalized Leibniz rule for the boost action on HOM(3)-valued momenta as

$$\delta(g(p)g(q)) = (\delta^{(1)}g(p))g(q) + g(p)(\delta^{(2)}g(q)), \qquad \delta = [\vec{v}\cdot\vec{K}, \cdot], \tag{4.95}$$

where

$$\delta^{(i)} = [\vec{v}\cdot\vec{K}^{(i)}, \cdot], \qquad i = 1, 2\,. \tag{4.96}$$

Eq. (4.95) can be rewritten

$$\begin{aligned}\delta(g(p)g(q)) = g(p + \delta^{(1)}p)g(q) + g(p)g(q + \delta^{(2)}q)\\ - g(p)g(q) - \mathcal{O}(\vec{v}^2)\end{aligned} \tag{4.97}$$

where $\delta^{(1)}p_\mu = \delta p_\mu = (\Omega_{\vec{v}\vec{K}}\big|_p)_\mu$ is known from Eq. (4.85) and

$$\begin{aligned}\delta^{(2)}q_\mu &= (\Omega_{\vec{v}'\vec{K}'}\big|_q)_\mu\\ &= \mathrm{e}^{-\varepsilon/\kappa}\delta q_\mu + \frac{1}{\kappa}(\vec{p}\times\vec{v})\cdot(\Omega_{\vec{J}}\big|_q)_\mu\,,\end{aligned} \tag{4.98}$$

i.e.

$$\begin{aligned}\delta^{(2)}q_0 &= \mathrm{e}^{-\varepsilon/\kappa}\,\vec{v}\cdot\vec{q}\,,\\ \delta^{(2)}\vec{q} &= \mathrm{e}^{-\varepsilon/\kappa}\left(\vec{q} + \frac{\kappa}{2}(1 - \mathrm{e}^{-2q_0/\kappa}) + \frac{\vec{q}^2}{2\kappa}\vec{v} - \frac{\vec{v}\cdot\vec{q}}{\kappa}\vec{q}\right) + \frac{1}{\kappa}(\vec{p}\cdot\vec{q}\,\vec{v} - \vec{v}\cdot\vec{q}\,\vec{p})\,.\end{aligned} \tag{4.99}$$

Here we used that $\Omega_{\vec{J}}$ is purely spatial and that

$$(\vec{p}\times\vec{v})\cdot\left(\vec{q}\times\frac{\partial}{\partial\vec{q}}\right) = (\vec{p}\cdot\vec{q}\,\vec{v} - \vec{v}\cdot\vec{q}\,\vec{p})\cdot\frac{\partial}{\partial\vec{q}}\,. \tag{4.100}$$

The counit is realized by ϱ such that it maps the coordinate functions on $\Xi \simeq \mathrm{HOM}(3)$ to their value at the identity, where they are zero,

$$\epsilon_\varrho : \varrho(P_\mu)(g) \mapsto \varrho(P_\mu)(e) = 0\,, \tag{4.101}$$

and as a consequence, the image under ϵ_ϱ of $\varrho(\vec{J})$ and $\varrho(\vec{K})$, when defined in terms of their action on (any of) the coordinate functions, are also zero:

$$\begin{aligned}\epsilon_\varrho(\varrho(\vec{K})) &:= (\Omega_{\vec{K}}\cdot\varrho(P_\mu))(g) = 0,\\ \epsilon_\varrho(\varrho(\vec{J})) &:= (\Omega_{\vec{J}}\cdot\varrho(P_\mu))(g) = 0,\end{aligned} \tag{4.102}$$

since the induced change in coordinates vanishes at the identity.

Hopf algebra	Geometric realization
$T = U(\mathbb{R}^4)$	Flat slicing coordinates on $\Xi \simeq \mathrm{HOM}(3)$
$L = U(\mathfrak{so}(1,3))$	Lorentz vector fields, restricted to Ξ
Bracket relations in L	Lie bracket of vector fields
Commutative product in T	Pointwise multiplication of functions
Cross relations $[L,T]$	Directional derivative
Unit element $\mathrm{Im}(\eta)$	Zero vector field / unit function
Coproduct Δ on T	Group multiplication
Coproduct Δ on L	Leibniz rule for directional derivative
Counit ϵ on T	Zero map (coordinates of unit element)
Counit ϵ on L	Directional derivative at origin / identity
Antipode S in T	Group inverse
Antipode S in L	Inverse flow of vector fields

Table 4.2.: Overview of the geometric interpretation of the κ-Poncaré algebra.

Finally, the antipode map can be realized as

$$S_\varrho(\varrho(P_\mu))(g) = \varrho(P_\mu)(g^{-1})\,, \tag{4.103}$$

and on $\varrho(L)$ as inverting the flow vector fields.

The geometric realization of the κ-Poincaré Hopf algebra is summarized in Table 4.2.

Mass as a characteristic, invariant property of a fundamental particle appears in the special-relativistic setting as the (square root of the) unique quadratic invariant ('Casimir') that only depends on the translationary momenta,

$$m^2 = \eta^{mn} P_m \otimes P_n = -H \otimes H + \vec{P} \dot{\otimes} \vec{P}. \tag{4.104}$$

It is an element of the universal envelope of the Poincaré Lie algebra, on which the bracket relations are extendable as derivations, i.e.

$$\begin{aligned}[M,&N_1 \otimes N_2 \otimes \cdots \otimes N_n] := \\ &[M, N_1] \otimes N_2 \otimes \cdots \otimes N_n + N_1 \otimes [M, N_2] \otimes \cdots \otimes N_n \\ &+ \cdots + N_1 \otimes N_2 \otimes \cdots \otimes [M, N_n]\end{aligned} \tag{4.105}$$

for all elements M, N_i [3]. In this sense, the invariance property reads

$$\begin{aligned}[\vec{J}, m^2] &= 0\,, \\ [\vec{K}, m^2] &= 0\,.\end{aligned} \tag{4.106}$$

The form of $\mathbf{P}_\kappa$ makes it easy to generalize the mass Casimir from here. Since the boosts act in a deformed, nonlinear way now, the difference will be however that the Casimir

[3] This is possible since, due to the Jacobi identity, this defintion yields zero on the ideal by which the quotient from the tensor algebra is taken.

will cease to be expressable in terms of quadratic expressions in the P_m as well. To be precise, since in the undeformed case one could just as well define any function of m^2 to be the central invariant, the defining feature that distinguishes the deformed case is that it is no longer possible to find an invariant that *scales homogeneously* with the momenta.

Finding Casimir elements for Lie algebras can be non-trivial in general and becomes truly challenging for quantum algebras. In the geometric setting we presented for the κ-Poincaré algebra however, it is the matter of one simple observation. There, we must look out for a function on Ξ which is constant along the flow of the vector fields belonging to $\vec{J}$ and $\vec{K}$. This task would still amount to solving a differential equation. Fortunately, we have already seen that it is also possible to analyse the situation from the extrinsic viewpoint, where the flow of the Lorentz vector fields is represented by matrices. In the way we set things up, Lorentz transformations reside in the upper left 4×4 block,

$$\begin{pmatrix} \Lambda & \vec{0} \\ \vec{0}^t & 1 \end{pmatrix} \in \mathrm{SO}_0(1,3) \subset \mathrm{SO}_0(1,4)\,, \tag{4.107}$$

and hence leave the fourth components $\Pi_4(p)$ of vectors untouched. This means that in principle we can choose any function of $\Pi_4(p)$ and it will represent an invariant element of $\mathbf{P}_\kappa$. Actually, regarding Eq. (4.36), this is equivalent to considering functions of

$$m_\kappa^2 = \Pi_0^2 - \vec{\Pi}^2 = \Pi_4^2 - \kappa^2\,, \tag{4.108}$$

which merely amounts to a renormalization by the deformation parameter κ. In terms of the intrinsic coordinates, this becomes

$$m^2 = \Pi_4^2(p) - \kappa^2 = \left(\kappa \cosh\frac{\varepsilon}{\kappa} - \frac{\vec{p}^2}{2\kappa}\mathrm{e}^{\varepsilon/\kappa}\right)^2 - \kappa^2\,. \tag{4.109}$$

In order to see the relation to the undeformed case, it is good to study the limiting behaviour for small κ. We do this in the so-called non-covariant form, where the energy is understood as a function of the spatial momenta, with parameter m [RRS11]. Note that the mass shell familiar from Special Relativity does not change in form, being the intersection of H_+ with a plane of $\Pi_4 = \text{const.}$, which is indeed a two-sheeted hyperboloid, now only coordinatized very differently.

The deformed dispersion relations one obtains are

$$\begin{aligned} \varepsilon_+(\vec{p}) &= \kappa \log \kappa \frac{\sqrt{m^2+\kappa^2} + \sqrt{m^2+\vec{p}^2}}{|\kappa^2 - \vec{p}^2|}\,, \\ \varepsilon_-(\vec{p}) &= \kappa \log \kappa \frac{\sqrt{m^2+\kappa^2} - \sqrt{m^2+\vec{p}^2}}{\kappa^2 - \vec{p}^2}\,, \end{aligned} \tag{4.110}$$

corresponding to particles and antiparticles, respectively, as in the undeformed case. If the magnitude of the spatial momentum is small compared with κ, differences to the special-relativistic relation are still under control and small as well:

$$\varepsilon_\pm = \pm\sqrt{m^2+\vec{p}^2} + \frac{\vec{p}^2}{2\kappa} + \mathcal{O}(\kappa^{-2})\,. \tag{4.111}$$

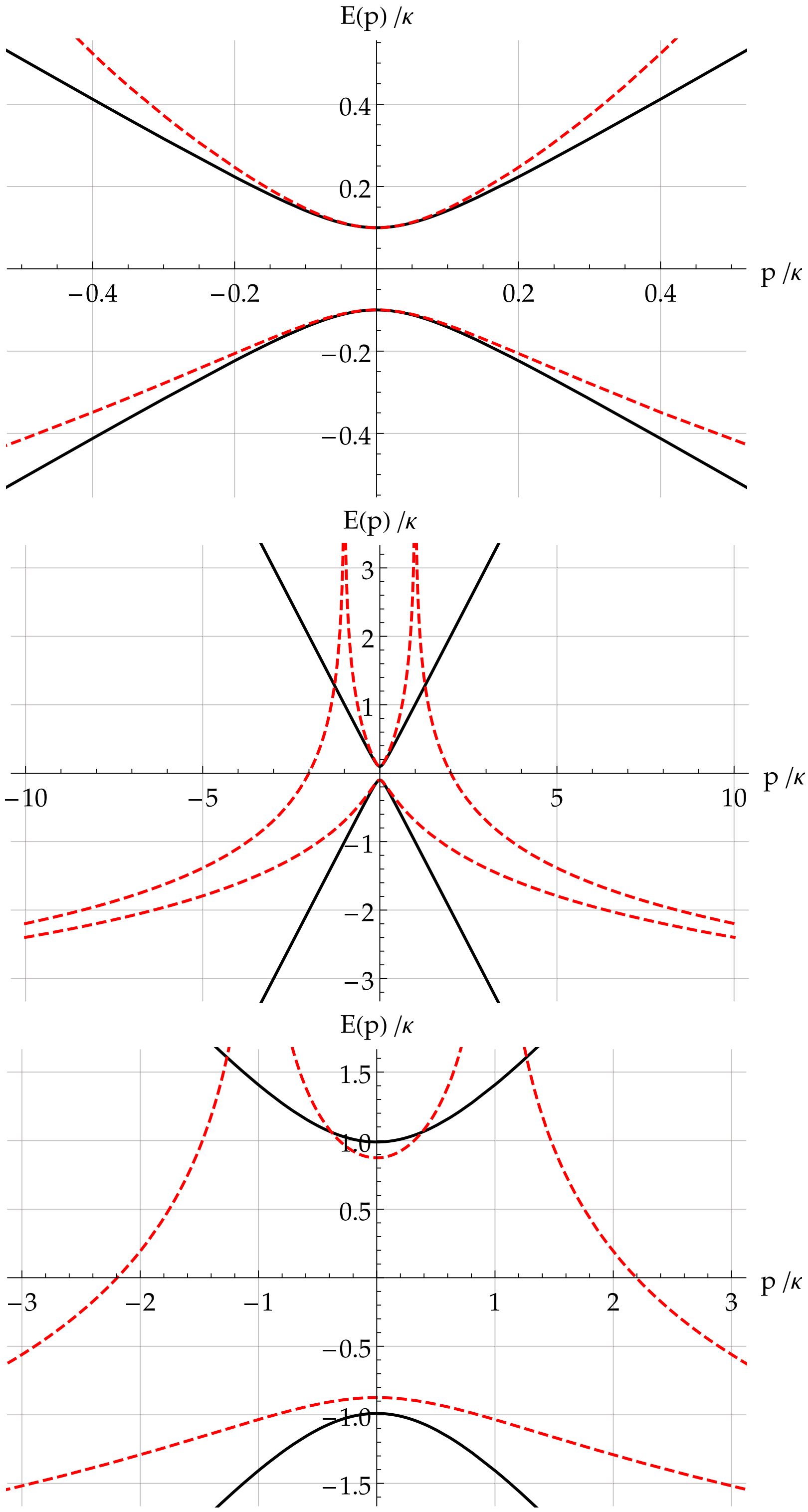


Figure 4.2.: The standard Lorentzian mass shell (black, solid) and its κ-deformation (red, dashed). In the *top* and *middle* graph, the ratio between the mass parameter m (classical rest mass) and the deformation parameter κ amounts to $m/\kappa = 0.1$, while in the *bottom* case it is 0.99.

There are however a few qualitative differences. Most notably, the energy diverges as $|\vec{p}|$ approaches κ. If the modulus grows even further, the band gap between ε_+ and ϵ_- shrinks again. Asymptotically, it closes completely, and both branches describe the same, logarithmically falling behaviour. This is obviously completely different from the undeformed case, where the mass shell approaches the light cone for high spatial momenta.

The last property of the deformation is the renormalization of mass. Being the zero-component of the four-momentum in the rest frame, we can no longer identify it with the Casimir itself. Rather, we need to evaluate the dispersion relation at zero. We find that in $\mathbf{P}_\kappa$ kinematics the bare mass corresponding to the Casimir is being modulated by an inverse sinh function.

All these limiting cases are summarized as

$$\begin{aligned}
&\epsilon_+(\vec{p}) \overset{|\vec{p}|\to\kappa}{\longrightarrow} \infty\,, \\
&\epsilon_\pm(\vec{p}) \overset{\vec{p}\to 0}{\longrightarrow} \kappa \log \frac{\sqrt{\kappa + m^2} \pm m}{\kappa} = \pm\kappa \operatorname{arsinh} \frac{m}{\kappa}\,, \\
&\epsilon_\pm(\vec{p}) \overset{|\vec{p}|\to\infty}{\longrightarrow} -\kappa \log|\vec{p}|\,.
\end{aligned} \tag{4.112}$$

Plots of the deformed dispersion relations displaying these three features are provided in Fig. 4.2. Note that since for $m = 0$, the renormalized mass is zero as well, we can infer the κ-deformed momentum space light cones from Eqs. (4.110) simply by letting $m \to 0$, which gives

$$\lim_{m\to 0} \varepsilon_\pm(\vec{p}) = -\kappa \log|1 - |\vec{p}|/\kappa|\,. \tag{4.113}$$

Being able to extract deformed dispersion relations is particularly valuable for phenomenological studies, since they allow to derive implications for the motion of particles. As will be discussed in a bit more detail in the following Chapter, this can be formalized within the framework of Hamilton Geometry [Bar+15; Bar+17].

