About Factorization of Quantum States with Few Qubits

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Abstract

Through the study of the factorization conditions of a wave function made up of two, three and four qubits, we propose an analytical expression which can characterize entangled states in terms of the coefficients of the wave function and density matrix elements.

Keywords

Entangled State, Factorized State, Decoherence, Quantum Computing and Information, Chain of Spins System

1. Introduction

In quantum mechanics, multiple dimensional or multiple particles systems are characterized by the tensor product of the Hilbert subspaces [1], where each subspace is associated to each element. It is well known [2] that when there is no interaction among these elements, the wave function is just the tensor product of the wave functions of each element; that is, the tensor product of the wave function associated to each element determines the non-interacting characteristic of the elements in a quantum system. If interaction occurs at some time among these elements, this tensor product disappears, and the wave function becomes entangled [3]. So, it is necessary to point out that if the wave function is not factorized, the wave function is entangled. In this way, the characterization of factorization is somewhat equivalent to the characterization of entanglement. In this paper, we will follow this line of ideas to determine the characterization of an entangled state [4]-[12].

2. Factorized State

When a quantum systems is made up of several quantum subsystems, where the

ith-subsystem is characterized by a Hamiltonian H_i and a Hilbert space ε_i, corresponding a two-states and having a basis $\left\{ \left| \xi_i \right\rangle \right\}_{\xi_i = 0,1}$, the Hilbert space is written as the tensorial product of each subsystem, $\varepsilon = \varepsilon_n \otimes \cdots \otimes \varepsilon_1$ (for n-subsystems), The state in each subsystems is defined as a qubit in quantum computation and information theory [13] and is given by

$$\left| \psi_i \right\rangle = a_i \left| 0 \right\rangle + b_i \left| 1 \right\rangle, \quad \left| a_i \right|^2 + \left| b_i \right|^2 = 1, \tag{1}$$

where $\left\{ \left| 0 \right\rangle, \left| 1 \right\rangle \right\}$ is the basis of the two-states Hilbert subspace ε_i. A general state $\left| \Psi \right\rangle$ in the Hilbert space ε can be written as

$$\left| \Psi \right\rangle = \sum_\xi C_\xi \left| \xi \right\rangle, \quad \text{with} \quad \sum_\xi \left| C_\xi \right|^2 = 1, \tag{2}$$

where $C'_\xi s$ are complex numbers, and $\left| \xi \right\rangle$ is an element of the basis of ε,

$$\left| \xi \right\rangle = \left| \xi_n \cdots \xi_1 \right\rangle = \left| \xi_n \right\rangle \otimes \cdots \otimes \left| \xi_1 \right\rangle, \quad \xi_k = 0,1 \quad k = 1, \cdots, n. \tag{3}$$

A full factorized state in this space is

$$\left| \Psi \right\rangle = \left| \psi_n \right\rangle \otimes \cdots \otimes \left| \psi_1 \right\rangle, \tag{4}$$

where $\left| \psi_k \right\rangle$ is given by (1).

For $n = 2$, one has a general state $\left| \Psi \right\rangle \in \varepsilon$,

$$\left| \Psi \right\rangle = C_1 \left| 00 \right\rangle + C_2 \left| 01 \right\rangle + C_3 \left| 10 \right\rangle + C_4 \left| 11 \right\rangle, \tag{5}$$

where we have chosen to use decimal notation for the coefficients. Let us assume that this state can be written as

$$\left| \Psi \right\rangle = \left| \psi_2 \right\rangle \otimes \left| \psi_1 \right\rangle, \tag{6}$$

with $\left| \psi_k \right\rangle, k = 1, 2$ given by (1). Then, substituting (1) in (6) and equaling coefficients with (5), it follows that

$$C_1 = a_1 a_2, \ C_2 = a_1 b_2, \ C_3 = b_1 a_2, \ C_4 = b_1 b_2. \tag{7}$$

From these expressions one obtains a single condition for factorization,

$$C_1 C_4 = C_2 C_3. \tag{8}$$

Thus, if this condition is not satisfied, the state (5) represents an entangled state. So, one can use the following known expression [14] as a characterization of an entangled state

$$C_\Psi^{(2)} = 2 \left| C_1 C_4 - C_2 C_3 \right|. \tag{9}$$

For $n = 3$, a general state in the Hilbert space ε is

$$\begin{aligned} \left| \Psi \right\rangle = &\ C_1 \left| 00 \right\rangle + C_2 \left| 001 \right\rangle + C_3 \left| 010 \right\rangle + C_4 \left| 011 \right\rangle \\ &+ C_5 \left| 100 \right\rangle + C_6 \left| 101 \right\rangle + C_7 \left| 110 \right\rangle + C_8 \left| 111 \right\rangle, \end{aligned} \tag{10}$$

where, we have chosen again decimal notation for the coefficients. Assuming that this wave function can be written as $\left| \Psi \right\rangle = \left| \psi_3 \right\rangle \otimes \left| \psi_2 \right\rangle \otimes \left| \psi_1 \right\rangle$ with $\left| \psi_k \right\rangle$ given by (1), and after some identifications (as before) and rearrangements, one gets

$$\frac{C_1}{C_2} = \frac{C_3}{C_4} = \frac{C_5}{C_6} = \frac{C_7}{C_8} = \frac{a_3}{b_3}, \quad b_3 \neq 0 \tag{11a}$$

$$\frac{C_1}{C_3} = \frac{C_2}{C_4} = \frac{C_5}{C_7} = \frac{C_6}{C_8} = \frac{a_2}{b_2}, \quad b_2 \neq 0 \tag{11b}$$

$$\frac{C_1}{C_5} = \frac{C_2}{C_6} = \frac{C_3}{C_7} = \frac{C_4}{C_8} = \frac{a_1}{b_1}, \quad b_1 \neq 0 \tag{11c}$$

