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Abstract

We classify all rational 2-reptiles in the plane. We also establish properties concerning rational reptiles in the plane in general.

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1 Introduction

A compact set $T \subseteq \mathbb{R}^d$ is a *tile* of $\mathbb{R}^d$ if there exists a countable collection of sets $\{T_1, T_2, \ldots\}$, where each T_j is congruent to T, such that their union is the whole of $\mathbb{R}^d$ and the intersection of any two distinct T_j's has zero Lebesgue measure. A tile T has non-empty interior (by Baire category theorem), and we will assume that it is the closure of its interior.

A tile T is a *reptile*, or more precisely an *n-reptile*, if T can be dissected into n compact subsets $\Omega_1, \ldots, \Omega_n$ with non-overlapping interiors such that all Ω_j's are congruent among each other using translations and rotations (but not reflections), and each Ω_j is similar to T (again, no reflections).

By comparing the volume of T with the sum of the volumes of the Ω_j's we easily see that each Ω_j is scaled down from T by a factor of $\sqrt[d]{n}$. So we can formulate T by

$$\sqrt[d]{n}\,(T) = \bigcup_{j=1}^{n} f_j(T), \tag{1.1}$$

where each f_j is an isometry in $\mathbb{R}^d$ (without reflection). Alternatively we may write equation

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(1.1) as

$$T = \bigcup_{j=1}^{n} \frac{1}{\sqrt[d]{n}} f_j(T). \tag{1.2}$$

In other words T is the attractor of the iterated function system (IFS) $\{n^{-\frac{1}{d}} f_j : 1 \le j \le n\}$ (see [Ba]).

In this paper we consider reptiles in the plane. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ we may formulate an n-reptile as

$$\sqrt{n}\,(T) = \bigcup_{j=1}^{n} (e^{i\theta_j} T + a_j), \quad \theta_j \in \mathbb{R} \text{ and } a_j \in \mathbb{C}. \tag{1.3}$$

It is well known that for any given $\theta_1, \ldots, \theta_n$ and $a_1, \ldots, a_n$ there exists a *unique* compact set T satisfying (1.3). We will call a set T satisfying (1.3) a *repset*, or more precisely, an n-*repset* in the plane. In most cases a repset fails to be a reptile because it has an empty interior. So our goal is to determine the values of $\theta_1, \ldots, \theta_n$ and $a_1, \ldots, a_n$ for which the corresponding repset T has non-empty interior. A straightforward inflation argument shows that if a repset T has non-empty interior then it must tile the plane $\mathbb{R}^2$.

Reptiles in which all isometries f_j in (1.1) use the same rotation (often called *self-similar tiles*) have been studied extensively, often as a special case of the so-called *self-affine tiles*. In the self-affine tile setting, all f_j in (1.1) have identical linear part, but they and the expansion factor are not required to be similarities. Self-affine tiles arise in many contexts, including radix expansions ([Gi], [O]), the construction of compactly supported wavelets ([GM], [LW2]), and Markov partitions ([Bo], [P]). They are also studied directly as interesting tiles ([B], [HSV], [K1], [LW1]). The introduction of different rotational angles in (1.3) makes reptiles far more difficult to study, especially since virtually all Fourier analytic techniques that are so useful for the study of self-affine tiles no longer apply. Although there have been some studies of reptiles in the plane, few definitive results are known, even for n-reptiles with small n's (see section C17 of [CFG], [G], and references therein).

In our study of 2-reptiles in the plane we focus on a class of reptiles called *rational reptiles* (defined below). Here we can make substantial progress, partly as a consequence of the studies of Bandt, Thurston, and Kenyon ([B], [T], [K2]).

Definition 1.1 *A compact set $T \subset \mathbb{C}$ is called a* rational n-repset *in the plane if it satisfies equation (1.3) for some $\theta_1, \ldots, \theta_n \in \mathbb{R}$ and $a_1, \ldots, a_n \in \mathbb{C}$, and in which all differences $\theta_i - \theta_j$ are rational multiples of π. A rational n-repset T is called a* rational n-reptile *if it is an n-reptile.*

The following theorem allows us to completely classify all rational 2-reptiles in the plane. The main ingredients for proving this theorem are results of Thurston [T] and Praggastis [P].

Theorem 1.1 *Let T be a rational n-reptile satisfying*

$$\sqrt{n}\,(T) = \bigcup_{j=1}^{n} (e^{i\theta_j} T + a_j), \quad \theta_j \in \mathbb{R} \text{ and } a_j \in \mathbb{C},$$

where $\theta_i - \theta_j$ are rational multiples of π for all i, j. Then for each j, $\sqrt{n}e^{-i\theta_j}$ is either an integer or a nonreal quadratic integer.

To state our classification of rational 2-reptiles we show first (Lemma 3.2) that each 2-reptile is similar (no reflection) to a reptile satisfying the following *canonical* equation:

$$\sqrt{2}e^{i\phi}\,(T) = T \cup (e^{i\theta} T + 1), \quad \phi,\, \theta \in \mathbb{R}. \tag{1.4}$$

Clearly, it is sufficient to restrict ϕ and θ to $[0, 2\pi)$.

Theorem 1.2 *Let T be a rational 2-repset satisfying the canonical equation*

$$\sqrt{2}e^{i\phi}\,(T) = T \cup (e^{i\theta} T + 1), \quad \phi, \theta \in [0, 2\pi) \text{ and } \theta/\pi \in \mathbb{Q}.$$

Then T is a 2-reptile if and only if (ϕ, θ) takes on one the following values:

(a) *$\phi = k\pi/4$ and $\theta = \ell\pi/2$ where k and ℓ are integers, and k odd;*

(b) *$\phi = k\pi/2$ where k is odd, and $\theta = 0$ or π;*

(c) *$\phi = k\pi \pm \tan^{-1}(\sqrt{7})$, where $k \in \{0, 1\}$ and $\theta = 0$ or π.*

Many of the pairs listed in this theorem yield equivalent tiles. We say that two tiles T_1 and T_2 are *equivalent* if one of them can be obtained from the other by a combination of scaling, translation, rotation, and reflection. In other words, T_1 and T_2 "look alike." We show:

Theorem 1.3 *There are exactly six non-equivalent rational 2-reptiles in the plane. The equivalence classes have (ϕ, θ) represented by:*

$$(\frac{\pi}{4}, 0),\ (\frac{\pi}{4}, \frac{\pi}{2}),\ (\frac{\pi}{4}, \frac{3\pi}{2}),\ (\frac{3\pi}{4}, \frac{3\pi}{2}),\ (\frac{\pi}{2}, 0),\ (\tan^{-1}(\sqrt{7}), 0).$$

The 2-reptiles in the six equivalence classes in Theorem 1.3 are shown in the following table.

