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GLOBALLY UNIFORMLY ULTIMATELY BOUNDED OBSERVER DESIGN FOR A CLASS OF NONLINEAR SYSTEMS WITH SAMPLED AND DELAYED MEASUREMENTS

Daoyuan Zhang, Yanjun Shen and Xiaohua Xia

In this paper, we consider two kinds of sampled-data observer design for a class of nonlinear systems. The system output is sampled and transmitted under two kinds of truncations. Firstly, we present definitions of the truncations and the globally uniformly ultimately bounded observer, respectively. Then, two kinds of observers are proposed by using the delayed measurements with these two truncations, respectively. The observers are hybrid in essence. For the first kind of observers, by constructing a Lyapunov–Krasovskii functional, sufficient conditions of globally uniformly ultimately bounded of the estimation errors are derived, and the maximum allowable sampling period and the maximum delay are also given. For the second ones, sufficient conditions are also given to ensure that the estimation errors are globally uniformly ultimately bounded. Finally, an example is provided to illustrate the design methods.

Keywords: nonlinear systems, continuous observers, sampled output, delayed measurements

Classification: 93C10, 93C57

1. INTRODUCTION

The problem of observer design for nonlinear systems has attracted more and more attention in the past years. The research interest comes from many engineering applications in practice. There are lots of results about observer design, for instance, an observer was constructed for a class of nonlinear systems under a global lipschitz condition and applied for a biological system [7]. Based on a high-gain observer, the author studied the output feedback stabilization for a class of lower triangular systems [13]. A class of high gain observers were also proposed for nonlinear systems by using a gain update law which is dependent on the system output in [4]. In [9, 10, 14, 15], the authors have studied the finite-time observers design for nonlinear systems with lower triangular form. However, the whole above results are studied based on continuous time analysis. In practice, more and more systems are controlled by digital computers. The system output, usually sampled by sampler at discrete time instants and transmitted over a shared band-limited digital communication network, is only available with a time delay.

Therefore, the research on sampled-data system is becoming a very hot topic. There exist two main methods to deal with these problems:

i) a discrete observer design based on a consistent approximation of the exact discretized model [5];

ii) a hybrid observer design based on a mixed continuous-time and discrete-time design [1, 2, 3, 6, 8, 12, 16, 17, 18].

For instance, a hybrid observer design method was presented in [6]. Then, by using a continuous design coupled with a sampled output predictor, a sampled-data observer was proposed in [8]. Similar results have been studied in [1, 12]. Moreover, by taking transmission delay into consideration, a high gain observer design was proposed for a class of nonlinear systems [16]. It is noted that the results of [16] can be easily extended to the case of sampled measurements. In [2], two kinds of different structure of continuous-discrete observers were proposed for a class of state affine systems. In addition, the sampled-data observer design for a wide classes of nonlinear systems has been studied via using a small gain approach [3]. Note that the observer designs in both [2] and [3] include a predictor part and a correction part. In [18], a global exponential sampled-data observer design has been considered for a class of nonlinear systems. The sampled measurements were directly used to update the observer.

It should be noted that, for a sampled-data system, the output data are sampled by data collector and transmitted through the output channels. Therefore, the values of sampled data from data collector may not be the real values of original data. However, in the papers [1, 2, 3, 6, 8, 12, 16, 17, 18], the values of sampled data are directly used for observers design, which may not be true for sampled-data systems. For this reason, we should specify the maximum byte of the sampled data and how the original data are truncated.

In this paper, we consider observers design for a class of nonlinear system with sampled and delayed measurements. The output of the system is sampled and transmitted through a network under two kinds of truncations. Firstly, we present definitions of the truncations and the globally uniformly ultimately bounded observer, respectively. Then, two kinds of observers are designed by using the delayed measurements with these two truncation, respectively. The observers are hybrid in essence. For the first observer, by constructing a Lyapunov–Krasovskii functional, sufficient conditions of globally uniformly ultimately bounded of the estimation errors are derived, and the maximum allowable sampling period and the maximum delay are also given. For the second one, sufficient conditions are also given to ensure that the estimation errors are globally uniformly ultimately bounded. The major contributions of this paper include: i) Two kinds of protocol functions are introduced, which represent the truncations of sampled data; ii) Two kinds of sampled-data observers design are studied for a class of nonlinear systems with sampled and delayed measurements.

This paper is organized as follows. In Section 2, we present our main results: two kinds of sampled-data observers are designed for a class nonlinear systems with two kinds of truncations. In Section 3, an example is used to illustrate the validity of the proposed design methods. Finally, the paper is concluded in Section 4.

Throughout this paper, let $\mathbb{R}^n$ denote n-dimension real space and $X^\top$ represent the

transposed matrix of X. For any $x \in \mathbb{R}^n$, let $\|x\| = (x^\top x)^{1/2}$. For a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, let $\lim_{s \to t^-} f(s) = \lim_{s \to t, s < t} f(s)$. $\lambda_{\min}(P)$ denotes the smallest eigenvalue of the symmetric matrix $P \in \mathbb{R}^{n \times n}$.

2. SAMPLED-DATA OBSERVER DESIGN FOR NONLINEAR SYSTEM

In this section, the problem of sampled-data observer design for a class of lower triangular nonlinear systems is firstly introduced. Then, two kinds of sampled-data observers are designed and sufficient conditions are given to guarantee the observation errors are globally uniformly ultimately bounded.

Consider the following system

$$
\begin{cases}
\dot{x}_1(t) & = x_2(t) + f_1(x_1(t)), \\
\dot{x}_2(t) & = x_3(t) + f_2(x_1(t), x_2(t)), \\
\quad \vdots \\
\dot{x}_{n-1}(t) & = x_n(t) + f_{n-1}(x_1(t), x_2(t), \ldots, x_{n-1}(t)), \\
\dot{x}_n(t) & = f_n(x_1(t), x_2(t), \ldots, x_n(t)) + u(t), \\
\tilde{y}(t) & = x_1(t), \quad t \in [t_k, t_{k+1}), \; k \geq 0, \\
y(t_k) & = h(\tilde{y}(t_k)),
\end{cases}
\tag{1}
$$

where the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}$, and $f_i(\cdot)$ $(i = 1, \ldots, n)$ satisfy the following globally Lipschitz condition

$$
\mid f_i(x_1, x_2, \ldots, x_i) - f_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_i) \mid \leq l(\mid x_1 - \hat{x}_1 \mid + \mid x_2 - \hat{x}_2 \mid + \cdots + \mid x_i - \hat{x}_i \mid),
\tag{2}
$$

where l is a positive constant. The function $h : \mathbb{R} \to \mathbb{R}$ is continuous. Note that since $f_i(\cdot)$ satisfy the global Lipschitz condition (2), all solutions of system (1) exist and are complete (i.e. defined on $[0, +\infty)$). We assume that the output $\tilde{y}(t)$ is sampled at instants t_k and is available for the observer at instants $t_k + \tau_k$, where $\{t_k\}$ denotes a strictly increasing sequence satisfying $\lim_{k \to \infty} t_k = \infty$, and $\tau_k \geq 0$ represents the transmission delay. The sampling interval $T = t_{k+1} - t_k$ is a positive constant. The transmission delays τ_k are unknown, but have an upper bound $\bar{\tau}$ and a lower bound $\underline{\tau}$, that is, $\max\{\tau_k\} \leq \bar{\tau}$ and $\min\{\tau_k\} \geq \underline{\tau} > 0$ for all $k = 0, 1, \ldots, \infty$. We also make the assumption: $\bar{\tau} < T$, that is, the output of the system (1) sampled at instants t_k is available for the observer before the next instants t_{k+1}. In addition, $h(\cdot)$ denotes a truncation of the sampled data, whose definition is given as follows.