At last, let us highlight again the model for spacetime that condenses from the presented deformation of Special Relativity, usually dubbed κ-Minkowski space M_κ. The geometric realization of the κ-Poincaré algebra suggests the identification

$$\mathrm{M}_\kappa \simeq \mathfrak{hom}(3)\,. \tag{4.114}$$

This result was already implicit in [KN04], where the Lie subalgebra closed by X^0 and $\vec{X}$ was recognized as the nilpotent part of the Iwasawa decomposition of $\mathbf{dS}_+$. But since then its identification with the three-dimensional homotheties $\mathfrak{hom}(3)$ seems to have been missing, and with it a physical and geometric interpretation. Indeed, if we consider the vector space structure of Minkowski spacetime (i.e. its tangent spaces), and imagine it to be endowed with an abelian Lie algebra structure, then we may consider $\mathfrak{hom}(3)$ the mildest possible modification of it. In its abstract structure,

$$\mathfrak{hom}(3) \simeq \mathbb{R} \ltimes \mathbb{R}^3\,, \tag{4.115}$$

it leaves space intact, but introduces a split from the time direction, which now acts on space by rescalings. This observation makes it appear reasonable to suspect some relevance of the constructions in this chapter within the programme of Shape Dynamics, which attempts a formulation of gravity based on the idea of spatial conformal invariance [Bar12].

As discussed however, M_κ cannot really be considered a deformation of true Minkowski spacetime M in the proper sense, since that would require the existence of a continuous map from one to the other. This is not present here, since for $\kappa \to \infty$, instead of an affine space, one obtains the vector space $\mathbb{R}^4$ with its abelian Lie algebra structure.

The general framework in which $\mathbf{P}_\kappa$-like deformations of spacetime symmetries are discussed in the literature of quantum gravity phenomenology is called 'Deformed', or, 'Doubly Special Relativity' (cf. [Kow05; IK06] for an overview). These names indicate the deformed action of the Lorentz generators on the translations, and the fact that, apart form c, now a second (therefore 'doubly') invariant scale κ determines the kinematics. What is usually considered to be a major conceptual as well as technical obstacle is known as the 'soccer-ball problem'. In essence, this is the observation that in order to apply Doubly Special Relativity to interaction processes, possibly of composed systems, or even to the dynamics of macroscopic objects like a soccer ball one would need to implement a renormalization procedure for the deformation scale κ. For otherwise, one would either be facing unacceptably large deviations at macroscopic physics—reflected by the divergence of $\varepsilon_+(\vec{p})$ at κ—or completely negligible corrections for elementary particles and hence bring into question the initial motivation for the model. This point continues to be under discussion [Ame+11].

Setting difficulties in associating a physical interpretation to the κ-Poincaré algebra aside, there might still be a mathematical lesson to be learned. The author finds it hard to believe that the possibility of finding a geometric realization of a contraction of a quantum deformation of the de Sitter *group* in terms of de Sitter *space* was a coincidence. This belief has been expressed elsewhere, too [KN04]. What seems peculiar however is the special choice of coordinates, which gave momentum space the structure of a Lie group. It is the author's conviction that this is not necessary, or in other words, that a different choice of coordinates on H_+ would correspond to a quantum algebra that differs from $\mathbf{P}_\kappa$ only by a Hopf algebra homomorphism. The latter, however, needs to be one which leaves the translationary momenta commutative, for otherwise they could no longer be interpreted as coordinate functions on a manifold. We close this section by making the hypothesis articulated here precise.

Hypothesis. *Let G be Lie group with closed subgroup*

$$H \subset G \tag{4.116}$$

such that the Lie algebra $\mathfrak{g}$ of G is simple and splits into the Lie subalgebra $\mathfrak{h}$ of H and a complement $\mathfrak{t}$ in the form of Eq. (3.125). *Denote the Wigner-Inönü contraction of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by*

$$c : \mathfrak{g} \to \mathfrak{h}_\mathfrak{t} := \mathfrak{h} \ltimes \mathbb{R}^d \,, \tag{4.117}$$

with $d = \dim G - \dim H$. Further, there is a quantum deformation $\mathcal{G}$ of $\mathfrak{g}$ according to the Drinfeld-Jimbo scheme. By suitably sending $q \to 1$, $\mathcal{G}$ admits a contraction $\mathcal{G}_c$ parallel to c (i.e. involving a rescaling of the same generators). The following statements await confirmation:

1. *There are Hopf algebra homomorphisms f which bring $\mathcal{G}_c$ into the from of a bicrossproduct*

$$\mathcal{B}(\mathfrak{h}, \mathbb{R}^d) := U(\mathfrak{h}) \triangleright\!\!\blacktriangleleft U(\mathbb{R}^d) .$$

2. *The Hopf algebra maps of $\mathcal{B}(\mathfrak{h}, \mathbb{R}^d)$ can be derived in terms of the infinitesimal action of G on the coset space G/H in the same way as it was done for $\mathbf{P}_\kappa$. In particular, the $U(\mathbb{R}^d)$ part can be represented as coordinate functions on G/H.*

3. *The residual freedom in the choice of f reflects the choice of coordinates on G/H.*

As a first step towards testing this hypothesis it would seem most promising to stay with the example of $\mathbf{P}_\kappa$, but apply a Hopf algebra homomorphism to it that only affects its $U(\mathbb{R}^4)$ part and then look out for coordinates on H_+ that re-install the correspondence. In the context of Planck scale phenomenology, such transformations have been considered, and there is an ongoing debate concerning the physical meaning of such coordinate transformations in momentum space.

A second direction in which tests of the above hypothesis appear to be in reach is given by low-dimensional examples. An immediate choice would be $\mathfrak{g} = \mathfrak{so}(3)$, for which it has already been shown that the procedures of quantum deformation commute as well [SWZ92].

It would also be interesting to start from the geometric side, and investigate under which circumstances group actions on homogeneous spaces can be given a Hopf structure in the sense observed here.

Although serious attempts at testing the hypothesis are beyond the scope of this dissertation, the next section at least expands upon the main conceptual leap that is necessary on the geometric side of the conjectured correspondence, namely the shift from a group to a coset structure for the manifold which is subject to the various transformations.

4.4. Nonlinear realizations

The technique that is going to be discussed here is quite universal. While its first main application to physics was in constructing phenomenological Lagrangians in quantum field theory [CWZ69] and continues to play a role for non-linear sigma models [Per86], it is also of abstract mathematical interest for the analysis of induced representations [Mac51]. Although far more general, and since we are aiming at explicit results, we limit ourselves here to the case of the de Sitter group acting on de Sitter spacetime. In order to facilitate a better comparability with existing results motivated by de Sitter gauge theory of gravity [SW80], at this point we return to the 'spacetime' instead of 'momentum space' interpretation (which largely means nothing but a relabelling of generators and coset coordinates).

As discussed earlier, we parametrize group elements of $\mathrm{SO}_0(1,4)$ as

$$g = \exp(\vec{a}\cdot\vec{P})\exp(bH)h \tag{4.118}$$

with $h \in \mathrm{SO}_0(1,3) \subset \mathrm{SO}_0(1,4)$ an element annihilating $o^M := (0,0,0,0,1)^t$, so that vectors on the de Sitter hyperboloid are given by

$$(g_0 o)^M = X^M = \begin{pmatrix} \sinh t \\ \cosh t \sin x\,\vec{n} \\ \cosh t \cos x \end{pmatrix}, \tag{4.119}$$

with $\vec{x} = x\vec{n}$,

$$g_0 = g|_{(b,\vec{a})=(t,\vec{x})} = \exp(\vec{x}\cdot\vec{P})\exp(tH)h_0\,, \tag{4.120}$$

and elements of H are parametrized as

$$h = \exp(\vec{v}\cdot\vec{K})R \tag{4.121}$$

with R as explained around Eq. (3.118). Multiplying from the left by another group element g, one can make the ansatz

$$g_0' = g g_0 = \exp(\vec{x}'\cdot\vec{P})\exp(t'H)h'\,. \tag{4.122}$$

Usually, two cases are distinguished: these are $\vec{a} = b = 0$ and $h = e$, corresponding to 'homogenous' and 'inhomogeneous' transformations, respectively. In the first case, Eq. (4.122) becomes, according to (3.43),

$$\begin{aligned} &h\exp(\vec{x}\cdot\vec{P})\exp(bH)h_0 \\ &\quad= \exp(\mathrm{Ad}_h(\vec{x}\cdot\vec{P}))\exp(\mathrm{Ad}_h(tH))hh_0 \\ &\quad= \exp\left(\mathrm{Ad}_{\exp(\vec{v}\vec{K})}((\mathbf{R}\vec{x})\cdot\vec{P})\right)\exp\left(\mathrm{Ad}_{\exp(\vec{v}\vec{K})}(tH)\right)hh_0 \\ &\quad= \exp\left(\exp(\mathrm{ad}_{\vec{v}\vec{K}})((\mathbf{R}\vec{x})\cdot\vec{P})\right)\exp\left(\exp(\mathrm{ad}_{\vec{v}\vec{K}})(tH)\right)hh_0 \end{aligned} \tag{4.123}$$

This is all we can say immediately. In order to extract $\vec{x}'$, b' and h' we would now need to evaluate the expressions in the exponents. Note that if we had chosen, as was done in [SW80], to represent group elements as $\exp(\vec{a}\cdot\vec{P} + bH)h$ we would now have been able to infer right away that h' is simply given by hh_0. But since translations in space and time mix under boosts ($c = 1$), the regrouping needed in order to arrive again at the parametrization we have chosen will spawn additional H-terms. A method of circumventing the tedious calculations that would be needed at this point in evaluating (4.123) will now be presented.

It is a lucky fact that the exponential map can be evaluated in a concise way for pseudo-orthogonal groups. In our case,

$$\begin{aligned}\exp(\vec{a}\cdot\vec{P}) &= \mathbb{1} + \sum_{n=1}^{\infty}\frac{(\vec{a}\cdot\vec{P})^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(\vec{a}\cdot\vec{P})^{2n+1}}{(2n+1)!}\\ &= \mathbb{1} + (\cos a - 1)\mathcal{P}_{\vec{a}} + \hat{a}\cdot\vec{P}\sin a\,,\end{aligned} \tag{4.124}$$

$$\exp(bH) = \mathbb{1} + (\cosh b - 1)\mathcal{T} + H\sinh b\,, \tag{4.125}$$

$$\begin{aligned}\exp(\vec{v}\cdot\vec{K}) &= \mathbb{1} + \sum_{n=1}^{\infty}\frac{(\vec{v}\cdot\vec{K})^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(\vec{v}\cdot\vec{K})^{2n+1}}{(2n+1)!}\\ &= \mathbb{1} + (\cosh v - 1)\mathcal{K}_{\vec{v}} + \hat{v}\cdot\vec{K}\sinh v\,,\end{aligned} \tag{4.126}$$

with

$$\mathcal{P}_{\vec{a}} = \begin{pmatrix} 0 & \vec{0}^t & 0\\ \vec{0} & \hat{a}\hat{a}^t & \vec{0}\\ 0 & \vec{0}^t & 1\end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 1 & \vec{0}^t & 0\\ \vec{0} & \mathbf{0} & \vec{0}\\ 0 & \vec{0}^t & 1\end{pmatrix}, \quad \mathcal{K}_{\vec{v}} = \begin{pmatrix} 1 & \vec{0}^t & 0\\ \vec{0} & \hat{v}\hat{v}^t & \vec{0}\\ 0 & \vec{0}^t & 0\end{pmatrix}. \tag{4.127}$$

Also, since we chose coordinates that cover the de Sitter hyperboloid completely, we can express them in terms of embedding coordinates everywhere, i.e., invert the embedding relation (4.119):

$$\begin{cases} t(X^M) & = \operatorname{artanh}\frac{X^0}{\sqrt{\vec{X}^2+(X^4)^2}}\\ x(X^M) & = \arctan\frac{\sqrt{\vec{X}^2}}{X^4}\\ \vartheta(X^M) & = \arctan\frac{\sqrt{(X^1)^2+(X^2)^2}}{X^3}\\ \varphi(X^M) & = \arctan\frac{X^2}{X^1}\end{cases} \tag{4.128}$$

The strategy that we pursue is then the following:

1. Use Eqs. (4.124)–(4.126) to see how the extrinsic coordinates X^M of a point in de Sitter space change under any transformation of $\mathrm{SO}_0(1,4)$.

2. Find the corresponding transformation of intrinsic coordinates by applying the relations (4.128).

3. Insert the result into Eq. (4.122) and calculate h' from it.

4. Optionally, perform a polar decomposition of h' to decompose it into a boost and a rotationary part.

Starting with the case of a pure Lorentz transformation, i.e. $\vec{a} = b = 0$, and with

$$R = \begin{pmatrix} 1 & \vec{0}^t & 0 \\ \vec{0} & \mathbf{R} & \vec{0} \\ 0 & \vec{0}^t & 1 \end{pmatrix} , \tag{4.129}$$

one receives

$$\begin{aligned} (hg_0o)^M &= (RX)^M + (\cosh\nu - 1) \begin{pmatrix} X^0 \\ \hat{\nu}\cdot(\mathbf{R}\vec{X})\,\hat{\nu} \\ 0 \end{pmatrix} + \sinh\nu \begin{pmatrix} \hat{\nu}\cdot(\mathbf{R}\vec{X}) \\ X^0\hat{\nu} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} X^0\cosh\nu + \hat{\nu}\cdot(\mathbf{R}\vec{X})_\| \sinh\nu \\ (\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu \\ X^4 \end{pmatrix} \\ &= \begin{pmatrix} \sinh t' \\ \cosh t' \sin x' \vec{n}' \\ \cosh t' \cos x' \end{pmatrix} = X'^M , \end{aligned} \tag{4.130}$$

where we introduced the symbols $\|$ and $\perp$ for the parallel and perpendicular components with respect to the boost vector $\vec{\nu} = \nu\hat{\nu}$, i.e.

$$\begin{aligned} \vec{X}_\| &= \vec{X}\cdot\hat{\nu}\,\hat{\nu} , \\ \vec{X}_\perp &= \vec{X} - \vec{X}_\| . \end{aligned} \tag{4.131}$$

Note that the transformed vector becomes a lengthy expression in trigonometric and hyperbolic functions when expressed in the original, intrinsic coordinates. The relation is given by ($\vec{x}' = x'\vec{n}' = x'\vec{n}(\vartheta', \varphi')$):

$$\begin{cases} t' &= \operatorname{artanh} \frac{X^0\cosh\nu + \hat{\nu}(\mathbf{R}\vec{X})_\| \sinh\nu}{\sqrt{((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu)^2 + (X^4)^2}} \\ x' &= \arctan \frac{\sqrt{((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu)^2}}{X^4} \\ \vartheta' &= \arctan \frac{\sqrt{\sum_{i=1,2}(\hat{e}_i\cdot((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu))^2}}{\hat{e}_3\cdot((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu))} \\ \varphi' &= \arctan \frac{\hat{e}_2\cdot((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu))}{\hat{e}_1\cdot((\mathbf{R}\vec{X})_\perp + (\mathbf{R}\vec{X})_\| \cosh\nu + X^0\hat{\nu}\sinh\nu))} \end{cases} \tag{4.132}$$

