Analysis of Multilevel Finite Volume Approximation of 2D Convective Cahn-Hilliard Equation

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Abstract

In this work, four finite volume methods have been constructed to solve the 2D convective Cahn-Hilliard equation with specified initial condition and periodic boundary conditions. We prove existence and uniqueness of solutions. The stability and convergence analysis of the numerical methods have been discussed thoroughly. The nonlinear terms are approximated by a linear expression based on Mickens' rule [1] of nonlocal approximations of nonlinear terms. Numerical experiments for a test problem have been carried out to test all methods.

Keywords: 2D convective Cahn-Hilliard equation, existence of solution, uniqueness, stability, convergence, finite volume, multilevel

1 Introduction

The general setting of this work is the 2D convective Cahn-Hilliard equation:

$$u_t - \gamma u(\boldsymbol{\beta} \cdot \nabla u) + \varepsilon^2 \Delta^2 u = \Delta f(u),\ (x,y) \in \mathcal{M}, t > 0, \tag{1.1}$$

with initial condition

$$u(x,y,0) = u^0(x,y), \quad (x,y) \in \overline{\mathcal{M}}, \tag{1.2}$$

and periodic boundary conditions

$$\frac{\partial^j u}{\partial x^j}(-L_1,y,t) = \frac{\partial^j u}{\partial x^j}(L_1,y,t), \quad y \in (-L_2,L_2) \text{ and } 0 \le t \le T, \tag{1.3}$$

$$\frac{\partial^j u}{\partial y^j}(x,-L_2,t) = \frac{\partial^j u}{\partial y^j}(x,L_2,t), \quad x \in (-L_1,L_1) \text{ and } 0 \le t \le T, \tag{1.4}$$

where

$$f(u) = u^3 - u,$$

γ is the driving force, $j = 0,1,2,3$, $\mathcal{M} = (-L_1,L_1) \times (-L_2,L_2)$, $\overline{\mathcal{M}}$ is the closure of $\mathcal{M}$, L_1 and L_2 are positive constants, $u^0 \in L^2(\mathcal{M})$, ε is a dimensionless interfacial width and $\boldsymbol{\beta}$ is a vector in 2D.

This equation is a successful model for the description of several physical phenomena: spinodal decomposition of phase separating systems in the presence of an external field (e.g. gravitational, magnetic and electronic) [2, 3, 4], formation of facets and corners in crystal growth [5, 6].

In the absence of the driving field, i.e. $\gamma = 0$, the system reduced to the well known Cahn-Hilliard equation

$$u_t + \varepsilon^2 \Delta^2 u = \Delta f(u), \tag{1.5}$$

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which is a model to describe the evolution of a concentration field for a binary mixture [7] and phase separation of binary liquids or binary alloys [8]. This reduced model has been studied by several authors (see [9, 10, 11] and the references therein). In [11], higher order schemes preserving the properties such as energy and large time behavior are constructed. The Cahn-Hilliard equation, (1.5), admits a Lyapunov (free energy) functional which guarantees that generically all solutions converge to an equilibrium.

The one-dimensional case of (1.1) has been studied by several researchers, theoretically and numerically. Analytical solutions have been obtained for a single interface in the presence of the driving force, i.e. $\gamma \neq 0$, in an infinite system [2]. The effect of this driving force on the coarsening dynamics of the one-dimensional Cahn-Hilliard equation at $T = 0$ has been studied by Emmott and Bray [3] when $\varepsilon = 1$. They observed that the driving force γ has an asymmetric effect on the solution of a single stationary domain wall. They also noted that the behavior of the kink-anti kink pair (bubble) depends on γ^{-1} and the separation of the interfaces. Later, Golovin et al. [12] demonstrated numerically that the one-dimensional convective Cahn-Hilliard equation exhibits a transition from coarsening to chaotic behaviour as γ increases. The presence of the driving force elucidates a fundamental asymmetry between kinks and anti-kinks which is not present in the Cahn-Hilliard theory [13]. In Podolny et al. [14], the dynamics of domain walls (kinks) governed by the convective Cahn-Hilliard equation is studied by means of asymptotic and numerical methods. The bifurcations of stationary solutions for different values of γ with $\varepsilon = 1$ has been studied by Zaks et. al [15]. Eden and Kalantarov [16] proved the existence of compact attractor and a finite inertial manifold that contains it and Zhao and Liu [17] proved the existence of optimal solutions for the one dimensional convective Cahn-Hilliard Equation. Aderogba et al. [18] solved the one dimensional convective Cahn-Hilliard equation numerically using fractional step-splitting methods for $\gamma = 0.1$ and $\varepsilon = 1$. The authors observe that the solution coarsens as t progresses and they tested numerically the transition of convective Cahn-Hilliard equation from coarsening to an order less pattern as γ increases, which is the behavior of Kuramoto Sivashinsky equation.

For multi-dimensional convective Cahn-Hilliard equation, the transition to roughening and the structure of the steady states are not well understood [3, 12, 19]. The existence of optimal solutions for the 2D convective Cahn-Hilliard equation has been proved by Zhao and Liu [20]. Eden and Kalantarov [19] considered the 3D convective Cahn-Hilliard equation with periodic boundary conditions and proved the existence of absorbing balls.

It is worth noting that (1.1) together with (1.3) and (1.4), for $j = 0$, leads to

$$\iint_{\mathcal{M}} u(x, y, t)\, dx\, dy = \iint_{\mathcal{M}} u^0(x, y)\, dx\, dy, \quad \forall t.$$

Hence for the analysis of (1.1)-(1.4), it is important to assume that [21]

$$\iint_{\mathcal{M}} u^0(x, y)\, dxdy = 0. \tag{1.6}$$

Our objective is to propose numerical techniques based on the work in [22, 23] to compute the numerical solution of (1.1)-(1.4).

In this work, we focus on the numerical solution of the 2D convective Cahn-Hilliard equation with $\gamma = 1$ and $\boldsymbol{\beta} = \langle 1, 1 \rangle$, using multilevel finite volume methods. Multilevel methods were introduced to improve calculation speed in the simulation of complex physical phenomena while maintaining an accurate solution [22, 23, 24, 25, 26, 27]. We construct two schemes associated with (1.1)-(1.4) based on the work of Bousquet et al. [23]. The schemes we construct are easy to implement and are respectively called:

(a) linear implicit multilevel approximation, and

(b) explicit multilevel approximation.

Our contribution can be regarded as extension to the works of Bousquet et al. [22, 23]. Indeed, in the latter 1D advection equation is analyzed and 2D shallow water linearized around a constant flow is proposed and implemented. In contrast, in our work we tackle fourth order 2D nonlinear partial differential equation. One of the challenge as mentioned earlier is to discretize the nonlinear term $u(\boldsymbol{\beta} \cdot \nabla u)$ in a linear way while maintaining basic properties, and as a consequence saving computational time.

For the sake of comparison, we also formulate two one-level methods associated to the multilevel methods. One of the difficulties is to design an appropriate linear expression for the nonlinear term. We achieve that thanks to the nonlocal approximation of nonlinear quantity introduced by Mickens [1] and Anguelov [28]. In particular, following [29], we approximate the nonlinear term $u(\boldsymbol{\beta} \cdot \nabla u)$ in a linear way such that the property

$$\iint\limits_{\mathcal{M}} u(\boldsymbol{\beta} \cdot \nabla u)\, u \, dxdy = 0 \tag{1.7}$$

is constructed at the discrete level.

After the construction of new schemes, we show the existence and uniqueness of the solution. At this step, we should bear in mind that since we are dealing with linear equations in finite dimension, existence of solutions is equivalent to uniqueness, thus, we provide conditions under which there is one solution. Of course, this analysis is only done for the one-level implicit scheme and easily extended to multilevel. The third contribution of this work is the stability of the new schemes. We show that the implicit multilevel method is conditionally stable with a region of stability smaller than one obtained from the one-level implicit method on the fine mesh. The fourth contribution of our study is the convergence analysis of the implicit methods. Indeed, we show that the implicit methods are first order accurate in time and second order accurate in space. Our last contribution is numerical result that supports our theoretical findings. We compute L_2-error and rate of convergence for the proposed numerical methods. We also demonstrate that in all numerical tests, the multilevel methods are faster than the one-level methods on the fine mesh.

The rest of this work is organized as follows: in the next section, we recall some preliminaries and introduce some standard notations. We also discuss, in Section 2, some properties of difference operators and the discrete analogue of L_2 space. In Sections 3 and 4, we construct one-level and multilevel finite volume methods and proved those methods are conditionally stable and conditionally convergent. In Section 5, we present some numerical results comparing computations done by one-level methods and computations done by the multilevel methods. Lastly, conclusions are given in Section 6.

2 Some Preliminaries and Space Discretizations

In this section, we recall some preliminaries which are helpful to our discussion and we present the space discretization in a 2D rectangular region. To develop finite volume approximations that satisfy the discrete analogue of (1.7), we first introduce some standard notations and results. We partition $\mathcal{M}$ into $N_1 \times N_2$ control volumes $(k_{i,j})_{1\le i\le N_1, 1\le j\le N_2}$ of uniform area $\Delta x \Delta y$, where Δx and Δy are the spatial step sizes in the $x-$ and $y-$ directions, respectively. It is assumed that the partition of the domain is conform, meaning that for two elements A and B one has, $A \cap B$ is either a face, a vertex or empty set. For $0 \le i \le N_1$ and $0 \le j \le N_2$,

$$x_{i+1/2} = i\Delta x - L_1,\; y_{j+1/2} = j\Delta y - L_2,$$

so that

$$k_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \text{ for } 1 \le i \le N_1, 1 \le j \le N_2.$$

(x_i, y_j) is the centre of the (i, j) control volume, which is given by the formula

$$(x_i, y_j) = ((i-1)\Delta x + \frac{\Delta x}{2} - L_1,\; (j-1)\Delta y + \frac{\Delta y}{2} - L_2), \quad 1 \le i \le N_1, 1 \le j \le N_2.$$

In the rest of this work, we take $h = (\Delta x, \Delta y)$. The approximate solution to the control volume average of the true solution at $t_n = n\Delta t$ is denoted by $u^n_{i,j}$, i.e.

$$u^n_{i,j} \approx \frac{1}{\Delta x \Delta y} \iint\limits_{k_{i,j}} u(x, y, t_n) dx dy, 1 \leq i \leq N_1, 1 \leq j \leq N_2,$$

where Δt is the temporal step size such that $\Delta t M = T$, which is obtained recursively by starting with the initial average value, $u^0_{i,j}$, given by

$$u^0_{i,j} = \frac{1}{\Delta x \Delta y} \iint\limits_{k_{i,j}} u^0(x, y) dx dy, 1 \leq i \leq N_1,\ 1 \leq j \leq N_2.$$

Define the space $\mathcal{H}_h$ as

$$\mathcal{H}_h = \left\{ \mathbf{u} = \left(u_{i,j}\right)_{i,j\in\mathbb{Z}}, u_{i,j} \in \mathbb{R} |\ u_{i+N_1,j} = u_{i,j} = u_{i,j+N_2},\ \text{and}\ \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j} = 0 \right\},$$

equipped with the inner product and discrete L^2 norm

$$(\mathbf{u}, \mathbf{v})_h = \Delta x \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j} v_{i,j} \quad \text{and} \quad \|\mathbf{u}\|_h = \left(\Delta x \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u^2_{i,j} \right)^{1/2},$$

respectively.