Definition 2.1. The function $h(\cdot)$ is said to be a truncation at the sampled instant t_k or a static truncation, if the following inequality

$$
|h(x_1(t_k)) - x_1(t_k)| \leq c_1,
\tag{3}
$$

holds, where c_1 is an absolute error limit. The function $h(\cdot)$ is said to be a uniform truncation on $[t_k, t_{k+1}]$, if the following inequality

$$
|h(x_1(t_k)) - x_1(t)| \leq c_2, \quad \forall t \in [t_k, t_{k+1}], \; k \geq 0,
\tag{4}
$$

holds, where $c_2 > 0$ is a constant.

Remark 2.2. For the system (1), let $f(x,u) = [x_2(t)+f_1(x_1(t)), x_3(t)+f_2(x_1(t),x_2(t)),$ $\ldots, f_n(x_1(t),\ldots,x_n(t)) + u(t)]^\top$. If $T = t_{k+1} - t_k$ is controlled, and $f(x,u)$ is globally bounded, that is, there exists a constant $b_1 > 0$ such that

$$|f(x,u)^\top f(x,u)| \leq b_1^2, \tag{5}$$

then, the static truncation implies the uniform truncation. In fact, from (1), we have

$$|x_1(t_k) - x_1(t)| \leq b_1 |t - t_k|, \forall t \in [t_k, t_{k+1}].$$

Thus,

$$|h(x_1(t_k)) - x_1(t)| \leq |h(x_1(t_k)) - x_1(t_k)| + |x_1(t_k) - x_1(t)| \leq c_1 + b_1 T.$$

Let $c_2 = c_1 + b_1 T$, then the inequality (4) is satisfied.

Remark 2.3. For system (1), the truncation function $h(\cdot)$, which indicates the existence of the error between the sampled data and the real data, is often classified as systematic error. Systematic error, which is an error in measurement arising from a defect, such as the mis-calibration of a meter or some physical effect not taken into account in the measurement, can be checked and corrected in principle. One point of this paper shows the effects of measuring accuracy to sampled data. It should be noticed that systematic error is distinguished from noise (also called random error or statistical uncertainty). Moreover, if both systematic error and noise are taken into account, the effect of truncation function $h(\cdot)$ to sampled data might be ambiguous. Therefore, noise measurement is not taken into consideration in this paper.

We also present the definitions of uniformly ultimately bounded observer and globally uniformly ultimately bounded observer, respectively.

Definition 2.4. Consider the following system:

$$\begin{aligned}
&\dot{z}(t) = g(z(t), h(\tilde{y}(t_k)), z(t_k), u(t)), \\
&z(t_k + \tau_k) = \lim_{t \to (t_k + \tau_k)^-} z(t), \quad t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), \ k \geq 0, \\
&z(t) = z_0, t \in [t_0, t_0 + \tau_0],
\end{aligned} \tag{6}$$

where $z_0, z(t) \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous. With loss of confusion, denote the solutions of (1) and (6) with respect to the corresponding input functions and the initial conditions x_0 and z_0 as $x(t)$ and $z(t)$, respectively. Let $e_0 = x_0 - z_0$.

i) If there exists a constant $b_2 > 0$ and a constant $T'(e_0, b_2)$ such that

$$|x(t) - z(t)| < b_2, \ \forall t > t_0 + T'(e_0, b_2), \tag{7}$$

then, the system (6) is said to be an uniformly ultimately bounded observer for the system (1).

ii) If for any $e_0 \in \mathbb{R}^n$, there exists a constant $b_2 > 0$ and a constant $T'(e_0, b_2)$ such that (7) holds, then, the system (6) is said to be a globally uniformly ultimately bounded observer for the system (1).

The following lemma is useful for our main results.

Lemma 2.5. (Liu et al. [11]) For any positive definite matrix $M \in \mathbb{R}^{n \times n}$, scalar $\sigma > 0$, vector function $w : [0, \sigma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^\sigma w(s) \, \mathrm{d}s \right)^\top M \int_0^\sigma w(s) \, \mathrm{d}s \leq \sigma \cdot \int_0^\sigma w(s)^\top M w(s) \, \mathrm{d}s.$$

Now, we design two kinds of observers for the system (1). The first one with the truncation (3) is given as follows:

$$
\begin{cases}
\dot{\hat{x}}_1(t) & = & \hat{x}_2(t) + k_1(y(t_k) - \hat{x}_1(t_k)) + f_1(\hat{x}_1(t)), \\
\dot{\hat{x}}_2(t) & = & \hat{x}_3(t) + k_2(y(t_k) - \hat{x}_1(t_k)) + f_2(\hat{x}_1(t), \hat{x}_2(t)), \\
& \vdots & \\
\dot{\hat{x}}_{n-1}(t) & = & \hat{x}_n(t) + k_{n-1}(y(t_k) - \hat{x}_1(t_k)) + f_{n-1}(\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_{n-1}(t)), \\
\dot{\hat{x}}_n(t) & = & u(t) + k_n(y(t_k) - \hat{x}_1(t_k)) + f_n(\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_n(t)), \\
\hat{x}_i(t_k + \tau_k) & = & \lim_{t \to (t_k + \tau_k)^-} \hat{x}_i(t), \quad t \in [t_k + \tau_k, \, t_k + T + \tau_{k+1}), \quad k \geq 0,
\end{cases}
$$

$$(8)$$

where $\hat{x}(t) = \hat{x}_0$ for $t \in [t_0, t_0 + \tau_0]$, the gains are given by $(k_1, k_2, \dots, k_n)^\top = S^{-1}(\theta) C^\top$, and $S(\theta)$ is the unique solution of the matrix equation

$$-\theta S(\theta) - A^\top S(\theta) - S(\theta) A + C^\top C = 0, \tag{9}$$

$A_{i,j} = \delta_{i,j-1}$, $1 \leq i, j \leq n$, and $C = (1, 0, \dots, 0)$.

As the result of (9), $S(\theta)$ has the following property.

Lemma 2.6. (Gauthier et al. [7]) The matrix $S(\theta)$ verifies the following property:

$$S(\theta) = S^\top(\theta) \geq \delta_0 I, \tag{10}$$

$$S(\theta)_{i,j} = S(1)_{i,j} \frac{1}{\theta^{i+j-1}}, \tag{11}$$

for any $\theta > 0$ ($1 \leq i, j \leq n$), where $\delta_0 > 0$ is a scalar.

Since $y(t_k) - \hat{x}_1(t_k)$ is a constant on $[t_k + \tau_k, t_k + T + \tau_{k+1})$ for $k \geq 0$ and $f_i(\cdot)$ ($i = 1, \dots, n$) are continuous and satisfy the condition (2), then, $\lim_{t \to (t_k + T + \tau_{k+1})^-} \hat{x}_i(t) = \lim_{t \to (t_k + T + \tau_{k+1})^+} \hat{x}_i(t) = \hat{x}_i(t_k + T + \tau_{k+1})$ ($i = 1, \dots, n$). Therefore, $\hat{x}_i(t)$ ($i = 1, \dots, n$) are continuous on $[t_0, \infty)$.

Remark 2.7. It should be noted that the evolution process $y(t_k) - \hat{x}_1(t_k)$ is only updated at instants $t_k + \tau_k$. Therefore, the dynamics of observer (8) is hybrid in essence.

Remark 2.8. Even though τ_k and τ_{k+1} are unknown, we can update $y(t_k) - \hat{x}_1(t_k)$ automatically whenever sampled and delayed measurements arrive. In [2], the sampled measurement is used for observer design at instants $t_k + \tau_k$ and $e_1(t_k)$ is updated at instants $t_k + \bar{\tau}$. Therefore, there exists a time delay $\bar{\tau} - \tau_k$ after the sampled measurements can be available.