Using these, one obtains an explicit expression for

$$\begin{aligned} h' &= \exp(-t'H)\exp(-\vec{x}'\cdot\vec{P})hg_0 \\ &= \exp(-t'H)\exp(-\vec{x}'\cdot\vec{P})h\exp(\vec{x}\cdot\vec{P})\exp(tH)h_0 . \end{aligned} \tag{4.133}$$

As a Lorentz group element, one could in principle decompose this further into a rotation and a boost part by applying the same procedure to the velocity hyperboloid $SO_0(1,3)/SO_0(1,3)$. We will however not pursue this here. Instead, for illustrative purposes, let us evaluate $(t', \vec{x}')$ and h' in the particularly simple case of a pure boost in the line of sight, i.e. $\mathbf{R} = \mathbb{1}$, $\hat{v} = \vec{n}$. In that case,

$$\begin{cases} t' &= t + v \sin x + \mathcal{O}(v^2)\,, \\ \vec{x}' &= \vec{x} + \vec{v} \cos x \tanh t + \mathcal{O}(v^2)\,. \end{cases} \tag{4.134}$$

Note how the linearized standard formulae for a Lorentz transformation is recovered when considering only small spacetime intervals t and x, and the Galilei limit is achieved by additionally neglecting $\mathcal{O}(\vec{v}^2)$ terms. This is exactly in keeping with the network of relations between the kinematical groups presented earlier in this work. Eq. (4.133) may be written, to linear order in v and in the chosen special situation of a pure and collinear boost,

$$\begin{aligned} h' &= (1 - \delta t H) \exp(-tH)(1 - \delta\vec{x}\cdot\vec{P}) \exp(-\vec{x}\cdot\vec{P}) \times \\ &\qquad \times (1 + \vec{v}\cdot\vec{K}) \exp(\vec{x}\cdot\vec{P}) \exp(tH) h_0 + \mathcal{O}(v^2) \\ &= \Big(1 - \delta t H - \mathrm{Ad}_{\exp(-tH)}(\delta\vec{x}\cdot\vec{P}) + \\ &\qquad + \mathrm{Ad}_{\exp(-tH)\exp(-\vec{x}\vec{P})}(\vec{v}\cdot\vec{K})\Big)\, h_0 + \mathcal{O}(v^2)\,, \end{aligned} \tag{4.135}$$

where $\delta t = t' - t$ and $\delta\vec{x} = \vec{x}' - \vec{x}$.

Let us now turn to the case where the considered transformation is of translationary type, i.e., $h = e$. Hoping to not cause any confusion, we stick to the same notation as in the first case. In particular, now $\perp$ and $\|$ refer to $\vec{a}$ instead of $\mathbf{R}\vec{v}$. In the same way as before, one finds this time

$$\begin{aligned} &(\exp(\vec{a}\cdot\vec{P}) \exp(bH) g_0 o)^M \\ &\quad = \tilde{X}^M + (\cos a - 1) \begin{pmatrix} 0 \\ \hat{a}\cdot\vec{\tilde{X}}\hat{a} \\ \tilde{X}^4 \end{pmatrix} + \sin a \begin{pmatrix} 0 \\ \tilde{X}^4 \hat{a} \\ -\hat{a}\cdot\vec{\tilde{X}} \end{pmatrix} \\ &\quad = \begin{pmatrix} \tilde{X}^0 \\ \vec{\tilde{X}}_\perp + \vec{\tilde{X}}_\| \cos a + \tilde{X}^4 \hat{a} \sin a \\ \tilde{X}^4 \cos a - \vec{\tilde{X}}_\| \cdot \hat{a} \sin a \end{pmatrix} \\ &\quad = \begin{pmatrix} \sinh t' \\ \cosh t' \sin x' \vec{n}' \\ \cosh t' \cos x' \end{pmatrix} = X'^M\,, \end{aligned} \tag{4.136}$$

where

$$\tilde{X}^M = \begin{pmatrix} X^0 \cosh b + X^4 \sinh b \\ \vec{X} \\ X^4 \cosh b + X^0 \sinh b \end{pmatrix}\,. \tag{4.137}$$

Again one infers in straightforward manner,

$$\begin{cases} t' &= \operatorname{artanh} \frac{X^0}{\sqrt{(\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a)^2 + (\tilde{X}^4 \cos a - \vec{X}_\parallel \hat{a} \sin a)^2}}\,, \\ x' &= \arctan \frac{\sqrt{(\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a)^2}}{\tilde{X}^4 \cos a - \vec{X}_\parallel \hat{a} \sin a}\,, \\ \vartheta' &= \arctan \frac{\sqrt{\sum_{i=1,2} (\hat{e}_i \cdot (\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a))^2}}{\hat{e}_3 \cdot (\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a)}\,, \\ \varphi' &= \arctan \frac{\hat{e}_2 \cdot (\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a)}{\hat{e}_1 \cdot (\vec{X}_\perp + \vec{X}_\parallel \cos a + \tilde{X}^4 \hat{a} \sin a)}\,, \end{cases} \tag{4.138}$$

as well as

$$\begin{aligned} h' &= \exp(-t'H) \exp(-\vec{x}' \cdot \vec{P}) \exp(\vec{a} \cdot \vec{P}) \exp(bH) g_0 \\ &= \exp(-t'H) \exp(-\vec{x}' \cdot \vec{P}) \exp(\vec{a} \cdot \vec{P}) \exp(bH) \exp(\vec{x} \cdot \vec{P}) \exp(tH) h_0\,. \end{aligned} \tag{4.139}$$

Once more, for the sake of illustration, consider the collinear case $\hat{a} = \vec{n}$. Additionally linearizing in a and b then gives

$$\begin{cases} t' &= t + b \cos x + \mathcal{O}(a^2,\, b^2,\, ab)\,, \\ \vec{x}' &= \vec{x} + \vec{a} - b\vec{n} \sin x \tanh t + \mathcal{O}(a^2,\, b^2,\, ab)\,, \end{cases} \tag{4.140}$$

and

$$\begin{aligned} h' &= (1 - \delta t H) \exp(-tH)(1 - \delta\vec{x} \cdot \vec{P}) \exp(-\vec{x} \cdot \vec{P})(1 + bH) \times \\ &\qquad \times (1 + \vec{a} \cdot \vec{P}) \exp(\vec{x} \cdot \vec{P}) \exp(tH) h_0 + \mathcal{O}(a^2, b^2, ab) \\ &= \Big(1 - \delta t H - \mathrm{Ad}_{\exp(-tH)}(\delta\vec{x} \cdot \vec{P}) + \\ &\qquad + \mathrm{Ad}_{\exp(-tH)\exp(\vec{x}\vec{P})}(bH + \vec{a} \cdot \vec{P})\Big)\, h_0 + \mathcal{O}(a^2, b^2, ab) \end{aligned} \tag{4.141}$$

As the last two and a half pages of calculations demonstrated, explicitly executing nonlinear realizations of symmetries leads to expressions that quickly grow in length and lose any intuitive meaning. Nevertheless it has hopefully been made clear how one should treat them in order to lose as little of their inherent structure as possible. The structure-preserving character of the technique we propose allows in particular a non-perturbative derivation of the transformation rules for both the coset parameters $(t', \vec{x}')$ and the induced shift in section through the space of cosets h'.

It should be stressed how our results for the particular case of $\mathrm{SO}_0(1,4)$ acting on its associated model spacetime $\mathbf{dS}_+$ generalize.

Firstly, a change to other coordinate systems is straightforward to perform via the equality of embedding coordinates. The transition functions are defined wherever the other coordinate system is defined, since the one we employed is complete. To give an example, in order to change from our coordinates to those used in [SW80] (here denoted

$y^\mu = (s, \vec{y})$, one would employ

$$\begin{aligned}\begin{pmatrix}\sinh t \\ \cosh t \sin x\vec{n} \\ \cosh t \cos x\end{pmatrix} &= \exp(\vec{x}\cdot\vec{P})\exp(tH)o^M = X^M \\ &= \exp(sH + \vec{y}\cdot\vec{P})o^M = \begin{pmatrix}\frac{\sinh\sqrt{s^2-\vec{x}^2}}{\sqrt{s^2-\vec{x}^2}}s \\ \frac{\sinh\sqrt{s^2-\vec{x}^2}}{\sqrt{s^2-\vec{x}^2}}y\vec{n} \\ \cosh\sqrt{s^2-\vec{x}^2}\end{pmatrix} .\end{aligned} \tag{4.142}$$

After evaluating the right hand side in order to obtain X^M as a function of the new coordinates, one must then solve the relations (4.142) for one set of coordinates. In this specific case, with

$$\begin{aligned}&\exp(sH + \vec{y}\cdot\vec{P}) = \\ &\mathbb{1} + \frac{\sinh\sqrt{s^2-\vec{y}^2}}{\sqrt{s^2-\vec{y}^2}}(sH + \vec{y}\cdot\vec{P}) + \left(\cosh\sqrt{s^2-\vec{y}^2} - 1\right)\mathcal{P}_y ,\end{aligned} \tag{4.143}$$

where

$$\mathcal{P}_y = \frac{1}{s^2-\vec{y}^2}\begin{pmatrix} yy^t & 0 \\ 0^t & s^2-\vec{y}^2\end{pmatrix} , \tag{4.144}$$

one finds

$$\begin{cases} t &= \operatorname{arsinh}\left(\frac{\sinh\sqrt{s^2-\vec{y}^2}}{\sqrt{s^2-\vec{y}^2}}s\right) \\ \vec{x} &= \vec{n}\arccos\left(\frac{\cosh\sqrt{s^2-\vec{y}^2}}{\cosh\left(\operatorname{arsinh}\left(\frac{\sinh\sqrt{s^2-\vec{y}^2}}{\sqrt{s^2-\vec{y}^2}}s\right)\right)}\right)\end{cases} . \tag{4.145}$$

Secondly, from the nonlinear realization of de Sitter symmetry in de Sitter spacetime it is possible to derive the nonlinear realization of any of the BLL groups on their respective model spacetimes by linearizing according to the contraction scheme of the kinematical cube. The negative curvature cases are derived from the Anti de Sitter analogue, i.e. the action of $SO_0(2,3)$ on dS_-.

Thirdly, the presented scheme can be expected to work in far more general situations. While the simplicity of the exponential map hinges on the specific properties of pseudo-orthogonal groups, the scheme itself stays applicable to other groups as well. Furthermore, other choices of subgroup, and hence different coset manifolds are possible. For the BLL groups, such different quotients do no longer have a spacetime interpretation, but instead they can be regarded as configuration spaces for the elementary systems ('particles') associated with the considered type of kinematics. In this sense we make contact with the general theory of induced representations. Such are representations of a group G that are derived from those of a subgroup H on a vector space V as automorphisms of a vector space Σ that can be seen as an infinite tensor sum of vector spaces $V_{[g]}$, labelled by representatives of the cosets of G/H, and each isomorphic to V,

$$\Sigma = \bigoplus_{[g]\,\in\, G/H} V_{[g]}\,. \tag{4.146}$$

G acts on this by permutation of the vector spaces $V_{[g_0]} \to V_{[gg_0]}$, and on vectors therein in terms of the H-representation of h', which is determined from $gg_0 = g_0' h'$ exactly as we discussed. In physics, induced representations play a role for the so-called orbit method [Kir04], a mathematical prescription for quantizing a classical system on the basis of its symmetry properties.

In conclusion to this chapter, let us recapitulate the main conceptual positions that have been occupied.

As a starting point we took the assumption that in principle, Hopf algebras, as generalizations of the theory of Lie algebras and Lie groups, qualify as possible mathematical objects in terms of which to describe spacetime symmetries. An outline of the basic definitions showed how this conceptual shift brings about a much greater freedom in the choice of kinematic structure. With the κ-Poincaré algebra however, we concentrated on the most-discussed candidate for a high-energy modification of the symmetries of Special Relativity, supposedly capturing remnants of a theory of quantum gravity. Its singular character in the multitude of Hopf deformations of the Poincaré algebra is that it features a bicrossproduct structure. This was then used to show how, in fact, Lie algebraic notions are completely sufficient for the definition of $\mathbf{P}_\kappa$, once the technique of nonlinear realizations is employed. This observation motivated conjecturing its validity for more general cases. To prepare for its possible verification, the last section then presented a framework to deal with nonlinear realizations in a non-perturbative way. As in the case of the κ-Poincaré algebra, an extrinsic view, in terms of embedding coordinates, was of great merit in dealing with the nonlinear transformation properties encountered.

5. RELATED TOPICS

Everything so far concerned alternative realizations of the Relativity Postulate. This means we investigated the nature of transformations between inertial frames that are conceivable beyond those of Special Relativity. Gravity, in its manifestation as a dynamical spacetime geometry, has only played a minor role. But it is clear that the generalized kinematical concepts that have been discussed also have implications for our understanding of the gravitational interaction. In order to capture these, the rigid model geometries for spacetime that arose from our considerations in Chapter 3 need to be embedded ('localized') in more general geometric structures, similarly to how Special Relativity remains only a local approximation in General Relativity. Also, the occurrence of intrinsically deformed particle dispersion as discussed towards the end of the previous chapter needs an understanding in the presence of dynamical spacetime curvature. In the following we comment on results and recent developments in these directions.

5.1. HAMILTON GEOMETRY AND OBSERVATIONS

As was briefly mentioned earlier, it is of much interest to perform astrophysical tests for possible modifications of the special-relativistic dispersion relation. Conceptually, a hurdle for efforts of this kind is to implement such deformations in a general curved background geometry. Using Hamilton geometry, a framework for this was provided in [Bar+15] with contributions from the author. The main ideas are reviewed in this section.

It is one of the basic assumption of General Relativity that test particles travel on geodesics, i.e. curves $\gamma : \mathbb{R} \to M$ which extremize the length functional

$$L[\gamma, \dot\gamma] = \int_\gamma \sqrt{-g(\dot\gamma(\tau), \dot\gamma(\tau))}\, d\tau\,. \tag{5.1}$$

This means that its variation with respect to γ must vanish for geodesics, and one finds that the corresponding equation of motion in an appropriate (affine) parametrization is the autoparallel equation

$$\nabla_{\dot\gamma}\dot\gamma = 0 \quad \Leftrightarrow \quad \ddot\gamma^\mu + \{{}^\mu{}_{\nu\rho}\}\dot\gamma^\nu\dot\gamma^\rho = 0 \tag{5.2}$$

for the Levi-Civita covariant derivative ∇. It is this Lagrangian point of view in which relativistic point particle motion is phrased most often. A Hamiltonian approach is however equally possible. In that case one does not start from a length functional on the tangent bundle TM over spacetime, but instead with a Hamiltonian H, which is a function on the cotangent bundle T^*M. Identifying the latter with the phase space of test particles, their motion is then determined via Hamilton's equations of motion,

$$\dot{\gamma}^\mu = \left.\frac{\partial H}{\partial p_\mu}\right|_{(\gamma,\pi)} \qquad \text{and} \qquad \dot{\pi}_\mu = -\left.\frac{\partial H}{\partial x^\mu}\right|_{(\gamma,\pi)} . \tag{5.3}$$

Here x^μ are coordinates on M, p_μ are manifold-induced coordinates in the fibres (i.e. cotangent spaces) of T^*M, saying that they transform like the components of a one-form on M, and $\mathbb{R} \to T^*M$, $\tau \mapsto (x, p) = (\gamma, \pi)$ is the particle's path in phase space; in particular π is the four-momentum of the particle. If

$$H(x, p) = g^{\mu\nu}(x) p_\mu p_\nu \tag{5.4}$$

with $g^{\mu\nu}$ being the components of the inverse metric, then Eqs. (5.3) are quickly seen to be identical to Eq. (5.2). In a local inertial frame we can write this *metric Hamiltonian* in terms of the Minkowski metric as

$$H(x, p) = \eta^{mn} \tilde{p}_m \tilde{p}_n = \eta^{\mu\nu} p_\mu p_\nu + \mathcal{O}(x^2)\,, \tag{5.5}$$

which shows how it generalizes the special-relativistic case in accord with the equivalence principle. In order to derive particle dynamics from arbitrary dispersion relations, we continue to interpret them as defining level sets of the Hamiltonian,

$$H(x, p) = \text{const.}\,, \tag{5.6}$$

where in the metric case, the constant on the right is interpreted as the mass of the particle, but now we also allow H to be non-quadratic in the momenta. The κ-Poincaré dispersion relation (4.109) is a particularly simple deformation, since the spacetime dependence of the Hamiltonian vanishes completely for it, expressing the circumstance that it captures a deformation from *special*-relativistic motion. In [Bar+17] it is shown, how it can be adapted to the cosmological setting.