For $\mathbf{u} \in \mathcal{H}_h$, we introduce the following difference operators:

$$\nabla^-_{1,h} u_{i,j} = \frac{1}{\Delta x}\left(u_{i,j} - u_{i-1,j}\right),\ \nabla^+_{1,h} u_{i,j} = \frac{1}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right), \tag{2.1}$$

$$\nabla^-_{2,h} u_{i,j} = \frac{1}{\Delta y}\left(u_{i,j} - u_{i,j-1}\right),\ \nabla^+_{2,h} u_{i,j} = \frac{1}{\Delta y}\left(u_{i,j+1} - u_{i,j}\right), \tag{2.2}$$

$$\Delta_{1,h} u_{i,j} = \frac{1}{\Delta x^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right), \tag{2.3}$$

$$\Delta_{2,h} u_{i,j} = \frac{1}{\Delta y^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right), \tag{2.4}$$

$$\Delta^2_{1,h} u_{i,j} = \frac{1}{\Delta x^2}\left(\Delta_{1,h} u_{i+1,j} - 2\Delta_{1,h} u_{i,j} + \Delta_{1,h} u_{i-1,j}\right), \tag{2.5}$$

$$\Delta^2_{2,h} u_{i,j} = \frac{1}{\Delta y^2}\left(\Delta_{2,h} u_{i,j+1} - 2\Delta_{2,h} u_{i,j} + \Delta_{2,h} u_{i,j-1}\right). \tag{2.6}$$

From (2.1)-(2.6), we have

$$\boldsymbol{\beta} \cdot \nabla^{\pm}_h = \nabla^{\pm}_{1,h} + \nabla^{\pm}_{2,h},\ \Delta_h = \Delta_{1,h} + \Delta_{2,h},\ \Delta^2_h = \Delta^2_{1,h} + \Delta_{1,h}\Delta_{2,h} + \Delta_{2,h}\Delta_{1,h} + \Delta^2_{2,h}. \tag{2.7}$$

The discrete analogue of the derivative of product of functions is given as follows: for $\mathbf{u}, \mathbf{v} \in \mathcal{H}_h$,

$$(\boldsymbol{\beta} \cdot \nabla^+_h)(u_{i,j} v_{i,j}) = (\nabla^+_{1,h} u_{i,j}) v_{i+1,j} + u_{i,j}(\nabla^+_{1,h} v_{i,j}) + (\nabla^+_{2,h} u_{i,j}) v_{i,j+1} + u_{i,j}(\nabla^+_{2,h} v_{i,j}), \tag{2.8}$$

$$(\boldsymbol{\beta} \cdot \nabla^-_h)(u_{i,j} v_{i,j}) = (\nabla^-_{1,h} u_{i,j}) v_{i-1,j} + u_{i,j}(\nabla^-_{1,h} v_{i,j}) + (\nabla^-_{2,h} u_{i,j}) v_{i,j-1} + u_{i,j}(\nabla^-_{2,h} v_{i,j}). \tag{2.9}$$

From the definition of $\mathcal{H}_h$ and the discrete product rules, (2.8) and (2.9), one obtains:

Lemma 2.1. *Let $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{H}_h$. Then for any vector $\boldsymbol{\beta} = \langle \beta_1, \beta_2 \rangle$ with $\beta_1, \beta_2 \in \mathbb{R}$*

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}(\boldsymbol{\beta} \cdot \nabla^+_h) u_{i,j} = -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\boldsymbol{\beta} \cdot \nabla^-_h) w_{i,j}.$$

Proof. To prove this, we use the definition of $\mathcal{H}_h$.

$$
\begin{aligned}
\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)u_{i,j} &= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}(\beta_1\,\nabla_{1,h}^+ u_{i,j} + \beta_2\,\nabla_{2,h}^+ u_{i,j}) \\
&= \frac{\beta_1}{\Delta x}\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i+1,j} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i,j}\right) + \frac{\beta_2}{\Delta y}\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i,j+1} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i,j}\right) \\
&= \frac{\beta_1}{\Delta x}\left(\sum_{i=2}^{N_1}\sum_{j=1}^{N_2} w_{i-1,j}u_{i,j} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i,j}\right) + \frac{\beta_2}{\Delta y}\left(\sum_{i=1}^{N_1}\sum_{j=2}^{N_2} w_{i,j-1}u_{i,j} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} w_{i,j}u_{i,j}\right) \\
&\quad + \frac{\beta_1}{\Delta x}\sum_{j=1}^{N_2} w_{N_1,j}u_{1,j} + \frac{\beta_2}{\Delta y}\sum_{i=1}^{N_1} w_{i,N_2}u_{i,1} \\
&= -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}\left[\beta_1\left(\frac{w_{i,j}-w_{i-1,j}}{\Delta x}\right) + \beta_2\left(\frac{w_{i,j}-u_{i,j-1}}{\Delta y}\right)\right] \\
&= -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)w_{i,j}.
\end{aligned}
$$

□

We define the following discrete semi-norms and norms for $\mathbf{u} = (u_{i,j})$, $1 \le i \le N_1$, $1 \le j \le N_2$.

$$
|\mathbf{u}|_{1,h} = \left(\Delta x\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[(\nabla_{1,h}^- u_{i,j})^2 + (\nabla_{2,h}^- u_{i,j})^2\right]\right)^{\frac{1}{2}}, \tag{2.10}
$$

$$
|\mathbf{u}|_{2,h} = \left(\Delta x\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\Delta_h u_{i,j})^2\right)^{\frac{1}{2}}, \tag{2.11}
$$

$$
\|\mathbf{u}\|_{\infty,h} = \max_{\substack{1\le i\le N_1\\ 1\le j\le N_2}} |u_{i,j}|, \qquad \|\mathbf{u}\|_{1,h}^2 = |\mathbf{u}|_{1,h}^2 + \|\mathbf{u}\|_h^2.
$$

In (2.10), $\nabla_{1,h}^-$ and $\nabla_{2,h}^-$ can be replaced by $\nabla_{1,h}^+$ and $\nabla_{2,h}^+$, respectively. Using (2.10) and (2.11), we have

$$
|\mathbf{u}|_{1,h}^2 \le 4\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\|\mathbf{u}\|_h^2, \tag{2.12}
$$

and the following are obtained by direct computations

$$
\|\mathbf{u}\|_h^2 \le 4L_1L_2\|\mathbf{u}\|_{\infty,h}^2
$$

and

$$
\|\mathbf{u}\|_{\infty,h}^2 \le \frac{1}{\Delta x\,\Delta y}\|\mathbf{u}\|_h^2. \tag{2.13}
$$

Moreover, it is important to note that if $\mathbf{u}$ belongs to $\mathcal{H}_h$, then the discrete Poincaré's inequality holds; this is to say that there is $\eta > 0$, independent of Δx and Δy such that

$$
\eta\,\|\mathbf{u}\|_h \le |\mathbf{u}|_{1,h}. \tag{2.14}
$$

Remark 2.1. *With (2.14), we conclude that the semi-norm $|\cdot|_{1,h}$ is a norm on $\mathcal{H}_h$ equivalent to $\|\cdot\|_{1,h}$.*

The following identities and inequalities will be helpful.

- For any $\mathbf{u}, \mathbf{v} \in \mathcal{H}_h$

$$2(\mathbf{u}-\mathbf{v},\mathbf{u})_h = \|\mathbf{u}\|_h^2 - \|\mathbf{v}\|_h^2 + \|\mathbf{u}-\mathbf{v}\|_h^2, \tag{2.15}$$
$$2(\mathbf{u}-\mathbf{v},\mathbf{v})_h = \|\mathbf{u}\|_h^2 - \|\mathbf{v}\|_h^2 - \|\mathbf{u}-\mathbf{v}\||_h^2. \tag{2.16}$$

- For $x \in [0, \frac{1}{2}]$,

$$\left(\frac{1}{2}\right)^{2x} \leq 1 - x. \tag{2.17}$$

- Young's inequality: For any $a, b \in \mathbb{R}$ and $\delta > 0$, we have

$$ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2. \tag{2.18}$$

- Cauchy-Schwarz's inequality: For $N \in \mathbb{N}$

$$\sum_{i=1}^{N} a_i b_i \leq \left(\sum_{i=1}^{N} a_i^2\right)^{1/2} \left(\sum_{i=1}^{N} b_i^2\right)^{1/2}. \tag{2.19}$$

In order to approximate the nonlinear term, we introduce the bilinear map: $C_h : \mathcal{H}_h \times \mathcal{H}_h \to \mathbb{R}^{N_1 \times N_2}$ in the form

$$\begin{aligned} C_h(\mathbf{u},\mathbf{v})_{i,j} = {} & \alpha_1[u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)v_{i,j} + v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)u_{i,j} + v_{i+1,j}\nabla_{1,h}^+u_{i,j} + v_{i,j+1}\nabla_{2,h}^+u_{i,j}] \\ & + \alpha_2[u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)v_{i,j} + v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)u_{i,j} + v_{i-1,j}\nabla_{1,h}^-u_{i,j} + v_{i,j-1}\nabla_{2,h}^-u_{i,j}], \end{aligned} \tag{2.20}$$

where α_1 and α_2 are constants. We use this bilinear map to approximate the nonlinear term $u(\boldsymbol{\beta}\cdot\nabla)u$ at t_{n+1} and t_n for the implicit and explicit methods, respectively.
Using (2.8), (2.9) and Lemma 2.1, we prove the following.

Lemma 2.2. *For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}_h$*

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(C_h(\boldsymbol{u},\boldsymbol{v}))_{i,j}u_{i,j} = 0. \tag{2.21}$$

Proof. For all $\mathbf{u}, \mathbf{v} \in \mathcal{H}_h$, we have

$$\begin{aligned} \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}((\boldsymbol{\beta}\cdot\nabla_h^+)v_{i,j})u_{i,j} &= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)(v_{i,j}u_{i,j}) - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{1,h}^+u_{i,j})v_{i+1,j} \\ &\quad - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{2,h}^+u_{i,j})v_{i,j+1} \qquad \text{using (2.8)} \\ &= -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)(u_{i,j}) - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{1,h}^+u_{i,j})v_{i+1,j} \\ &\quad - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{2,h}^+u_{i,j})v_{i,j+1} \qquad \text{using Lemma 2.1.} \end{aligned}$$

Similarly

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}((\boldsymbol{\beta}\cdot\nabla_h^-)v_{i,j})u_{i,j} = -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)(u_{i,j}) - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{1,h}^- u_{i,j})v_{i-1,j}$$
$$-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}(\nabla_{2,h}^- u_{i,j})v_{i,j-1}.$$

Thus we have

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}\left[u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)v_{i,j} + v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)u_{i,j} + v_{i+1,j}\nabla_{1,h}^+u_{i,j} + v_{i,j+1}\nabla_{2,h}^+u_{i,j}\right] = 0$$

and

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j}\left[u_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^-)v_{i,j} + v_{i,j}(\boldsymbol{\beta}\cdot\nabla_h^+)u_{i,j} + v_{i-1,j}\nabla_{1,h}^-u_{i,j} + v_{i,j-1}\nabla_{2,h}^-u_{i,j}\right] = 0.$$

Therefore, the proof is complete. □

Remark 2.2. *For any d-dimensional space problem with $d \geq 3$, we can easily extend (2.20) such that an analogous of (2.21) holds. That is*

$$C_h(\boldsymbol{u},\boldsymbol{v})_{i_1,i_2,\cdots,i_d} = \alpha_1\left[u_{i_1,i_2,\cdots,i_d}(\boldsymbol{\beta}\cdot\nabla_h^+)v_{i_1,i_2,\cdots,i_d} + v_{i_1,i_2,\cdots,i_d}(\boldsymbol{\beta}\cdot\nabla_h^-)u_{i_1,i_2,\cdots,i_d} + \sum_{s=1}^{d} v_{s+}\nabla_{s,h}^+u_{i_1,i_2,\cdots,i_d}\right]$$
$$+\alpha_2\left[u_{i_1,i_2,\cdots,i_d}(\boldsymbol{\beta}\cdot\nabla_h^-)v_{i_1,i_2,\cdots,i_d} + v_{i_1,i_2,\cdots,i_d}(\boldsymbol{\beta}\cdot\nabla_h^+)u_{i_1,i_2,\cdots,i_d} + \sum_{s=1}^{d} v_{s-}\nabla_{s,h}^-u_{i_1,i_2,\cdots,i_d}\right],$$

where $v_{s\pm} = v_{i_1,\cdots,i_s\pm1,\cdots,i_d}$, for $s = 1,2,\cdots,d$ and i_s is the position of the vector at the s^{th} coordinate.