Remark 2.9. A sampled-data observer for system (1) has been investigated in [18]. The values of the sampled measurements are considered to be the true values and used for the sampled-data observer design. In this paper, the truncation function h is introduced to denote the distinction between the sampled data and the real data. When $c_1 = 0$, the system (1) is essentially reduced to the one considered in [18]. However, based on the truncated data, the results in [18] can not be obtained again, because the estimations we got are for the truncated data, not the real one. There is nothing to do to improve the precision of estimated data from [18]. In additions, from the view of this paper, we can intuitively see the influence of the absolute error limit c_1 on the bound of errors. That is, the errors between real data and estimated data based on the truncated data can be controlled by changing c_1. Moreover, the less c_1, the less errors. However, it should be noted that the value of the parameter c_1 depends on the transmission network and measuring devices. In general, it is given as a certain known constant.

Remark 2.10. In [13], the nonlinear functions $f_i(\cdot)$ are assumed to admit an incremental rate depending on the measured output $y(t)$, i.e.,

$$f_i(x_1, x_2, \ldots, x_i) - f_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_i) \mid \leq \ l(y)(\mid x_1 - \hat{x}_1 \mid + \mid x_2 - \hat{x}_2 \mid + \cdots + \mid x_i - \hat{x}_i \mid).$$

It is interesting to research sampled-data observer design under such a case. A time-variant gain may be needed to guarantee that the observation errors are convergent. We will discuss this issue elsewhere.

From $(1) - (8)$, for $k \geq 0$, we obtain the observation error:

$$\begin{cases} \dot{e}_1(t) & = & e_2(t) - k_1(y(t_k) - \hat{x}_1(t_k)) + \tilde{f}_1, \\ \dot{e}_2(t) & = & e_3(t) - k_2(y(t_k) - \hat{x}_1(t_k)) + \tilde{f}_2, \\ & \vdots & \\ \dot{e}_{n-1}(t) & = & e_n(t) - k_{n-1}(y(t_k) - \hat{x}_1(t_k)) + \tilde{f}_{n-1}, \\ \dot{e}_n(t) & = & -k_n(y(t_k) - \hat{x}_1(t_k)) + \tilde{f}_n, \quad t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \end{cases} \quad (12)$$

where $e_i(t) = x_i(t) - \hat{x}_i(t)$, $\tilde{f}_i = f_i(x_1(t), x_2(t), \ldots, x_i(t)) - f_i(\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_i(t))$, $1 \leq i \leq n$. Further, (12) can be rewritten as follows:

$$\begin{cases} \dot{e}_1(t) & = & e_2(t) - k_1 e_1(t) + k_1(x_1(t_k) - h(x_1(t_k))) + k_1(e_1(t) - e_1(t_k)) + \tilde{f}_1, \\ \dot{e}_2(t) & = & e_3(t) - k_2 e_1(t) + k_2(x_1(t_k) - h(x_1(t_k))) + k_2(e_1(t) - e_1(t_k)) + \tilde{f}_2, \\ & \vdots & \\ \dot{e}_{n-1}(t) & = & e_n(t) - k_{n-1} e_1(t) + k_{n-1}(x_1(t_k) - h(x_1(t_k))) + k_{n-1}(e_1(t) - e_1(t_k)) + \tilde{f}_{n-1}, \\ \dot{e}_n(t) & = & -k_n e_1(t) + k_n(x_1(t_k) - h(x_1(t_k))) + k_n(e_1(t) - e_1(t_k)) + \tilde{f}_n, \\ & & t \in [t_k + \tau_k, t_k + T + \tau_{k+1}). \end{cases}$$

$$(13)$$

Now, we give one of our main results.

Theorem 2.11. Consider the system (1) with the condition (2) and the truncation (3). The output $y(t)$ is assumed to be sampled at instants t_k and is available for the observer

at instants $t_k + \tau_k$. If there exists $\theta \geq \max\{1, c_4^2\}$ such that

$$T \leq \frac{c_4 + 2}{2c_4} T_1, \quad \bar{\tau} \leq \left(\frac{c_4 - 2}{c_4 + 2}\right) T \tag{14}$$

and

$$3\theta/4 - \sqrt{\theta} - 2nl\sqrt{c_3 S} - 1 > 0, \tag{15}$$

hold, then, the system (8) is a globally uniformly ultimately bounded observer for the system (1), where $T_1 = \min\{\frac{3\theta/4 - \sqrt{\theta} - 2nl\sqrt{c_3 S} - 1}{c_3 \theta}, \frac{1}{24c_3(l+1)^2 + \sqrt{\theta} + 1}, \frac{1}{48c_3 k_1^2 \sqrt{\theta}}\}$, $S = \sup_{i,j} |S(1)_{i,j}|$, $c_3 = \frac{1}{\lambda_{\min}(S(1))}$, and $c_4 > 2$ is a constant.

P r o o f . From (3), it is easy to obtain that

$$|y(t_k) - \hat{x}_1(t_k)| \leq |x_1(t_k) - \hat{x}_1(t_k)| + |y(t_k) - x_1(t_k)| = |e_1(t_k)| + c_1. \tag{16}$$

Consider the following Lyapunov–Krasovskii functional

$$V(t) = V_1(t) + V_2(t), \tag{17}$$

where

$$V_1(t) = e(t)^\top S(\theta) e(t), \tag{18}$$

and

$$V_2(t) = \int_{t-T-\bar{\tau}}^t \int_\rho^t [e_1(s)^2 + e_2(s)^2 + \cdots + e_n(s)^2] \, \mathrm{d}s \mathrm{d}\rho, t \in [t_{k_0}, \infty), \tag{19}$$

and $e(t) = [e_1(t), \ldots, e_n(t)]^\top$, $k_0 = \min\{k : T + \bar{\tau} \leq t_k\}$.
The derivative of $V_1(t)$ along the system (13) is given as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t)|_{(13)} = \dot{e}(t)^\top S(\theta) e(t) + e(t)^\top S(\theta) \dot{e}(t) \leq -\theta e(t)^\top S(\theta) e(t)$$

$$+ 2(x_1(t_k) - h(x_1(t_k))) e_1(t) + 2e_1(t)(e_1(t) - e_1(t_k)) + 2F^\top S(\theta) e(t),$$

$$t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0,$$

where $F = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)^\top$.
   Note that

$$2(x_1(t_k) - h(x_1(t_k))) e_1(t) \leq \frac{8}{S(1)_{1,1}} (x_1(t_k) - h(x_1(t_k)))^2 + \frac{S(1)_{1,1}}{8} e_1(t)^2$$
$$\leq 8c_3 c_1^2 + \frac{1}{8} \theta e^T(t) S(\theta) e(t),$$

and

$$2e_1(t)(e_1(t) - e_1(t_k)) \leq \frac{1}{8} \theta e^T(t) S(\theta) e(t) + 8c_3(e_1(t) - e_1(t_k))^2.$$

   Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t)|_{(13)} \leq -\frac{3\theta}{4} e^\top(t) S(\theta) e(t) + 8c_3(e_1(t) - e_1(t_k))^2 + 2F^\top S(\theta) e(t) + 8c_3 c_1^2,$$

$$t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0. \tag{20}$$

By Lemma 2.5, we have

$$| e_1(t) - e_1(t_k) |^2 = \left| \int_{t_k}^t \dot{e}_1(s) \, \mathrm{d}s \right|^2 \leq (t - t_k) \int_{t_k}^t | \dot{e}_1(s) |^2 \, \mathrm{d}s \leq (t - t_k) \int_{t_k}^t [e_2(s)$$

$$- k_1(y(t_k) - \hat{x}_1(t_k)) + \tilde{f}_1]^2 \, \mathrm{d}s \leq 3(l+1)^2 (t - t_k) \int_{t_k}^t [e_1(s)^2 + e_2(s)^2$$

$$+ \frac{k_1^2}{(l+1)^2} (|e_1(t_k)| + c_1)^2] \, \mathrm{d}s, \ t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0. \tag{21}$$