(ϕ, θ)	2-reptile
$(\frac{\pi}{4}, 0)$	twindragon
$(\frac{\pi}{4}, \frac{\pi}{2})$	Lévy dragon
$(\frac{\pi}{4}, \frac{3\pi}{2})$	Heighway dragon
$(\frac{3\pi}{4}, \frac{3\pi}{2})$	triangle
$(\frac{\pi}{2}, 0)$	rectangle
$(\tan^{-1}(\sqrt{7}), 0)$	tame twindragon

One of the many unsolved problems concerning reptiles, proposed by Grunbaum and listed in [CFG], is whether there exists a 2-reptile that is also a 3-reptile. We prove:

Theorem 1.4 *There exists no rational 2-reptile that is also a rational 3-reptile.*

The techniques used in this paper do not seem to apply to irrational repsets. This raises a question: are there any irrational reptiles? Our numerical calculations indicate that there are no irrational 2-reptiles. The answer is less clear for n-reptiles where $n > 2$. Conway's Pinwheel Tiling (see [R]) involves rotations by some irrational multiples of π, but it also involves reflections, which are prohibited in our setting. We conjecture:

Conjecture 1.5 *All reptiles are rational.*

The rest of this paper is organized as follows: In §2, we establish several preliminary results on self-affine multi-tiles, which are needed to prove our main theorems. In §3, we prove Theorem 1.1 and one direction of Theorem 1.2 (the necessary condition). Complete proofs of Theorems 1.2 and 1.3 are given in §4. In the last section §5, we prove Theorem 1.4 and the connectedness of 2-repsets in $\mathbb{R}^d$.

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2 Preliminaries: Self-Affine Multi-tiles

In this section we introduce an extension of self-affine tiles, known as self-affine multi-tiles ([FW]). Our study of reptiles is based largely on the fact that a rational reptile can be reformulated as a self-affine multi-tile.

We adopt the following notation and terminology: Let X, Y be subsets of $\mathbb{R}^d$. We use $X + Y$ to denote the Minkowski sum of X and Y, $X + Y = \{x + y : \ x \in X, y \in Y\}$. The union $X \cup Y$ is said to be *essentially disjoint* if $X \cap Y$ has zero Lebesgue measure.

Let $A \in M_d(\mathbb{R})$ be an *expanding* matrix, i.e. all its eigenvalues λ have $|\lambda| > 1$. Let $T_1, T_2, \ldots, T_r$ be compact sets in $\mathbb{R}^d$ with nonempty interiors. Then we call the r-tuple of compact sets $(T_1, T_2, \ldots, T_r)$ a *self-affine multi-tile* (with expansion factor A) if there exist finite (possibly empty) subsets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $1 \le i, j \le r$ such that

$$A(T_i) = \bigcup_{j=1}^{r} \bigcup_{d \in \mathcal{D}_{ij}} (T_j + d) = \bigcup_{j=1}^{r} (T_j + \mathcal{D}_{ij}), \quad 1 \le i \le r, \tag{2.1}$$

where all unions on the righthand side are essentially disjoint. An important object associated to (2.1) is the $r \times r$ matrix $S = [s_{ij}]$ given by $s_{ij} = |\mathcal{D}_{ij}|$, the cardinality of $\mathcal{D}_{ij}$. We call this matrix S the *subdivision matrix* of (2.1). Equation (2.1) (without the assumption of essential disjointness of the unions) actually defines a *graph-directed IFS*, following the terminology of Mauldin and Williams [MW]. Suppose that for each i at least one of the sets $\mathcal{D}_{ij}$ is nonempty for $1 \leq j \leq r$. Then there always exists a unique r-tuple of nonempty compact sets $(T_1, T_2, \ldots, T_r)$ satisfying (2.1) (see [MW]). But only in very special cases (2.1) yields a solution $(T_1, T_2, \ldots, T_r)$ in which all T_j have nonempty interiors.

Example. Let $A = [\sigma]$, $T_1 = [0,1]$ and $T_2 = [0,\sigma]$ where $\sigma = (\sqrt{5}+1)/2$. Then (T_1, T_2) is a self-affine multi-tile with expansion factor σ. In fact,

$$\begin{aligned} A(T_1) &= [0,\sigma] = T_2, \\ A(T_2) &= [0,\sigma^2] = [0,1] \cup ([0,\sigma]+1) = T_1 \cup (T_2+1). \end{aligned}$$

Here $\mathcal{D}_{11} = \emptyset$, $\mathcal{D}_{12} = \{0\}$, $\mathcal{D}_{21} = \{0\}$ and $\mathcal{D}_{22} = \{1\}$. ■

Proposition 2.1 *Let $(T_1, \ldots, T_r)$ be a self-affine multi-tile in $\mathbb{R}^d$ satisfying (2.1). Then $\overline{T_i^o} = T_i$ for all $1 \leq i \leq r$, and there exist discrete sets $\mathcal{J}_1, \ldots, \mathcal{J}_r$ in $\mathbb{R}^d$ such that*

$$\bigcup_{i=1}^{r} (T_i + \mathcal{J}_i) = \mathbb{R}^d$$

is a tiling of $\mathbb{R}^d$.

Proof. Let $T_i' = \overline{T_i^o}$ for each i. Since each T_i has nonempty interior, $T_i' \neq \emptyset$. But $(T_1', \ldots, T_r')$ also satisfies (2.1). It follows from the uniqueness that $T_i' = T_i$.

The existence of the tiling sets $\mathcal{J}_1, \ldots, \mathcal{J}_r$ is proved in Flaherty and Wang [FW] (in it A is assumed to be an integral matrix, but the argument clearly holds for nonintegral matrices). ■

Many useful results concerning self-affine multi-tiles can be derived by iterating equation (2.1). Assume that

$$A^m(T_i) = \bigcup_{j=1}^{r} (T_j + \mathcal{D}_{ij}^{(m)}), \tag{2.2}$$

where m is an arbitrary positive integer. Then for each $1 \leq i \leq r$,

$$\begin{aligned} A^{m+1}(T_i) &= A^m \bigcup_{k=1}^{r} (T_k + \mathcal{D}_{ik}) \\ &= \bigcup_{k=1}^{r} (A^m(T_k) + A^m \mathcal{D}_{ik}) \\ &= \bigcup_{k=1}^{r} \Big(\bigcup_{j=1}^{r} (T_j + \mathcal{D}_{kj}^{(m)}) + A^m \mathcal{D}_{ik} \Big) \\ &= \bigcup_{j=1}^{r} \Big(T_j + \bigcup_{k=1}^{r} (A^m \mathcal{D}_{ik} + \mathcal{D}_{kj}^{(m)}) \Big). \end{aligned}$$

This yields

$$\mathcal{D}_{ij}^{(m+1)} = \bigcup_{k=1}^{r} (A^m \mathcal{D}_{ik} + \mathcal{D}_{kj}^{(m)}).$$

By iterating the above equation we now obtain

$$\mathcal{D}_{ij}^{(m)} = \bigcup_{k_1,\, \ldots,\, k_{m-1}=1}^{r} (A^{m-1}\mathcal{D}_{ik_1} + A^{m-2}\mathcal{D}_{k_1k_2} + \cdots + A\mathcal{D}_{k_{m-2}k_{m-1}} + \mathcal{D}_{k_{m-1}j}). \tag{2.3}$$