These expression reflex a parallelism between the complex vectors (C_1, C_3, C_5, C_7) and (C_2, C_4, C_6, C_8), the vectors (C_1, C_2, C_5, C_6) and (C_3, C_4, C_7, C_8), and the vectors (C_1, C_2, C_3, C_4) and (C_5, C_6, C_7, C_8). In addition, they bring about the following eight independent relations

$$C_1 C_4 - C_2 C_3 = 0, \; C_1 C_6 - C_2 C_5 = 0, \; C_1 C_8 - C_2 C_7 = 0, \; C_3 C_6 - C_4 C_5 = 0 \tag{12}$$

$$C_3 C_8 - C_4 C_7 = 0, \; C_5 C_8 - C_6 C_7 = 0, \; C_1 C_7 - C_3 C_5 = 0, \; C_2 C_8 - C_4 C_6 = 0. \tag{13}$$

If one of these expressions is not satisfied, the wave function (10) represents an entangled state. Therefore, one can propose the following expression to characterize an entangled state

$$\begin{aligned} C_\Psi^{(3)} = {} & 2|C_1 C_4 - C_2 C_3| + 2|C_1 C_6 - C_2 C_5| + 2|C_1 C_8 - C_2 C_7| + 2|C_3 C_6 - C_4 C_5| \\ & + 2|C_3 C_8 - C_4 C_7| + 2|C_5 C_8 - C_6 C_7| + 2|C_1 C_7 - C_3 C_5| + 2|C_2 C_8 - C_4 C_6|. \end{aligned} \tag{14}$$

For $n = 4$, a general state in the Hilbert space ε is of the form

$$\begin{aligned} |\Psi\rangle = {} & C_1 |0000\rangle + C_2 |0001\rangle + C_3 |010\rangle + C_4 |0011\rangle \\ & + C_5 |0100\rangle + C_6 |0101\rangle + C_7 |0110\rangle + C_8 |0111\rangle \\ & + C_9 |100\rangle + C_{10} |1001\rangle + C_{11} |110\rangle + C_{12} |1011\rangle \\ & + C_{13} |1100\rangle + C_{14} |1101\rangle + C_{15} |1110\rangle + C_{16} |1111\rangle. \end{aligned} \tag{15}$$

Assuming this function can be expressed as $|\Psi\rangle = |\psi_4\rangle \otimes |\psi_3\rangle \otimes |\psi_2\rangle \otimes |\psi_1\rangle$ with $|\psi_k\rangle, k = 1, 2, 3, 4$ given by (1), and after some identifications and rearrangements, one gets

$$\frac{C_1}{C_2} = \frac{C_3}{C_4} = \frac{C_5}{C_6} = \frac{C_7}{C_8} = \frac{C_9}{C_{10}} = \frac{C_{11}}{C_{12}} = \frac{C_{13}}{C_{14}} = \frac{C_{15}}{C_{16}}$$

$$\frac{C_1}{C_3} = \frac{C_2}{C_4} = \frac{C_5}{C_7} = \frac{C_6}{C_8} = \frac{C_9}{C_{11}} = \frac{C_{10}}{C_{12}} = \frac{C_{13}}{C_{15}} = \frac{C_{14}}{C_{16}}$$

$$\frac{C_1}{C_5} = \frac{C_2}{C_6} = \frac{C_3}{C_7} = \frac{C_4}{C_8} = \frac{C_9}{C_{13}} = \frac{C_{10}}{C_{14}} = \frac{C_{11}}{C_{15}} = \frac{C_{12}}{C_{16}}$$

$$\frac{C_1}{C_9} = \frac{C_2}{C_{10}} = \frac{C_3}{C_{11}} = \frac{C_4}{C_{12}} = \frac{C_5}{C_{13}} = \frac{C_6}{C_{14}} = \frac{C_7}{C_{15}} = \frac{C_8}{C_{16}},$$

expressing similar parallelism we mentioned before. Each row gives us 28 relations, having a total of 112 possible relations, and from these relations, one can get the following 36 independent conditions

$$C_1 C_4 - C_2 C_3 = 0 \quad C_4 C_{13} - C_7 C_{10} = 0 \quad C_1 C_6 - C_2 C_5 = 0 \quad C_4 C_{14} - C_6 C_{12} = 0 \tag{16a}$$

$$C_1 C_8 - C_3 C_6 = 0 \quad C_4 C_{15} - C_3 C_{16} = 0 \quad C_1 C_{10} - C_2 C_9 = 0 \quad C_4 C_{16} - C_8 C_{12} = 0 \tag{16b}$$

$$C_1 C_{11} - C_3 C_9 = 0 \quad C_5 C_8 - C_6 C_7 = 0 \quad C_1 C_{12} - C_2 C_{11} = 0 \quad C_5 C_{14} - C_6 C_{13} = 0 \tag{16c}$$

$$C_1 C_{14} - C_9 C_6 = 0 \quad C_5 C_{15} - C_7 C_{13} = 0 \quad C_1 C_{15} - C_5 C_{11} = 0 \quad C_5 C_{16} - C_7 C_{14} = 0 \tag{16d}$$

$$C_2 C_8 - C_4 C_6 = 0 \quad C_6 C_{11} - C_5 C_{12} = 0 \quad C_2 C_{12} - C_4 C_{10} = 0 \quad C_6 C_{15} - C_{16} C_8 = 0 \tag{16e}$$

$$C_2 C_{13} - C_5 C_{10} = 0 \quad C_6 C_{16} - C_8 C_{14} = 0 \quad C_2 C_{14} - C_6 C_{10} = 0 \quad C_7 C_{16} - C_8 C_{15} = 0 \tag{16f}$$

$$C_2 C_{16} - C_{10} C_8 = 0 \quad C_7 C_{12} - C_8 C_{11} = 0 \quad C_3 C_8 - C_4 C_7 = 0 \quad C_9 C_{12} - C_{10} C_{11} = 0 \tag{16g}$$

$$C_3 C_{15} - C_7 C_{11} = 0 \quad C_9 C_{14} - C_{10} C_{11} = 0 \quad C_3 C_{13} - C_{11} C_5 = 0 \quad C_9 C_{15} - C_{11} C_{13} = 0 \tag{16h}$$

$$C_{10} C_{16} - C_{12} C_{14} = 0 \quad C_{11} C_{16} - C_{12} C_{15} = 0 \quad C_{10} C_{15} - C_{11} C_{14} = 0 \quad C_{13} C_{16} - C_{14} C_{15} = 0. \tag{16i}$$