It follows from Lemma 2.6 that

$$F^\top S(\theta) F \leq l^2 \sum_{i,j} |S(1)_{i,j}| \cdot ||\bar{e}_i|| \cdot ||\bar{e}_j|| \frac{1}{\theta^{i+j-1}} \leq l^2 S \theta \sum_{i,j} \frac{||\bar{e}_i||}{\theta^i} \frac{||\bar{e}_j||}{\theta^j},$$

where $||\bar{e}_i|| = (e_1^2 + e_2^2 + \cdots + e_i^2)^{\frac{1}{2}}$. Let $\xi_i = \frac{e_i}{\theta^i}$, we have

$$||\bar{\xi}_i|| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_i^2)^{\frac{1}{2}},$$

and

$$||\xi|| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{\frac{1}{2}}.$$

Then,

$$\frac{||\bar{e}_i||}{\theta^i} \leq ||\bar{\xi}_i|| \leq ||\xi||, \ F^\top S(\theta) F \leq l^2 S \theta n^2 ||\xi||^2.$$

Note that

$$\theta ||\xi||^2 \leq \theta c_3 \xi^\top S(1) \xi \leq \theta c_3 \frac{1}{\theta} e(t)^\top S(\theta) e(t),$$

and

$$2 F^\top S(\theta) e(t) \leq 2 \sqrt{e(t)^\top S(\theta) e(t)} \sqrt{F^\top S(\theta) F}.$$

Thus,

$$2 F^\top S(\theta) e(t) \leq 2nl \sqrt{c_3 S} e(t)^\top S(\theta) e(t).$$

If follows from (20) and (21) that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t)|_{(13)} \leq -\left(\frac{3\theta}{4} - 2nl \sqrt{c_3 S} \right) e(t)^\top S(\theta) e(t) + 24 c_3 (l+1)^2 (t - t_k) \int_{t_k}^t [e_1(s)^2 + e_2(s)^2] \, \mathrm{d}s$$

$$+ 8 c_3 c_1^2 + 48 c_3 (t - t_k)^2 k_1^2 (e_1(t_k)^2 + c_1^2), t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0.$$

Note that when $t \in [t_k + \tau_k, t_k + T + \tau_{k+1})$, we have $t - t_k - \tau_k \leq T + \tau_{k+1} - \tau_k$. That is, $t - T - \tau_{k+1} < t_k$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t)|_{(13)} \leq -(\frac{3\theta}{4} - 2nl \sqrt{c_3 S}) e(t)^\top S(\theta) e(t)$$

$$+24c_3(l+1)^2(t-t_k) \int_{t-T-\tau_{k+1}}^{t} [e_1(s)^2 + e_2(s)^2]\,\mathrm{d}s \quad + 8c_3 c_1^2$$

$$+ 48c_3(t-t_k)^2 k_1^2(e_1(t_k)^2 + c_1^2), t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0. \tag{22}$$

From (19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2(t) = (T+\bar{\tau})e(t)^\top e(t) - \int_{t-T-\bar{\tau}}^{t} e(s)^\top e(s)\,\mathrm{d}s. \tag{23}$$

Therefore, from (22) and (23), the derivative of $V(t)$ defined in (17) along the system (13) is given as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t)|_{(13)} \leq -\left(\frac{3\theta}{4} - 2nl\sqrt{c_3 S}\right) e^\top(t)S(\theta)e(t)$$

$$+24c_3(l+1)^2(t-t_k)\int_{t-T-\tau_{k+1}}^{t} [e_1(s)^2 + e_2(s)^2]\,\mathrm{d}s \quad - \int_{t-T-\bar{\tau}}^{t} [e(s)^\top e(s)]\,\mathrm{d}s$$

$$+(T+\bar{\tau})e(t)^\top e(t) + 48c_3 k_1^2(T+\bar{\tau})^2 e_1(t_k)^2 + 56c_3 c_1^2$$

$$\leq -\left(\frac{3\theta}{4} - 2nl\sqrt{c_3 S} - c_3\theta(T+\bar{\tau})\right) e^\top(t)S(\theta)e(t)$$

$$+[24c_3(l+1)^2(t-t_k) - 1]\int_{t-T-\bar{\tau}}^{t} [e(s)^\top e(s)]\,\mathrm{d}s + 48c_3 k_1^2(T+\bar{\tau})^2 e_1(t_k)^2$$

$$+ 56c_3 c_1^2, \quad t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0. \tag{24}$$

Note that

$$V_2(t) \leq (T+\bar{\tau})\int_{t-T-\bar{\tau}}^{t} [e(s)^\top e(s)]\,\mathrm{d}s, \ t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0. \tag{25}$$

(24) and (25) imply

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t)|_{(13)} &\leq -(\tfrac{3\theta}{4} - 2nl\sqrt{c_3 S} - c_3\theta(T+\bar{\tau}))e(t)^\top S(\theta)e(t) \\ &+ \tfrac{24c_3(l+1)^2(T+\bar{\tau})-1}{T+\bar{\tau}} V_2(t) + 48c_3 k_1^2(T+\bar{\tau})^2 e_1(t_k)^2 + 56c_3 c_1^2 \\ &\leq -(\tfrac{3\theta}{4} - 2nl\sqrt{c_3 S} - c_3\theta(T+\bar{\tau}))V_1(t) + \tfrac{24c_3(l+1)^2(T+\bar{\tau})-1}{T+\bar{\tau}} V_2(t) \\ &+ 48\theta c_3 k_1^2(T+\bar{\tau})^2 V_1(t_k)^2 + 56c_3 c_1^2, \ t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0. \end{aligned} \tag{26}$$

Let $\mathcal{V}_1(c_1) = \{e(t) : V(e(t)) \leq 56c_3 c_1^2\}$. Note that $T + \bar{\tau}$ satisfies (14). When $e(t) \in \mathbb{R}^n/\mathcal{V}_1(c_1)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t)|_{(13)} \leq -\sqrt{\theta}V(t) + (T+\bar{\tau})\sqrt{\theta}V(t_k), t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \ k \geq k_0.$$

From the above differential inequality, we obtain

$$V(t) \leq e^{-\sqrt{\theta}(t-t_k-\tau_k)}V(t_k + \tau_k) + (T+\bar{\tau})V(t_k) - (T+\bar{\tau})e^{-\sqrt{\theta}(t-t_k-\tau_k)}V(t_k),$$
$$t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0. \tag{27}$$

Let $t = t_k + T + \tau_{k+1}$ and $t = t_k + T$, respectively, from (27), we have

$$
\begin{aligned}
V(t_k + T + \tau_{k+1}) &\leq e^{-\sqrt{\theta}(T + \tau_{k+1} - \tau_k)} V(t_k + \tau_k) \\
&+ (T + \bar{\tau}) V(t_k) - (T + \bar{\tau}) e^{-\sqrt{\theta}(T + \tau_{k+1} - \tau_k)} V(t_k),
\end{aligned}
\tag{28}
$$

and,

$$
V(t_k + T) \leq e^{-\sqrt{\theta}(T - \tau_k)} V(t_k + \tau_k) + (T + \bar{\tau}) V(t_k) - (T + \bar{\tau}) e^{-\sqrt{\theta}(T - \tau_k)} V(t_k). \tag{29}
$$

From (14), it follows that

$$
2(T + \bar{\tau}) \leq c_4(T - \bar{\tau}) < \sqrt{\theta}(T - \bar{\tau}).
$$