Lemma 2.2 *Let $(T_1, \ldots, T_r)$ be a self-affine multi-tile satisfying (2.1), with expansion factor A and subdivision matrix S. Then for any positive integer m, $(T_1, \ldots, T_r)$ is a self-affine multi-tile satisfying (2.2), with expansion factor A^m and subdivision matrix S^m.*

Proof. It is clear that $(T_1, \ldots, T_r)$ satisfies (2.2). The question is whether the unions on the righthand side of (2.2) are essentially disjoint. But this is clearly so, because it is obtained by iterating (2.1). The subdivision matrix for (2.2) is given by $[|\mathcal{D}_{ij}^{(m)}|]$. By (2.3) and the essential disjointness,

$$|\mathcal{D}_{ij}^{(m)}| = \sum_{k_1,\, k_2,\, \ldots,\, k_{m-1}=1}^{r} s_{ik_1}s_{k_1k_2}\cdots s_{k_{m-1}j} = s_{ij}^{(m)},$$

where $[s_{ij}^{(m)}] = S^m$. This proves the lemma. ■

Lemma 2.3 *Let $T_1, \ldots, T_r$ be compact sets in $\mathbb{R}^d$ satisfying (2.1). Then for each $1 \le i \le r$ we have*

$$T_i = \bigcup_{k_1,\, k_2,\, k_3,\, \ldots=1}^{r} \Big(A^{-1}\mathcal{D}_{ik_1} + A^{-2}\mathcal{D}_{k_1k_2} + A^{-3}\mathcal{D}_{k_2k_3} + \cdots \Big). \tag{2.4}$$

Proof. By (2.2) we have

$$T_i = \bigcup_{j=1}^{r} \Big(A^{-m}(T_j) + A^{-m}\mathcal{D}_{ij}^{(m)} \Big).$$

The lemma now follows by applying (2.3) and letting $m\to\infty$. Observe that $A^{-m}(T_j) \to \{0\}$ as $m\to\infty$. ■

For the rest of this section, we focus on self-affine multi-tiles in $\mathbb{R}^2$ (identified with $\mathbb{C}$) in which the expansion factor is a complex number τ with $|\tau| > 1$:

$$\tau(T_i) = \bigcup_{j=1}^{r} (T_j + \mathcal{D}_{ij}), \quad 1 \le i \le r. \tag{2.5}$$

The following result, essentially due to Thurston [T], serves as a basis for most of our results. Recall that a complex number τ is a *complex Perron number* if it is an algebraic integer and all its Galois conjugates other than $\overline{\tau}$ have moduli strictly smaller than that of τ. Complex Perron numbers include real Perron numbers.

Proposition 2.4 (Thurston) *Let $(T_1, \ldots, T_r)$ be a self-affine multi-tile in $\mathbb{C}$ satisfying (2.5), where $\tau \in \mathbb{C}$ and $|\tau| > 1$. Suppose the following hold:*

(i) *The subdivision matrix S is primitive, i.e. $S^k > 0$ for some $k \geq 1$.*

(ii) *For some $1 \leq i \leq r$ we have $0 \in \mathcal{D}_{ii}$ and $0 \in T_i^o$.*

Then τ is a complex Perron number.

Proof. Without loss of generality we assume that $0 \in \mathcal{D}_{11}$ and $0 \in T_1^o$. Since $0 \in T_1^o$, $\bigcup_{m=1}^{\infty} \tau^m(T_1) = \mathbb{C}$. Now by (2.2) we must have $\mathcal{D}_{1j}^{(m-1)} \subseteq \mathcal{D}_{1j}^{(m)}$ for all j. This means $\tau^{m-1}(T_1) \subseteq \tau^m(T_1)$. Let

$$\mathcal{D}_{1j}^{(\infty)} := \bigcup_{m=1}^{\infty} \mathcal{D}_{1j}^{(m)}.$$

It follows that

$$\bigcup_{j=1}^{r}(T_j + \mathcal{D}_{1j}^{(\infty)}) = \bigcup_{m=1}^{\infty}(T_j + \mathcal{D}_{1j}^{(m)}) = \mathbb{C} \tag{2.6}$$

is a tiling of $\mathbb{C}$. This tiling satisfies the first three hypotheses of a *self-similar tiling* defined in Kenyon [K2]. We need only to verify that the tiling is *quasiperiodic* (the fourth hypothesis in [K2]). But this follows from the fact that $0 \in T_1^o$ and S is primitive, by a result of Praggastis [P] (see also Lemma 4 in [K2]). Hence the tiling (2.6) is a self-similar tiling. Now it follows from Thurston's theorem (see [T]) that τ is a complex Perron number. ∎

Lemma 2.5 *Suppose that $\tau \in \mathbb{C}$ has the property that τ^k is a complex Perron number for all sufficiently large k. Then τ must itself be a complex Perron number.*

Proof. We prove the lemma by contradiction. Assume that τ is not a complex Perron number. Then τ has a Galois conjugate λ such that $\lambda \neq \overline{\tau}$ and $|\lambda| \geq |\tau|$. Fix any large k such that τ^k is complex Perron. Let $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of τ^k, and let $\mu_1, \ldots, \mu_{m-1}$ be the Galois conjugates of τ^k. Then $f(\mu_i) = 0$ by definition. Since $f(\tau^k) = 0$ and λ is a Galois conjugate of τ, we also have $f(\lambda^k) = 0$. This leads to

$$a_0\mu_i^0 + a_1\mu_i^1 + \cdots + \mu_i^m = 0, \quad 1 \leq i \leq m+1,$$

where we set $\mu_m = \tau^k$ and $\mu_{m+1} = \lambda^k$. But this can happen only if the Vandermonde matrix $[\mu_i^j]$ is singular, which is equivalent to $\mu_i = \mu_j$ for some $i \neq j$. Since $\mu_1, \ldots, \mu_{m-1}, \mu_m$ are all distinct, we therefore have $\mu_{m+1} = \lambda^k = \mu_i$ for some $i \leq m$. But $\mu_m = \tau^k$ is complex Perron and $|\lambda| \geq |\tau|$; it follows that we must have $\lambda^k = \tau^k$ or $\lambda^k = \overline{\tau}^k$. Since k is arbitrary, so long as it is sufficiently large, we conclude that $(\tau/\lambda)^k = 1$ or $(\overline{\tau}/\lambda)^k = 1$ for all sufficiently large k. Hence we can find two sufficiently large coprime integers k_1 and k_2 such that $(\tau/\lambda)^{k_1} = 1$ and $(\tau/\lambda)^{k_2} = 1$ at the same time, or $(\overline{\tau}/\lambda)^{k_1} = 1$ and $(\overline{\tau}/\lambda)^{k_2} = 1$ at the same time. The fact that k_1, k_2 are coprime now yields either $\tau/\lambda = 1$ or $\overline{\tau}/\lambda = 1$, contradicting the assumption that $\lambda \neq \tau$ and $\lambda \neq \overline{\tau}$. This proves the lemma. ∎

We now prove the following extension of Proposition 2.4.