Again, if one of these expression fail to happen, (15) will represents an entangled state. Thus, one can propose the following expression to characterize an entangled state made up of 4-qubits basis

$$
\begin{aligned}
C_\Psi^{(4)} = {} & 2\left|C_1C_4 - C_2C_3\right| + 2\left|C_4C_{13} - C_7C_{10}\right| + 2\left|C_1C_6 - C_2C_5\right| + 2\left|C_4C_{14} - C_6C_{12}\right| + 2\left|C_1C_8 - C_3C_6\right| + 2\left|C_4C_{15} - C_3C_{16}\right| \\
& + 2\left|C_1C_{10} - C_2C_9\right| + 2\left|C_4C_{16} - C_8C_{12}\right| + 2\left|C_1C_{11} - C_3C_9\right| + 2\left|C_5C_8 - C_6C_7\right| + 2\left|C_1C_{12} - C_2C_{11}\right| \\
& + 2\left|C_5C_{14} - C_6C_{13}\right| + 2\left|C_1C_{14} - C_9C_6\right| + 2\left|C_5C_{15} - C_7C_{13}\right| + 2\left|C_1C_{15} - C_5C_{11}\right| + 2\left|C_5C_{16} - C_7C_{14}\right| \\
& + 2\left|C_2C_8 - C_4C_6\right| + 2\left|C_6C_{11} - C_5C_{11}\right| + 2\left|C_2C_{12} - C_4C_{10}\right| + 2\left|C_6C_{15} - C_{16}C_8\right| + 2\left|C_2C_{13} - C_5C_{10}\right| \\
& + 2\left|C_6C_{16} - C_8C_{14}\right| + 2\left|C_2C_{14} - C_6C_{10}\right| + 2\left|C_7C_{16} - C_8C_{15}\right| + 2\left|C_2C_{16} - C_{10}C_8\right| + 2\left|C_7C_{12} - C_8C_{11}\right| \\
& + 2\left|C_3C_8 - C_4C_7\right| + 2\left|C_9C_{12} - C_{10}C_{11}\right| + 2\left|C_3C_{15} - C_7C_{11}\right| + 2\left|C_9C_{14} - C_{10}C_{11}\right| + 2\left|C_3C_{13} - C_{11}C_5\right| \\
& + 2\left|C_9C_{15} - C_{11}C_{13}\right| + 2\left|C_{10}C_{16} - C_{12}C_{14}\right| + 2\left|C_{11}C_{16} - C_{12}C_{15}\right| + 2\left|C_{10}C_{15} - C_{11}C_{14}\right| + 2\left|C_{13}C_{16} - C_{14}C_{15}\right|.
\end{aligned}
\tag{17}
$$

As we can see from these examples, the number of conditions needed to characterize a factorized state (or entangled state) grows exponentially with the number of qubits. So, characterization of an entangled state for n-qubits in general becomes a very hard work. Now, in terms of the density matrix elements, one could have the characterization of entangled states made up of 2, 3 and 4 qubits as

$$
C_\rho^{(2)} = 2\sqrt{\rho_{11}\rho_{44} + \rho_{22}\rho_{33} - 2\,\text{Re}\left(\rho_{12}\rho_{43}\right)}.
\tag{18}
$$