Lemma 2.3. *For $\boldsymbol{u},\boldsymbol{w}\in\mathcal{H}_h$*

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Delta_{1,h}(\Delta_{1,h}u_{i,j})w_{i,j} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Delta_{1,h}(u_{i,j})\Delta_{1,h}(w_{i,j}).$$

Proof. For all $\mathbf{u},\mathbf{w}\in\mathcal{H}_h$, we have

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Delta_{1,h}(\Delta_{1,h}u_{i,j})w_{i,j} = \sum_{j=1}^{N_2}\left[\sum_{i=1}^{N_1}\frac{1}{\Delta x^4}(u_{i+2,j}-2u_{i+1,j}+u_{i,j})w_{i,j} - 2\sum_{i=1}^{N_1}\frac{1}{\Delta x^4}(u_{i+1,j}-2u_{i,j}+u_{i-1,j})w_{i,j}\right.$$
$$\left.+\sum_{i=1}^{N_1}\frac{1}{\Delta x^4}(u_{i,j}-2u_{i-1,j}+u_{i-2,j})w_{i,j}\right]. \tag{2.22}$$

From the periodic boundary conditions, $\sum_{i=1}^{N_1}u_{i,j} = \sum_{i=1}^{N_1}u_{i-1,j} = \sum_{i=1}^{N_1}u_{i+1,j}$ for each $j = 1,\ldots,N_2$ and hence (2.22) yields

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Delta_{1,h}(\Delta_{1,h}u_{i,j})w_{i,j} = \sum_{j=1}^{N_2}\left[\sum_{i=1}^{N_1}\frac{1}{\Delta x^2}(\Delta_{1,h}u_{i,j})w_{i-1,j} - 2\sum_{i=1}^{N_1}\frac{1}{\Delta x^2}(\Delta_{1,h}u_{i,j})w_{i,j} + \sum_{i=1}^{N_1}\frac{1}{\Delta x^2}(\Delta_{1,h}u_{i,j})w_{i+1,j}\right]$$
$$= \sum_{j=1}^{N_2}\sum_{i=1}^{N_1}(\Delta_{1,h}u_{i,j})\left(\frac{1}{\Delta x^2}(w_{i-1,j}-2w_{i,j}+w_{i+1,j})\right)$$
$$= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\Delta_{1,h}u_{i,j})(\Delta_{1,h}w_{i,j}).$$

$\square$

Lemma 2.3 also holds when one (or two) of the operator(s) $\Delta_{1,h}$ is (are) replaced by $\Delta_{2,h}$.

Lemma 2.4. *For $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{H}_h$*

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_h^2(u_{i,j})w_{i,j} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_h(u_{i,j})\Delta_h(w_{i,j}).$$

Proof. For any $\mathbf{u}, \mathbf{w} \in \mathcal{H}_h$ using (2.7), we have

$$\begin{aligned}
\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_h^2(u_{i,j})w_{i,j} &= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left[(\Delta_{1,h}^2 + \Delta_{1,h}\Delta_{2,h} + \Delta_{2,h}\Delta_{1,h} + \Delta_{2,h}^2)(u_{i,j})\right] w_{i,j} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_{1,h}u_{i,j}\Delta_{1,h}w_{i,j} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_{2,h}u_{i,j}\Delta_{1,h}w_{i,j} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_{1,h}u_{i,j}\Delta_{2,h}w_{i,j} \\
&\quad + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_{2,h}u_{i,j}\Delta_{2,h}w_{i,j} \qquad \text{using Lemma 2.3} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} (\Delta_{1,h}u_{i,j} + \Delta_{2,h}u_{i,j})(\Delta_{1,h}w_{i,j} + \Delta_{2,h}w_{i,j}) \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \Delta_h u_{i,j}\Delta_h w_{i,j}.
\end{aligned}$$

$\square$

The following lemma will be used later.

Lemma 2.5. *For $\boldsymbol{u} \in \{\boldsymbol{u} = (u)_{i,j}, u_{i,j} \in \mathbb{R} |\ u_{i+N_1,j} = u_{i,j} = u_{i,j+N_2},\ i,j \in \mathbb{Z}\}$, the following inequality holds true*

$$|\boldsymbol{u}|_{1,h}^2 \le |\boldsymbol{u}|_{2,h}\|\boldsymbol{u}\|_h.$$

Proof. Using (2.7), for $\mathbf{u} \in \{\mathbf{u} = (u)_{i,j}, u_{i,j} \in \mathbb{R} |\ u_{i+N_1,j} = u_{i,j} = u_{i,j+N_2},\ i,j \in \mathbb{Z}\}$, we have

$$\begin{aligned}
(\Delta_h\mathbf{u}, \mathbf{u})_h &= (\Delta_{1,h}\mathbf{u} + \Delta_{2,h}\mathbf{u}, \mathbf{u})_h \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}\right] u_{i,j} \\
&= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x^2}\right) u_{i,j} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x^2}\right) u_{i,j} \\
&\quad + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y^2}\right) u_{i,j} - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y^2}\right) u_{i,j}.
\end{aligned} \tag{2.23}$$

Using periodicity, we obtain

$$(\Delta_h\mathbf{u}, \mathbf{u})_h = -\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right)^2 - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right)^2 = -|\mathbf{u}|_{1,h}^2.$$

Using Cauchy-Schwarz's inequality, we have

$$|\mathbf{u}|_{1,h}^2 =\leq \|\Delta_h \mathbf{u}\|_h \|\mathbf{u}\|_h = |\mathbf{u}|_{2,h}\|\mathbf{u}\|_h.$$

□

For $\mathbf{u} \in \mathcal{H}_h$, Lemma 2.5 and Young's inequality implies the existence of η, positive constant independent of both Δy, and Δx such that

$$\eta|\mathbf{u}|_{1,h} \leq |\mathbf{u}|_{2,h}. \tag{2.24}$$

3 One-level Finite Volume Methods

In this section, we present two traditional one-level finite volume methods: namely implicit finite volume method and explicit finite volume method. The existence, uniqueness and convergence of solution for the implicit method are proved and stability analysis is examined for both schemes. For both methods thirteen point stencils are used to approximate (1.1)-(1.4), as shown in Fig. 1. The introduction of these classical schemes is important at least for three reasons:

(a) comparison with multilevel methods;

(b) these schemes that are categorized as classical present significant challenges for their analysis as we will see;

(c) the analysis of these schemes will shed lights in the analysis of multilevel methods.

3.1 Implicit one-level finite volume method

The nonlinear term $u(\boldsymbol{\beta} \cdot \nabla u)$ at t_{n+1} is approximated linearly using the bilinear map defined in section 2, (2.20), and is given by

$$[u(\boldsymbol{\beta} \cdot \nabla)u]\,|_{i,j}^{n+1} \approx \left(C_h(\mathbf{u}^{n+1}, \tilde{\mathbf{u}}^n)\right)_{i,j}, \tag{3.1}$$

where $\tilde{\mathbf{u}}^n$ is the approximation of $\mathbf{u}^{n+1}$, given by

$$\tilde{\mathbf{u}}^n = a_1\mathbf{u}^n + a_2\mathbf{u}^{n-1} + a_3\mathbf{u}^{n-2} + \cdots + a_{m_0}\mathbf{u}^{n-m_0+1}, \tag{3.2}$$

where $m_0 \in \{1, 2, \ldots, n\}$ and $a_1, a_2, \ldots,$ and a_{m_0} are coefficients that determine the approximation with $3(\alpha_1 + \alpha_2)(a_1 + a_2 + \cdots + a_{m_0}) = 1$, ensuring consistency of the approximation. For $m < m_0 - 1$, the term $\tilde{\mathbf{u}}^m$ is given by the relation

$$\tilde{\mathbf{u}}^m = \mathbf{u}^m. \tag{3.3}$$

We approximate the nonlinear term on the right hand side of (1.1) at t_{n+1} by a linear second order accurate in space as follows:

$$\Delta f(u)|_{i,j}^{n+1} \approx \nabla_{1,h}^+(\varphi_{i-1/2,j}^n \nabla_{1,h}^- u_{i,j}^{n+1}) + \nabla_{2,h}^+(\varphi_{i,j-1/2}^n \nabla_{2,h}^- u_{i,j}^{n+1}). \tag{3.4}$$

where

$$\varphi_{i-1/2,j}^n = \frac{f'(u_{i,j}^n) + f'(u_{i-1,j}^n)}{2} \text{ and } \varphi_{i,j-1/2}^n = \frac{f'(u_{i,j}^n) + f'(u_{i,j-1}^n)}{2}.$$

Lemma 3.1. *For $\boldsymbol{u}^n, \boldsymbol{u}^{n+1} \in \mathcal{H}_h$*

$$\Delta x \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left[\nabla_{1,h}^+(\varphi_{i-1/2,j}^n \nabla_{1,h}^- u_{i,j}^{n+1}) + \nabla_{2,h}^+(\varphi_{i,j-1/2}^n \nabla_{2,h}^- u_{i,j}^{n+1})\right] u_{i,j}^{n+1} \leq |\boldsymbol{u}^{n+1}|_{1,h}^2.$$

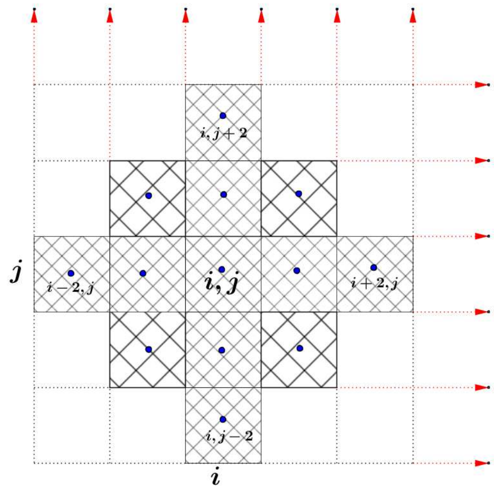


Figure 1: Finite volume discretization in 2D

Proof. For $\mathbf{u}^n, \mathbf{u}^{n+1} \in \mathcal{H}_h$, applying Lemma 2.1, we have

$$\begin{aligned}
\Delta x\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} &\nabla^+_{1,h}(\varphi^n_{i-1/2,j}\,\nabla^-_{1,h}u^{n+1}_{i,j})u^{n+1}_{i,j} + \nabla^+_{2,h}(\varphi^n_{i,j-1/2}\,\nabla^-_{2,h}u^{n+1}_{i,j})u^{n+1}_{i,j} \\
&= -\Delta x\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[\varphi^n_{i-1/2,j}\,(\nabla^-_{1,h}u^{n+1}_{i,j})^2 + \varphi^n_{i,j-1/2}\,(\nabla^-_{2,h}u^{n+1}_{i,j})^2\Big] \\
&= -\frac{3}{2}\Delta x\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[\big((u^n_{i,j})^2 + (u^n_{i-1,j})^2\big)\,(\nabla^-_{1,h}u^{n+1}_{i,j})^2 + \big((u^n_{i,j})^2 + (u^n_{i,j-1})^2\big)\,(\nabla^-_{2,h}u^{n+1}_{i,j})^2\Big] \\
&\qquad + \Delta x\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[(\nabla^-_{1,h}u^{n+1}_{i,j})^2 + (\nabla^-_{2,h}u^{n+1}_{i,j})^2\Big] \\
&\le \Delta x\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[(\nabla^-_{1,h}u^{n+1}_{i,j})^2 + (\nabla^-_{2,h}u^{n+1}_{i,j})^2\Big] = |\mathbf{u}^{n+1}|^2_{1,h}.
\end{aligned}$$