Theorem 2.6 *Let $(T_1, \ldots, T_r)$ be a self-affine multi-tile in $\mathbb{C}$ satisfying*

$$\tau(T_i) = \bigcup_{j=1}^{r} (T_j + \mathcal{D}_{ij}), \quad 1 \le i \le r,$$

where $\tau \in \mathbb{C}$, $|\tau| > 1$ and all unions are essentially disjoint. Suppose that the subdivision matrix S is primitive. Then τ is a complex Perron number.

Proof. We first consider the case where $S > 0$, i.e. all $\mathcal{D}_{ij}$ are nonempty. Since $T_1^o \neq \emptyset$, by Lemma 2.3 we can find a sequence $(k_1, k_2, k_3, \ldots)$ such that $\sum_{m=1}^{\infty} \tau^{-m} d_m \in T_1^o$, where $d_1 \in \mathcal{D}_{1k_1}$, $d_2 \in \mathcal{D}_{k_1 k_2}$, ... Because the sets $\mathcal{D}_{ij}$ are uniformly bounded and $|\tau| > 1$, for sufficiently large $K > 0$ we have

$$\sum_{m=1}^{K-1} \tau^{-m} d_m + \sum_{m=K}^{\infty} \tau^{-m} e_m \in T_1^o \tag{2.7}$$

for all $e_m \in \mathcal{D}_{ij}$, $1 \le i, j \le r$. Set

$$x_0 = \sum_{m=1}^{K-1} \tau^{-m} d_m + \tau^{-K} e_K, \quad \text{where } e_K \in \mathcal{D}_{k_{K-1}1}.$$

Then $\tau^K x_0 \in \mathcal{D}_{11}^{(K)}$, and by (2.7),

$$x^* = x_0 + \tau^{-K} x_0 + \tau^{-2K} x_0 + \cdots \ \in T_1^o.$$

We now let $\tilde{T}_1 = T_1 - x^*$ and $\tilde{T}_j = T_j$ for $j > 1$. Note that

$$x^* = \frac{x_0}{1 - \tau^{-K}} = \frac{\tau^K x_0}{\tau^K - 1}.$$

Thus

$$\begin{aligned} \tau^K(\tilde{T}_1) &= \tau^K(T_1) - \tau^K x^* = \bigcup_{j=1}^{r} (T_j + \mathcal{D}_{1j}^{(K)}) - \tau^K x^* \\ &= \Big(\tilde{T}_1 + \mathcal{D}_{11}^{(K)} + x^* - \tau^K x^*\Big) \cup \Big(\bigcup_{j=2}^{r} (\tilde{T}_j + \mathcal{D}_{1j}^{(K)} - \tau^K x^*)\Big). \end{aligned}$$

Similar expressions can be derived for $\tau^K(\tilde{T}_i)$ with $i > 1$. It is therefore clear that $(\tilde{T}_1, \ldots, \tilde{T}_r)$ is a self-affine multi-tile with expansion factor τ^K satisfying

$$\tau^K(\tilde{T}_i) = \bigcup_{j=1}^{r} (\tilde{T}_j + \tilde{\mathcal{D}}_{ij}^{(K)}), \quad 1 \le i \le r,$$

in which $\tilde{\mathcal{D}}_{11}^{(K)} = \mathcal{D}_{11}^{(K)} + x^* - \tau^K x^*$. But $\tau^K x^* - x^* = \tau^K x_0 \in \mathcal{D}_{11}^{(K)}$, so $0 \in \tilde{\mathcal{D}}_{11}^{(K)}$. Combining this fact with $0 \in \tilde{T}_1^o$ we prove that τ^K is complex Perron. But K is an arbitrary integer provided it is sufficiently large. Thus τ is a complex Perron number by Lemma 2.5.

Finally, in the general case in which S is not positive we have $S^k > 0$ for sufficiently large k. Since $(T_1, \ldots, T_r)$ is also a self-affine multi-tile with expansion factor τ^k and subdivision matrix S^k, we conclude that τ^k is a complex Perron number for sufficiently large k. Therefore τ must be a complex Perron number. ∎

3 Necessary Conditions

Let T be a rational n-reptile in the plane $\mathbb{C}$. So it satisfies

$$\sqrt{n}\,(T) = \bigcup_{j=1}^{n} (e^{i\theta_j} T + a_j) \tag{3.1}$$

in which $(\theta_i - \theta_j)/\pi \in \mathbb{Q}$ for all $1 \le i, j \le n$. Therefore there exists a positive integer r and $\phi_0 = \frac{2\pi}{r}$ such that

$$\sqrt{n} e^{-i\theta_1}\,(T) = \bigcup_{j=1}^{n} (e^{ip_j\phi_0} T + b_j), \tag{3.2}$$

where $b_j = e^{-i\theta_1} a_j$ and $p_j \in \mathbb{Z}$ for all j with $p_0 = 0$. We shall assume that r is the *minimal* of such integers, and without loss of generality we can always require all $0 \le p_j < r$. The minimality of r implies that

$$\text{g.c.d.}\,(r,\, p_1,\, \ldots,\, p_n) = 1. \tag{3.3}$$

A key observation is that T can be reformulated in terms of self-affine multi-tiles. Let $\tau = \sqrt{n} e^{-i\theta_1}$ and $\omega = e^{i\phi_0}$. If we denote

$$T_0 = T,\ T_1 = \omega T,\ \ \ldots,\ T_{r-1} = \omega^{r-1} T,$$

then $(T_0, T_1, \ldots, T_{r-1})$ is a self-affine multi-tile with expansion factor τ:

$$\tau\,(T_k) = \bigcup_{k=1}^{n} (T_{p_j+k} + \omega^k b_j), \quad 0 \le k < r, \tag{3.4}$$

where $T_k := T_{k-r}$ for $k \ge r$. This self-affine multi-tile formulation allows us to prove Theorem 1.1. First we establish the following lemma concerning complex Perron numbers:

Lemma 3.1 *Let $\tau \in \mathbb{C}$ with $|\tau| = \sqrt{n}$. Then τ is a complex Perron number if and only if τ is an integer, or a nonreal quadratic integer.*

Proof. If $\tau \in \mathbb{R}$ then $\tau = \pm\sqrt{n}$. Unless $\tau \in \mathbb{Z}$, it is not a complex Perron number because its Galois conjugate is $-\tau$. Suppose that τ is not real. Observe that $\overline{\tau} = n/\tau$ is also complex Perron, and its Galois conjugates are n/λ for all Galois conjugates λ of τ. If some $|\lambda| < |\tau|$ then $|n/\lambda| > |\overline{\tau}|$, contradicting $\overline{\tau}$ being complex Perron. So all Galois conjugates λ of τ satisfy $|\lambda| \ge |\tau|$, which implies that the only Galois conjugate of τ is $\overline{\tau}$. Of course, this means that τ is a quadratic integer. ∎