$$
\begin{aligned}
C_\rho^{(3)} = {} & 2\sqrt{\rho_{11}\rho_{44} + \rho_{22}\rho_{33} - 2\,\text{Re}\left(\rho_{12}\rho_{43}\right)} + 2\sqrt{\rho_{11}\rho_{66} + \rho_{22}\rho_{55} - 2\,\text{Re}\left(\rho_{12}\rho_{65}\right)} + 2\sqrt{\rho_{11}\rho_{88} + \rho_{22}\rho_{77} - 2\,\text{Re}\left(\rho_{12}\rho_{87}\right)} \\
& + 2\sqrt{\rho_{33}\rho_{66} + \rho_{44}\rho_{55} - 2\,\text{Re}\left(\rho_{34}\rho_{65}\right)} + 2\sqrt{\rho_{33}\rho_{88} + \rho_{44}\rho_{77} - 2\,\text{Re}\left(\rho_{34}\rho_{87}\right)} + 2\sqrt{\rho_{55}\rho_{88} + \rho_{66}\rho_{77} - 2\,\text{Re}\left(\rho_{56}\rho_{87}\right)} \\
& + 2\sqrt{\rho_{11}\rho_{77} + \rho_{33}\rho_{55} - 2\,\text{Re}\left(\rho_{13}\rho_{75}\right)} + 2\sqrt{\rho_{22}\rho_{88} + \rho_{44}\rho_{66} - 2\,\text{Re}\left(\rho_{24}\rho_{86}\right)}.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
C_\rho^{(4)} = {} & 2\sqrt{\rho_{11}\rho_{44} + \rho_{22}\rho_{33} - 2\,\text{Re}\left(\rho_{12}\rho_{43}\right)} + 2\sqrt{\rho_{44}\rho_{13,13} + \rho_{77}\rho_{10,10} - 2\,\text{Re}\left(\rho_{47}\rho_{13,10}\right)} + 2\sqrt{\rho_{11}\rho_{66} + \rho_{22}\rho_{55} - 2\,\text{Re}\left(\rho_{12}\rho_{65}\right)} \\
& + 2\sqrt{\rho_{44}\rho_{14,14} + \rho_{66}\rho_{12,12} - 2\,\text{Re}\left(\rho_{46}\rho_{14,12}\right)} + 2\sqrt{\rho_{11}\rho_{88} + \rho_{33}\rho_{66} - 2\,\text{Re}\left(\rho_{13}\rho_{86}\right)} + 2\sqrt{\rho_{44}\rho_{15,15} + \rho_{33}\rho_{16,16} - 2\,\text{Re}\left(\rho_{43}\rho_{15,16}\right)} \\
& + 2\sqrt{\rho_{11}\rho_{10,10} + \rho_{22}\rho_{99} - 2\,\text{Re}\left(\rho_{12}\rho_{10,9}\right)} + 2\sqrt{\rho_{44}\rho_{16,16} + \rho_{88}\rho_{12,12} - 2\,\text{Re}\left(\rho_{48}\rho_{16,12}\right)} + 2\sqrt{\rho_{11}\rho_{11,11} + \rho_{33}\rho_{99} - 2\,\text{Re}\left(\rho_{13}\rho_{11,9}\right)} \\
& + 2\sqrt{\rho_{55}\rho_{88} + \rho_{66}\rho_{77} - 2\,\text{Re}\left(\rho_{56}\rho_{87}\right)} + 2\sqrt{\rho_{11}\rho_{12,12} + \rho_{22}\rho_{11,11} - 2\,\text{Re}\left(\rho_{12}\rho_{12,11}\right)} + 2\sqrt{\rho_{55}\rho_{14,14} + \rho_{66}\rho_{13,13} - 2\,\text{Re}\left(\rho_{56}\rho_{14,13}\right)} \\
& + 2\sqrt{\rho_{11}\rho_{14,14} + \rho_{99}\rho_{66} - 2\,\text{Re}\left(\rho_{19}\rho_{14,6}\right)} + 2\sqrt{\rho_{55}\rho_{15,15} + \rho_{77}\rho_{13,13} - 2\,\text{Re}\left(\rho_{57}\rho_{15,13}\right)} + 2\sqrt{\rho_{11}\rho_{15,15} + \rho_{55}\rho_{11,11} - 2\,\text{Re}\left(\rho_{15}\rho_{15,11}\right)} \\
& + 2\sqrt{\rho_{55}\rho_{16,16} + \rho_{77}\rho_{14,14} - 2\,\text{Re}\left(\rho_{57}\rho_{16,14}\right)} + 2\sqrt{\rho_{22}\rho_{88} + \rho_{44}\rho_{66} - 2\,\text{Re}\left(\rho_{24}\rho_{86}\right)} + 2\sqrt{\rho_{66}\rho_{11,11} + \rho_{55}\rho_{12,12} - 2\,\text{Re}\left(\rho_{6,5}\rho_{11,12}\right)} \\
& + 2\sqrt{\rho_{22}\rho_{12,12} + \rho_{44}\rho_{10,10} - 2\,\text{Re}\left(\rho_{24}\rho_{12,10}\right)} + 2\sqrt{\rho_{66}\rho_{15,15} + \rho_{16,16}\rho_{88} - 2\,\text{Re}\left(\rho_{6,16}\rho_{15,8}\right)} \\
& + 2\sqrt{\rho_{22}\rho_{13,13} + \rho_{55}\rho_{10,10} - 2\,\text{Re}\left(\rho_{25}\rho_{13,10}\right)} + 2\sqrt{\rho_{66}\rho_{16,16} + \rho_{88}\rho_{14,14} - 2\,\text{Re}\left(\rho_{68}\rho_{16,14}\right)} \\
& + 2\sqrt{\rho_{22}\rho_{14,14} + \rho_{66}\rho_{10,10} - 2\,\text{Re}\left(\rho_{2,6}\rho_{14,10}\right)} + 2\sqrt{\rho_{77}\rho_{16,16} + \rho_{88}\rho_{15,15} - 2\,\text{Re}\left(\rho_{78}\rho_{16,15}\right)} \\
& + 2\sqrt{\rho_{22}\rho_{16,16} + \rho_{10,10}\rho_{88} - 2\,\text{Re}\left(\rho_{2,10}\rho_{16,8}\right)} + 2\sqrt{\rho_{77}\rho_{12,12} + \rho_{88}\rho_{11,11} - 2\,\text{Re}\left(\rho_{78}\rho_{12,11}\right)} \\
& + 2\sqrt{\rho_{33}\rho_{88} + \rho_{44}\rho_{77} - 2\,\text{Re}\left(\rho_{34}\rho_{87}\right)} + 2\sqrt{\rho_{99}\rho_{12,12} + \rho_{10,10}\rho_{11,11} - 2\,\text{Re}\left(\rho_{9,10}\rho_{12,11}\right)} + 2\sqrt{\rho_{33}\rho_{15,15} + \rho_{77}\rho_{11,11} - 2\,\text{Re}\left(\rho_{37}\rho_{15,11}\right)} \\
& + 2\sqrt{\rho_{99}\rho_{14,14} + \rho_{10,10}\rho_{11,11} - 2\,\text{Re}\left(\rho_{9,10}\rho_{14,11}\right)} + 2\sqrt{\rho_{33}\rho_{13,13} + \rho_{11,11}\rho_{55} - 2\,\text{Re}\left(\rho_{3,11}\rho_{13,5}\right)} \\
& + 2\sqrt{\rho_{99}\rho_{15,15} + \rho_{11,11}\rho_{13,13} - 2\,\text{Re}\left(\rho_{9,11}\rho_{1513}\right)} + 2\sqrt{\rho_{10,10}\rho_{16,16} + \rho_{12,12}\rho_{14,14} - 2\,\text{Re}\left(\rho_{10,12}\rho_{1614}\right)} \\
& + 2\sqrt{\rho_{11,11}\rho_{16,16} + \rho_{12,12}\rho_{15,15} - 2\,\text{Re}\left(\rho_{11,12}\rho_{16,15}\right)} \\
& + 2\sqrt{\rho_{10,10}\rho_{15,15} + \rho_{11,11}\rho_{14,14} - 2\,\text{Re}\left(\rho_{10,11}\rho_{15,14}\right)} + 2\sqrt{\rho_{13,13}\rho_{16,16} + \rho_{14,14}\rho_{15,15} - 2\,\text{Re}\left(\rho_{13,14}\rho_{16,15}\right)}
\end{aligned}
\tag{20}
$$