□

The fourth order derivative is discretized using the central difference method and combining together with (3.1) and (3.4), the implicit one-level finite volume discretization of (1.1)-(1.4) is given as follows:

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - C_h(\mathbf{u}^{n+1}, \tilde{\mathbf{u}}^n)_{i,j} + \varepsilon^2\Delta_h^2 u^{n+1}_{i,j} = \nabla^+_{1,h}(\varphi^n_{i-1/2,j}\nabla^-_{1,h}u^{n+1}_{i,j}) + \nabla^+_{2,h}(\varphi^n_{i,j-1/2}\nabla^-_{2,h}u^{n+1}_{i,j}), \quad (3.5a)$$

$$u^n_{i,j} = u^n_{i+N_1,j} = u^n_{i,j+N_2}, \quad (3.5b)$$

$$u^0_{i,j} = \frac{1}{\Delta x\Delta y}\iint\limits_{k_{i,j}} u^0(x)dxdy. \quad (3.5c)$$

Remark 3.1. *It is worth noting thanks to* (2.20), *that* (3.5a)-(3.5c) *is a linear system of equations, while* (1.1)-(1.4) *is nonlinear.*

Before discussing some qualitative properties of the solution of (3.5a)-(3.5c), we first address its feasibility.

Theorem 3.1. *If $\Delta t < 4\varepsilon^2$, then the approximate solution $\boldsymbol{u}^n$ of* (3.5a)-(3.5c) *is unique.*

Noting that Equations (3.5a)-(3.5c) is a linear system in finite dimensional space, its existence is equivalent to uniqueness of solution [30]. Thus, we only show that the approximations $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^M$ satisfying (3.5a)-(3.5c), are unique.

Proof. For $n = 0, 1, 2, \ldots, M$, let $\mathbf{v}^n$ and $\mathbf{u}^n$ be two sequences of solutions of (3.5a)-(3.5c) with $\mathbf{v}_0 = \mathbf{u}_0$. Let $\mathbf{z}^n = \mathbf{u}^n - \mathbf{v}^n$ and clearly $\mathbf{z}^0 = 0$. We shall prove by induction that $\mathbf{z}^n = 0$ for all $n = 1, 2, \ldots, M$. We observe that $\mathbf{z}^{n+1}$ is a solution of

$$\begin{aligned}\frac{1}{\Delta t}(z_{i,j}^{n+1} - z_{i,j}^{n}) - (C_h(\mathbf{u}^{n+1}, \tilde{\mathbf{u}}^n))_{i,j} &+ (C_h(\mathbf{v}^{n+1}, \tilde{\mathbf{v}}^n))_{i,j} + \varepsilon^2 \Delta_h^2 z_{i,j}^{n+1} \\ &= \nabla_{1,h}^{+}(\varphi_{i-1/2,j}^{n} \nabla_{1,h}^{-} u_{i,j}^{n+1}) + \nabla_{2,h}^{+}(\varphi_{i,j-1/2}^{n} \nabla_{2,h}^{-} u_{i,j}^{n+1}) \\ &\quad - \nabla_{1,h}^{+}(\psi_{i-1/2,j}^{n} \nabla_{1,h}^{-} v_{i,j}^{n+1}) + \nabla_{2,h}^{+}(\psi_{i,j-1/2}^{n} \nabla_{2,h}^{-} v_{i,j}^{n+1}),\end{aligned} \tag{3.6}$$

for $i = 1, \cdots, N_1,\ j = 1, \cdots, N_2$ and

$$\psi_{i-1/2,j}^{n} = \frac{f'(v_{i,j}^n) + f'(v_{i-1,j}^n)}{2} \text{ and } \psi_{i,j-1/2}^{n} = \frac{f'(v_{i,j}^n) + f'(v_{i,j-1}^n)}{2}.$$

By induction, we assume that $\mathbf{z}^n = 0$ and we want to show that $\mathbf{z}^{n+1} = 0$. It follows then that

$$z_{i,j}^{n+1} - \Delta t\,(C_h(\mathbf{z}^{n+1}, \tilde{\mathbf{u}}^n))_{i,j} + \Delta t\,\varepsilon^2 \Delta_h^2 z_{i,j}^{n+1} = \Delta t\,\left[\nabla_{1,h}^{+}(\varphi_{i-1/2,j}^{n} \nabla_{1,h}^{-} z_{i,j}^{n+1}) + \nabla_{2,h}^{+}(\varphi_{i,j-1/2}^{n} \nabla_{2,h}^{-} z_{i,j}^{n+1})\right]. \tag{3.7}$$

Multiplying (3.7) by $\Delta t \Delta x \Delta y\, z_{i,j}^{n+1}$ and summing the resulting equalities for $i = 1, \ldots, N_1,\ j = 1, \ldots, N_2$, with help of (2.21) and Lemma 2.1, we obtain

$$\|\mathbf{z}^{n+1}\|_h^2 + \Delta t \varepsilon^2 |\mathbf{z}^{n+1}|_{2,h}^2 = -\Delta t \Delta x \Delta y \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \left[\varphi_{i-1/2,j}^{n} (\nabla_{1,h}^{-} z_{i,j}^{n+1})^2 + \varphi_{i,j-1/2}^{n} (\nabla_{2,h}^{-} z_{i,j}^{n+1})^2\right],$$

which after the application of Lemma 3.1 gives

$$\|\mathbf{z}^{n+1}\|_h^2 + \Delta t \varepsilon^2 |\mathbf{z}^{n+1}|_{2,h}^2 \le \Delta t |\mathbf{z}^{n+1}|_{1,h}^2. \tag{3.8}$$

Applying Lemma 2.5 and Young's inequality, (3.8) implies that

$$(1 - \frac{\Delta t}{4\varepsilon^2}) \|\mathbf{z}^{n+1}\|_h^2 \le 0. \tag{3.9}$$

For $\dfrac{\Delta t}{4\varepsilon^2} < 1$, we get

$$\|\mathbf{z}^{n+1}\|_h^2 \le 0.$$

Therefore, $\mathbf{z}^{n+1} = 0$. This completes the proof of uniqueness, hence the existence of solution. □

With the existence of solution being conditioned, it is quite clear that all possible results will be under at least the same condition, that is $\Delta t < 4\varepsilon^2$. About the stability of the method (3.5a)-(3.5c), we have the following theorem.

Theorem 3.2. *The finite volume method defined by* (3.5a)-(3.5c)*, is conditionally stable in* $L^\infty(0, T; \mathcal{H}_h)$*, that is, for $\Delta t \le \varepsilon^2$ and $1 \le n \le M$,*

$$\|\boldsymbol{u}^n\|_h^2 \le 2^{\frac{2T}{\varepsilon^2}} \|\boldsymbol{u}^0\|_h^2.$$

Proof. By multiplying (3.5a) with $2\Delta x\Delta y\Delta t\ u_{i,j}^{n+1}$ and summing from $i = 1$ to N_1 and $j = 1$ to N_2, we obtain

$$\begin{aligned}\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^n\|_h^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_h^2 + 2\Delta t\,\varepsilon^2(\Delta_h^2\mathbf{u}^{n+1}, \mathbf{u}^{n+1})_h \\ = 2\Delta x\Delta y\Delta t\sum_{i=1}^{N_1}\sum_{i=1}^{N_2}\left[\nabla_{1,h}^+(\varphi_{i-1/2,j}^n\nabla_{1,h}^- u_{i,j}^{n+1}) + \nabla_{2,h}^+(\varphi_{i,j-1/2}^n\nabla_{2,h}^- u_{i,j}^{n+1})\right]u_{i,j}^{n+1}.\end{aligned} \tag{3.10}$$

Using Lemmas 2.4 and 3.1 together with (3.10), we have

$$\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^n\|_h^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_h^2 + 2\Delta t\varepsilon^2|\mathbf{u}^{n+1}|_{2,h}^2 \le 2\Delta t|\mathbf{u}^{n+1}|_{1,h}^2. \tag{3.11}$$

Due to Lemma 2.5, Young's inequality and dropping the term $\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_h^2$, (3.11) gives

$$\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^n\|_h^2 \le \frac{\Delta t}{2\varepsilon^2}\|\mathbf{u}^{n+1}\|_h^2,$$

which is re-written as follows

$$\left[1 - \frac{\Delta t}{2\varepsilon^2}\right]\|\mathbf{u}^{n+1}\|_h^2 \le \|\mathbf{u}^n\|_h^2. \tag{3.12}$$

Based on (2.17), for $\dfrac{\Delta t}{2\varepsilon^2} \le \dfrac{1}{2}$, (3.12) gives

$$\|\mathbf{u}^{n+1}\|_h^2 \le 2^{\frac{2\Delta t}{\varepsilon^2}}\|\mathbf{u}^n\|_h^2.$$

By induction over n, we obtain

$$\|\mathbf{u}^n\|_h^2 \le 2^{\frac{2n\,\Delta t}{\varepsilon^2}}\|\mathbf{u}^0\|_h^2 \le 2^{\frac{2T}{\varepsilon^2}}\|\mathbf{u}^0\|_h^2.$$

Therefore, the proof is complete. □

Remark 3.2. *Starting with* (3.11) *and Lemma 2.5, one has*

$$\|\boldsymbol{u}^{n+1}\|_h^2 - \|\boldsymbol{u}^n\|_h^2 + \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\|_h^2 + 2\Delta t\varepsilon^2\eta^2|\boldsymbol{u}^{n+1}|_{1,h}^2 \le 2\Delta t|\boldsymbol{u}^{n+1}|_{1,h}^2. \tag{3.13}$$

Hence, we obtain

$$\|\boldsymbol{u}^{n+1}\|_h^2 + 2\Delta t(\varepsilon^2\eta^2 - 1)|\boldsymbol{u}^{n+1}|_{1,h}^2 \le \|\boldsymbol{u}^n\|_h^2, \tag{3.14}$$

from which one deduces that if ε is big enough, then

$$\|\boldsymbol{u}^{n+1}\|_h^2 \le \|\boldsymbol{u}^n\|_h^2 \le \|\boldsymbol{u}^{n-1}\|_h^2 \le \cdots \le \|\boldsymbol{u}^0\|_h^2.$$

This alternative stability result requires both that $\Delta t < 4\varepsilon^2$ and the viscosity constant ε big enough.