Proof of Theorem 1.1. We prove that $\tau = \sqrt{n} e^{-i\theta_j}$ is a complex Perron number for $j = 1$. Others follow from the same argument. By Theorem 2.6 we only need to show that the subdivision matrix $S = [s_{ij}]$ is primitive. The matrix S has the property that each row is the cyclical shift of the previous row to the right by one position. If $r = 1$ then the primitivity of S is trivially true because $s_{00} > 0$. For such a cyclical matrix it is well known (and easy to verify) that S has r eigenvectors given by

$$\mathbf{v} = [1, \nu, \ldots, \nu^{r-1}]^T, \quad \text{where } \nu = 1,\, \omega,\, \ldots,\, \omega^{r-1}.$$

Since ω is a primitive r-th root of unity, the standard property of a Vandermonde matrix implies that the above r eigenvectors are independent. The eigenvalues corresponding to the eigenvectors are $f(\nu)$, where $f(x)$ is the polynomial

$$f(x) = \sum_{j=1}^{n} x^{p_j}.$$

To show that S is primitive, we first note that (3.3) implies that $|f(\omega^k)| < |f(1)|$ for $0 < k < r$ ($f(1)$ is the Perron-Frobenius eigenvalue). So we now only need to show that S is irreducible (see Theorem 1.7 of [BP]). For irreducibility, we observe that $[1, 1, \ldots, 1]^T > 0$ is the (unique up to a scalar multiple) Perron-Frobenius eigenvector for both S and S^T. So S is irreducible (see Corollary 3.15 of [BP]), and hence primitive. This proves the theorem.

■

We now consider 2-reptiles in the plane. It is convenient to consider only 2-reptiles in the canonical form, as a result of the following lemma:

Lemma 3.2 *Let T' be a 2-reptile in the plane. Then there exists a 2-reptile T similar to T' (via translation, rotation and scaling) such that T satisfies the following equation*

$$\sqrt{2}e^{i\phi}\,(T) = T \cup (e^{i\theta}T + 1), \tag{3.5}$$

for some ϕ, $\theta \in \mathbb{R}$. Furthermore, T' is rational if and only if T is.

Proof. Suppose that T' satisfies

$$\sqrt{2}\,(T') = (e^{i\theta_1}T' + a_1) \cup (e^{i\theta_2}T' + a_2).$$

A simple translation $T_1 = T' - \frac{a_1}{\sqrt{2}-e^{i\theta_1}}$ yields

$$\sqrt{2}(T_1) = e^{i\theta_1}T_1 \cup (e^{i\theta_2}T_1 + b_2), \quad \text{where } b_2 = a_2 - \tfrac{\sqrt{2}-e^{i\theta_2}}{\sqrt{2}-e^{i\theta_1}}a_1.$$

Note that $b_2 \neq 0$, for otherwise $T_1 = \{0\}$ would be the solution to the above equation. Now let $T = e^{i\theta_1}T_1/b_2$. Then it is easy to check that T satisfies (3.5) with $\phi = -\theta_1$ and $\theta = \theta_2 - \theta_1$. The last assertion now follows from the identity

$$\sqrt{2}\,(T) = e^{i\theta_1}T \cup (e^{i\theta_2}T + e^{i\theta_1}).$$

■

Lemma 3.2 in fact holds for any 2-repset T' that is not degenerated (i.e. not a single point). For the rest of this paper it is convenient to introduce the following terminology:

Definition 3.1 *We say that a 2-repset T is the* canonical 2-repset *corresponding to (ϕ, θ) if T is given by (3.5). A 2-repset T' is said to have a* canonical representation *(ϕ, θ) if T' is similar (via translation, scaling, rotation) to the canonical 2-repset T corresponding to (ϕ, θ).*

As we shall see in §4, a 2-repset may have more than one canonical representations. In Proposition 4.3 we list several canonical representations that yield equivalent 2-repsets.

Now let T be a rational 2-reptile satisfying the canonical equation (3.5), in which $\theta/\pi \in \mathbb{Q}$. We apply Theorem 1.1 to reduce the number of admissible pairs (ϕ, θ). First:

Lemma 3.3 *Let τ be a complex Perron number such that $|\tau| = \sqrt{2}$. Then τ must be one of the following complex numbers:*

$$\pm 1 \pm i, \quad \pm\sqrt{2}i, \quad \textit{or} \quad \pm\frac{1}{2} \pm \frac{\sqrt{7}}{2}i.$$

Proof. By Lemma 3.1 τ must be a nonreal quadratic integer (it cannot be an integer), which is equivalent to $\tau = \pm\frac{a}{2} \pm \frac{\sqrt{b}}{2}i$ for integers $a \geq 0$ and $b > 0$. The lemma follows immediately from $|\tau|^2 = 2$. ■

Lemma 3.4 *Let T be a rational 2-reptile having a canonical representation (ϕ, θ) with $\theta/\pi \in \mathbb{Q}$. Then (ϕ, θ) takes on one the following values:*

(a) *$\phi = k\pi/4$ and $\theta = \ell\pi/2$ for some integer ℓ and odd integer k;*

(b) *$\phi = k\pi/2$ with k odd, and $\theta = 0$ or π;*

(c) *$\phi = k\pi \pm \tan^{-1}(\sqrt{7})$ where $k \in \{0, 1\}$, and $\theta = 0$ or π.*

Proof. Let $\tau = \sqrt{2}e^{i\phi}$ and $\omega = e^{i\theta}$. By Theorem 1.1 both τ and $\tau\omega^{-1}$ are nonreal quadratic integers. This fact together with the assumption $\theta/\pi \in \mathbb{Q}$ immediately yield the following constraints on τ and ω:

(i) $\tau = \pm 1 \pm i$ or $\tau = \pm\sqrt{2}i$, and $\tau\omega^{-1} = \pm 1 \pm i$ or $\tau\omega^{-1} = \pm\sqrt{2}i$;

(ii) $\tau = \pm\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$ and $\omega^{-1} = \pm 1$.

To prove our lemma we would have to show that some of the combinations listed in (i) and (ii) are not possible. These are $\tau = \pm 1 \pm i$ and $\tau\omega^{-1} = \pm\sqrt{2}i$, or $\tau = \pm\sqrt{2}i$ and $\tau\omega^{-1} = \pm 1 \pm i$. Here we show that T cannot have $\tau = 1 + i$ and $\tau\omega^{-1} = \sqrt{2}i$. The impossibility of other combinations are proved by an identical argument.