In treating T^*M as a manifold in its own right, it is possible to derive from the dispersion relation alone a notion of parallel transport for momentum dependent tensors on M (so-called d-tensors). This is achieved via the introduction of connection coefficients $N_{\mu\nu}$ which generalize the Christoffel symbols of metric geometry in the sense that they reduce to $N_{\mu\nu} = -q_\rho\{^\rho{}_{\mu\nu}\}$ for a metric Hamiltonian (5.4), but acquire nonlinear momentum dependence in the general case. The autoparallel equation then generalizes to

$$\dot{p}_\mu + N_{\mu\nu}\dot{x}^\nu = 0\,. \tag{5.7}$$

For the Hamiltonian motion of test particles however, it is found that it corresponds to autoparallel motion only in the homogeneous case, in which $H(x, \lambda p) = \lambda^r$ for some real parameter λ and a rational number r (for $r = 2$ this is the metric case). Otherwise a source term appears on the right hand side which can be interpreted as an additional effective force:

$$\dot{\pi}_\mu + N_{\mu\nu}\dot{\gamma}^\nu = -\left.\left(\partial_\mu H + N_{\mu\nu}\frac{\partial H}{\partial p_\nu}\right)\right|_{(\gamma,\pi)} \tag{5.8}$$

For further details, in particular the definition of curvature in this setting, we refer to the original article [Bar+15]. Here we only mention that dispersion relations of the κ-Poincaré type, which are independent of the spacetime coordinates, can be understood as dual to the general-relativistic ones: the presence of curvature is shifted entirely from spacetime to momentum space.

The observational feature implied by curvature in momentum space and the drag force associated with it is called the 'lateshift' effect. This term is chosen in duality to the gravitational redshift, which describes the change in frequency of light signals when travelling through a curved spacetime. A lateshift, in turn, occurs between two particles of already different energy, and describes the difference in travel times for a fixed distance in spacetime. In terms of Hamilton geometry, both effects are treated on an equal footing, which makes it well-suited for the analysis of results from astrophysical observations. Only recently, hopes of being able to measure lateshifts for high-energetic photons have risen again with the observation of a particularly long-lasting gamma ray burst by the Fermi telescope in June 2016 [Wei+17; Zha+16].

5.2. Gauge theory of gravitation and Hořava-Lifshitz gravity

The successful application of gauge symmetries in quantum field theory and in particular their unifying character in the Standard Model of particle physics has since inspired efforts to describe gravity as a gauge theory, too. The immediate choice one is facing then is of course that of the gauge group. The Poincaré group would appear as a canonical choice in regard of the equivalence principle, and it was the one taken by Kibble when he laid the foundations for the subject (improving on earlier work by Utiyama) [Kib61; Uti56]. Since then also other groups have been taken into consideration; most notably the de Sitter groups, for which [SW80] is the first comprehensive treatment. Only recently, they have re-appeared as viable candidates motivated from quantum field theoretic approaches to gravity [CM10; CM13; CM16]. An overview over gauge theories of gravity as a whole is given in [BH12].

It is important to realize that gauge theories of gravity really are generalized theories of gravity. They include General Relativity only upon imposing suitable constraints, as e.g. the vanishing of torsion, which is a natural curvature component already in Poincaré gauge theory. The fundamentally different character compared to Einstein's theory becomes manifest also in the abstract geometrical framework. This changes from Riemannian geometry to Cartan geometry—see [Wis07] for an accessible introduction, or [Sha96] for a comprehensive and formal account of the subject, but also [Tse82] for an emphasis of the geometrical features the (A)dS case shares with the Poincaré one.

When asking how the topics of this dissertation translate to a theory of dynamical spacetime, gauge theories of gravity provide a natural framework.

Naively, it seems reasonable to assume that viable gauge theories for all the BLL groups follow from the existing theory for the de Sitter and Anti de Sitter groups. Indeed it has been shown that they do reduce in a straightforward manner to Poincaré gauge theory, and even the Carroll group has recently been used as gauge group in order to obtain an ultra-relativistic gravity theory with possible applications in the holographic scenario of string theory [Har15]. In order to obtain a (generalization of the) Newtonian theory however, it has been shown that one must not gauge the Galilei group but instead its central extension, the Bargmann group [And+11].

Most notably, Hořava-Lifshitz gravity was found to fit into the geometrical setting of Bargmann gauge theory [HO15], which improved the understanding of some of its structural features. Hořava-Lifshitz gravity had been proposed as a possible ultraviolet completion of General Relativity [Hoř09]. Its regular behaviour relies on the absence of Lorentz symmetry at very small length scales that have not been experimentally probed so far. In this sense it is a continuation of the old idea that Lorentz symmetry is only an approximate symmetry emerging at low energies, or, relatively long distances when compared to the Planck scale [CN83]. Leaving aside the fact that it actually features the centrally extended Galilei group instead of the Galilei group itself as the local symmetry group, from our point of view we can understand Hořava-Lifshitz gravity as a possible dynamical realization of the concept of group contraction.

These remarks show that the 'flat' contractions of $\mathbf{dS}_\pm$, namely **P**, **G** and **C**, have received considerable attention as candidate gauge symmetries in the description of gravitational phenomena—the latter two most recently, and with motivation in high energy physics. In the abstract BLL analysis however, their 'curved' versions $\mathbf{NH}_\pm$, **pE** and **pP** appear just as naturally. In fact they have the potential of filling a gap between the mentioned developments in non-relativistic gauge theory of gravity and its (anti-) de Sitter ancestor. In particular, including our remarks from Sec. 3.3 on the topic, their introduction could clarify the role of curvature in the non-relativistic setting. This has been a source of ambiguity in the mentioned recent works when considering the coupling of the geometry to different sorts of matter.

Our observations concerning the κ-Poincaré algebra may be viewed from the perspective of (A)dS gauge theory, too. While it is difficult to imagine how general Hopf algebraic spacetime symmetries might be localized to fit into the gauge framework for gravitational theories, the geometric nature of $\mathbf{P}_\kappa$ should make it possible. Indeed we clarified that it essentially amounts to a particular choice of coordinates in a de Sitter momentum space. This conviction was substantiated from a phase space point of view in [GG09], although with rather disappointing phenomenological conclusions. What is however missing in that study is the issue of coordinate choices for nonlinear realizations, which is rather central as we saw.

Finally, we wish to mention the fundamental issue of (A)dS gauge theory of gravity that has to be addressed when interpreting the theory as an extension of General Relativity that becomes noticeable at small length scales.

We mentioned earlier already that it reduces to Poincaré gauge theory in the group contraction limit. This is apparent also from its Lagrangian, which apart from a torsional and a topologically invariant contribution is proportional to

$$\frac{1}{L^2}\left(R - \frac{6}{L^2}\right) \tag{5.9}$$

in the positive curvature case, with R being the Ricci scalar here, and where L specifies the degree of non-commutativity between momenta (cf. our definition of $\mathbf{dS}_+$ in Eq. (3.31)). This can be made equal to the Einstein-Hilbert Lagrangian by interpreting L, which is now assumed to be small, to determine the strength of gravity, i.e. if one sets $L^2 = 16\pi G$. One then realizes two things: Firstly, a cosmological constant terms appears naturally in de Sitter gauge theory. This is contrary to General Relativity (and Poincaré gauge theory), where it must be inserted by hand. Secondly however, it is unacceptably large from the point of view of the classical theory. (This is, given that all observations are pointing to a tiny and positive value.) Since however one is in this setting assuming the existence of an underlying quantum theory of gravity anyhow, and one knows that quantum field theory predicts a large gravitational constant, one can maintain the hope that the two terms cancel. This possibility was first articulated in [Tow77].

6. Summary and conclusions

Asking for possible alternative realizations of the Relativity Postulate, we presented, in Sec. 2.1, the theory of Lie algebra cohomology, and the well-known result that the only deformation of the Poincaré Lie algebra **P** is given by the de Sitter Lie algebras $\mathbf{dS}_{\pm}$. In addition to the speed c of causal propagation, their interpretation in terms of kinematical transformations necessitates the introduction of an additional fundamental length constant L, which endows spacetime with curvature. From de Sitter and Anti de Sitter space ($\mathrm{dS}_{\pm}$), i.e. the spacetime models associated with $\mathbf{dS}_{\pm}$, seven limiting geometries were defined in Sec. 3.1. Minkowski space is one of them, distinguished by its flatness (infinite L) and a finite c. When reduced to two dimensions, the model spacetimes fall into the Cayley-Klein classification of possible planar geometries. Their explicit construction as limits from $\mathrm{dS}_{\pm}$ is new to the literature, and yields a new perspective on the work of Cayley and Klein. This is also true for our discussion of coordinates in these spaces, suggesting the complete systems (3.23)-(3.26) (but also see the Appendix A.3, in particular Tab. A.1).

Algebraically, the geometric limiting procedures from $\mathrm{dS}_{\pm}$ are reflected in Wigner-Inönü contractions, the theory of which has been introduced in Sec. 2.2. Given the relation to the Cayley-Klein scheme, we spelled out a second correspondence that had been proposed by Fernández Sanjuan [Fer84], by finding in Sec. 3.2 that the automorphism Lie algebras of the limiting geometries coincide with nine of the eleven kinematical Lie algebras derived by Bacry and Lévy-Leblond [BL68]. Throughout we emphasized the dimensionality of all quantities involved, which supported the physical interpretation. In particular, this made c and L the parameters for the Wigner-Inönü contractions of $\mathbf{dS}_{\pm}$, and in this way clarified the relation between the abstract concept of Wigner-Inönü contractions and their appearance in physical applications.

The discussion has been supported by providing, in some cases for the first time, representations for all the considered kinematical Lie algebras. These were derived from the fundamental representation of $\mathbf{dS}_{\pm}$, and should prove useful in applications. All Lie algebras were additionally characterized in their abstract structure as semidirect sums.

Compared with the original discussion by Bacry and Lévy-Leblond, our focus on a spacetime interpretation of the derived kinematical groups led us also to the infinite curvature ($L \to 0$) limits of de Sitter and Anti de Sitter spacetime. These are singular only in the sense that while the abstract kinematical algebras remain the same, they now assume the role of conformal isometries for the spatial part of the geometry.

The relaxed notions of causality and curvature that are suggested by the discussed, alternative realizations of the Relativity Postulate were made precise in Sec. 3.3. While the techniques employed to do so are standard in differential-geometric analyses of Lie groups, their application to the problem is new, and we hope that it can help in interpretations of models utilizing alternative kinematical groups. In essence, we observed that curvature is determined by the non-commutativity between translationary generators in the kinematical Lie algebra, and that the causal limits $c \to \infty$ (Galilean) and $c \to 0$ (Carrollian) let the metric degenerate to a purely-time and a purely-space metric, respectively, and reversely for the dual metric. While in the Galilean limit the light cones open up completely, they collapse to a line in the Carrollian case. As an example for the far-reaching consequences of these generalized concepts one may consider para-Galilean space, which is one of the two spacetime models not covered by the Cayley-Klein classification. This is because it exhibits curvature even in the absence of a metric. Its existence suggests that, from an abstract relativistic standpoint, causality and curvature really should be regarded as mutually independent.

In Sec. 3.4, we showed that Cayley-Klein kinematics appear more generally also in the context of Bargmann structures in General Relativity, namely in terms of light-like projections and embeddings, as was shown by Duval et al. [Duv+14]. While, apart from the flat cases Gal and Car, the explicit construction had also been performed for $\mathrm{NH}_\pm$ in the literature, we demonstrated that the corresponding Bargmann structures are unique in this case. Additionally, we obtained the Bargmann structures leading to their Carrollian analogues pEuc and pM. As a negative result, we remark that these are not the same as for $\mathrm{NH}_\pm$, which is in contrast to the flat case.

In Chapter 4, we allowed kinematical transformations to be mathematically represented by Hopf algebras instead of Lie groups. A brief overview over the mathematical apparatus was followed by a discussion of the conceptional challenges concerning the interpretation of spacetime and momentum space that arise from it. These include predominantly the introduction of non-standard addition rules for momenta (formalized by the coproduct), and the change of spacetime structure to that of a vector space.

In Sec. 4.3, a particular Hopf deformation of $\mathbf{P}$, the κ-Poincaré algebra, has been analysed in detail, and its geometric nature was reconstructed. We were able to recognize that the corresponding model for momentum space assumes the shape of the Lie group $\mathrm{HOM}(3)$, the three-dimensional homotheties, which form a subgroup of the de Sitter group. The derived model for spacetime, κ-Minkowski space, was identified with its Lie algebra $\mathfrak{hom}(3)$. In this model, spacetime loses the passive role that it has in Special Relativity, but instead must be interpreted as acting in terms of translations in momentum space.

The nature of this geometric realization of the κ-Poincaré algebra led us to a proposal to generalize the construction to other quantum algebras of the same, bicrossproduct type.

In Sec. 4.4 we presented a scheme for the nonperturbative treatment of nonlinear realizations, which we explicitly worked out for the de Sitter case. With the acquired knowledge of how coordinates for the presented kinematical Lie groups relate to each other, this opens up the possibility to study their nonlinear realizations from a single, unified point of view.

Finally, in Chapter 5, we commented on the relevance of alternative kinematics in gravity theory. In particular, we made reference to a new framework for treating modified dispersion relations for particles in the presence of gravity, to whose development the author of this dissertation contributed.

In conclusion, we can say that it seems worthwhile to promote applications of the introduced relatives of special-relativistic symmetries. Just as the Newtonian limit of General Relativity was illuminated by Newton-Cartan geometry, one may envisage similar merit in the use of BLL-type kinematics in other limiting situations—within gravity but also quantum theory. Only some instances were mentioned in the text, touching upon topics that are currently attracting a considerable amount of interest. Among them is the holographic principle in string theory, Hořava-Lifshitz gravity and de Sitter gauge theory. Unrestricted to high-energy physics, the underlying expectation is that it should be practically as well as conceptionally advantageous to treat extremal regimes of a theory in an adapted structural form. A hint in favour of this was our distinction between rapidities, velocities and paces, which reflected the division into three fundamentally different causal structures. What might at first seem like a mere choice of terminology, can on the other hand be taken as a precursor for insights into nonlinear realizations of symmetries. With respect to these, we hope that our emphasis on the role of coset parametrizations might help to avoid some confusion in applications. Their possible relation to Hopf algebras and quantum aspects of gravity remains a fascinating prospect.