Theorem 3.3. *Suppose that the solution $u(x,t)$ of* (1.1)-(1.4) *is sufficiently smooth.*
Assume that $\Delta t < \min(4\varepsilon^2, c)$, with c given by (3.43)*, independent of Δx and Δy.*
Assume that Δt, Δx and Δy satisfy the relation (3.45).
Then, the solution of the finite volume discretization (3.5a)-(3.5c) *converges to the solution of the problems* (1.1)-(1.4) *in the discrete L^2-norm with rate of convergence $\mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2)$.*

Proof. For $i = 1, \dots, N_1$ and $j = 1, \dots, N_2$, let

$$\upsilon_{i,j}^n = \frac{1}{\Delta x \Delta y} \iint\limits_{k_{i,j}} u(x, y, t_n) dx dy,$$

be the cell average of the exact solution u of (1.1)-(1.4) at time t_n, for $0 \leq n \leq M$, on the cell $k_{i,j}$. Since u is smooth enough by assumption, (hence at least continuous on $[-L_1, L_1] \times [-L_2, L_2]$) we let

$$s = \max_{-L_1 \leq x \leq L_1, -L_2 \leq y \leq L_2, 0 \leq t \leq T} |u(x, y, t)|. \tag{3.15}$$

Also, the smoothness of u gives

$$\upsilon_{i,j}^n = u(x_i, y_j, t_n) + \mathcal{O}(\Delta x^2 + \Delta y^2), \quad 1 \leq i \leq N_1, 1 \leq j \leq N_2 \text{ and } 0 \leq n \leq M.$$

Making use of Taylor's expansion (see Appendix A), we obtain

$$\frac{\upsilon_{i,j}^{n+1} - \upsilon_{i,j}^n}{\Delta t} = u_t|_{i,j}^n + \mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2), \tag{3.16}$$

$$\Delta_h^2 \upsilon_{i,j}^{n+1} = \Delta^2 u|_{i,j}^n + \mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2), \tag{3.17}$$

$$\nabla_{1,h}^+(\psi_{i-\frac{1}{2},j}^n \nabla_{1,h}^- \upsilon_{i,j}^{n+1}) + \nabla_{2,h}^+(\psi_{i,j-\frac{1}{2}}^n \nabla_{2,h}^- \upsilon_{i,j}^{n+1}) = \Delta f(u)|_{i,j}^n + \mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2), \tag{3.18}$$

and

$$\begin{aligned}
(C_h(\boldsymbol{\upsilon}^{n+1}, \tilde{\boldsymbol{\upsilon}}^n))_{i,j} = {} & (\alpha_1 - \alpha_2)(a_1 + a_2 + \cdots + a_{m_0})(\Delta x\, (u_x^2)|_{i,j}^n + \Delta y\, (u_y^2)|_{i,j}^n) \\
& + \frac{1}{2}(\alpha_1 - \alpha_2)(a_1 + a_2 + \cdots + a_{m_0})(\Delta x\, (u\, u_{xx})|_{i,j}^n + \Delta y\, (u\, u_{yy})|_{i,j}^n) \\
& + 3(\alpha_1 + \alpha_2)(a_1 + a_2 + \cdots + a_{m_0})\, (u\, (\boldsymbol{\beta} \cdot \nabla)u)|_{i,j}^n + \mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2).
\end{aligned} \tag{3.19}$$

One observes that the numerical scheme is first order accurate when $\alpha_1 \neq \alpha_2$ and it is second order accurate in space if $\alpha_1 = \alpha_2$, from which (3.19) gives

$$C_h(\boldsymbol{\upsilon}^{n+1}, \tilde{\boldsymbol{\upsilon}}^n)_{i,j} = (u(\boldsymbol{\beta} \cdot \nabla)u)|_{i,j}^n + \mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2). \tag{3.20}$$

In this study, we consider the case $\alpha_1 = \alpha_2$ to obtain second order accurate method. Combining (3.16)-(3.18) and (3.20), we obtain

$$\begin{cases}
\dfrac{\upsilon_{i,j}^{n+1} - \upsilon_{i,j}^n}{\Delta t} - (C_h(\boldsymbol{\upsilon}^{n+1}, \tilde{\boldsymbol{\upsilon}}^n))_{i,j} + \varepsilon^2 \Delta_h^2 \upsilon_{i,j}^{n+1} = \nabla_{1,h}^+(\psi_{i-\frac{1}{2},j}^n \nabla_{1,h}^- \upsilon_{i,j}^{n+1}) + \nabla_{2,h}^+(\psi_{i,j-\frac{1}{2}}^n \nabla_{2,h}^- \upsilon_{i,j}^{n+1}) + r_{i,j}^n, \\
\upsilon_{i,j}^0 = \dfrac{1}{\Delta x \Delta y} \iint_{k_{i,j}} u^0(x, y) dx dy,
\end{cases} \tag{3.21}$$

where $r_{i,j}^n$ is the truncation error of the finite volume discretization (3.5a) for $0 \leq n \leq M-1$, $1 \leq i \leq N_1$ and $1 \leq j \leq N_2$. There exists a positive constant c_1 such that

$$\max_{i,j,n} |r_{i,j}^n| \leq c_1(\Delta t + \Delta x^2 + \Delta y^2),\ 0 \leq n \leq M-1,\ 1 \leq i \leq N_1,\ 1 \leq j \leq N_2. \tag{3.22}$$

Let $\mathbf{e}^n = \boldsymbol{\upsilon}^n - \mathbf{u}^n, 0 \leq n \leq M$, where $u_{i,j}^n$ is the solution of (3.5a)-(3.5c). Clearly $\mathbf{e}^n \in \mathcal{H}_h$ and $\mathbf{e}^0 = 0$. Substituting $u_{i,j}^n = \upsilon_{i,j}^n - e_{i,j}^n$ into (3.5a), and using (3.21), we obtain

$$\begin{aligned}
\frac{e_{i,j}^{n+1} - e_{i,j}^n}{\Delta t} & - (C_h(\mathbf{e}^{n+1}, \tilde{\boldsymbol{\upsilon}}^n - \tilde{\mathbf{e}}^n))_{i,j} + \varepsilon^2 \Delta_h^2 e_{i,j}^{n+1} = \nabla_{1,h}^+(\varphi_{i-\frac{1}{2},j}^n \nabla_{1,h}^- e_{i,j}^{n+1}) + \nabla_{2,h}^+(\varphi_{i,j-\frac{1}{2}}^n \nabla_{2,h}^- e_{i,j}^{n+1}) \\
& + \nabla_{1,h}^+ \left[\left(3(\upsilon_{i,j}^n\, e_{i,j}^n + \upsilon_{i-1,j}^n\, e_{i-1,j}^n) - \frac{3}{2}[(e_{i,j}^n)^2 + (e_{i-1,j}^n)^2] \right) \nabla_{1,h}^- \upsilon_{i,j}^{n+1} \right] + (C_h(\boldsymbol{\upsilon}^{n+1}, \tilde{\mathbf{e}}^n))_{i,j} \\
& + \nabla_{2,h}^+ \left[\left(3(\upsilon_{i,j}^n\, e_{i,j}^n + \upsilon_{i,j-1}^n\, e_{i,j-1}^n) - \frac{3}{2}[(e_{i,j}^n)^2 + (e_{i,j-1}^n)^2] \right) \nabla_{2,h}^- \upsilon_{i,j}^{n+1} \right] + r_{i,j}^n,
\end{aligned} \tag{3.23}$$

where

$$\psi^n_{i-1/2,j} = \frac{f'(\upsilon^n_{i,j}) + f'(\upsilon^n_{i-1,j})}{2} \text{ and } \psi^n_{i,j-1/2} = \frac{f'(\upsilon^n_{i,j}) + f'(\upsilon^n_{i,j-1})}{2}.$$

Multiplying (3.23) by $2\Delta t\,\Delta x\,\Delta y\, e^{n+1}_{i,j}$ and summing the corresponding equalities for $i = 1,\ldots,N_1$ and $j = 1,\ldots,N_2$, together with (2.15) and Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}
&\|\mathbf{e}^{n+1}\|_h^2 - \|\mathbf{e}^n\|_h^2 + \|\mathbf{e}^{n+1}-\mathbf{e}^n\|_h^2 + 2\Delta t\varepsilon^2|\mathbf{e}^{n+1}|^2_{2,h} \le 2\Delta t|\mathbf{e}^{n+1}|^2_{1,h}\\
&\quad - 6\Delta t\,\Delta x\,\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[\big(\upsilon^n_{i,j}\,e^n_{i,j} + \upsilon^n_{i-1,j}\,e^n_{i-1,j}\big)\,\nabla^-_{1,h}\upsilon^{n+1}_{i,j}\nabla^-_{1,h}e^{n+1}_{i,j} + \big(\upsilon^n_{i,j}\,e^n_{i,j} + \upsilon^n_{i,j-1}\,e^n_{i,j-1}\big)\,\nabla^-_{2,h}\upsilon^{n+1}_{i,j}\nabla^-_{2,h}e^{n+1}_{i,j}\Big]\\
&\quad + 3\Delta t\,\Delta x\,\Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Big[[(e^n_{i,j})^2 + (e^n_{i-1,j})^2]\nabla^-_{1,h}\upsilon^{n+1}_{i,j}\nabla^-_{1,h}e^{n+1}_{i,j} + [(e^n_{i,j})^2 + (e^n_{i,j-1})^2]\nabla^-_{2,h}\upsilon^{n+1}_{i,j}\nabla^-_{2,h}e^{n+1}_{i,j}\Big]\\
&\quad + 2\Delta t(C_h(\boldsymbol{\upsilon}^{n+1},\tilde{\mathbf{e}}^n),\mathbf{e}^{n+1})_h + 2\Delta t(\mathbf{r}^n,\mathbf{e}^{n+1})_h\\
&\quad = \sum_{m=1}^{7} I_m.
\end{aligned} \tag{3.24}$$

We estimate each of the terms $I_2,\ldots,I_7$ (3.24) as follows:

$$I_2 \le 6\Delta t\left[\left(\max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\upsilon^n_{i,j}|\right)\left(\max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla^-_{1,h}\upsilon^n_{i,j}|\right)\right]\left(\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|e^n_{i,j}||\nabla^-_{1,h}e^{n+1}_{i,j}| + |e^n_{i-1,j}||\nabla^-_{1,h}e^{n+1}_{i,j}|\right) \tag{3.25}$$

Since u is smooth then there exist constants c_2 and c_3 such that

$$\max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla^-_{1,h}\upsilon^{n+1}_{i,j}| \le c_2 \text{ and } \max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla^-_{2,h}\upsilon^{n+1}_{i,j}| \le c_3 \quad \forall n = 0,1,\cdots,M-1.$$

Thus,

$$I_2 \le 12sc_2\,\Delta t\,\|\mathbf{e}^n\|_h\left(\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\nabla^-_{1,h}e^{n+1}_{i,j})^2\right)^{1/2} \tag{3.26}$$

In a similar way, one obtains

$$I_3 \le 12sc_3\,\Delta t\,\|\mathbf{e}^n\|_h\left(\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\nabla^-_{2,h}e^{n+1}_{i,j})^2\right)^{1/2}. \tag{3.27}$$

Combining (3.26) and (3.27), and applying Young's inequality, we obtain

$$I_2 + I_3 \le 6\ \Delta t\ \left(\frac{2s^2\,c_4^2}{\delta_1}\|\mathbf{e}^n\|_h^2 + \delta_1\,|\mathbf{e}^{n+1}|^2_{1,h}\right), \tag{3.28}$$

where $c_4 = \max\{c_2, c_3\}$.

$$\begin{aligned}
I_4 &\le 3c_2\,\Delta t\left(\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[(e^n_{i,j})^2 + (e^n_{i-1,j})^2\right]|\nabla^-_{1,h}e^{n+1}_{i,j}|\right)\\
&\le 6c_2\,\Delta t\,\|\mathbf{e}^n\|_{\infty,h}\|\mathbf{e}^n\|_h\left(\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\nabla^-_{1,h}e^{n+1}_{i,j})^2\right)^{1/2}.
\end{aligned} \tag{3.29}$$