Suppose that T does have $\tau = 1 + i$ and $\tau\omega^{-1} = \sqrt{2}i$. Without loss of generality we may assume T is the canonical 2-reptile. Then $\omega = e^{-i\frac{\pi}{4}}$ and

$$\sqrt{2}\,(T) = e^{-i\frac{\pi}{4}}T \cup (e^{-i\frac{\pi}{2}}T + \omega) = \omega T \cup (\omega^2 T + \omega).$$

Let $T_j := \omega^j T$ for $j = 0, 1, 2, 3$. Then (T_0, T_1, T_2, T_3) is a self-affine multi-tile with expansion factor $\sqrt{2}$:

$$\sqrt{2}(T_k) = T_{k+1} \cup (T_{k+2} + \omega^{k+1}), \quad k = 0, 1, 2, 3,$$

where $T_j := T_{j-4}$ for $j \geq 4$. The subdivision matrix is

$$S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

This matrix is primitive. It follows from Theorem 2.6 that $\sqrt{2}$ must be a complex Perron number, which it is a contradiction. By the same argument, it is impossible to have $\tau = \pm 1 \pm i$ and $\tau\omega^{-1} = \pm\sqrt{2}i$, or vice versa. ∎

4 Sufficient Conditions

In this section, we complete the proof of Theorem 1.2 by showing that the pairs (ϕ, θ) listed in the theorem indeed give rise to 2-reptiles. We will also study the equivalences of these tiles and classify them into six equivalence classes.

We begin by stating a theorem of Bandt [B, Theorem 2]. Let $A \in M_d(\mathbb{Z})$ be an expanding matrix with $q = |\det(\mathrm{A})|$. A finite group $\mathcal{S}$ of integer matrices with determinant ± 1 is called a *symmetry group* of A if $A\mathcal{S} = \mathcal{S}A$.

Theorem 4.1 (Bandt) *Let $A \in M_d(\mathbb{Z})$ be an expanding matrix with $q = |\det(\mathrm{A})|$. Suppose that $\{s_1, \ldots, s_q\}$ is contained in a symmetry group of A such that*

$$\mathbb{Z}^d = \bigcup_{i=1}^{q} s_i^{-1}(A\mathbb{Z}^d + b_i). \tag{4.1}$$

Then the attractor T of the IFS $\{f_i(x) = s_i A^{-1}x + b_i : \ 1 \leq i \leq q\}$ has nonempty interior and $T = \overline{T^o}$. Furthermore, T tiles $\mathbb{R}^d$.

Theorem 4.1 leads to the following lemma:

Lemma 4.2 *Let T be a 2-repset with a canonical representation (ϕ, θ). Suppose that $\phi = k\pi/4$ for an odd integer k and $\theta = \ell\pi/2$ for an integer ℓ. Then T is a 2-reptile.*

Proof. Clearly we only need to consider $k \in \{1, 3, 5, 7\}$ and $\ell \in \{0, 1, 2, 3\}$. For $k \in \{1, 3, 5, 7\}$ the corresponding τ's are $\tau = \pm 1 \pm i$. The maps $x \mapsto \tau x$ in $\mathbb{C}$ correspond to the maps $x \mapsto Ax$ in $\mathbb{R}^2$ with A being one of the following four matrices in $\mathcal{A}$:

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

Define

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Then $\mathcal{S}$ is a symmetry group for each $A \in \mathcal{A}$; in fact $A\mathcal{S} = \mathcal{A} = \mathcal{S}A$. It remains to be shown that for each $A \in \mathcal{A}$ and each $s \in \mathcal{S}$,

$$\mathbb{Z}^2 = A\mathbb{Z}^2 \cup \left(sA\mathbb{Z}^2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right). \tag{4.2}$$

Since $A\mathbb{Z}^2$ dilates $\mathbb{Z}^2$ by a factor of $\sqrt{2}$ and then rotates it through $k\pi/4$, the resulting lattice is invariant under a $\pi/2$-rotation. Hence, $sA\mathbb{Z}^2 = A\mathbb{Z}^2$ for all $s \in \mathbb{S}$, and (4.2) follows. ■

The proof for the sufficiency of Theorem 1.2 can be made easier after we explore the equivalences of the reptiles. Recall that two reptiles, or repsets in general, are *equivalent* if they can be obtained from each other by any combination of translation, rotation, dilation, and reflection. Since each 2-repset (other than the degenerate ones made up by a single point) is similar to a canonical 2-repset corresponding to some (ϕ, θ), we shall use the notation $(\phi, \theta) \sim (\phi', \theta')$ to denote the equivalence of two 2-repsets having the respective canonical representations.

Proposition 4.3 *Using the canonical representation (ϕ, θ) we have the following equivalent 2-repsets:*

(a) $(\phi, \theta) \sim (\phi - \theta, -\theta) \sim (-\phi, -\theta) \sim (-\phi + \theta, \theta)$.

(b) $(\phi, 0) \sim (\phi + \pi, 0) \sim (\phi, \pi) \sim (\phi + \pi, \pi)$.

Proof. Observe that

$$\sqrt{2}e^{i\phi}(T) = T \cup (e^{i\theta}T + 1) \tag{4.3}$$

is identical to

$$\sqrt{2}e^{i(\phi-\theta)}(T) = (T + e^{-i\theta}) \cup (e^{-i\theta}T),$$

which has canonical representation $(\phi - \theta, -\theta)$ by Lemma 3.2. So $(\phi, \theta) \sim (\phi - \theta, -\theta)$.

By taking complex conjugates we see that (4.3) is also identical to

$$\sqrt{2}e^{-i\phi}(\bar{T}) = \bar{T} \cup (e^{-i\theta}\bar{T} + 1),$$

where $\bar{T} := \{\bar{z} : \ z \in T\}$. Consequently the canonical 2-repset corresponding to $(-\phi, -\theta)$ is simply the complex conjugate (i.e. reflection about the real axis) of the canonical 2-repset corresponding to (ϕ, θ). This fact and the fact $(\phi, \theta) \sim (\phi - \theta, -\theta)$ immediately yield (a).

We now prove (b). Write $\tau = \sqrt{2}e^{i\phi}$. Since $\theta = 0$, (4.3) is

$$\tau(T) = T \cup (T + 1). \tag{4.4}$$

Let T_1 be the canonical 2-repset corresponding to $(\phi + \pi, 0)$:

$$-\tau(T_1) = T_1 \cup (T_1 + 1). \tag{4.5}$$

Iterating (4.4) and (4.5) yields

$$\begin{aligned}\tau^2(T) &= T \cup (T+1) \cup (T+\tau) \cup (T+1+\tau), &(4.6)\\ \tau^2(T_1) &= T_1 \cup (T_1+1) \cup (T_1-\tau) \cup (T_1+1-\tau). &(4.7)\end{aligned}$$

Set $T_2 = T_1 - \tau/(\tau^2-1)$. Then (4.7) becomes

$$\tau^2(T_2) = (T_2+\tau) \cup (T_2+1+\tau) \cup T_2 \cup (T_2+1), \tag{4.8}$$

The uniqueness now implies that $T_2 = T$. Since T_2 is a translate of T_1, this proves $(\phi, 0) \sim (\phi+\pi, 0)$.