For other considerations of entanglement multiqubit entanglement see [6] [10] [15] [16]. **Figure 1** below shows the values of the expressions (14) and (19) for 50 entangled states made up of 3-qubits basis and with values C_j, with $j = 1, \cdots, 6$ randomly generated. As we can see, the values obtained with the coefficients $C_j's$ and with the density matrix elements ρ_{nm} are the same. **Figure 2** shows for the $|W\rangle$ state,

$$|W\rangle = C_2 |001\rangle + C_3 |010\rangle + C_5 |100\rangle, \quad |C_2|^2 + |C_3|^2 + |C_5|^2 = 1, \tag{21}$$

the possible values of $C^{(3)}(\Psi)$. As we can see, there are four possible maxima corresponding to the values $C_2 = C_3 = C_5 = \pm 1/\sqrt{3}$ and four maxima corres-
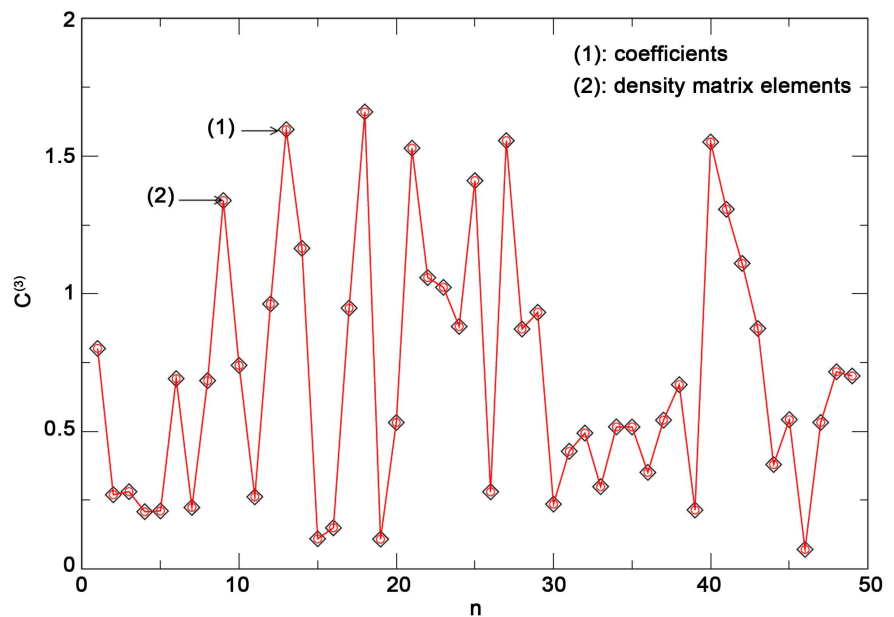
Figure 1. $C^{(3)}$ for arbitrary entangled state. Diamonds (density matrix) and circles (coefficients).
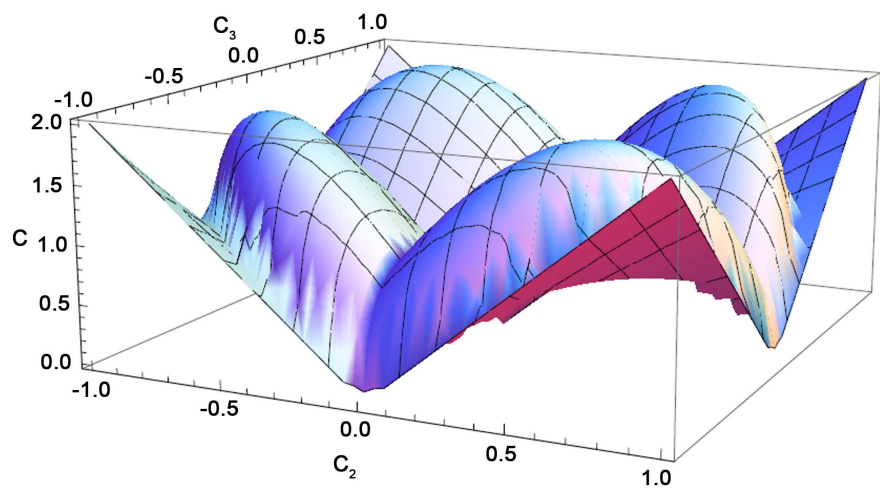
Figure 2. $C^{(3)}$ for the state $|W\rangle = C_1 |001\rangle + C_2 |010\rangle + C_3 |100\rangle$ such that $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$.

ponding to the values $C_5 = 0$, $C_2 = C_3 = \pm 1/\sqrt{2}$, related to semi-factorized state $|0\rangle \otimes (C_2 |01\rangle + C_3 |10\rangle)$. **Figure 3** shows the possible values of $C^{(3)}(\Psi)$ for the state

$$|GHZ\rangle = C_1 |000\rangle + C_8 |111\rangle, \quad |C_1|^2 + |C_8|^2 = 1. \tag{22}$$

The maximum value of $C^{(3)}(\Psi)$ is gotten for $C_1 = C_8 = 1/\sqrt{2}$, as one would expect.

For a Hilbert space ε generated by n-qubits, $C^{(n)}$ defines a continuous function $C^{(n)} : \varepsilon \rightarrow \mathfrak{R}^+$ with $\mathfrak{R}^+ = [0, +\infty)$, $c \geq 1$. Since the coefficients of the wave function defines a compact set on the real space $\mathfrak{R}^{2n}$ due to the relation $\sum_i |C_i|^2 = 1$, the image of this compact set is a compact set in $\mathfrak{R}^+$ [17]. Thus, a normalization factor is possible to introduce on this function to define any compact set $[0, c]$, which it is not important.