Similarly, we get

$$I_5 \le 6c_3\,\Delta t\,\|\mathbf{e}^n\|_{\infty,h}\|\mathbf{e}^n\|_h \left(\Delta x\,\Delta y\,\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(\nabla_{2,h}^{-}e_{i,j}^{n+1})^2\right)^{1/2}. \tag{3.30}$$

Adding (3.29) and (3.30), one obtains

$$\begin{aligned} I_4 + I_5 &\le 6\sqrt{2}\,c_4\,\Delta t\,\|\mathbf{e}^n\|_{\infty,h}\,\|\mathbf{e}^n\|_h\,|\mathbf{e}^{n+1}|_{1,h} \\ &\le 3\Delta t\left(\frac{2\,c_4^2}{\delta_2}\,\|\mathbf{e}^n\|_h^2 + \delta_2\|\mathbf{e}^n\|_{\infty,h}^2\,\,|\mathbf{e}^{n+1}|_{1,h}^2\right). \end{aligned} \tag{3.31}$$

From the definition of the bilinear map C_h, we have

$$\begin{aligned} &(C_h(\boldsymbol{\upsilon}^{n+1},\tilde{\mathbf{e}}^n),\mathbf{e}^{n+1})_h \\ &= \Delta x\,\Delta y\,\alpha\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[\upsilon_{i,j}^{n+1}(\boldsymbol{\beta}\cdot\nabla_h^{+})\tilde{e}_{i,j}^n + \tilde{e}_{i,j}^n(\boldsymbol{\beta}\cdot\nabla_h^{-})\upsilon_{i,j}^{n+1} + \tilde{e}_{i+1,j}^n\nabla_{1,h}^{+}\upsilon_{i,j}^{n+1} + \tilde{e}_{i,j+1}^n\nabla_{2,h}^{+}\upsilon_{i,j}^{n+1}\right]e_{i,j}^{n+1} \\ &+ \Delta x\,\Delta y\,\alpha\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[\upsilon_{i,j}^{n+1}(\boldsymbol{\beta}\cdot\nabla_h^{-})\tilde{e}_{i,j}^n + \tilde{e}_{i,j}^n(\boldsymbol{\beta}\cdot\nabla_h^{+})\upsilon_{i,j}^{n+1} + \tilde{e}_{i-1,j}^n\nabla_{1,h}^{-}\upsilon_{i,j}^{n+1} + \tilde{e}_{i,j-1}^n\nabla_{2,h}^{-}\upsilon_{i,j}^{n+1}\right]e_{i,j}^{n+1}. \end{aligned} \tag{3.32}$$

We now estimate each of the terms in (3.32).

$$\begin{aligned} \Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\upsilon_{i,j}^{n+1}(\boldsymbol{\beta}\cdot\nabla_h^{-}\tilde{e}_{i,j}^n)\,e_{i,j}^{n+1} &= -\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\tilde{e}_{i,j}^n(\boldsymbol{\beta}\cdot\nabla_h^{+})(\upsilon_{i,j}^{n+1}\,e_{i,j}^{n+1}) \\ &\le (c_2+c_3)\|\tilde{\mathbf{e}}^n\,\|_h\|\mathbf{e}^{n+1}\|_h + \sqrt{2}s\,\|\tilde{\mathbf{e}}^n\|_h\,|\mathbf{e}^{n+1}|_{1,h} \\ &= c_5\|\tilde{\mathbf{e}}^n\,\|_h\|\mathbf{e}^{n+1}\|_h + \sqrt{2}s\,\|\tilde{\mathbf{e}}^n\|_h\,|\mathbf{e}^{n+1}|_{1,h}, \end{aligned} \tag{3.33}$$

where $c_5 = c_2 + c_3$.

$$\begin{aligned} \Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\tilde{e}_{i,j}^n(\boldsymbol{\beta}\cdot\nabla_h^{-}\upsilon_{i,j}^{n+1})\,e_{i,j}^{n+1} &\le \max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|(\boldsymbol{\beta}\cdot\nabla_h^{-})\upsilon_{i,j}^{n+1}|\left[\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|\tilde{e}_{i,j}^n|\,|e_{i,j}^{n+1}|\right] \\ &\le \left[\max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla_{1,h}^{-}\upsilon_{i,j}^{n+1}| + \max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla_{2,h}^{-}\upsilon_{i,j}^{n+1}|\right]\left[\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|\tilde{e}_{i,j}^n|\,|e_{i,j}^{n+1}|\right] \\ &\le c_5\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h. \end{aligned} \tag{3.34}$$

$$\begin{aligned} \Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[\tilde{e}_{i+1,j}^n\nabla_{1,h}^{+}\upsilon_{i,j}^{n+1} + \tilde{e}_{i,j+1}^n\nabla_{2,h}^{+}\upsilon_{i,j}^{n+1}\right]e_{i,j}^{n+1} &\le \max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla_{1,h}^{+}\upsilon_{i,j}^{n+1}|\left[\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|\tilde{e}_{i+1,j}^n|\,|e_{i,j}^{n+1}|\right] \\ &\quad + \max_{\substack{1\le i\le N_1\\1\le j\le N_2}}|\nabla_{2,h}^{+}\upsilon_{i,j}^{n+1}|\left[\Delta x\,\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|\tilde{e}_{i,j+1}^n|\,|e_{i,j}^{n+1}|\right] \\ &\le c_2\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h + c_3\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h \\ &= c_5\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h. \end{aligned} \tag{3.35}$$

In a similar fashion, we obtain

$$\Delta x\, \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \upsilon_{i,j}^{n+1}(\boldsymbol{\beta}\cdot\nabla_h^- \tilde{e}_{i,j}^n)\, e_{i,j}^{n+1} \le c_5\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h + \sqrt{2}s\,\|\tilde{\mathbf{e}}^n\|_h\,|\mathbf{e}^{n+1}|_{1,h}. \tag{3.36}$$

$$\Delta x\, \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \tilde{e}_{i,j}^n(\boldsymbol{\beta}\cdot\nabla_h^+ \upsilon_{i,j}^{n+1})\, e_{i,j}^{n+1} \le c_5\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h. \tag{3.37}$$

$$\Delta x\, \Delta y \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \left[\tilde{e}_{i-1,j}^n\nabla_{1,h}^-\upsilon_{i,j}^{n+1} + \tilde{e}_{i,j-1}^n\nabla_{2,h}^-\upsilon_{i,j}^{n+1}\right] e_{i,j}^{n+1} \le c_5\|\tilde{\mathbf{e}}^n\|_h\|\mathbf{e}^{n+1}\|_h. \tag{3.38}$$

Combining the inequalities from (3.33)-(3.38), and using Young's inequality, we obtain

$$2\Delta t(C_h(\boldsymbol{\upsilon}^{n+1},\tilde{\mathbf{e}}^n),\mathbf{e}^{n+1})_h \le \Delta t\left[\frac{8s^2\alpha^2}{\delta_4}\|\tilde{\mathbf{e}}^n\|_h^2 + \delta_4|\mathbf{e}^{n+1}|_{1,h}^2\right] + \Delta t\Big[\frac{36c_5^2\alpha^2}{\delta_5}\|\tilde{\mathbf{e}}^n\|_h^2 + \delta_5\|\mathbf{e}^{n+1}\|_h^2\Big]. \tag{3.39}$$

Finally, using (3.22) and Young's inequality, we obtain

$$(\mathbf{r}^n,\mathbf{e}^{n+1})_h = \Delta x\Delta y\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} r_{i,j}^n e_{i,j}^{n+1} \le \frac{16L_1^2L_2^2c_1^2}{2\delta_3}\left(\Delta t + \Delta x^2 + \Delta y^2\right)^2 + \frac{\delta_3}{2}\|\mathbf{e}^{n+1}\|_h^2. \tag{3.40}$$

Thus, from (3.28), (3.31), (3.39) and (3.40), we get

$$\begin{aligned}
\|\mathbf{e}^{n+1}\|_h^2 - \|\mathbf{e}^n\|_h^2 + 2\Delta t\varepsilon^2|\mathbf{e}^{n+1}|_{2,h}^2 &\le 2\Delta t|\mathbf{e}^{n+1}|_{1,h}^2 + 6\ \Delta t\ \left(\frac{2s^2\,c_4^2}{\delta_1}\|\mathbf{e}^n\|_h^2 + \delta_1\,|\mathbf{e}^{n+1}|_{1,h}^2\right)\\
&+ 3\Delta t\ \left(\frac{2\,c_4^2}{\delta_2}\,\|\mathbf{e}^n\|_h^2 + \delta_2\|\mathbf{e}^n\|_{\infty,h}^2\ |\mathbf{e}^{n+1}|_{1,h}^2\right) + \Delta t\left[\frac{8s^2\alpha^2}{\delta_4}\|\tilde{\mathbf{e}}^n\|_h^2 + \delta_4|\mathbf{e}^{n+1}|_{1,h}^2\right]\\
&+ \Delta t\Big[\frac{36c_5^2\alpha^2}{\delta_5}\|\tilde{\mathbf{e}}^n\|_h^2 + \delta_5\|\mathbf{e}^{n+1}\|_h^2\Big] + \frac{16\Delta t\,L_1^2L_2^2c_1^2}{\delta_3}(\Delta t + \Delta x^2 + \Delta y^2)^2 + \Delta t\,\delta_3\|\mathbf{e}^{n+1}\|_h^2,
\end{aligned}$$

which after using (2.13) and applying Lemma 2.5 and Young's inequality gives

$$\begin{aligned}
[1-\Delta t(\delta_3+\delta_5)]\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\varepsilon^2|\mathbf{e}^{n+1}|_{2,h}^2 &\le \frac{\Delta t(2+6\delta_1+\delta_4)^2}{4\varepsilon^2}\|\mathbf{e}^{n+1}\|_h^2 + \left[1 + 6\ \Delta t\ \left(\frac{2s^2\,c_4^2}{\delta_1} + \frac{c_4^2}{\delta_2}\right)\right]\|\mathbf{e}^n\|_h^2\\
+ \frac{3\Delta t\,\delta_2}{\Delta x\Delta y}\|\mathbf{e}^n\|_h^2\ |\mathbf{e}^{n+1}|_{1,h}^2 &+ 4\Delta t\,\alpha^2\left[\frac{2s^2}{\delta_4} + \frac{9c_5^2}{\delta_5}\right]\|\tilde{\mathbf{e}}^n\|_h^2 + \frac{16\Delta t\,L_1^2L_2^2c_1^2}{\delta_3}(\Delta t + \Delta x^2 + \Delta y^2)^2.
\end{aligned} \tag{3.41}$$

Using (2.24), (3.41) gives

$$\begin{aligned}
[1-\Delta t\,c]\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 &\le (1+\Delta t c_9)\,\|\mathbf{e}^n\|_h^2 + \Delta t c_8\|\tilde{\mathbf{e}}^n\|_h^2 + \frac{12\Delta t\,\delta_2}{\Delta x\Delta y}\|\mathbf{e}^n\|_h^2\ |\mathbf{e}^{n+1}|_{1,h}^2\\
&+ \Delta t c_7(\Delta t + \Delta x^2 + \Delta y^2)^2,
\end{aligned} \tag{3.42}$$

where

$$c_7 = \frac{16\,L_1^2L_2^2c_1^2}{\delta_3},\quad c = \delta_3 + \delta_5 + \frac{(2+6\delta_1+\delta_4)^2}{4\varepsilon^2}\quad c_9 = 6\ \left(\frac{2s^2\,c_4^2}{\delta_1} + \frac{c_4^2}{\delta_2}\right),\quad c_8 = 4\alpha^2\left[\frac{2s^2}{\delta_4} + \frac{9c_5^2}{\delta_5}\right].$$