Further, there exists $0 < \rho < 1$ such that

$$
2(T + \bar{\tau}) < \rho < \sqrt{\theta}(T - \bar{\tau}) < e^{\sqrt{\theta}(T - \bar{\tau})} - 1.
$$

Thus,

$$
e^{-\sqrt{\theta}(T - \bar{\tau})}(1 + \rho) < 1, \quad \frac{2(T + \bar{\tau})}{\rho} < 1. \tag{30}
$$

Then, it follows from (28) and (29) that

$$
\begin{aligned}
V(t_k + T + \tau_{k+1}) + \rho V(t_k + T) &\leq e^{-\sqrt{\theta}T}(e^{-\sqrt{\theta}(\tau_{k+1} - \tau_k)} + \rho e^{\sqrt{\theta}\tau_k}) V(t_k + \tau_k) \\
&+ (T + \bar{\tau})(1 + \rho - e^{-\sqrt{\theta}(T + \tau_{k+1} - \tau_k)} - \rho e^{-\sqrt{\theta}(T - \tau_k)}) V(t_k) \\
&\leq e^{-\sqrt{\theta}(T - \tau_k)}(e^{-\sqrt{\theta}\tau_{k+1}} + \rho) V(t_k + \tau_k) + 2(T + \bar{\tau}) V(t_k) \\
&\leq e^{-\sqrt{\theta}(T - \bar{\tau})}(1 + \rho) V(t_k + \tau_k) + \frac{2(T + \bar{\tau})}{\rho} \rho V(t_k).
\end{aligned}
\tag{31}
$$

Let $\eta = \max\{e^{-\sqrt{\theta}(T - \bar{\tau})}(1 + \rho), \frac{2(T + \bar{\tau})}{\rho}\}$, from (30), we have $0 < \eta < 1$. Then, from (31), it follows that

$$
\begin{aligned}
V(t_{k+1} + \tau_{k+1}) + \rho V(t_k + T) &= V(t_k + T + \tau_{k+1}) + \rho V(t_k + T) \\
&\leq \eta[V(t_k + \tau_k) + \rho V(t_k)], \quad k \geq k_0.
\end{aligned}
\tag{32}
$$

Applying iteratively (32), for $k \geq k_0$, we have

$$
\begin{aligned}
V(t_k + \tau_k) + \rho V(t_k) &\leq \eta^{k - k_0}[V(t_{k_0} + \tau_{k_0}) + \rho V(t_{k_0})], \\
t &\in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0.
\end{aligned}
\tag{33}
$$

It follows from (27) and (33) that

$$
\begin{aligned}
V(t) &\leq V(t_k + \tau_k) + \rho V(t_k) \leq \eta^{k - k_0}[V(t_{k_0} + \tau_{k_0}) \\
&+ \rho V(t_{k_0})], t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \quad k \geq k_0.
\end{aligned}
$$

For any $t > t_{k_0} + \tau_{k_0}$, there exists $k \geq k_0$ such that $t \in [t_k + \tau_k, t_k + T + \tau_{k+1})$. Note that $\frac{t - t_{k_0} - \bar{\tau}}{T} - 1 \leq k$. Then,

$$
V(e(t)) \leq e^{\frac{t}{T} \ln \eta} \eta^{\frac{-t_{k_0} - \bar{\tau}}{T} - k_0 - 1}[V(e(t_{k_0} + \tau_{k_0})) + \rho V(e(t_{k_0}))], \quad k \geq k_0. \tag{34}
$$

Let $\eta_1 = \eta^{\frac{-t_{k_0} - \bar{\tau}}{T} - k_0 - 1}[V(e(t_{k_0} + \tau_{k_0})) + \rho V(e(t_{k_0}))]$ and $T' = [\ln 224 c_3 c_1^2 - \ln \eta_1] T / \ln \eta$, when $t > T'$, $V(e(t)) < 224 c_3 c_1^2$, or $e(t) \in \mathcal{V}_1(2c_1)$. That is, the error system (13) is globally uniformly ultimately bounded. $\qquad \square$

Remark 2.12. If the initial conditions $e(0) \notin \mathcal{V}_1(2c_1)$, there exists a finite time T' in which the state observation errors reach the boundary $\mathcal{V}_1(2c_1)$ and stay in it for all time thereafter. On the other hand, if the initial conditions $e(0) \in \mathcal{V}_1(2c_1)$, the state observation errors will be uniformly bounded and they cannot escape from $\mathcal{V}_1(2c_1)$. If the absolute error limit $c_1 = 0$, then, the system (13) is globally exponentially stable.

Remark 2.13. In [7], a case of an academic bioreactor is considered, where $x'_1(t)$ and $x'_2(t)$ denote the concentrations of microorganisms and substrate, respectively. After a state transformation, the standard equations for the bioreactor can be rewritten as the form of (1):

$$\begin{cases} \dot{x}_1(t) &= x_2(t) - ux_1(t), \\ \dot{x}_2(t) &= [x_2^3(t) + (x_1(t) - x_2(t))^2(0.1u - x_2(t))]/x_1^2(t) - x_2(t)u, \\ y(t) &= x_1(t), \end{cases} \quad (35)$$

where the input u is a positive constant. By using a data collector, we can obtain the system output $y(t)$ which is a truncated data, not the real one. Thus, based on the observer (8), only the truncated data of $x_1(t)$ and $x_2(t)$ can be estimated. As we all know, the values of the concentrations microorganisms and substrate have a strong influence on each other. Therefore, how to improve the precision of the concentrations makes sense. From the process of Theorem 2.11, more accurate data of $x_1(t)$ and $x_2(t)$ can be obtained with a smaller c_1, that is, we should improve the precision of data collector.

For the system (1), the second kind of observer design is proposed as follows.

$$\begin{cases} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + k_1(y(t_k) - \hat{x}_1(t)) + f_1(\hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) &= \hat{x}_3(t) + k_2(y(t_k) - \hat{x}_1(t)) + f_2(\hat{x}_1(t), \hat{x}_2(t)), \\ &\vdots \\ \dot{\hat{x}}_{n-1}(t) &= \hat{x}_n(t) + k_{n-1}(y(t_k) - \hat{x}_1(t)) + f_{n-1}(\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_{n-1}(t)), \\ \dot{\hat{x}}_n(t) &= u(t) + k_n(y(t_k) - \hat{x}_1(t)) + f_n(\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t)), \\ & \quad t \in [t_k + \tau_k, \ t_k + T + \tau_{k+1}), \quad k \geq 0, \end{cases} \quad (36)$$

where $\hat{x}(t) = \hat{x}_0$ for $t \in [t_0, t_0 + \tau_0]$, the gains are given by $(k_1, k_2, \ldots, k_n)^\top = S^{-1}(\theta)C^\top$, where $S(\theta)$ is the unique solution of the matrix equation (9). Compared to the first kind of the observer, the main difference is that $\hat{x}_1(t)$ instead of $\hat{x}_1(t_k)$ is used to update the observer.