3. Dynamical Consideration

Following Lloyd's idea [18], consider a linear chain of nuclear spin one half, separated by some distance and inside a magnetic in a direction z, $B(z) = (0, 0, B_0(z))$, and making and angle θ with respect this linear chain. Choosing this angle such that $\cos\theta = 1/\sqrt{3}$, the dipole-dipole interaction is canceled, the Larmore's frequency for each spin is different, $\omega_k = \gamma B_0(z_k)$ with γ the gyromagnetic ratio. The magnetic moment of the nucleus μ_k is related with its spin through the relation $\mu_k = \gamma S_k$, and the interaction energy between the magnetic field and magnetic moments is $H_{\text{int}} = -\sum_k \mu_k \cdot B(z_k) = -\sum_k \omega_k S_k^z$. If in addition, one has first and second neighbor Ising interaction, the Hamiltonian of the system is just [19] [20].
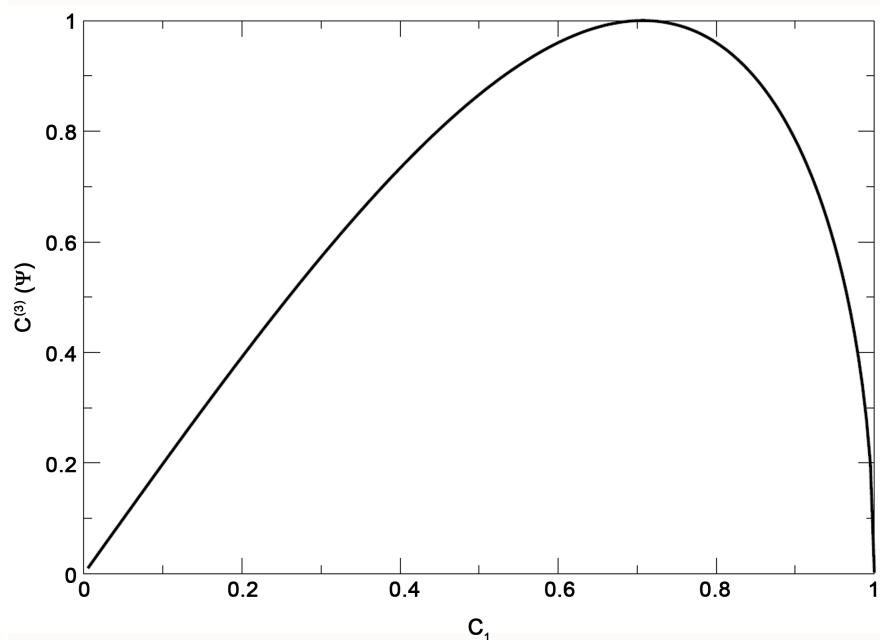

Figure 3. $C^{(3)}$ for the state $|GHZ\rangle = C_1 |000\rangle + C_8 |111\rangle$ such that $|C_1|^2 + |C_8|^2 = 1$.

$$H_s = -\sum_{k=1}^{N} \omega_k S_k^z - \frac{2J}{\hbar} \sum_{k=1}^{N-1} S_k^z S_{k+1}^z - \frac{2J'}{\hbar} \sum_{k=1}^{N-2} S_k^z S_{k+2}^z, \tag{23}$$

where N is the number of nuclear spins in the chain (or qubits), J and J' are the coupling constant of the nucleus at first and second neighbor. Using the basis of the register of N-qubits, $\left\{ \left| \xi_N, \cdots, \xi_1 \right\rangle \right\}$ with $\xi_k = 0,1$, one has that $S_k^z \left| \xi_k \right\rangle = (-1)^{\xi_k} \hbar \left| \xi_k \right\rangle / 2$. Therefore, the Hamiltonian is diagonal on this basis, and its eigenvalues are

$$E_\xi = -\frac{\hbar}{2} \sum_{k=1}^{N} (-1)^{\xi_k} \omega_k - \frac{J\hbar}{2} \sum_{k=1}^{N-1} (-1)^{\xi_k + \xi_{k+1}} - \frac{J'\hbar}{2} \sum_{k=1}^{N-2} (-1)^{\xi_k + \xi_{k+2}}. \tag{24}$$

Consider now that the environment is characterized by a Hamiltonian H_e and its interacting with the quantum system with Hamiltonian H_s. Thus, the total Hamiltonian would be $H = H_s + H_e + H_{se}$, where H_{se} is the part of the Hamiltonian which takes into account the interaction system-environment, and the equation one would need to solve, in terms of the density matrix, is [21] [22]

$$i\hbar \frac{\partial \rho_t}{\partial t} = [H, \rho_t], \tag{25}$$

where $\rho_t = \rho_t(s,e)$ is the density matrix which depends on the system and environment coordinates. The evolution of the system is unitary, but it is not possible to solve this equation due to a lot of degree of freedom. Therefore, under some approximations and tracing over the environment coordinates [23], it is possible to arrive to a Lindblad type of equation [24] [25] for the reduced density matrix $\rho(s) = tr_e(\rho_t)$,

$$i\hbar \frac{\partial \rho}{\partial t} = [H_s, \rho] + \sum_{i=1}^{I} \left\{ V_i \rho V_i^\dagger - \frac{1}{2} V_i^\dagger V_i \rho - \frac{1}{2} \rho V_i V_i^\dagger \right\} \tag{26}$$

where V_i are called Kraus' operators. This equation is not unitary and Markovian (without memory of the dynamical process). This equation can be written in the interaction picture, through the transformation $\tilde{\rho} = U \rho U^\dagger$ with $U = e^{iH_s t/\hbar}$, as

$$i\hbar \frac{\partial \tilde{\rho}}{\partial t} = \tilde{\mathcal{L}}(\tilde{\rho}), \tag{27}$$

where $\tilde{\mathcal{L}}(\tilde{\rho})$ is the Lindblad operator

$$\tilde{\mathcal{L}}(\tilde{\rho}) = \sum_{i=1}^{I} \left\{ \tilde{V}_i \tilde{\rho} \tilde{V}_i^\dagger - \frac{1}{2} \tilde{V}_i^\dagger \tilde{V}_i \tilde{\rho} - \frac{1}{2} \tilde{\rho} \tilde{V}_i \tilde{V}_i^\dagger \right\} \tag{28}$$

with $\tilde{V} = UVU^\dagger$. The explicit form of Lindblad operator is determined by the type of environment to consider [26] at zero temperature. So, the operators can be $V_i = S_i^-$ (for dissipation) for the model independent with the environment. In this case, each qubit of the chain acts independently with the environment, and one has local decoherence of the system. The Lindblad operator is

$$\tilde{\mathcal{L}}(\tilde{\rho}) = \frac{1}{2i\hbar} \sum_{k}^{N} \gamma_k \left(2\tilde{S}_k^- \tilde{\rho} \tilde{S}_k^+ - S_k^+ S_k^- \tilde{\rho} - \tilde{\rho} \tilde{S}_k^+ \tilde{S}_k^- \right) \tag{29}$$

where $\tilde{S}_k^+$ and $\tilde{S}_k^-$ are the ascend and descend operators such that