Finally, if T satisfies the equation (4.4) then so does $T_3 = -T + a$ for $a = 1/(\tau-1)$, a fact that is easy to verify. Hence $T = -T + a$. Substituting this into (4.4) yields

$$\tau(T) = T \cup (-T + a + 1).$$

Note that $a + 1 \neq 0$. It follows from Lemma 3.2 that T has a canonical representation (ϕ, π). This proves $(\phi, 0) \sim (\phi, \pi)$. Of course, it also yields $(\phi, 0) \sim (\phi+\pi, \pi)$ because $(\phi, \pi) \sim (\phi+\pi, \pi)$ by (a). ∎

Lemma 4.4 *A 2-repset T having a canonical representation (ϕ, θ) in the form of $\phi = k\pi/2$ for $k \in \{1, 3\}$ and $\theta \in \{0, \pi\}$ is a 2-reptile. In fact, T is a rectangle whose long and short sides have a length ratio of $\sqrt{2}$.*

Proof. Consider the case $(\phi, \theta) = (\pi/2, 0)$. It follows from a direct check that the rectangle $[2\sqrt{2}/3, \sqrt{2}/3] \times [-4/3, 2/3]$ satisfies

$$\sqrt{2}e^{i\pi/2}(T) = T \cup (T+1).$$

Hence for $(\phi, \theta) = (\pi/2, 0)$ the corresponding canonical 2-repset T is the aforementioned rectangle. By Proposition 4.3 (b) all the other cases yield equivalent 2-repsets. This proves the lemma. ∎

Lemma 4.5 *A 2-repset T having a canonical representation in the form of $\phi = k\pi \pm \tan^{-1}(\sqrt{7})$ for $k \in \{0, 1\}$ and $\theta \in \{0, \pi\}$ is a 2-reptile.*

Proof. Again, by Proposition 4.3 it suffices to consider the case $(\phi, \theta) = (\tan^{-1}(\sqrt{7}), 0)$; all other cases yield equivalent 2-repsets. $\tau = 1/2 + i\sqrt{7}/2$ corresponds to the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{7} \\ \sqrt{7} & 1 \end{bmatrix}.$$

With respect to the basis $\{(1,0), (1/2, -\sqrt{7}/2)\}$, A takes the form $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. This gives a self-affine 2-reptile known as the tame twindragon (see [B]). ∎

Proof of Theorem 1.2. The theorem is now the consequence of the combination of Lemma 3.4 and Lemmas 4.2, 4.4, and 4.5. ∎

Proof of Theorem 1.3. Among the sixteen 2-reptiles stated in Theorem 1.2 (a), $(\pi/4, \theta)$ and $(7\pi/4, 2\pi - \theta)$ are reflections of each other, so are $(3\pi/4, \theta)$ and $(5\pi/4, 2\pi - \theta)$. We need only consider the remaining eight 2-reptiles. By Proposition 4.3, $(\frac{\pi}{4}, 0)$, $(\frac{\pi}{4}, \pi)$, $(\frac{3\pi}{4}, 0)$, and $(\frac{3\pi}{4}, \pi)$ are all equivalent. The same is true for $(\frac{\pi}{4}, \frac{3\pi}{2})$ and $(\frac{3\pi}{4}, \frac{\pi}{2})$. Hence, there are only four equivalence classes in this collection. They are represented by

$$(\frac{\pi}{4}, 0), \quad (\frac{\pi}{4}, \frac{\pi}{2}), \quad (\frac{\pi}{4}, \frac{3\pi}{2}), \quad (\frac{3\pi}{4}, \frac{3\pi}{2}).$$

They correspond respectively to the well-known twindragon, Lévy dragon, Heighway dragon (see [DK], [E]), and the 45° right-angled triangle.

By Lemma 4.4 the four 2-reptiles in Theorem 1.2 (b) are rectangles and hence equivalent. The reader can easily check by using Proposition 4.3 and Lemma 4.5 that the eight reptiles in Theorem 1.2 (c) are all equivalent; they are all tame twindragons.

Altogether the twenty-eight 2-reptiles account for all six equivalence classes listed in Theorem 1.3.

To show that the six 2-reptiles are mutually non-equivalent, we first notice that two reptiles can be equivalent only when their boundaries have the same Hausdorff dimension. For each of the six 2-reptiles here, the dimension of its boundary is known. For the triangle and the rectangle the dimension is 1; for the others the dimension is given by $\dim_H(\partial T) = 2 \log \lambda_{\max} / \log 2$, where $\lambda_{\max}$ is the largest eigenvalue of some characteristic polynomial associated to the tile (see [V], [SW], [DKe], [DKV], [E], [KLSW]). We summarized the results below:

2-reptile	Characteristic polynomial	Dimension of boundary
twindragon	$\lambda^3 - \lambda^2 - 2$	1.5236270862 . . .
Lévy dragon	$\lambda^9 - 3\lambda^8 + 3\lambda^7 - 3\lambda^6 + 2\lambda^5 + 4\lambda^4 - 8\lambda^3 + 8\lambda^2 - 16\lambda + 8$	1.9340071829 . . .
Heighway dragon	$\lambda^3 - \lambda^2 - 2$	1.5236270862 . . .
tame twindragon	$\lambda^3 - \lambda - 2$	1.2107605332 . . .

Since the triangle and the rectangle are clearly non-equivalent, it suffices to show that the twindragon and the Heighway dragon are also non-equivalent. Let T_t be the twindragon satisfying

$$(1 + i)T_t = T_t \cup (T_t + 1). \tag{4.9}$$

Then $T_t' = -T_t - i$ also satisfies (4.9). So $T_t' = T_t$ and hence the twindragon is centrally symmetric. We show that the Heighway dragon is not. Assume that the Heighway dragon T_H satisfies

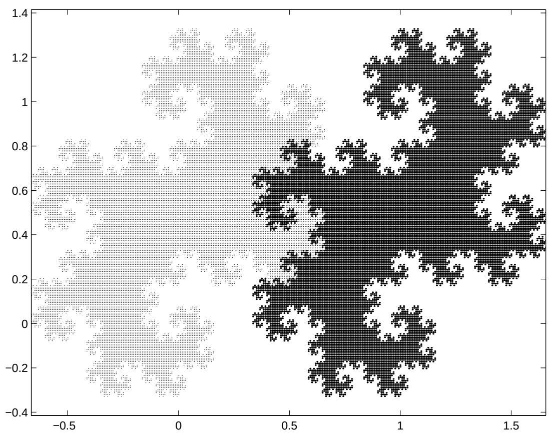
$$(1 + i)T_H = T_H \cup (-iT_H + 1)$$

and is centrally symmetric, i.e. $T_H = -T_H + a$ for some $a \in \mathbb{C}$. Then