From $(1) - (36)$, for $k \geq 0$, we obtain the observation error:

$$\begin{cases} \dot{e}_1(t) &= e_2(t) - k_1(y(t_k) - \hat{x}_1(t)) + \tilde{f}_1, \\ \dot{e}_2(t) &= e_3(t) - k_2(y(t_k) - \hat{x}_1(t)) + \tilde{f}_2, \\ &\vdots \\ \dot{e}_{n-1}(t) &= e_n(t) - k_{n-1}(y(t_k) - \hat{x}_1(t)) + \tilde{f}_{n-1}, \\ \dot{e}_n(t) &= -k_n(y(t_k) - \hat{x}_1(t)) + \tilde{f}_n, t \in [t_k + \tau_k, t_k + T + \tau_{k+1}), \end{cases} \quad (37)$$

where $e_i(t) = x_i(t) - \hat{x}_i(t)$, $\tilde{f}_i = f_i(x_1(t), x_2(t), \ldots, x_i(t)) - f_i(\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_i(t))$, $1 \leq i \leq n$. Further, (37) can be rewritten as follows:

$$
\begin{cases}
\dot{e}_1(t) & = & e_2(t) - k_1 e_1(t) + k_1(x_1(t) - h(x_1(t_k))) + \tilde{f}_1, \\
\dot{e}_2(t) & = & e_3(t) - k_2 e_1(t) + k_2(x_1(t) - h(x_1(t_k))) + \tilde{f}_2, \\
& \vdots & \\
\dot{e}_{n-1}(t) & = & e_n(t) - k_{n-1} e_1(t) + k_{n-1}(x_1(t) - h(x_1(t_k))) + \tilde{f}_{n-1}, \\
\dot{e}_n(t) & = & -k_n e_1(t) + k_n(x_1(t) - h(x_1(t_k))) + \tilde{f}_n, \\
& & t \in [t_k + \tau_k, t_k + T + \tau_{k+1}).
\end{cases}
\tag{38}
$$

The other main results are given as follows.

Theorem 2.14. Consider the system (1) with the condition (2) and the truncation (4). For any $T > 0$, if $\bar{\tau} < T$, then, the system (36) is a globally uniformly ultimately bounded observer for the system (1).

Proof. Consider the following Lyapunov function

$$
V(t) = e(t)^\top S(\theta) e(t).
\tag{39}
$$

We obtain the derivative of $V(t)$ along the system (38):

$$
\frac{\mathrm{d}}{\mathrm{d}t} V(t)|_{(38)} = \dot{e}(t)^\top S(\theta) e(t) + e(t)^\top S(\theta) \dot{e}(t) \leq -\theta e(t)^\top S(\theta) e(t)
$$

$$
+2(x_1(t) - h(x_1(t_k))) e_1(t) + 2F^\top S(\theta) e(t), \ t \in [t_k + \tau_k, t_k + T + \tau_{k+1}).
$$

If $t \in [t_k + \tau_k, t_{k+1}]$, we have

$$
|x_1(t) - h(x_1(t_k))| \leq c_2.
$$

If $t \in [t_{k+1}, t_{k+1} + \tau_{k+1}]$, we have

$$
|x_1(t) - h(x_1(t_k))| \leq |x_1(t) - h(x_1(t_{k+1})) + h(x_1(t_{k+1})) - x_1(t_{k+1}) + x_1(t_{k+1}) - h(x_1(t_k))|
$$
$$
\leq |x_1(t) - h(x_1(t_{k+1})) + |h(x_1(t_{k+1})) - x_1(t_{k+1})| + |x_1(t_{k+1}) - h(x_1(t_k))| \leq 3c_2.
$$

Note that

$$
2(x_1(t) - h(x_1(t_k))) e_1(t) \leq 4c_3(x_1(t) - h(x_1(t_k)))^2 + \frac{1}{4c_3} e_1(t)^2 \leq 36 c_3 c_2^2 + \frac{1}{4} \theta e^T(t) S(\theta) e(t),
$$

where $c_3 = 1/\lambda_{\min}(S(1))$. Then,

$$
\frac{\mathrm{d}}{\mathrm{d}t} V(t)|_{(38)} \leq -\frac{3\theta}{4} e^\top(t) S(\theta) e(t) + 2F^\top S(\theta) e(t) + 36 c_3 c_2^2, t \in [t_k + \tau_k, t_k + T + \tau_{k+1}).
$$

From Lemma 2.6, it follows that

$$
2F^\top S(\theta) e(t) \leq 2nl\sqrt{c_3 S} e(t)^\top S(\theta) e(t).
$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t)|_{(38)} \leq -\left(\frac{3\theta}{4} - 2nl\sqrt{c_3 S}\right)e(t)^{\top}S(\theta)e(t) + 36c_3c_2^2, t \in [t_k + \tau_k, t_k + T + \tau_{k+1}).$$

Let $\mathcal{V}_2(c_2) = \{e(t) : V(e(t)) \leq 36c_3c_2^2\}$. When $e(t) \in \mathbb{R}^n/\mathcal{V}_2(c_2)$, θ can be chosen such that $-(3/4\theta - 2nl\sqrt{c_3 S}) < -\frac{1}{2}\theta$, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t)|_{(38)} \leq -(\frac{3\theta}{4} - 2nl\sqrt{c_3 S})V(t) < -\frac{1}{2}\theta V(t).$$

Let $\eta_2 = V(t_0+\tau_0)/(72c_3c_2^2)$ and $T' = (\ln \eta_2^2)/\theta+t_0+\tau_0$, when $t > T'$, $V(e(t)) < 72c_3c_2^2$, or $e(t) \in \mathcal{V}_2(2c_2)$. That is, the system (38) is globally uniformly ultimately bounded. By using the same method as Theorem 1, the system (36) is a globally uniformly ultimately bounded observer for the system (1). □

Remark 2.15. From the proof of Theorem 2.11, we can see that the sampling period T and the maximum delay $\bar{\tau}$ are dependent on the Lipschitz constant l, the matrix $S(\theta)$ and θ. The boundary of the observation errors depends on c_1. For Theorem 2.14, the sampling period $T > 0$ can be selected arbitrarily, and the maximum delay $\bar{\tau}$ just needs to satisfy $\bar{\tau} < T$. However, the bound c_2 in (4) is usually dependent on T. For example, in Remark 2.2, $c_2 = c_1 + b_1 T$, which has an effect on the boundary of the observation errors. The larger value of T, the larger boundary of the observation errors.

We also have the following corollary.

Corollary 2.16. Consider the system (1) with the condition (2). If there exist $b_1 > 0$, $c_1 > 0$ and $c_2 > c_1$ such that (5) and (3) hold, then, the system (36) is a globally uniformly ultimately bounded observer for the sampled-data system (1).

P r o o f . Using the same method as Theorem 2.14, we can obtain the result. □

3. EXAMPLE AND SIMULATION

Example 3.1. In this section, we use the following example to show the effectiveness of our high gain observer design for nonlinear system (1) with the truncation (3) and (4),

$$\begin{cases} \dot{x}_1(t) &= x_2(t) - l\sin(x_1(t)), \\ \dot{x}_2(t) &= -l(x_1(t) + \cos(x_2(t))) + u(t), \\ y(t) &= x_1(t). \end{cases} \tag{40}$$

The trajectories of the states $x_1(t)$ and $x_2(t)$ $(x(0) = (0.7, 0.3)$, $u(t) = 2\sin(t))$ are shown in Figure 1 with $l = 3$ and $l = 0.01$, respectively. The matrix $S(\theta)$ is given as follows:

$$S(\theta) = \begin{pmatrix} \frac{1}{\theta} & -\frac{1}{\theta^2} \\ -\frac{1}{\theta^2} & \frac{2}{\theta^3} \end{pmatrix}, \ S^{-1}(\theta) = \begin{pmatrix} 2\theta & \theta^2 \\ \theta^2 & \theta^3 \end{pmatrix}.$$
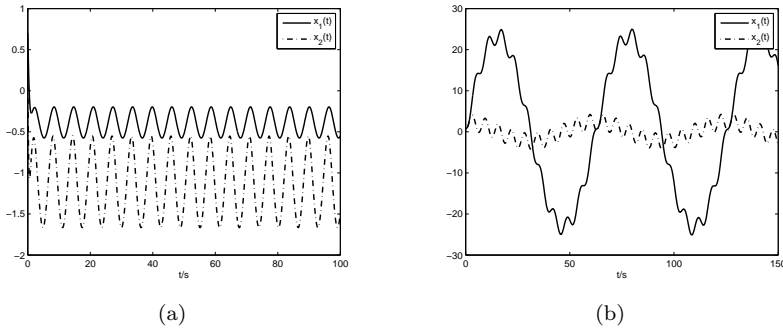
Fig. 1. (a) Trajectories of states $x_i(t)$ $(i = 1, 2)$ with $l = 3$; (b)
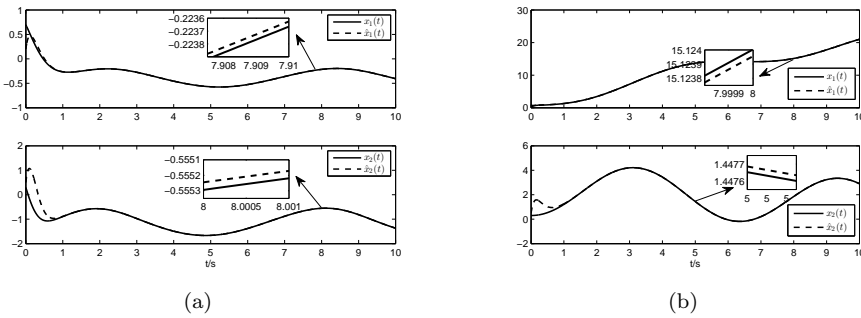Trajectories of states $x_i(t)$ $(i = 1, 2)$ with $l = 0.01$.

Fig. 2. (a) Trajectories of the states $x_i(t)$ and the estimates $\hat{x}_i(t)$
with $l = 3$; (b) Trajectories of the states $x_i(t)$ and the estimates $\hat{x}_i(t)$
with $l = 0.01$.

Construct the following observer:

$$\begin{cases} \dot{\hat{x}}_1(t) & = & \hat{x}_2(t) - l\sin(\hat{x}_1(t)) + k_1(h(y(t_k)) - \hat{x}_1(t_k)), \\ \dot{\hat{x}}_2(t) & = & u(t) - l(\hat{x}_1(t) + \cos(\hat{x}_2(t))) + k_2(h(y(t_k)) - \hat{x}_1(t_k)), \\ & & t \in [t_k + \tau_k, \ t_k + T + \tau_{k+1}), \ k \geq 0. \end{cases}$$

In the following simulation, we set $u(t) = 2\sin(t)$ and the initial conditions of the whole
system are $(x_1(0), x_2(0)) = (0.7, 0.3)$ and $(\hat{x}_1(0), \hat{x}_2(0)) = (0.2, 0.5)$, $S = 2$, $c_3 = 2.618$,
$(k_1, k_2) = (2\theta, \theta^2)$. We choose $c_4 = \sqrt{6}$, $\theta = 6$ and the absolute error limit is given
as $c_1 = 10^{-4}$. For simplicity, the following algorithm is used to truncate the sampled
data, $h(x) = [10^4 x]/10^4$, where $[x]$ denotes the integer part of x. It is easy to check that
$|h(x) - x| < 10^{-4}$. By using (14), we can obtain the sampling period $T = 2 \cdot 10^{-5}s$ and
the delay τ_k satisfies $0 \leq \tau_k \leq 2 \cdot 10^{-6}s$, the simulation results are also shown in Figure 2
and Figure 3. When $c_1 = 0$ and $c_1 = 10^{-3}$, the errors trajectories are shown in Figure 4
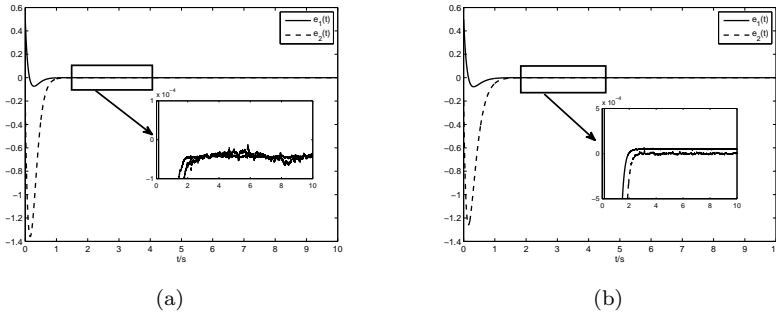and Figure 5 with $l = 3$ and $l = 0.01$, respectively.

Fig. 3. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 3$; (b) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 0.01$.
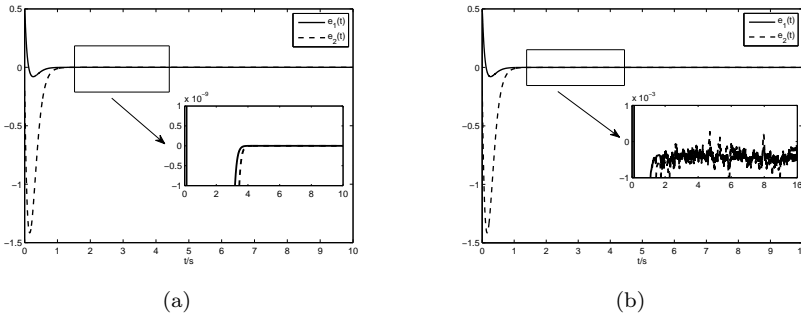

Fig. 4. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 3$ and $c_1 = 0$; (b) Trajectories of the errors $e_i(t)$ with $l = 3$ and $c_1 = 10^{-3}$.

Note that the states $x_1(t)$ and $x_2(t)$ are globally bounded. Then, from Remark 2.2, the static truncation $h(\cdot)$ is also a uniform truncation. Thus, the following observer can be constructed:

$$\begin{cases} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) - l\sin(\hat{x}_1(t)) + k_1(h(y(t_k)) - \hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) &= u(t) - l(\hat{x}_1(t) + \cos(\hat{x}_2(t))) + k_2(h(y(t_k)) - \hat{x}_1(t)), \\ & t \in [t_k + \tau_k, \ t_k + T + \tau_{k+1}), \ k \geq 0. \end{cases}$$

By taking the same values of the sampling period and the bound of time delay as the first observer, the simulation results are shown in Figure 6 and Figure 7.

Example 3.2. Now, we give the following observer design for the bioreactor system (35):

$$\begin{cases} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) - u\hat{x}_1(t) + k_1(h(y(t_k)) - \hat{x}_1(t_k)), \\ \dot{\hat{x}}_2(t) &= [\hat{x}_2^3(t) + (\hat{x}_1(t) - \hat{x}_2(t))^2(0.1u - \hat{x}_2(t))]/\hat{x}_1^2(t) - \hat{x}_2(t)u + k_2(h(y(t_k)) - \hat{x}_1(t_k)), \\ & t \in [t_k + \tau_k, \ t_k + T + \tau_{k+1}), \ k \geq 0, \end{cases}$$
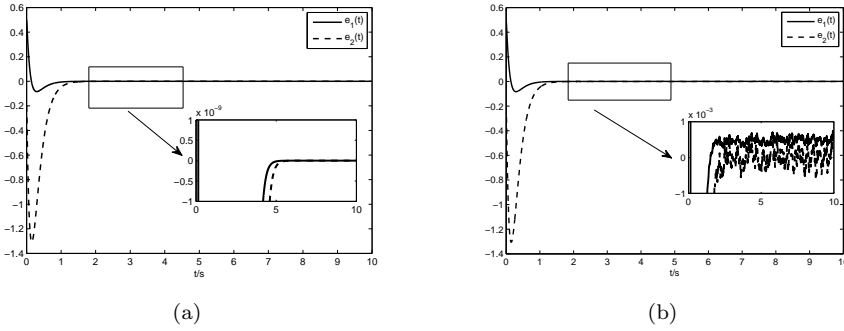
$$(41)$$