$$\tilde{S}_k^\pm = U S_k^\pm U^\dagger = S_k^\pm e^{\pm i \hat{\Omega}_k t}$$

where $\hat{\Omega}_k$ is defined as $\hat{\Omega}_k = w_k + \dfrac{J}{\hbar}\left(S_{k+1}^z + S_{k-1}^z\right) + \dfrac{J'}{\hbar}\left(S_{k+2}^z + S_{k-2}^z\right)$.. The solutions of the equations are

$$\begin{aligned}
\rho_{11}(t) = &\ \rho_{11}(0) + \rho_{22}(0) + \rho_{33}(0) + \rho_{44}(0) + \rho_{55}(0) + \rho_{66}(0) + \rho_{77}(0) + \rho_{88}(0) \\
&- \left(\rho_{55}(0) + \rho_{66}(0) + \rho_{77}(0) + \rho_{88}(0)\right) e^{-\gamma_1 t} \\
&- \left(\rho_{33}(0) + \rho_{44}(0) + \rho_{77}(0) + \rho_{88}(0)\right) e^{-\gamma_2 t} \\
&- \left(\rho_{22}(0) + \rho_{44}(0) + \rho_{66}(0) + \rho_{88}(0)\right) e^{-\gamma_3 t} \\
&+ \left(\rho_{77}(0) + \rho_{88}(0)\right) e^{-(\gamma_1+\gamma_2)t} \\
&+ \left(\rho_{66}(0) + \rho_{88}(0)\right) e^{-(\gamma_1+\gamma_3)t} + \left(\rho_{44}(0) + \rho_{88}(0)\right) e^{-(\gamma_2+\gamma_3)t} \\
&- \rho_{88}(0) e^{-(\gamma_1+\gamma_2+\gamma_3)t};
\end{aligned}$$

$$\rho_{14}(t) = \left[\rho_{14}(0) + \frac{\gamma_1 \rho_{58}(0) e^{-i\phi_{14}}}{\sqrt{\gamma_1^2 + (j+j')^2}} \left(1 - e^{(i(j+j')-\gamma_1)t}\right) \right] e^{-\frac{1}{2}(\gamma_2+\gamma_3)t};$$

$$\rho_{16}(t) = \left[\rho_{16}(0) + \frac{\gamma_2 \rho_{38}(0) e^{-i\phi_{16}}}{\sqrt{\gamma_2^2 + 4j^2}} \left(1 - e^{(2ij-\gamma_2)t}\right) \right] e^{-\frac{1}{2}(\gamma_1+\gamma_3)t};$$

$$\rho_{17}(t) = \left[\rho_{17}(0) + \frac{\gamma_3 \rho_{28}(0) e^{-i\phi_{17}}}{\sqrt{\gamma_3^2 + (j+j')^2}} \left(1 - e^{[i(j+j')-\gamma_3]t}\right) \right] e^{-\frac{1}{2}(\gamma_1+\gamma_2)t};$$

$$\rho_{18}(t) = \rho_{18}(0) e^{-\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\begin{aligned}
\rho_{22}(t) = &\ \left(\rho_{22}(0) + \rho_{44}(0) + \rho_{66}(0) + \rho_{88}(0)\right) e^{-\gamma_3 t} - \left(\rho_{66}(0) + \rho_{88}(0)\right) e^{-(\gamma_1+\gamma_3)t} \\
&- \left(\rho_{44}(0) + \rho_{88}(0)\right) e^{-(\gamma_2+\gamma_3)t} + \rho_{88}(0) e^{-(\gamma_1+\gamma_2+\gamma_3)t};
\end{aligned}$$

$$\rho_{23}(t) = \left[\rho_{23}(0) + \frac{\gamma_1 \rho_{67}(0) e^{-i\phi_{23}}}{\sqrt{\gamma_1^2 + (j-j')^2}} \left(1 - e^{(i(j-j')-\gamma_1)t}\right) \right] e^{-\frac{1}{2}(\gamma_2+\gamma_3)t};$$

$$\rho_{25}(t) = \left[\rho_{25}(0) + \rho_{47}(0)\left(1 - e^{-\gamma_2 t}\right) \right] e^{-\frac{1}{2}(\gamma_1+\gamma_3)t};$$

$$\rho_{27}(t) = \rho_{27}(0) e^{-\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{28}(t) = \rho_{28}(0) e^{-\frac{1}{2}(\gamma_1+\gamma_2+2\gamma_3)t};$$

$$\begin{aligned}
\rho_{33}(t) = &\ \left(\rho_{33}(0) + \rho_{44}(0) + \rho_{77}(0) + \rho_{88}(0)\right) e^{-\gamma_2 t} - \left(\rho_{77}(0) + \rho_{88}(0)\right) e^{-(\gamma_1+\gamma_2)t} \\
&- \left(\rho_{44}(0) + \rho_{88}(0)\right) e^{-(\gamma_2+\gamma_3)t} + \rho_{88}(0) e^{-(\gamma_1+\gamma_2+\gamma_3)t};
\end{aligned}$$

$$\rho_{35}(t) = \left[\rho_{35}(0) + \frac{\gamma_3 \rho_{46}(0) e^{-i\phi_{35}}}{\sqrt{\gamma_3^2 + (j-j')^2}} \left(1 - e^{-[i(j-j')+\gamma_3]t}\right) \right] e^{-\frac{1}{2}(\gamma_1+\gamma_2)t};$$

$$\rho_{36}(t) = \rho_{36}(0) e^{-\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{38}(t) = \rho_{38}(0)e^{-\frac{1}{2}(\gamma_1+2\gamma_2+\gamma_3)t};$$

$$\rho_{44}(t) = (\rho_{44}(0)+\rho_{88}(0))e^{-(\gamma_2+\gamma_3)t} - \rho_{88}(0)e^{-(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{45}(t) = \rho_{45}(0)e^{-\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{46}(t) = \rho_{46}(0)e^{-\frac{1}{2}(\gamma_1+\gamma_2+2\gamma_3)t};$$