„ ... denn der ‚Sinn' ist ja eben jene Einheit des Vielfältigen,
oder doch jene Fähigkeit des Geistes, den Wirrwarr der Welt
als Einheit und Harmonie zu ahnen. "
Hermann Hesse (Über das Glück)

ACKNOWLEDGEMENTS

My deepest gratitude belongs to my supervisor Domenico Giulini, who introduced me to an intriguing topic and raised my awareness for many more. Under his guidance, my time as a PhD student felt like a great privilege.

For an always enjoyable time sharing the office and including many exciting discussions I would like to thank Steffen Aksteiner, Isak Buhl-Mortensen, Michael Fennen and Christian Pfeifer. Michael, I owe you at least many hours which I would have spent reading software documentations without your advice. Christian, I am truly grateful for your inclusive scientific enthusiasm and initiative.

For his encouragement to take up doctoral studies, his continuous motivation along the way, and countless inspiring arguments and ideas I thank Amel Duraković.

I am indepted to Niels Obers for allowing me to spend an exciting time at the Niels Bohr Institute, for introducing me to his subject, and for being interested in my work.

Among all others who also shared their knowledge with me and gave me helpful advice, I would like to explicitly mention Peter Schupp and Sven Herrmann.

To everyone who proof-read the manuscript for this dissertation, thank you very much. Since he has not yet been addressed, and because he made the most amusing comments, Florian Knoop be mentioned in particular.

Lastly, I wish to express my gratitude to my family and friends. For being there. And for their understanding.

Among the institutions which supported me during my time as a PhD student I would like to highlight the DFG-funded Research Training Group *Models of Gravity*, the Center for Applied Space Technology and Microgravity in Bremen, the Niels Bohr Institute in Copenhagen and the Institute for Theoretical Physics in Hannover.

A. Appendix

A.1. Notation

- Usage of indices, when not specified further:

Indices for Cartesian coordinates:	K, L, M, N	$\in$	$\{0,1,2,3,4\}$
	l, m, n	$\in$	$\{0,1,2,3\}$
	i, j, k	$\in$	$\{1,2,3\}$
Indices for arbitrary coordinates:	$\mu, \nu, \rho, ...$	$\in$	$\{0,1,2,3\}$
	i, j, k	$\in$	$\{1,2,3\}$

  Abstract indices taking context-dependent values: $a, b, c, ...$

  Note: Primed indices take values as if unprimed. Repeated indices are summed over (no matter their position, for spatial ones).

- Other shorthand notations:

$$\mathrm{d}x\mathrm{d}y = \frac{1}{2}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x)$$

$$\vec{x} \cdot \vec{y} = \delta_{ij} x^i y^j$$

$[x]$ = physical dimension of x (e.g., time, length, etc.)

$$\vec{x} \overset{\cdot}{\otimes} \vec{y} = \delta_{ij} x^i \otimes y^j$$

$$\vec{x} \overset{\times}{\otimes} \vec{y} = (\epsilon_{1jk} x^j \otimes y^k,\ \epsilon_{2jk} x^j \otimes y^k,\ \epsilon_{3jk} x^j \otimes y^k)$$

  We use the $(\vec{\ }, \cdot)$-notation shown here both for vectors and covectors.

- Further conventions:
  - Structures are denoted by the same letters as the manifolds on which they are defined. This means we would write, for instance, $\mathrm{M} = (M, g)$ for a manifold M with metric g.
  - Anti-symmetrization of indices: $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$.
  - Metric signature: mostly '+'.

A.2. Mathematical Glossary

The following standard mathematical notions were used without further explanation in the text. Brief definitions are given here in alphabetical order. For the cases in which there exist multiple equivalent definitions, not all are given; we refer to the literature for the rest. In this glossary, $\mathfrak{g}$, $\mathfrak{g}'$ denote generic Lie algebras (and at the same time their underlying vector spaces), $[\cdot,\cdot]$, $[\cdot,\cdot]'$ their Lie brackets, A any associative algebra (and its underlying vector space) and $m(\cdot,\cdot)$ the multiplication in A.

In an **abelian** Lie algebra, $[e,e'] = 0\ \forall e,e' \in \mathfrak{g}$. For a Lie algebra **anti-homomorphism** $f : \mathfrak{g} \to \mathfrak{g}'$, $f([e,e']) = -[f(e), f(e')]'$ holds $\forall e,e' \in \mathfrak{g}$. The **centre** Z of $\mathfrak{g}$ is the subset $Z(\mathfrak{g}) = \{c \in \mathfrak{g} \mid [c,e] = 0\ \forall\ e \in \mathfrak{g}\}$. **Conf**$(M)$ for a manifold M with metric g consists of all vector fields Z on M which satisfy $\mathrm{Lie}_Z g = \lambda g$ for some function λ, which is occasionally restricted to be positive. Lie algebra **derivations** $\mathrm{Der}(\mathfrak{g})$ are linear maps $\mathfrak{g} \to \mathfrak{g}$ s.t. $f([e,e']) = [f(e),e'] + [e, f(e')]\ \forall e,e' \in \mathfrak{g}$. Derivations generalize to associative algebras by replacing the Lie bracket by the associative product. For **exact sequences**, see below. An **ideal** $I \subseteq \mathfrak{g}$ satisfies $[\mathfrak{g}, I] \subseteq I$. Similarly, if $I \subseteq A$, it must satisfy $m(A, I) \subseteq I$. The **Killing form** of a Lie algebra is $K(e,e') = \mathrm{tr}(\mathrm{ad}_e \circ \mathrm{ad}_{e'})$. A **nilpotent** operator N is one for which $\exists n \in \mathbb{N}_{>0}$ s.t. its n-fold composition becomes the zero map, $N^n = 0$. A **normal subgroup** $N \subseteq G$ of a group G is one for which $gN = Ng = N\ \forall g \in G$. **ODer**$(\mathfrak{g})$, where O stands for 'outer', are those derivations $f \in \mathrm{Der}(\mathfrak{g})$ for which there is no $e \in \mathfrak{g}$ s.t. $f = \mathrm{ad}_e$. A **semisimple** Lie algebra is a direct sum of simple Lie algebras $\Leftrightarrow$ has, except $\{0\}$, only non-abelian ideals $\Leftrightarrow$ has non-degenerate Killing form. A **simple** Lie algebra has no ideals except $\{0\}$ and itself (and is at least two-dimensional).

Exact sequences:

Exact sequences are sequences of structure-preserving maps $(f_n)_{n\in\mathbb{N}}$ between mathematical objects of equal type which have the property that $\mathrm{im}(f_n) = \ker(f_{n+1})$. The objects are in our case Lie algebras, Lie groups and Hopf algebras, and the structure-preserving maps are the corresponding homomorphisms. Note that while the kernel of a Lie algebra homomorphism comprises all those elements that map to the *zero* element, kernels of Lie group and Hopf algebra homomorphism consist of those elements which are mapped to the corresponding *unit* element.

Short exact sequences consist of only two non-trivial maps. In the Lie algebra case, they have the form $0 \xrightarrow{l} \mathfrak{n} \overset{\iota}{\hookrightarrow} \mathfrak{g} \overset{\pi}{\twoheadrightarrow} \mathfrak{h} \xrightarrow{r} 0$, where, as in the main text, we denote the zero Lie algebra by 0. The hooked arrow indicates that ι is injective, which is necessarily so due to $\ker(\iota) = \mathrm{im}(l) = 0$. Often, one identifies $\mathfrak{n}$ with its copy $\mathrm{im}(\iota) = \iota(\mathfrak{n})$ in $\mathfrak{g}$, i.e considers it as a subalgebra. This is an ideal of $\mathfrak{g}$, which is implied by $\pi([\iota(n), g]) = [(\pi \circ \iota)(n), \pi(g)]_{\mathfrak{h}} = 0\ \ \forall g \in \mathfrak{g},\ n \in \mathfrak{n}$ and $\ker(\pi) = \mathrm{im}(\iota)$. The two-headed arrow indicates the surjectivity of π, which follows from $\mathrm{im}(\pi) = \ker(r) = \mathfrak{h}$. One usually identifies $\mathfrak{h}$ with the quotient $\mathfrak{g}/\mathrm{im}(\iota)$, which, given the previous identification, may be written $\simeq \mathfrak{g}/\mathfrak{n}$. The quotient has Lie algebra relations $[h + \mathrm{im}(\iota), h' + \mathrm{im}(\iota)] = [h,h'] + \mathrm{im}(\iota)$, for $\mathfrak{g} \ni h, h' \notin \mathrm{im}(\iota)$. Hence π can be taken to consist in 'setting $\mathrm{im}(\iota)$ to zero', in which case it is called the 'canonical projection'. Note that the self-evident homomorphisms l and r are usually omitted. With the mentioned identifications $\mathfrak{n} \simeq \mathrm{im}(\iota)$ and $\mathfrak{h} \simeq \mathfrak{g}/\mathfrak{n}$, also ι and π may be left understood. Instead of referring to the whole sequence, one can call $\mathfrak{g}$ itself an *extension*—an *up*ward extension of $\mathfrak{n}$ by $\mathfrak{h}$ or a *down*ward extension of $\mathfrak{h}$ by $\mathfrak{n}$.

These remarks on short exact sequences of Lie algebras generalize in the straightforward way to Lie groups and Hopf algebras, replacing the Lie bracket by the group or Hopf multiplication. For Lie groups, the concept of an ideal becomes that of a normal subgroup. The left and right ends of the sequence become the trivial Lie group (consisting only of the identity element) or, respectively, the trivial Hopf algebra (consisting only of the span of the identity element). Both were denoted by 1 in the text.

Hadamard's Lemma is the following identity for two operators A, B:

$$\mathrm{Ad}_{\mathrm{e}^A} B = \mathrm{e}^{\mathrm{ad}_A} B\,. \tag{A.1}$$

It can be proven by considering the function $f(t) = \mathrm{Ad}_{\mathrm{e}^{tA}} B = \mathrm{e}^{tA} B \mathrm{e}^{-tA}$, and noting that its derivative is given by $f'(t) = [A, f(t)] = \mathrm{ad}_A f(t)$. The right hand side of Eq. (A.1) is then simply the Taylor expansion of $f(t)$ at $t = 0$.

As a consequence of the Hadamard Lemma,

$$\mathrm{Ad}_{\mathrm{e}^A} \mathrm{e}^B = \mathrm{e}^{\mathrm{Ad}_{\mathrm{e}^A} B} = \mathrm{e}^{\mathrm{e}^{\mathrm{ad}_A} B}\,. \tag{A.2}$$

A.3. Coordinate systems

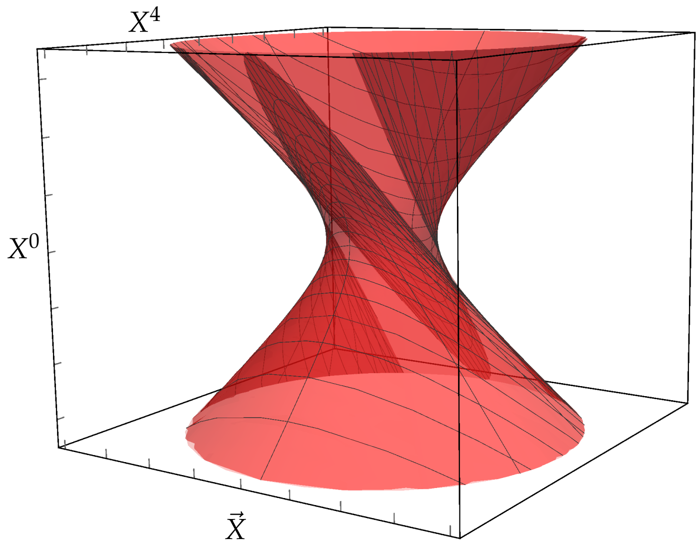


Figure A.1.: The two coordinate patches on de Sitter spacetime that arise from its 'flat slicing' by planes of constant $X^0 + X^4$.