Fig. 5. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 0.01$ and
$c_1 = 0$; (b) Trajectories of the errors $e_i(t)$ with $l = 0.01$ and
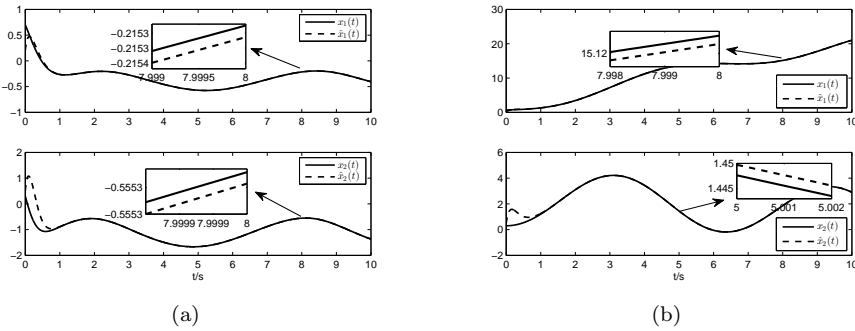$c_1 = 10^{-3}$.

Fig. 6. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 3$; (b)
Trajectories of the errors $e_i(t)$ with $l = 0.01$.

where the control input $u = 0.08$ is a constant. The initial conditions of the system (35)
and (41) are $(x_1(0), x_2(0)) = (0.03, 0.018)$ and $(\hat{x}_1(0), \hat{x}_2(0)) = (0.05, 0.029)$, and the
gain $\theta = 3$ and the absolute error limit is given as $c_1 = 10^{-4}$, $T = 2 \cdot 10^{-4}s$ and the
delay τ_k satisfies $0 \le \tau_k \le 2 \cdot 10^{-5}s$, the simulation results are also shown in Figure 8
and Figure 9.

Also, we noticed that the states $x_i(t)$ are globally bounded. Then, the second observer
design with a static truncation $h(\cdot)$ is given as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) & = & \hat{x}_2(t) - u\hat{x}_1(t) + k_1(h(y(t_k)) - \hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) & = & [\hat{x}_2^3(t) + (\hat{x}_1(t) - \hat{x}_2(t))^2(0.1u - \hat{x}_2(t))]/\hat{x}_1^2(t) - \hat{x}_2(t)u + k_2(h(y(t_k)) - \hat{x}_1(t)), \\ & & t \in [t_k + \tau_k, \ t_k + T + \tau_{k+1}), \ k \ge 0, \end{cases}$$

$$(42)$$

then, the trajectories of the errors $e_i(t)$, the states $x_i(t)$ and the estimations $\hat{x}_i$ $(i = 1, 2)$
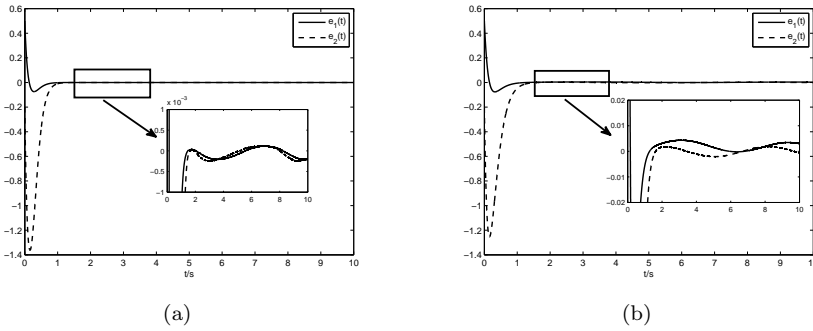are shown in Figure 10.

Fig. 7. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 3$; (b) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $l = 0.01$.
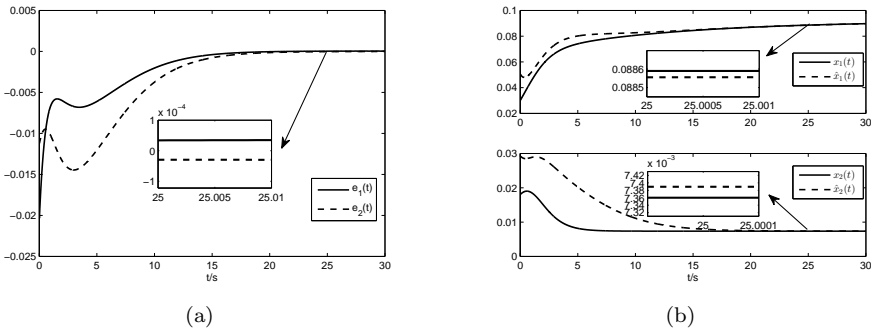

Fig. 8. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$; (b) Trajectories of the states $x_i(t)$ and the estimations $\hat{x}_i$ $(i = 1, 2)$.

4. CONCLUSIONS

In this paper, we considered two kinds of sampled-data observer design for a class of nonlinear systems. The system output was sampled and transmitted under two kinds of truncations. Firstly, we proposed definitions of the truncations and the globally uniformly ultimately bounded observer, respectively. Then, two kinds of observers were proposed by using the delayed measurements with these two truncations, respectively. The observers were hybrid in essence. For the first kind of observers, by constructing a Lyapunov–Krasovskii functional, sufficient conditions of globally uniformly ultimately bounded of the estimation errors were derived, and the maximum allowable sampling period and the maximum delay were also given. For the second ones, sufficient conditions were also given to ensure that the estimation errors were globally uniformly ultimately bounded. Finally, an example was provided to illustrate the design methods.
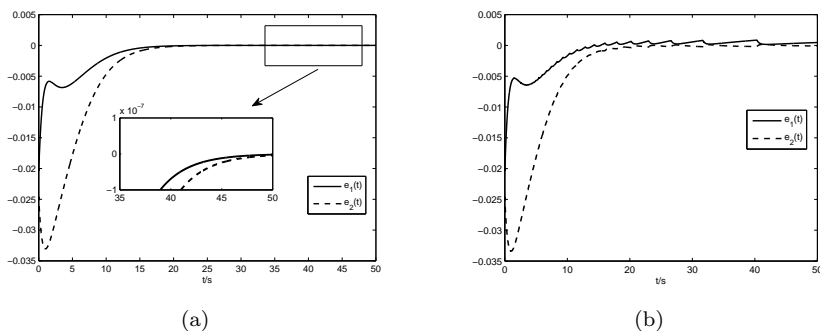
Fig. 9. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $c_1 = 0$; (b) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$ with $c_1 = 10^{-3}$.
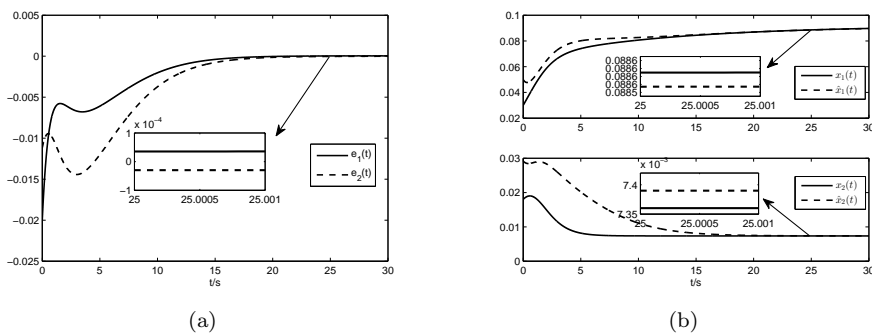

Fig. 10. (a) Trajectories of the errors $e_i(t)$ $(i = 1, 2)$; (b) Trajectories of the states $x_i(t)$ and the estimations $\hat{x}_i$ $(i = 1, 2)$.

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