$$\rho_{47}(t) = \rho_{47}(0)e^{-\frac{1}{2}(\gamma_1+2\gamma_2+\gamma_3)t};$$

$$\rho_{48}(t) = \rho_{48}(0)e^{-\frac{1}{2}(\gamma_1+2\gamma_2+2\gamma_3)t};$$

$$\rho_{55}(t) = (\rho_{55}(0)+\rho_{66}(0)+\rho_{77}(0)+\rho_{88}(0))e^{-\gamma_1 t} - (\rho_{77}(0)+\rho_{88}(0))e^{-(\gamma_1+\gamma_2)t}$$
$$- (\rho_{66}(0)+\rho_{88}(0))e^{-(\gamma_1+\gamma_3)t} + \rho_{88}(0)e^{-(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{58}(t) = \rho_{58}(0)e^{-\frac{1}{2}(2\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{66}(t) = (\rho_{66}(0)+\rho_{88}(0))e^{-(\gamma_1+\gamma_3)t} - \rho_{88}(0)e^{-(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{67}(t) = \rho_{67}(0)e^{-\frac{1}{2}(2\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{68}(t) = \rho_{68}(0)e^{-\frac{1}{2}(2\gamma_1+\gamma_2+2\gamma_3)t};$$

$$\rho_{77}(t) = (\rho_{77}(0)+\rho_{88}(0))e^{-(\gamma_1+\gamma_2)t} - \rho_{88}(0)e^{-(\gamma_1+\gamma_2+\gamma_3)t};$$

$$\rho_{78}(t) = \rho_{78}(0)e^{-\frac{1}{2}(2\gamma_1+2\gamma_2+\gamma_3)t};$$

$$\rho_{88}(t) = \rho_{88}(0)e^{-(\gamma_1+\gamma_2+\gamma_3)t}.$$

where ϕ_{ij} are given by

$$\phi_{14} = \tan^{-1}\left(\frac{j+j'}{\gamma_1}\right); \quad \phi_{16} = \tan^{-1}\left(\frac{2j}{\gamma_2}\right); \quad \phi_{17} = \tan^{-1}\left(\frac{j+j'}{\gamma_3}\right);$$

$$\phi_{23} = \tan^{-1}\left(\frac{j-j'}{\gamma_1}\right); \quad \phi_{35} = \tan^{-1}\left(\frac{j-j'}{\gamma_3}\right).$$

In our case, we have three qubits space $\{|\xi_3\xi_2\xi_1\rangle\}_{\xi_i=0,1}$, and our parameter in units 2π MHz are

$$\omega_1 = 400; \quad \omega_2 = 200; \quad \omega_3 = 100 \quad J = 10; \quad J' = 0.4$$

$$\gamma_1 = 0.05; \quad \gamma_2 = 0.05; \quad \gamma_3 = 0.05.$$

the time is normalized by the same factor of 2π MHz, and we include in this study the entangle state

$$|\Psi_1\rangle = \frac{1}{2}(|000\rangle + |111\rangle + |001\rangle + |110\rangle). \tag{30}$$

Figure 4 shows the behavior of the entangled states $|W\rangle$ and $|GHZ\rangle$ as a function of time when this entangled state interact with the environment. Purity behavior, $tr(\rho^2)$, is also shown. The system starts as a pure entangled state, it
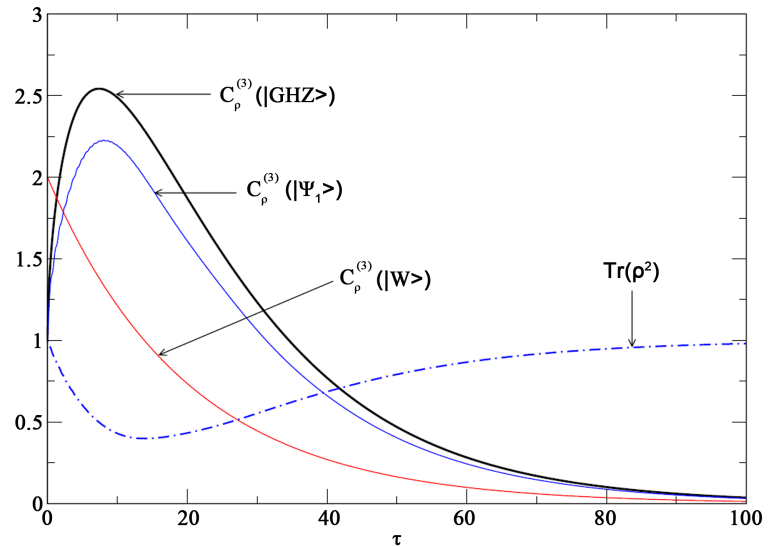
Figure 4. $C_{\rho}^{(3)}$ and Purity for the entangled state $|W\rangle$, $|GHZ\rangle$, and $|\Psi_1\rangle$.

evolves in a mixed state and finishes in the pure ground state $(|000\rangle)$, after sharing energy with the environment. The states $|GHZ\rangle$ and $|\Psi_1\rangle$ behave more robust than the state $|W\rangle$, $C_{\rho}^{(3)}$ grows since other entangled stated contribute to this function. In contrast, starting with the entangled state $|W\rangle$, there are not other entangled states which make contribution to the function $C_{\rho}^{(3)}$ in the dynamics, and one sees an exponential decay.

4. Conclusion

We have studied the full factorization of a state made up of up to 4-qubits basic states. We have seen that there is an indication that the number of conditions to characterize a factorized state grows exponentially with the number of qubits. For two, three and four basic qubits, we showed the conditions in order to have a factorized state, and if any of one of these conditions fails, one gets instead an entangled state. Therefore, an entangled state is also characterized by the complement of each of this conditions, and the resulting expression has been denoted by $C^{(n)}(n=2,3,4)$. This non-negative function expressed in terms of the coefficients of the wave function or in terms of the density matrix elements represents a measurement of the entanglement of any wave function made up of basic n-qubits $(n=2,3,4)$. Using this function, we study the decay of entangled states $|W\rangle$, $|GHZ\rangle$, and $|\Psi_1\rangle$ due to interaction with environment, and we noticed a great different behavior of the function $C_{\rho}^{(3)}$, indicating some type of robustness behavior of the states $|GHZ\rangle$ and $|\Psi_1\rangle$. The main reason for this different behavior is that the entangled states $|GHZ\rangle$ and $|\Psi_1\rangle$ contain the ground state $|000\rangle$, which is the final state in the dynamics.

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