For

$$\Delta t \le \frac{1}{2c} \equiv c \tag{3.43}$$

then it follows from (3.42) that

$$\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 \leq \; 4^{\Delta tc}\Big[(1+\Delta tc_9)\,\|\mathbf{e}^n\|_h^2 + \Delta tc_8\|\tilde{\mathbf{e}}^n\|_h^2 + \frac{3\Delta t\,\delta_2}{\Delta x\Delta y}\|\mathbf{e}^n\|_h^2\;|\mathbf{e}^{n+1}|_{1,h}^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\Big]. \tag{3.44}$$

Now for

$$\frac{3\delta_2\,c_7}{\Delta x\Delta y}\left[\,(\Delta t+\Delta x^2+\Delta y^2)^2\right] \leq \frac{c_{10}}{2}\varepsilon^2\eta^2 4^{-Tc}\exp\Big(-Tc_{10}\Big), \tag{3.45}$$

where $c_{10} = c_9 + m_0\,A\,c_8$ and $A = \sum_{i=1}^{m_0}|a_i|^2$, we prove by inductive method that

$$\|\mathbf{e}^{n+1}\|_h^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 \leq 4^{\Delta t\,c}\Big[(1+\Delta tc_9)\,\|\mathbf{e}^n\|_h^2 + \Delta tc_8\|\tilde{\mathbf{e}}^n\|_h^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\Big]. \tag{3.46}$$

For $n = 0$ from (3.44), one obtains

$$\|\mathbf{e}^1\|_h^2 + \Delta t\varepsilon^2\eta^2|\mathbf{e}^1|_{1,h}^2 \leq 4^{\Delta t\,c}\left[(1+\Delta t(c_9+c_8))\,\|\mathbf{e}^0\|_h^2 + \frac{3\Delta t\,\delta_2}{\Delta x\Delta y}\|\mathbf{e}^0\|_h^2\;|\mathbf{e}^1|_{1,h}^2\right] + \Delta tc_7(4^{\Delta t\,c})\left[(\Delta t+\Delta x^2+\Delta y^2)^2\right],$$

and hence,

$$\|\mathbf{e}^1\|_h^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^1|_{1,h}^2 \leq 4^{\Delta t\,c}\left[(1+\Delta t(c_9+c_8))\,\|\mathbf{e}^0\|_h^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\right],$$

which is (3.46) for $n = 0$. Now suppose that (3.46) is true up to the order $n-1$. Thus, for $s = 0, 1, \ldots, n-1$,

$$\|\mathbf{e}^{s+1}\|_h^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{s+1}|_{1,h}^2 \leq 4^{\Delta t\,c}\left[(1+\Delta tc_9)\,\|\mathbf{e}^s\|_h^2 + \Delta tc_8\|\tilde{\mathbf{e}}^s\|_h^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\right]. \tag{3.47}$$

It is now remaining to treat the term $\|\tilde{\mathbf{e}}^s\|_h^2$. For $s < m_0 - 1$, it is clear from (3.3) that $\|\tilde{\mathbf{e}}^s\|_h = \|\mathbf{e}^s\|_h$. Thus, (3.47) becomes

$$\|\mathbf{e}^{s+1}\|_h^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{s+1}|_{1,h}^2 \leq 4^{\Delta t\,c}\left[(1+\Delta t(c_9+c_8))\,\|\mathbf{e}^s\|_h^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\right]. \tag{3.48}$$

It follows from (3.48) that

$$\|\mathbf{e}^{s+1}\|_h^2 \leq 4^{\Delta t\,c}\left[(1+\Delta t(c_9+c_8))\,\|\mathbf{e}^s\|_h^2 + \Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\right].$$

After $s+1$ iterations, we get

$$\|\mathbf{e}^{s+1}\|_h^2 \leq 4^{\Delta t(s+1)\,c}\Big[\Delta tc_7(\Delta t+\Delta x^2+\Delta y^2)^2\sum_{j=0}^{s}(1+\Delta t(c_9+c_8))^j\Big]. \tag{3.49}$$

For the case $s \geq m_0 - 1$ and $m_0 > 1$, it follows from (3.2) that

$$\|\tilde{\mathbf{e}}^s\|_h \leq \sum_{i=1}^{m_0}|a_i|\|\tilde{\mathbf{e}}^{s-i+1}\|_h,$$

which by Cauchy-Schwartz's inequality gives

$$\|\tilde{\mathbf{e}}^s\|_h^2 \leq A\Big[\|\mathbf{e}^s\|_h^2 + \|\mathbf{e}^{s-1}\|_h^2 + \cdots + \|\mathbf{e}^{s-m_0+1}\|_h^2\Big]. \tag{3.50}$$

Hence, (3.47) gives

$$\|\mathbf{e}^{s+1}\|_h^2 \leq 4^{\Delta t\, c}\left[(1+\Delta t c_9)\,\|\mathbf{e}^s\|_h^2 + \Delta t c_8 A\Big(\|\mathbf{e}^s\|_h^2 + \|\mathbf{e}^{s-1}\|_h^2 + \cdots + \|\mathbf{e}^{s-m_0+1}\|_h^2\Big)\right] + 4^{\Delta t\, c}\left[\Delta t c_7(\Delta t + \Delta x^2 + \Delta y^2)^2\right]. \tag{3.51}$$

It follows easily that

$$\max\Big\{\|\mathbf{e}^{s+1}\|_h^2, \|\mathbf{e}^{s}\|_h^2, \ldots, \|\mathbf{e}^{s-m_0+2}\|_h^2\Big\} \leq 4^{\Delta tc}\Bigg[(1+\Delta t\, c_{10})\,\max\Big\{\|\mathbf{e}^{s}\|_h^2, \|\mathbf{e}^{s-1}\|_h^2, \ldots, \|\mathbf{e}^{s-m_0+1}\|_h^2\Big\} + \Delta t\, c_7(\Delta t + \Delta x^2 + \Delta y^2)^2\Bigg]. \tag{3.52}$$

which after $s - m_0 + 2$ iterations gives

$$\begin{aligned}
&\max\Big\{\|\mathbf{e}^{s+1}\|_h^2, \|\mathbf{e}^{s}\|_h^2, \ldots, \|\mathbf{e}^{s-m_0+2}\|_h^2\Big\} \\
&\quad\leq 4^{(s-m_0+2)\Delta tc}\Bigg[(1+\Delta t\, c_{10})^{s-m_0+2}\,\max\Big\{\|\mathbf{e}^{m_0-1}\|_h^2, \|\mathbf{e}^{m_0-2}\|_h^2, \ldots, \|\mathbf{e}^{0}\|_h^2\Big\} \\
&\qquad + \Delta t\, c_7(\Delta t + \Delta x^2 + \Delta y^2)^2 \sum_{j=0}^{s-m_0+1}(1+\Delta t c_{10})^j\Bigg].
\end{aligned} \tag{3.53}$$

Combining (3.49) and (3.53), we get

$$\begin{aligned}
\|\mathbf{e}^{s+1}\|_h^2 &\leq 4^{(s+1)\Delta tc}\Big[\Delta t\, c_7(\Delta t + \Delta x^2 + \Delta y^2)^2 \sum_{j=0}^{s}(1+\Delta t c_{10})^j\Big] \\
&= 4^{(s+1)\Delta tc}\Big[\Delta t\, c_7(\Delta t + \Delta x^2 + \Delta y^2)^2 \left(\frac{(1+\Delta t c_{10})^{s+1}-1}{\Delta t c_{10}}\right)\Big] \\
&\leq 4^{(s+1)\Delta tc}\exp((s+1)\Delta t\, c_{10})\Big[\frac{c_7}{c_{10}}(\Delta t + \Delta x^2 + \Delta y^2)^2\Big]
\end{aligned} \tag{3.54}$$

and

$$\|\tilde{\mathbf{e}}^{s+1}\|_h^2 \leq m_0\, A\, 4^{(s+1)\Delta tc}\exp((s+1)\Delta t\, c_{10})\Big[\frac{c_7}{c_{10}}(\Delta t + \Delta x^2 + \Delta y^2)^2\Big], \tag{3.55}$$

for $s = 0, 1, \ldots, n-1$.
Going back to (3.44), and using (3.54) we obtain

$$\begin{aligned}
\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 \leq{}& 4^{\Delta tc}\Big[(1+\Delta t c_9)\,\|\mathbf{e}^{n}\|_h^2 + \Delta t c_8|\tilde{\mathbf{e}}^{n}|_h^2 + \Delta t c_7(\Delta t + \Delta x^2 + \Delta y^2)^2\Big] \\
&+ \frac{3\Delta t\,\delta_2}{\Delta x\Delta y}\,4^{n\Delta tc}\exp(n\Delta t\, c_{10})\Big[\frac{c_7}{c_{10}}(\Delta t + \Delta x^2 + \Delta y^2)^2\Big]\,|\mathbf{e}^{n+1}|_{1,h}^2,
\end{aligned}$$

which by (3.45) gives

$$\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 \leq 4^{\Delta tc}\Big[(1+\Delta t c_9)\,\|\mathbf{e}^{n}\|_h^2 + \Delta t c_8\|\tilde{\mathbf{e}}^{n}\|_h^2 + \Delta t c_7(\Delta t + \Delta x^2 + \Delta y^2)^2\Big] + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2.$$

Thus, we have

$$\|\mathbf{e}^{n+1}\|_h^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+1}|_{1,h}^2 \leq 4^{\Delta tc}\Big[(1+\Delta t c_9)\,\|\mathbf{e}^{n}\|_h^2 + \Delta t c_8\|\tilde{\mathbf{e}}^{n}\|_h^2 + \Delta t c_7\,(\Delta t + \Delta x^2 + \Delta y^2)^2\Big]. \tag{3.56}$$

Using (3.54) and (3.55), (3.56) gives

$$\begin{aligned}\|\mathbf{e}^n\|_h^2 &\le 4^{n\Delta tc}\exp\left(n\,\Delta tc_{10}\right)\,\left[\frac{c_7}{c_{10}}(\Delta t+\Delta x^2+\Delta y^2)^2\right]\\ &\le 4^{Tc}\exp(T\,c_{10})\left[\frac{c_7}{c_{10}}(\Delta t+\Delta x^2+\Delta y^2)^2\right],\end{aligned}\tag{3.57}$$

for $n = 1, 2, \ldots, M$. Thus, it follows from (3.57) that

$$\|\mathbf{e}^n\|_h \le C(\Delta t + \Delta x^2 + \Delta y^2),$$

for constant C independent of Δt, Δx and Δy. This completes the proof. □

Remark 3.3. *The assumption $\Delta t < 4\varepsilon^2$ in Theorem 3.3 is to ensure the existence of solution for* (3.5a)-(3.5c). *It is worth noting that the convergence result Theorem 3.3 is conducted for $\alpha_1 = \alpha_2$. The case when $\alpha_1 \ne \alpha_2$ is treated in the same way but the rate of convergence will change.*

3.2 Explicit Finite Volume Method

In this section, we approximate the solution of (1.1) using an explicit finite volume method:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^n}{\Delta t}-(C_h(\mathbf{u}^n,\mathbf{u}^n))_{i,j}+\varepsilon^2\Delta_h^2 u_{i,j}^n=\nabla_{1,h}^+(\varphi_{i-1/2,j}^n\nabla_{1,h}^- u_{i,j}^n)+\nabla_{2,h}^+(\varphi_{i,j-1/2}^n\nabla_{2,h}^- u_{i,j}^n),\tag{3.58a}$$

$$u_{i,j}^n = u_{i+N_1,j}^n = u_{i,j+N_2}^n,\tag{3.58b}$$

$$u_{i,j}^0 = \frac{1}{\Delta x\Delta y}\iint\limits_{k_{i,j}} u^0(x,y)dxdy,\tag{3.58c}$$

for $1 \le n \le M-1$ and $1 \le i \le N_1, 1 \le j \le N_2$. (3.58) is explicit, hence the solution $\mathbf{u}^n$ is computed at each time step. One important feature of this scheme is stated in the following.