	dS_+	dS_-	NH_+	NH_-	pEuc	pM	$\{M, Gal, Car\}$
(Λ, c)	$(1,1)$	$(-1,1)$	$(1,\infty)$	$(-1,\infty)$	$(1,0)$	$(-1,0)$	$(0,\{1,\infty,0\})$
'Time-Space' ($\mathrm{e}^{\vec{x}\cdot\vec{P}}\mathrm{e}^{tH}o$)	$\begin{pmatrix}\sinh t\\ \cosh t\sin x\,\vec{n}\\ \cosh t\cos x\end{pmatrix}$	$\begin{pmatrix}\sin t\\ \cos t\sinh x\,\vec{n}\\ \cos t\cosh x\end{pmatrix}$	$\begin{pmatrix}\sinh t\\ x\cosh t\,\vec{n}\\ \cosh t\end{pmatrix}$	$\begin{pmatrix}\sin t\\ x\cos t\,\vec{n}\\ \cos t\end{pmatrix}$	$\begin{pmatrix}t\\ \sin x\,\vec{n}\\ \cos x\end{pmatrix}$	$\begin{pmatrix}t\\ \sinh x\,\vec{n}\\ \cosh x\end{pmatrix}$	$\begin{pmatrix}t\\ x\,\vec{n}\\ 1\end{pmatrix}$
Key property	complete	x-independent metric	=	=	adapted	adapted	spherical polar
Metric	$-\mathrm{d}t^2+\cosh^2 t\,\mathrm{d}\Omega_3^2$	$-\mathrm{d}t^2+\cos^2 t\,\mathrm{d}\Sigma_3^2$	$-\mathrm{d}t^2$	$-\mathrm{d}t^2$	$\mathrm{d}\Omega_3^2$	$\mathrm{d}\Sigma_3^2$	$\{-\mathrm{d}t^2+\mathrm{d}\vec{x}^2, -\mathrm{d}t^2, \mathrm{d}\vec{x}^2\}$
Domain	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$
'Space-Time' ($\mathrm{e}^{tH}\mathrm{e}^{\vec{x}\cdot\vec{P}}o$)	$\begin{pmatrix}\cos x\sinh t\\ \sin x\,\vec{n}\\ \cos x\cosh t\end{pmatrix}$	$\begin{pmatrix}\cosh x\sin t\\ \sinh x\,\vec{n}\\ \cosh x\cos t\end{pmatrix}$	$\begin{pmatrix}\sinh t\\ x\,\vec{n}\\ \cosh t\end{pmatrix}$	$\begin{pmatrix}\sin t\\ x\,\vec{n}\\ \cos t\end{pmatrix}$	$\begin{pmatrix}t\cos x\\ \sin x\,\vec{n}\\ \cos x\end{pmatrix}$	$\begin{pmatrix}t\cosh x\\ \sinh x\,\vec{n}\\ \cosh x\end{pmatrix}$	$\begin{pmatrix}t\\ x\,\vec{n}\\ 1\end{pmatrix}$
Key property	static	complete	adapted	adapted	=	=	spherical polar
Metric	$-\cos^2 x\,\mathrm{d}t^2+\mathrm{d}\Omega_3^2$	$-\cosh^2 x\,\mathrm{d}t^2+\mathrm{d}\Sigma_3^2$	$-\mathrm{d}t^2$	$-\mathrm{d}t^2$	$\mathrm{d}\Omega_3^2$	$\mathrm{d}\Sigma_3^2$	$\{-\mathrm{d}t^2+\mathrm{d}\vec{x}^2, -\mathrm{d}t^2, \mathrm{d}\vec{x}^2\}$
Domain	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$
'Stelle-West' ($\mathrm{e}^{tH+\vec{x}\cdot\vec{P}}o$)	$\begin{pmatrix}\frac{t}{\sqrt{t^2-x^2}}\sinh\sqrt{t^2-x^2}\\ \frac{x}{\sqrt{t^2-x^2}}\sinh\sqrt{t^2-x^2}\,\vec{n}\\ \cosh\sqrt{t^2-x^2}\end{pmatrix}$	$\begin{pmatrix}\frac{t}{\sqrt{t^2-x^2}}\sin\sqrt{t^2-x^2}\\ \frac{x}{\sqrt{t^2-x^2}}\sin\sqrt{t^2-x^2}\,\vec{n}\\ \cos\sqrt{t^2-x^2}\end{pmatrix}$	$\begin{pmatrix}\sinh t\\ \frac{x}{t}\sinh t\,\vec{n}\\ \cosh t\end{pmatrix}$	$\begin{pmatrix}\sin t\\ \frac{x}{t}\sin t\,\vec{n}\\ \cos t\end{pmatrix}$	$\begin{pmatrix}\frac{t}{x}\sin x\\ \sin x\,\vec{n}\\ \cos x\end{pmatrix}$	$\begin{pmatrix}\frac{t}{x}\sinh x\\ \sinh x\,\vec{n}\\ \cosh x\end{pmatrix}$	$\begin{pmatrix}t\\ x\,\vec{n}\\ 1\end{pmatrix}$
Key property	algebraically handy	algebraically handy	=	=	=	=	spherical polar
Metric	see Eq. (A.3)	see Eq. (A.4)	$-\mathrm{d}t^2$	$-\mathrm{d}t^2$	$\mathrm{d}\Omega_3^2$	$\mathrm{d}\Sigma_3^2$	$\{-\mathrm{d}t^2+\mathrm{d}\vec{x}^2, -\mathrm{d}t^2, \mathrm{d}\vec{x}^2\}$
Domain	$0\le\lvert x\rvert\le\lvert t\rvert\le\infty$	$0\le\lvert x\rvert\le\lvert t\rvert\le 2\pi$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$
'Null' ($\mathrm{e}^{\frac{1}{2}((u+v)H+(u-v)\vec{n}\cdot\vec{P})}o$)	$\begin{pmatrix}\frac{u+v}{2\sqrt{uv}}\sinh\sqrt{uv}\\ \frac{u-v}{2\sqrt{uv}}\sinh\sqrt{uv}\,\vec{n}\\ \cosh\sqrt{uv}\end{pmatrix}$	$\begin{pmatrix}\frac{u+v}{2\sqrt{uv}}\sin\sqrt{uv}\\ \frac{u-v}{2\sqrt{uv}}\sin\sqrt{uv}\,\vec{n}\\ \cos\sqrt{uv}\end{pmatrix}$	/	/	/	/	/
Key property	conformal flatness manifest	conformal flatness manifest	/	/	/	/	/
Metric	see Eq. (A.5)	see Eq. (A.6)	$-\mathrm{d}t^2$	$-\mathrm{d}t^2$	$\mathrm{d}\Omega_3^2$	$\mathrm{d}\Sigma_3^2$	$\{-\mathrm{d}t^2+\mathrm{d}\vec{x}^2, -\mathrm{d}t^2, \mathrm{d}\vec{x}^2\}$
Domain	$0\le uv\le\infty$	$0\le u,v\le 2\pi$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in[0,2\pi)$	$x\in[0,2\pi),\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$	$x\in\mathbb{R},\ t\in\mathbb{R}$

Table A.1.: Different Coordinate systems on Cayley-Klein manifolds. $\mathrm{d}\Omega_3^2 = \mathrm{d}x^2 + \sin^2 x\,\mathrm{d}\Omega_2^2$, $\mathrm{d}\Sigma_3^2 = \mathrm{d}x^2 + \sinh^2 x\,\mathrm{d}\Omega_2^2$, $\mathrm{d}\vec{x}^2 = \mathrm{d}x^2 + x^2\,\mathrm{d}\Omega_2^2$, $\mathrm{d}\Omega_2^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$, $\vec{n} = \vec{x}/x = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)^t$. '=' means that there is nothing special except the property inherited from the non-contracted case. The metricless Para-Galilean and Static spacetimes are missing. Since they are described by the same subsets of $\mathbb{R}^5$ as all the geometries of the rightmost column, they admit the same coordinatizations. Λ is the (normalized) curvature constant.

Metrics of de Sitter and Anti de Sitter spacetime $\mathrm{dS}_\pm$ in the coordinates of the third and fourth row of Tab. A.1:

- Metric of $\mathrm{dS}_\pm$ in 'Stelle-West' coordinates:

$$\begin{aligned} g_+ = \frac{1}{2(t^2-x^2)^2}\Big[&\left(-2t^4+2t^2x^2-x^2+x^2\cosh(2\sqrt{t^2-x^2})\right)\mathrm{d}t^2 \\ &+2tx\left(1+2t^2-2x^2-\cosh(2\sqrt{t^2-x^2})\right)\mathrm{d}t\mathrm{d}x \\ &+\left(2x^4-t^2-2t^2x^2+t^2\cosh(2\sqrt{t^2-x^2})\right)\mathrm{d}x^2\Big] \\ &+\frac{x^2\sinh^2\sqrt{t^2-x^2}}{t^2-x^2}\mathrm{d}\Omega_3^2 \end{aligned} \tag{A.3}$$

$$\begin{aligned} g_- = \frac{1}{2(t^2-x^2)^2}\Big[&\left(x^2-(2t^4-2t^2x^2+x^2)\cos(2\sqrt{t^2-x^2})\right)\mathrm{d}t^2 \\ &+\left(tx\left((2t^2-2x^2+1)\cos(2\sqrt{t^2-x^2})-1\right)\right)\mathrm{d}t\mathrm{d}x \\ &+\left(t^2-(t^2(2x^2+1)-2x^4)\cos(2\sqrt{t^2-x^2})\right)\mathrm{d}x^2\Big] \\ &+\frac{x^2\sin^2\sqrt{t^2-x^2}}{t^2-x^2}\mathrm{d}\Omega_3^2 \end{aligned} \tag{A.4}$$

- Metric of $\mathrm{dS}_\pm$ in 'Null' coordinates, which follow from 'Stelle-West' coordinates via $u=t+x,\ v=t-x$:

$$g_+ = \frac{\sinh^2\sqrt{uv}}{uv}\left(-\frac{1}{2}2\mathrm{d}u\mathrm{d}v+\frac{(u-v)^2}{4}\mathrm{d}\Omega_3^2\right) = \frac{\sinh^2\sqrt{uv}}{uv}\eta \tag{A.5}$$

$$g_- = \frac{\sin^2\sqrt{uv}}{uv}\left(-\frac{1}{2}2\mathrm{d}u\mathrm{d}v+\frac{(u-v)^2}{4}\mathrm{d}\Omega_3^2\right) = \frac{\sin^2\sqrt{uv}}{uv}\eta \tag{A.6}$$

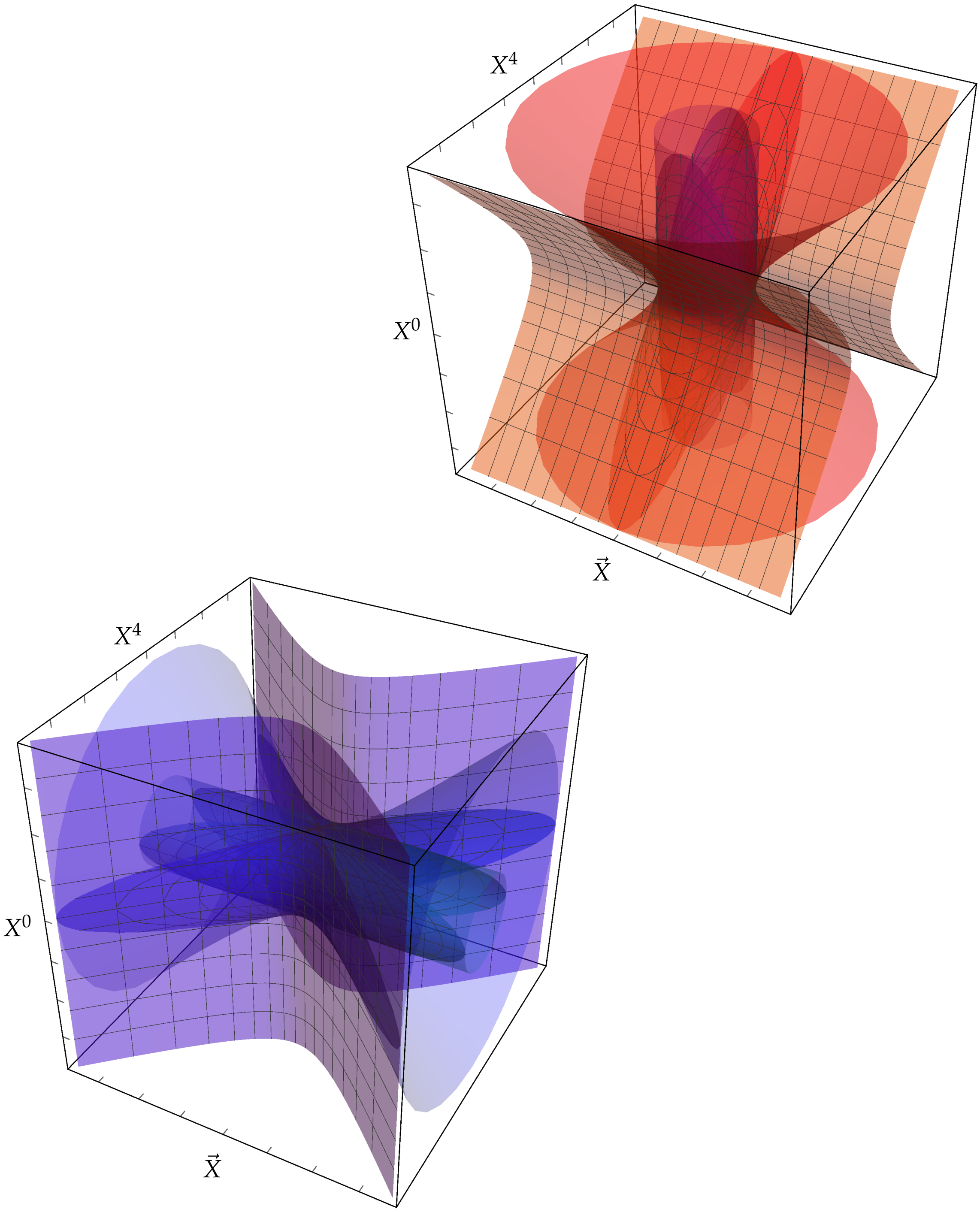


Figure A.2.: *Top:* Static coordinates on de Sitter spacetime and its Galilean and Carrollian descendants. *Bottom:* Analogue in the negative curvature case. Cf. first two rows of Tab. A.1.

h

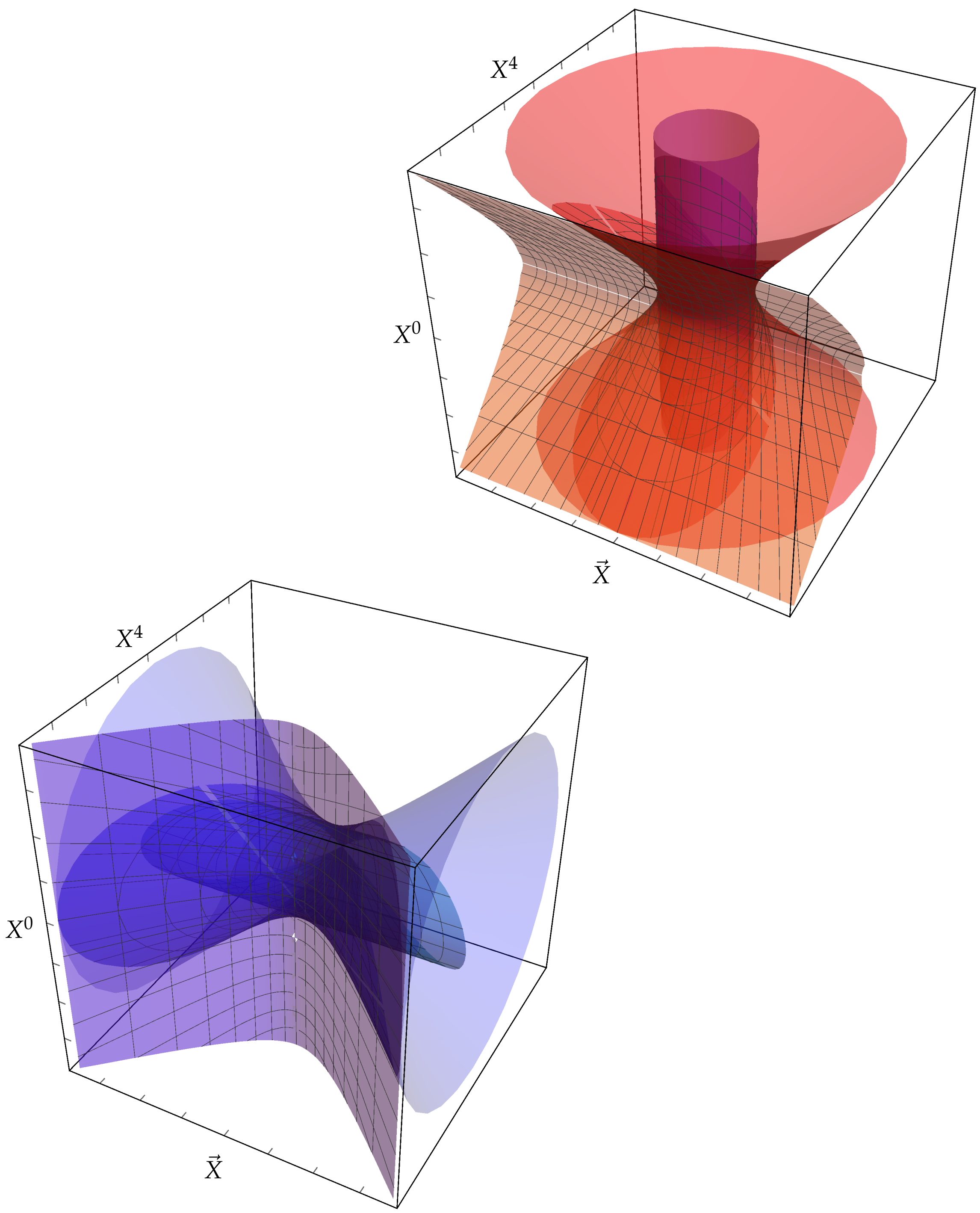


Figure A.3.: Conformally flat coordinates on de Sitter (*top*) and Anti de Sitter (*bottom*) spacetime, as they were used in [SW80], as well as their Galilean and Carrollian descendants. Cf. third row in Tab. A.1.