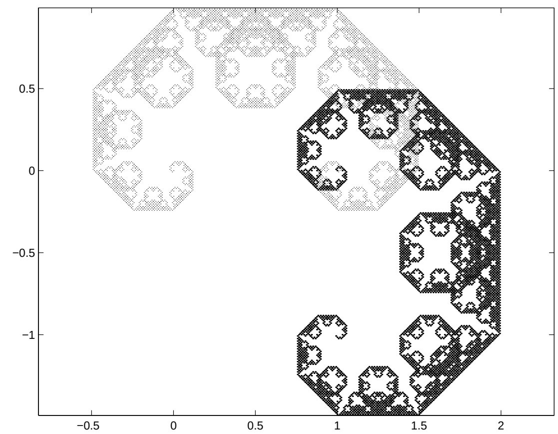
$$(1 + i)T_H = T_H \cup (iT_H + 1 - ia). \tag{4.10}$$

But (4.10) is satisfied by the Lévy dragon. This is a contradiction. Therefore the twindragon and the Heighway dragon are non-equivalent. This proves the theorem. ∎

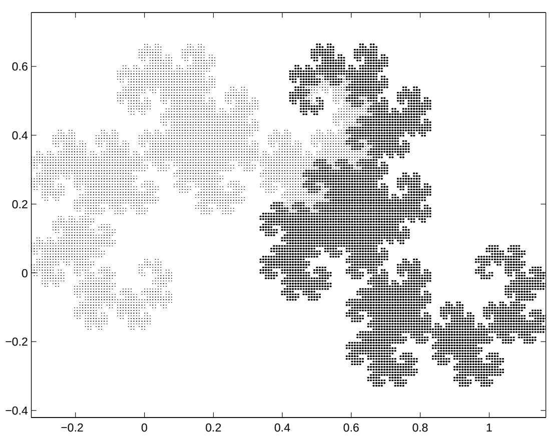
For convenience of the reader, we include in Figure 4.1 pictures of four of the 2-reptiles.



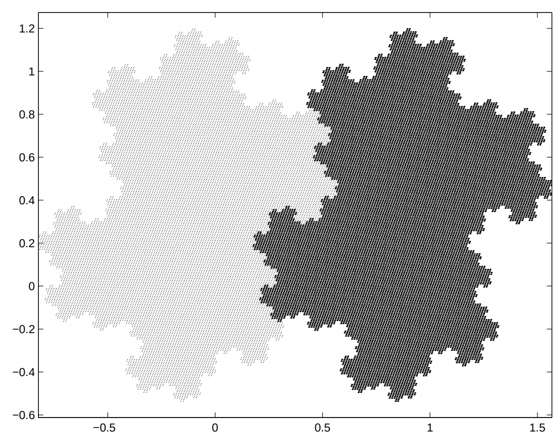
(a) Twindragon



(b) Lévy dragon



(c) Heighway dragon



(d) Tame twindragon

Figure 4.1: 2-reptiles with (ϕ, θ) values given by (a) $(\frac{\pi}{4}, 0)$, (b) $(\frac{\pi}{4}, \frac{\pi}{2})$, (c) $(\frac{\pi}{4}, \frac{3\pi}{2})$, (d) $(\tan^{-1}(\sqrt{7}), 0)$.

5 Some Geometric and Topological Properties

In this section we prove that a rational 2-reptile cannot be at the same time a rational 3-reptile. We then show that all 2-repsets in $\mathbb{R}^d$ are connected.

Proof of Theorem 1.4. Suppose that the theorem is false. Then there exists a T that is both a rational 2-reptile and a rational 3-reptile such that

$$\tau(T) = T \cup (\omega T + 1) \quad \text{and} \quad \sigma(T) = (T + a_1) \cup (\omega_2 T + a_2) \cup (\omega_3 T + a_3), \tag{5.1}$$

where $|\omega| = |\omega_1| = |\omega_2| = 1$, $|\tau| = \sqrt{2}$ and $|\sigma| = \sqrt{3}$. Note that T must also be a rational 6-reptile satisfying

$$\tau\sigma(T) = \bigcup_{j=1}^{6} (e^{i\theta_j} T + b_j)$$

in which $\theta_1 = 0$. It follows from Theorem 1.1 that τ, σ and $\tau\sigma$ are all nonreal quadratic integers.

Since σ is a nonreal quadratic integer of modulus $\sqrt{3}$, we have

$$\sigma = \frac{a \pm i\sqrt{12 - a^2}}{2}, \quad \text{where } a \in \{0, \pm 1, \pm 2, \pm 3\}.$$

The only combinations τ and σ that make $\tau\sigma$ a nonreal quadratic integer are $\tau = \pm\sqrt{2}i$ and $\sigma = \pm 1 \pm \sqrt{2}i$. But in these cases the corresponding 2-reptile is a rectangle. Since the angle of rotation by σ is not a multiple of $\pi/2$, $\sigma(T)$ can never be tiled by copies of T in which one of the copies is a translation of T, for some corners will not fit. So the second equation of (5.1) cannot be satisfied. This is a contradiction. ■

To prove the connectedness of 2-repsets in $\mathbb{R}^d$, we first state a proposition. The proof of it is essentially a direct generalization of [KL, Theorem 4.3].

Proposition 5.1 *Let $A_1, \ldots, A_n \in M_d(\mathbb{R})$ be expanding matrices and $d_1, \ldots, d_n \in \mathbb{R}^d$. Let T be the unique compact set given by*

$$T = \bigcup_{j=1}^{n} A_j^{-1}(T + d_j).$$

Denote $\mathcal{T} = \{A_j^{-1}(T + d_j) : \ 1 \le j \le n\}$. Suppose that for each pair $U, V \in \mathcal{T}$ there exists a finite subcollection $\{U_1, \ldots, U_\ell\} \subseteq \mathcal{T}$ such that $U = U_1$, $V = U_\ell$, and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, \ell - 1$. Then T is connected.

It is known that all self-affine tiles in $\mathbb{R}^d$ with two digits are connected (see [HSV]). The following corollary generalizes this result to 2-repsets. For $E \subseteq \mathbb{R}^d$ and $\epsilon > 0$, we call the set $E_\epsilon := \{x : \mathrm{dist}(x, E) < \epsilon\}$ the *(open) ϵ-neighborhood* of E.

Corollary 5.2 *A 2-repset in $\mathbb{R}^d$ is connected.*

Proof. Let T be defined by

$$T = A_1^{-1}(T + d_1) \cup A_2^{-1}(T + d_2), \tag{5.2}$$

where both A_1^{-1} and A_2^{-1} have contraction ratio $2^{-1/d}$. According to Proposition 5.1 we need only to show that

$$A_1^{-1}(T + d_1) \cap A_2^{-1}(T + d_2) \neq \emptyset.$$

Let us suppose that the intersection is empty. By iterating (5.2) we see that T is totally disconnected. Therefore it must have Lebesgue measure zero. Since the two sets on the righthand side of (5.2) are compact, there must be an $\epsilon > 0$ such that their ϵ-neighborhoods do not intersect either. Thus the IFS given by (5.2) satisfies the open set condition [H]. Hence the Hausdorff dimension of T must be d, and the Hausdorff measure in its dimension, which is the Lebesgue measure, must be strictly positive. This is a contradiction. ∎

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