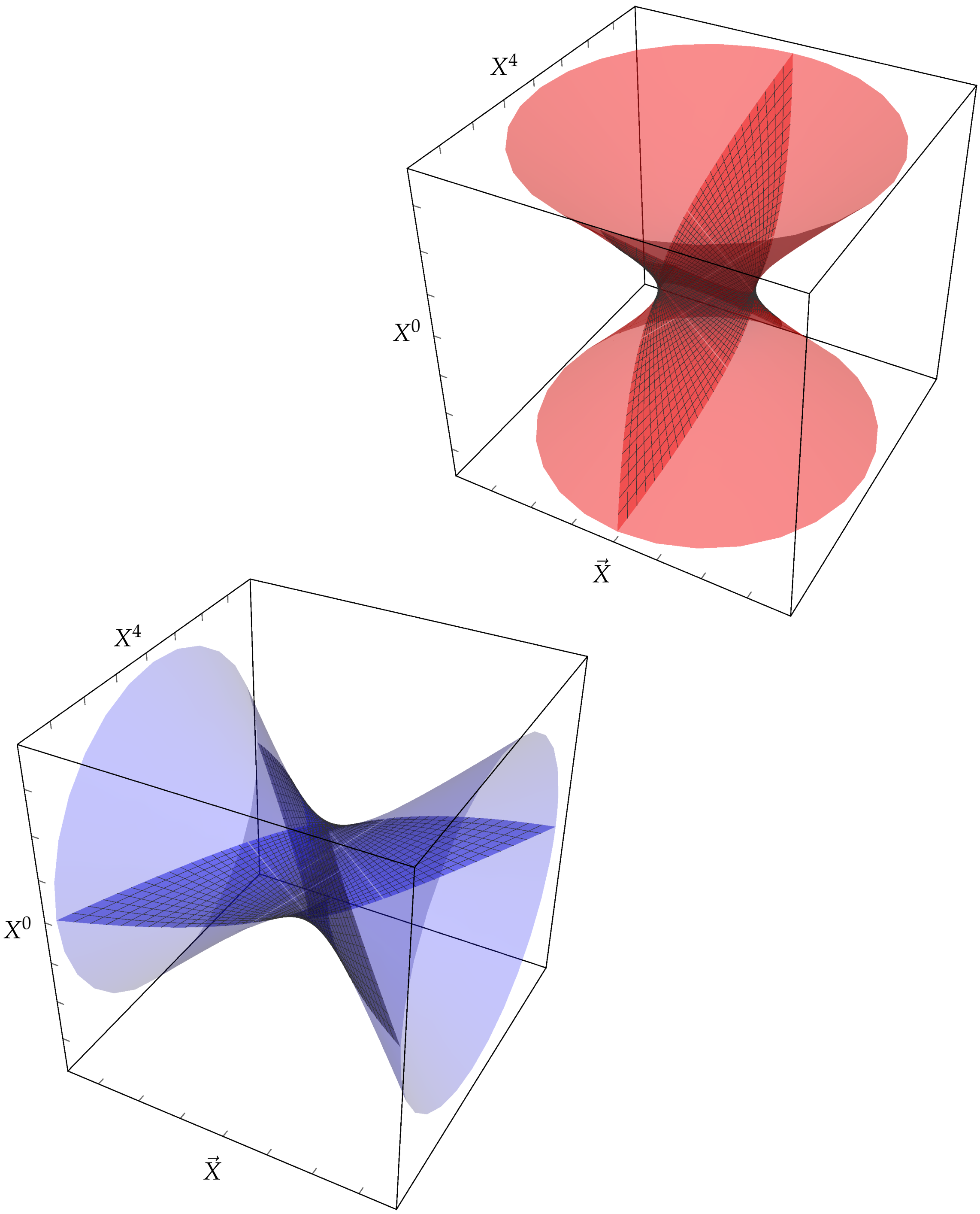


Figure A.4.: Null-adapted, conformally flat coordinates on de Sitter (*top*) and Anti de Sitter (*bottom*) spacetime. Cf. fourth row in Tab. A.1.

BIBLIOGRAPHY

[Ame+11] G. Amelino-Camelia et al. 'Relative locality and the soccer ball problem'. In: *Phys. Rev. D* 84 (8 Oct. 2011), p. 087702. DOI: 10.1103/PhysRevD.84.087702. URL: http://link.aps.org/doi/10.1103/PhysRevD.84.087702.

[Ame+98] G. Amelino-Camelia et al. 'Tests of quantum gravity from observations of γ-ray bursts'. In: *Nature* 393.6687 (June 1998), pp. 763–765. URL: http://dx.doi.org/10.1038/31647.

[And+11] R. Andringa et al. 'Newtonian gravity and the Bargmann algebra'. In: *Classical and Quantum Gravity* 28.10 (2011), p. 105011. URL: http://stacks.iop.org/0264-9381/28/i=10/a=105011.

[Bac93a] H. Bacry. 'Which deformations of the Poincare group?' In: *Journal of Physics A: Mathematical and General* 26.20 (1993), p. 5413. URL: http://stacks.iop.org/0305-4470/26/i=20/a=024.

[Bac93b] H. Bacry. 'Reflections on the evolution of physical theories'. In: *International Journal of Theoretical Physics* 32.8 (1993), pp. 1281–1292. ISSN: 1572-9575. DOI: 10.1007/BF00675195. URL: http://dx.doi.org/10.1007/BF00675195.

[Bar+15] L. Barcaroli et al. 'Hamilton geometry: Phase space geometry from modified dispersion relations'. In: *Phys. Rev. D* 92 (8 Oct. 2015), p. 084053. DOI: 10.1103/PhysRevD.92.084053. URL: http://link.aps.org/doi/10.1103/PhysRevD.92.084053.

[Bar+17] L. Barcaroli et al. 'Planck-scale-modified dispersion relations in homogeneous and isotropic spacetimes'. In: *Phys. Rev. D* 95 (2 Jan. 2017), p. 024036. DOI: 10.1103/PhysRevD.95.024036. URL: http://link.aps.org/doi/10.1103/PhysRevD.95.024036.

[Bar12] J. Barbour. 'Shape Dynamics. An Introduction'. In: *Quantum Field Theory and Gravity: Conceptual and Mathematical Advances in the Search for a Unified Framework*. Ed. by F. Finster et al. Basel: Springer Basel, 2012, pp. 257–297. ISBN: 978-3-0348-0043-3. DOI: 10.1007/978-3-0348-0043-3_13. URL: http://dx.doi.org/10.1007/978-3-0348-0043-3_13.

[BBM62] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner. 'Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems'. In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 269.1336 (1962), pp. 21–52. ISSN: 00804630. URL: http://www.jstor.org/stable/2414436.

[BFP02] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos. 'Penrose limits, supergravity and brane dynamics'. In: *Classical and Quantum Gravity* 19.18 (2002), p. 4753. URL: `http://stacks.iop.org/0264-9381/19/i=18/a=310`.

[BG69] V. Berzi and V. Gorini. 'Reciprocity Principle and the Lorentz Transformations'. In: *Journal of Mathematical Physics* 10.8 (1969).

[BH12] M. Blagojevic and F. W. Hehl. 'Gauge Theories of Gravitation'. In: (2012). arXiv: `1210.3775 [gr-qc]`.

[BL68] H. Bacry and J.-M. Lévy-Leblond. 'Possible Kinematics'. In: *Journal of Mathematical Physics* 9.10 (1968), pp. 1605–1614.

[Bla11] M. Blau. *Plane Waves and Penrose Limits*. 2011. URL: `http://www.blau.itp.unibe.ch/Lecturenotes.html`.

[Bor38] M. Born. 'A suggestion for unifying quantum theory and relativity'. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. Vol. 165. 921. The Royal Society. 1938, pp. 291–303.

[Bri25] H. W. Brinkmann. 'Einstein spaces which are mapped conformally on each other'. In: *Math. Ann.* 94 (1925), pp. 119–145. DOI: `10.1007/BF01208647`.

[Cay59] A. Cayley. 'A Sixth Memoir upon Quantics'. In: *Philosophical Transactions of the Royal Society of London* 149 (1859), pp. 61–90. DOI: `10.1098/rstl.1859.0004`. eprint: `http://rstl.royalsocietypublishing.org/content/149/61.full.pdf+html`. URL: `http://rstl.royalsocietypublishing.org/content/149/61.short`.

[CG06] A. G. Cohen and S. L. Glashow. 'Very Special Relativity'. In: *Physical Review Letters* (2006).

[Cla48] S. E. Claude Chevalley. 'Cohomology Theory of Lie Groups and Lie Algebras'. In: *Transactions of the American Mathematical Society* 63.1 (1948), pp. 85–124. ISSN: 00029947. URL: `http://www.jstor.org/stable/1990637`.

[CM10] A. H. Chamseddine and V. Mukhanov. 'Gravity with de Sitter and unitary tangent groups'. In: *Journal of High Energy Physics* 2010.3 (2010), p. 33. ISSN: 1029-8479. DOI: `10.1007/JHEP03(2010)033`. URL: `http://dx.doi.org/10.1007/JHEP03(2010)033`.

[CM13] A. H. Chamseddine and V. Mukhanov. 'Who ordered the anti-de Sitter tangent group?' In: *Journal of High Energy Physics* 2013.11 (2013), p. 95. ISSN: 1029-8479. DOI: `10.1007/JHEP11(2013)095`. URL: `http://dx.doi.org/10.1007/JHEP11(2013)095`.

[CM16] A. H. Chamseddine and V. Mukhanov. 'On unification of gravity and gauge interactions'. In: *Journal of High Energy Physics* 2016.3 (2016), p. 20. ISSN: 1029-8479. DOI: `10.1007/JHEP03(2016)020`. URL: `http://dx.doi.org/10.1007/JHEP03(2016)020`.

[CN83] S. Chadha and H. Nielsen. 'Lorentz invariance as a low energy phenomenon'. In: *Nuclear Physics B* 217.1 (1983), pp. 125–144. ISSN: 0550-3213. DOI: 10.1016/0550-3213(83)90081-0. URL: http://www.sciencedirect.com/science/article/pii/0550321383900810.

[Con+85] J. H. Conway et al. *Atlas of finite groups*. Oxford University Press, 1985.

[CWZ69] S. Coleman, J. Wess and B. Zumino. 'Structure of Phenomenological Lagrangians. I'. In: *Phys. Rev.* 177 (5 Jan. 1969), pp. 2239–2247. DOI: 10.1103/PhysRev.177.2239. URL: http://link.aps.org/doi/10.1103/PhysRev.177.2239.

[Dau98] G. Dautcourt. 'On the ultrarelativistic limit of general relativity'. In: *Acta Phys. Polon. B* 29.4 (1998).

[DGH14a] C. Duval, G. W. Gibbons and P. A. Horvathy. 'Conformal Carroll groups'. In: *Journal of Physics A: Mathematical and Theoretical* 47.33 (2014), p. 335204. URL: http://stacks.iop.org/1751-8121/47/i=33/a=335204.

[DGH14b] C. Duval, G. W. Gibbons and P. A. Horvathy. 'Conformal Carroll groups and BMS symmetry'. In: *Classical and Quantum Gravity* 31.9 (2014), p. 092001. URL: http://stacks.iop.org/0264-9381/31/i=9/a=092001.

[DGH91] C. Duval, G. Gibbons and P. Horváthy. 'Celestial mechanics, conformal structures, and gravitational waves'. In: *Phys. Rev. D* 43 (12 June 1991), pp. 3907–3922. DOI: 10.1103/PhysRevD.43.3907. URL: http://link.aps.org/doi/10.1103/PhysRevD.43.3907.

[Dri87] V. G. Drinfeld. 'Quantum groups'. In: AMS. Rhode Island, 1987, pp. 798–820.

[Duv+14] C. Duval et al. 'Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time'. In: *Classical and Quantum Gravity* 31.8 (2014), p. 085016. URL: http://stacks.iop.org/0264-9381/31/i=8/a=085016.

[Duv+85] C. Duval et al. 'Bargmann structures and Newton-Cartan theory'. In: *Physical Review D* 31.8 (1985).

[Dys72] F. J. Dyson. 'Missed opportunities'. In: *Bull. Amer. Math. Soc.* 78.5 (Sept. 1972), pp. 635–652. URL: http://projecteuclid.org/euclid.bams/1183533964.

[Ehl97] J. Ehlers. 'Examples of Newtonian limits of relativistic spacetimes'. In: *Classical and Quantum Gravity* 14.1A (1997), A119. URL: http://stacks.iop.org/0264-9381/14/i=1A/a=010.

[EL06] J. Ehlers and C. Lämmerzahl, eds. *Special Relativity: Will it Survive the Next 101 Years?* Vol. 702. Lecture Notes in Physics. 2006.

[Fer84] M. Fernández Sanjuan. 'Group contraction and the nine Cayley-Klein geometries'. English. In: *International Journal of Theoretical Physics* 23.1 (1984), pp. 1–14. ISSN: 0020-7748. DOI: 10.1007/BF02080668. URL: http://dx.doi.org/10.1007/BF02080668.

[GD53] G. Galilei and S. Drake. *Dialogue concerning the two chief world systems, Ptolemaic and Copernican*. Random House Digital, Inc., 1953.

[Ger64] M. Gerstenhaber. 'On the Deformation of Rings and Algebras'. In: *Annals of Mathematics* 79.1 (1964), pp. 59–103. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970484.

[GG09] G. W. Gibbons and S. Gielen. 'Deformed General Relativity and Torsion'. In: *Class. Quant. Grav.* 26 (2009), p. 135005. DOI: 10.1088/0264-9381/26/13/135005. arXiv: gr-qc/0902.2001 [gr-qc].

[Giu10] D. Giulini. 'The Rich Structure of Minkowski Space'. In: *Minkowski Spacetime: A Hundred Years Later*. Ed. by V. Petkov. Vol. 165. Fundamental Theories of Physics. Dordrecht: Springer, 2010, pp. 88–132.

[Giu15] D. Giulini. 'Energy-Momentum Tensors and Motion in Special Relativity'. In: *Equations of Motion in Relativistic Gravity*. Ed. by D. Pützfeld, C. Lämmerzahl and B. Schutz. Vol. 179. Fundamental Theories of Physics. Springer, 2015.

[GP03] G. W. Gibbons and C. E. Patricot. 'Newton-Hooke spacetimes, Hpp waves and the cosmological constant'. In: *Classical and Quantum Gravity* (2003).

[Hal] G. S. Hall. *Symmetries and Curvature Structure in General Relativity*.

[Hal15] B. C. Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*. Vol. 222. Springer, 2015.

[Har15] J. Hartong. 'Gauging the Carroll algebra and ultra-relativistic gravity'. In: *Journal of High Energy Physics* 2015.8 (2015), p. 69. ISSN: 1029-8479. DOI: 10.1007/JHEP08(2015)069. URL: http://dx.doi.org/10.1007/JHEP08(2015)069.

[HO15] J. Hartong and N. A. Obers. 'Hořava-Lifshitz gravity from dynamical Newton-Cartan geometry'. In: *Journal of High Energy Physics* 2015.7 (2015), p. 155. ISSN: 1029-8479. DOI: 10.1007/JHEP07(2015)155. URL: http://dx.doi.org/10.1007/JHEP07(2015)155.

[Hoř09] P. Hořava. 'Quantum gravity at a Lifshitz point'. In: *Phys. Rev. D* 79 (8 Apr. 2009), p. 084008. DOI: 10.1103/PhysRevD.79.084008. URL: http://link.aps.org/doi/10.1103/PhysRevD.79.084008.

[HPS16] S. W. Hawking, M. J. Perry and A. Strominger. 'Soft Hair on Black Holes'. In: *Phys. Rev. Lett.* 116 (23 June 2016), p. 231301. DOI: 10.1103/PhysRevLett.116.231301. URL: http://link.aps.org/doi/10.1103/PhysRevLett.116.231301.

[IK06] K. Imiłkowska and J. Kowalski-Glikman. 'Doubly Special Relativity as a Limit of Gravity'. In: *Special Relativity: Will it Survive the Next 101 Years?* Ed. by J. Ehlers and C. Lämmerzahl. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 279–298. ISBN: 978-3-540-34523-7. DOI: 10.1007/3-540-34523-X_10. URL: http://dx.doi.org/10.1007/3-540-34523-X_10.

[ISS70] C. Isham, A. Salam and J. Strathdee. 'Spontaneous breakdown of conformal symmetry'. In: *Physics Letters B* 31.5 (1970), pp. 300–302. ISSN: 0370-2693. DOI: 10.1016/0370-2693(70)90177-2. URL: http://www.sciencedirect.com/science/article/pii/0370269370901772.

[IW88] E. Inonu and E. P. Wigner. 'On the Contraction of Groups and Their Representations'. In: *Special Relativity and Quantum Theory: A Collection of Papers on the Poincaré Group*. Ed. by M. E. Noz and Y. S. Kim. Dordrecht: Springer Netherlands, 1988, pp. 357–371. ISBN: 978-94-009-3051-3. DOI: 10.1007/978-94-009-3051-3_32. URL: http://dx.doi.org/10.1007/978-94-009-3051-3_32.

[Jac62] N. Jacobson. *Lie algebras*. Interscience Tracts in Pure and Applied Mathematics 10. Interscience Publishers, 1962.

[Jim85] M. Jimbo. 'A q-difference analogue of U(g) and the Yang-Baxter equation'. In: *Letters in Mathematical Physics* 10.1 (1985), pp. 63–69. ISSN: 1573-0530. DOI: 10.1007/BF00704588. URL: http://dx.doi.org/10.1007/BF00704588.

[Kib61] T. W. B. Kibble. 'Lorentz Invariance and the Gravitational Field'. In: *Journal of Mathematical Physics* 2.2 (1961), pp. 212–221. DOI: 10.1063/1.1703702. eprint: http://dx.doi.org/10.1063/1.1703702. URL: http://dx.doi.org/10.1063/1.1703702.

[Kir04] A. A. Kirillov. *Lectures on the orbit method*. Vol. 64. Graduate Studies in Mathematics. American Mathematical Society Providence, 2004.

[Kle71] F. Klein. 'Über die sogenannte Nicht-Euklidische Geometrie'. In: *Mathematische Annalen* 4.4 (1871), pp. 573–625. ISSN: 1432-1807. DOI: 10.1007/BF02100583. URL: http://dx.doi.org/10.1007/BF02100583.

[Kle72a] F. Klein. 'A comparative review of recent researches in geometry''. In: *Bull. N. Y. Math. Soc* 2 (1892–1893 [1872]), pp. 215–249. URL: https://arxiv.org/abs/0807.3161.

[Kle72b] F. Klein. 'Vergleichende Betrachtungen über neuere geometrische Forschungen'. In: *Math. Ann.* 43 (1893 [1972]), pp. 63–100. URL: http://name.umdl.umich.edu/ABN7632.0001.001.

[Kle73] F. Klein. 'Über die sogenannte Nicht-Euklidische Geometrie'. In: *Mathematische Annalen* 6.2 (1873), pp. 112–145. ISSN: 1432-1807. DOI: 10.1007/BF01443189. URL: http://dx.doi.org/10.1007/BF01443189.

[KN04] J. Kowalski-Glikman and S. Nowak. 'Quantum κ-Poincaré Algebra from de Sitter Space of Momenta'. In: (2004). URL: https://arxiv.org/abs/hep-th/0411154.

[Kos04] V. A. Kostelecký. 'Gravity, Lorentz violation, and the standard model'. In: *Phys. Rev. D* 69 (10 May 2004), p. 105009. DOI: 10.1103/PhysRevD.69.105009. URL: http://link.aps.org/doi/10.1103/PhysRevD.69.105009.

[Kow02] J. Kowalski-Glikman. 'De Sitter space as an arena for doubly special relativity'. In: *Physics Letters B* 547.3–4 (2002), pp. 291–296. ISSN: 0370-2693. DOI: 10.1016/S0370-2693(02)02762-4. URL: http://www.sciencedirect.com/science/article/pii/S0370269302027624.

[Kow05] J. Kowalski-Glikman. 'Introduction to Doubly Special Relativity'. In: *Planck Scale Effects in Astrophysics and Cosmology*. Ed. by J. Kowalski-Glikman and G. Amelino-Camelia. Berlin, Heidelberg: Springer Berlin Heidelberg, 2005, pp. 131–159. ISBN: 978-3-540-31527-8. DOI: 10.1007/11377306_5. URL: http://dx.doi.org/10.1007/11377306_5.

[KR11] V. A. Kostelecký and N. Russell. 'Data tables for Lorentz and CPT violation'. In: *Rev. Mod. Phys.* 83 (1 Mar. 2011), pp. 11–31. DOI: 10.1103/RevModPhys.83.11. URL: http://link.aps.org/doi/10.1103/RevModPhys.83.11.

[KS89] V. A. Kostelecký and S. Samuel. 'Spontaneous breaking of Lorentz symmetry in string theory'. In: *Phys. Rev. D* 39 (2 Jan. 1989), pp. 683–685. DOI: 10.1103/PhysRevD.39.683. URL: http://link.aps.org/doi/10.1103/PhysRevD.39.683.

[Kue72] H. P. Kuenzle. 'Galilei and Lorentz structures on space-time - comparison of the corresponding geometry and physics'. In: *Annales Poincare Phys.Theor.* 17 (1972), pp. 337–362.

[Lév65] J.-M. Lévy-Leblond. 'Une nouvelle limite non-relativiste du groupe de Poincaré'. In: *Annales de l'institut Henri Poincaré (A) Physique théorique* 30.1 (1965), pp. 1–12.

[Lév67] M. Lévy-Nahas. 'Deformation and Contraction of Lie Algebras'. In: *Journal of Mathematical Physics* 8.6 (1967), pp. 1211–1222.

[Lév76] J. Lévy-Leblond. 'Quantum fact and classical fiction: Clarifying Landé's pseudo-paradox'. In: *American Journal of Physics* 44.11 (1976), pp. 1130–1132. DOI: 10.1119/1.10206. URL: http://scitation.aip.org/content/aapt/journal/ajp/44/11/10.1119/1.10206.

[LP79] J. Lévy-Leblond and J. Provost. 'Additivity, rapidity, relativity'. In: *American Journal of Physics* 47.12 (1979), pp. 1045–1049. DOI: 10.1119/1.11972. URL: http://scitation.aip.org/content/aapt/journal/ajp/47/12/10.1119/1.11972.

[Luk+91] J. Lukierski et al. 'q-deformation of Poincaré algebra'. In: *Physics Letters B* 264.3 (1991), pp. 331–338. ISSN: 0370-2693. DOI: 10.1016/0370-2693(91)90358-W. URL: http://www.sciencedirect.com/science/article/pii/037026939190358W.

[Mac51] G. W. Mackey. 'On Induced Representations of Groups'. In: *American Journal of Mathematics* 73.3 (1951), pp. 576–592. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372309.

[Maj95] S. Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, 1995.

[Mat05] D. Mattingly. 'Modern Tests of Lorentz Invariance'. In: *Living Reviews in Relativity* 8.5 (2005). URL: http://www.livingreviews.org/lrr-2005-5.

[Mav07] N. E. Mavromatos. 'Lorentz Invariance Violation from String Theory'. In: *PoS* QG-PH (2007), p. 027. arXiv: 0708.2250 [hep-th].

[Mic64] L. Michel. 'The group of automorphisms of the Poincaré group'. In: *Lectures in Theoretical Physics*. University of Colorado Press, 1964. URL: http://pagesperso.ihes.fr/~vergne/LouisMichel/publications/LectTheorPhysBoulder.117_1965.pdf.

[MR94] S. Majid and H. Ruegg. 'Bicrossproduct structure of κ-Poincare group and non-commutative geometry'. In: *Physics Letters B* 334.3–4 (1994), pp. 348–354. ISSN: 0370-2693. DOI: 10.1016/0370-2693(94)90699-8. URL: http://www.sciencedirect.com/science/article/pii/0370269394906998.

[Nag+15] Nagel Moritz et al. 'Direct terrestrial test of Lorentz symmetry in electrodynamics to 10^{-18}'. In: *Nature Communications* 6 (Sept. 2015), p. 8174. DOI: 10.1038/ncomms9174. URL: http://www.nature.com/articles/ncomms9174%5C#supplementary-information.

[Pan03] A. Pankiewicz. 'Strings in plane wave backgrounds'. In: *arXiv preprint hep-th/0307027* (2003).

[Pen72] R. Penrose. *Techniques of Differential Topology in Relativity*. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1972.

[Pen76] R. Penrose. 'Any spacetime has a plane wave as a limit'. In: *Differential Geometry and Relativity*. D. Reidel Publishing Company, 1976, pp. 271–275.

[Per04] V. Perlick. 'Gravitational Lensing from a Spacetime Perspective'. In: *Living Reviews in Relativity* 7.9 (2004). DOI: 10.1007/lrr-2004-9. URL: http://www.livingreviews.org/lrr-2004-9.

[Per86] R. Percacci. *Geometry of Nonlinear Field Theories*. World Scientific, 1986.

[Pod96] Podleś, P. and Woronowicz, S. L. 'On the classification of quantum Poincaré groups'. In: *Communications in Mathematical Physics* 178.1 (1996), pp. 61–82.

[Rad12] D. E. Radford. *Hopf Algebras*. Vol. 49. Series on Knots and Everything. World Scientific, 2012.

[Ric67] R. W. Richardson. 'On the rigidity of semi-direct products of Lie algebras.' In: *Pacific J. Math.* 22.2 (1967), pp. 339–344. URL: http://projecteuclid.org/euclid.pjm/1102992204.

[RRS11] D. Rätzel, S. Rivera and F. P. Schuller. 'Geometry of physical dispersion relations'. In: *Phys. Rev. D* 83 (4 Feb. 2011), p. 044047. DOI: 10.1103/PhysRevD.83.044047. URL: http://link.aps.org/doi/10.1103/PhysRevD.83.044047.

[Sac62] R. Sachs. 'Asymptotic Symmetries in Gravitational Theory'. In: *Phys. Rev.* 128 (6 Dec. 1962), pp. 2851–2864. DOI: 10.1103/PhysRev.128.2851. URL: http://link.aps.org/doi/10.1103/PhysRev.128.2851.

[Sha96] R. W. Sharpe. *Differential Geometry – Cartan's Generalization of Klein's Erlangen Program*. Vol. 166. Graduate Texts in Mathematics. Springer New York, 1996.

[Sin72] W. M. Singer. 'Extension theory for connected Hopf algebras'. In: *Journal of Algebra* 21.1 (1972), pp. 1–16. ISSN: 0021-8693. DOI: 10.1016/0021-8693(72)90031-2. URL: http://www.sciencedirect.com/science/article/pii/0021869372900312.

[Smo06] L. Smolin. 'Falsifiable predictions from semiclassical quantum gravity'. In: *Nuclear Physics B* 742.1–3 (2006), pp. 142–157. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2006.02.017. URL: http://www.sciencedirect.com/science/article/pii/S0550321306001623.

[Sny47] H. S. Snyder. 'Quantized Space-Time'. In: *Phys. Rev.* 71 (1 Jan. 1947), pp. 38–41. DOI: 10.1103/PhysRev.71.38. URL: http://link.aps.org/doi/10.1103/PhysRev.71.38.

[Som09] D. M. Y. Sommerville. 'Classification of Geometries with Projective Metric'. In: *Proceedings of the Edinburgh Mathematical Society* 28 (Feb. 1909), pp. 25–41. DOI: 10.1017/S0013091500034763. URL: https://www.cambridge.org/core/article/div-class-title-classification-of-geometries-with-projective-metric-div/78DE4BEEF6536DF6F03BF383957C086A.

[SW80] K. S. Stelle and P. C. West. 'Spontaneously broken de Sitter symmetry and the gravitational holonomy group'. In: *Phys. Rev. D* 21 (6 Mar. 1980), pp. 1466–1488. DOI: 10.1103/PhysRevD.21.1466. URL: http://link.aps.org/doi/10.1103/PhysRevD.21.1466.

[SWZ92] P. Schupp, P. Watts and B. Zumino. 'The two-dimensional quantum Euclidean algebra'. In: *Letters in Mathematical Physics* 24.2 (1992), pp. 141–145. ISSN: 1573-0530. DOI: 10.1007/BF00402677. URL: http://dx.doi.org/10.1007/BF00402677.

[Tow77] P. K. Townsend. 'Small-scale structure of spacetime as the origin of the gravitational constant'. In: *Phys. Rev. D* 15 (10 May 1977), pp. 2795–2801. DOI: 10.1103/PhysRevD.15.2795. URL: http://link.aps.org/doi/10.1103/PhysRevD.15.2795.

[Tra05] A. Trautman. 'On eight kinds of spinors'. In: *Acta Phys. Polon.* B36 (2005), pp. 121–130.

[Tse82] A. A. Tseytlin. 'Poincaré and de Sitter gauge theories of gravity with propagating torsion'. In: *Phys. Rev. D* 26 (12 Dec. 1982), pp. 3327–3341. DOI: 10.1103/PhysRevD.26.3327. URL: http://link.aps.org/doi/10.1103/PhysRevD.26.3327.

[Uti56] R. Utiyama. 'Invariant Theoretical Interpretation of Interaction'. In: *Phys. Rev.* 101 (5 Mar. 1956), pp. 1597–1607. DOI: 10.1103/PhysRev.101.1597. URL: http://link.aps.org/doi/10.1103/PhysRev.101.1597.

[Wei+17] J.-J. Wei et al. 'A New Test of Lorentz Invariance Violation: The Spectral Lag Transition of GRB 160625B'. In: *The Astrophysical Journal Letters* 834.2 (2017), p. L13. URL: http://stacks.iop.org/2041-8205/834/i=2/a=L13.

[Wei74] S. Weinberg. 'Gauge and global symmetries at high temperature'. In: *Phys. Rev. D* 9 (12 June 1974), pp. 3357–3378. DOI: 10.1103/PhysRevD.9.3357. URL: http://link.aps.org/doi/10.1103/PhysRevD.9.3357.

[Wig39] E. Wigner. 'On Unitary Representations of the Inhomogeneous Lorentz Group'. In: *Annals of Mathematics* 40.1 (1939), pp. 149–204. ISSN: 0003486X. URL: http://www.jstor.org/stable/1968551.

[Wis07] D. K. Wise. 'Topological Gauge Theory, Cartan Geometry, and Gravity'. PhD thesis. Universtiy of California, Riverside, 2007.

[Yag21] I. M. Yaglom. *A simple non-Euclidean geometry and its physical basis*. Heidelberg Science Library. Springer, 1921.

[Zee63] E. C. Zeeman. 'Causality Implies the Lorentz Group'. In: *Journal of Mathematical Physics* 5.4 (1963), pp. 490–493.

[Zha+16] B.-B. Zhang et al. 'Transition from Fireball to Poynting-flux-dominated Outflow in Three-Episode GRB 160625B'. In: (2016). arXiv: 1612.03089 [astro-ph.HE].