Theorem 3.4. *We assume that the following are satisfied for some $\delta, 0 < \delta < 1$:*

$$\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^2 \le \frac{1-\delta}{64\varepsilon^2},\tag{3.59}$$

$$16\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) \le \varepsilon^2\delta\eta^2(1-\delta),\tag{3.60}$$

$$\frac{72\Delta t}{\Delta x\Delta y}\left((|\alpha_1|+|\alpha_2|)^2+\frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\boldsymbol{u}^0\|_h^2\right)\|\boldsymbol{u}^0\|_h^2 \le \varepsilon^2\delta^2\eta^2\exp\left(\frac{-2T}{\varepsilon^2}\right),\tag{3.61}$$

where α_1 and α_2 are the constants in (2.20). *Then the finite volume method defined by* (3.58) *is $L^\infty(0,T;\mathcal{H}_h)$ stable in the following sense:*

$$\|\boldsymbol{u}^n\|^2 \le \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\boldsymbol{u}^{n-1}\|^2 \le \cdots \le \exp\left(\frac{n\Delta t}{\varepsilon^2}\right)\|\boldsymbol{u}^0\|^2 \le \exp\left(\frac{T}{\varepsilon^2}\right)\|\boldsymbol{u}^0\|^2, n=1,2\ldots,M,$$

$$\frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\sum_{n=0}^{M-1}\exp\left(\frac{(M-n)\Delta t}{\varepsilon^2}\right)|\boldsymbol{u}^n|_{1,h}^2 \le \exp\left(\frac{T}{\varepsilon^2}\right)\|\boldsymbol{u}^0\|^2$$

Proof. To prove this assertion we use the approach of Temam [31]. Multiplying (3.58a) by $2\Delta t\Delta x\Delta y u_{i,j}^n$ and summing the equalities for $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$, together with (2.16), (2.18) and Lemma 2.5, we arrive at

$$\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2 + \Delta t\varepsilon^2|\mathbf{u}^n|_{2,h}^2 \le \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2.\tag{3.62}$$

Now we have to estimate the term $\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2$. By multiplying (3.58a) by $2\Delta t\Delta x\Delta y(u_{i,j}^{n+1}-u_{i,j}^n)$ and adding the corresponding equalities for $i=1,\dots,N_1$ and $j=1,\dots,N_2$, we obtain

$$\begin{aligned}2\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2 &= 2\Delta t(C_h(\mathbf{u}^n,\mathbf{u}^n),\mathbf{u}^{n+1}-\mathbf{u}^n)_h - 2\Delta t\varepsilon^2(\Delta_h\mathbf{u}^n,\Delta_h(\mathbf{u}^{n+1}-\mathbf{u}^n))\\ &\quad - 2\Delta t\Delta x\Delta y\sum_{i=1}^{N_1}\sum_{i=1}^{N_2}\left[\varphi_{i-1/2,j}^n\nabla_{1,h}^- u_{i,j}^{n+1}\right]\nabla_{1,h}^-(u_{i,j}^{n+1}-u_{i,j}^n)\\ &\quad - 2\Delta t\Delta x\Delta y\sum_{i=1}^{N_1}\sum_{i=1}^{N_2}\left[\varphi_{i,j-1/2}^n\nabla_{2,h}^- u_{i,j}^{n+1}\right]\nabla_{2,h}^-(u_{i,j}^{n+1}-u_{i,j}^n)\end{aligned}\tag{3.63}$$

Using (2.18) and (2.19), we majorize all the terms on the right hand side of (3.63) as follows:

$$2\Delta t(C_h(\mathbf{u}^n,\mathbf{u}^n),\mathbf{u}^{n+1}-\mathbf{u}^n)_h \le \frac{36\Delta t^2}{\Delta x\Delta y}(|\alpha_1|+|\alpha_2|)^2\|\mathbf{u}^n\|_h^2|\mathbf{u}^n|_{1,h}^2+\frac{1}{4}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2;\tag{3.64}$$

$$-2\Delta t\varepsilon^2(\Delta_h\mathbf{u}^n,\Delta_h(\mathbf{u}^{n+1}-\mathbf{u}^n))_h \le 64\varepsilon^4\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^2|\mathbf{u}^n|_{2,h}^2+\frac{1}{4}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2;\tag{3.65}$$

$$\begin{aligned}-2\Delta t\Delta x\Delta y\sum_{i=1}^{N_1}\sum_{i=1}^{N_2}&\left[\left(\varphi_{i-1/2,j}^n\nabla_{1,h}^- u_{i,j}^{n+1}\right)\nabla_{1,h}^-(u_{i,j}^{n+1}-u_{i,j}^n)+\left(\varphi_{i,j-1/2}^n\nabla_{2,h}^- u_{i,j}^{n+1}\right)\nabla_{2,h}^-(u_{i,j}^{n+1}-u_{i,j}^n)\right]\\ &\le \frac{144\Delta t^2}{\Delta x^2\Delta y^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^n\|_h^4|\mathbf{u}^n|_{1,h}^2+\frac{1}{4}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2\\ &\quad + 16\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)|\mathbf{u}^n|_{1,h}^2+\frac{1}{4}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2.\end{aligned}\tag{3.66}$$

Thus using (3.64)-(3.66), (3.63) becomes

$$\begin{aligned}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2 &\le \frac{36\Delta t^2}{\Delta x\Delta y}(|\alpha_1|+|\alpha_2|)^2\|\mathbf{u}^n\|_h^2|\mathbf{u}^n|_{1,h}^2+64\varepsilon^4\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^2|\mathbf{u}^n|_{2,h}^2\\ &\quad+\frac{144\Delta t^2}{\Delta x^2\,\Delta y^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^n\|_h^4|\mathbf{u}^n|_{1,h}^2+16\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)|\mathbf{u}^n|_{1,h}^2,\end{aligned}$$

which by (3.59) gives

$$\begin{aligned}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_h^2 &\le \frac{36\Delta t^2}{\Delta x\Delta y}(|\alpha_1|+|\alpha_2|)^2\|\mathbf{u}^n\|_h^2|\mathbf{u}^n|_{1,h}^2+\Delta t\varepsilon^2(1-\delta)|\mathbf{u}^n|_{2,h}^2\\ &\quad+\frac{144\Delta t^2}{\Delta x^2\,\Delta y^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^n\|_h^4|\mathbf{u}^n|_{1,h}^2+16\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)|\mathbf{u}^n|_{1,h}^2.\end{aligned}\tag{3.67}$$

On substitution of (3.67) back to (3.62), we arrive at

$$\begin{aligned}\|\mathbf{u}^{n+1}\|_h^2-\exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2+\Delta t\varepsilon^2\delta|\mathbf{u}^n|_{2,h}^2 &\le 16\Delta t^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)|\mathbf{u}^n|_{1,h}^2\\ &\quad+\frac{36\Delta t^2}{\Delta x\Delta y}\left[(|\alpha_1|+|\alpha_2|)^2+\frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^n\|_h^2\right]\|\mathbf{u}^n\|_h^2|\mathbf{u}^n|_{1,h}^2.\end{aligned}\tag{3.68}$$

Using (2.24) and (3.60), (3.68) gives

$$\begin{aligned}\|\mathbf{u}^{n+1}\|_h^2&-\exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2+\Delta t\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2\\ &\le \frac{36\Delta t^2}{\Delta x\Delta y}\left((|\alpha_1|+|\alpha_2|)^2+\frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^n\|_h^2\right)\|\mathbf{u}^n\|_h^2|\mathbf{u}^n|_{1,h}^2.\end{aligned}\tag{3.69}$$

We then need to show by induction on n, that

$$\|\mathbf{u}^{n+1}\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2. \tag{3.70}$$

For $n = 0$, from (3.69), we obtain

$$\begin{aligned}
&\|\mathbf{u}^1\|_h^2 + \Delta t\varepsilon^2\delta^2\eta^2|\mathbf{u}^0|_{1,h}^2 \\
&\quad\leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2 + \frac{36\Delta t^2}{\Delta x\Delta y}\left[(|\alpha_1| + |\alpha_2|)^2 + \frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\|\mathbf{u}^0\|_h^2\right]\|\mathbf{u}^0\|_h^2|\mathbf{u}^0|_{1,h}^2,
\end{aligned}$$

which with (3.61) leads to

$$\|\mathbf{u}^1\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^0|_{1,h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2,$$

which is (3.70) for $n = 0$. Assuming now that (3.70) is true up to the order $n-1$, this is to say that for $s = 0, 2, \ldots, n-1$, we have

$$\|\mathbf{u}^s\|_h^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^{s-1}\|_h^2 \text{ and } \|\mathbf{u}^s\|_h^2 \leq \exp\left(\frac{s\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2. \tag{3.71}$$

Using (3.71) in (3.69), one obtains

$$\begin{aligned}
\|\mathbf{u}^{n+1}\|_h^2 + \Delta t\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 &\leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2 \\
&\quad + \frac{36\Delta t}{\Delta x\Delta y}\exp\left(\frac{2n\Delta t}{\varepsilon^2}\right)\left[(|\alpha_1| + |\alpha_2|)^2 + \frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\|\mathbf{u}^0\|_h^2\right]\|\mathbf{u}^0\|_h^2|\mathbf{u}^n|_{1,h}^2,
\end{aligned}$$

which by (3.61) gives

$$\|\mathbf{u}^{n+1}\|_h^2 + \Delta t\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2,$$

re-written also as follows

$$\|\mathbf{u}^{n+1}\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2.$$

We then have

$$\begin{aligned}
\|\mathbf{u}^{n+1}\|_h^2 &\leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^n\|_h^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \\
&\leq \exp\left(\frac{2\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^{n-1}\|_h^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\exp\left(\frac{\Delta t}{\varepsilon^2}\right)|\mathbf{u}^{n-1}|_{1,h}^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \\
&\leq \exp\left(\frac{3\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^{n-2}\|_h^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\exp\left(\frac{2\Delta t}{\varepsilon^2}\right)|\mathbf{u}^{n-2}|_{1,h}^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\exp\left(\frac{\Delta t}{\varepsilon^2}\right)|\mathbf{u}^{n-1}|_{1,h}^2 \\
&\qquad - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^n|_{1,h}^2 \\
&\ \ \vdots \\
&\leq \exp\left(\frac{(n+1)\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2 - \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\sum_{s=0}^{n}\exp\left(\frac{(n-s)\Delta t}{\varepsilon^2}\right)|\mathbf{u}^s|_{1,h}^2.
\end{aligned}$$

Hence we get

$$\|\mathbf{u}^{n+1}\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2\sum_{s=0}^{n}\exp\left(\frac{(n-s)\Delta t}{\varepsilon^2}\right)|\mathbf{u}^s|_{1,h}^2 \leq \exp\left(\frac{(n+1)\Delta t}{\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2.$$

Therefore, the proof is complete. $\square$

4 Multilevel Finite Volume Approximation

Multilevel methods were introduced to improve calculation speed in the simulation of complex physical phenomena while maintaining a good level of accuracy, see [22, 23, 24, 25, 26, 27]. This section is an application of the work presented in [23], in which the shallow water equations is analyzed. Here, we are concerned with the two dimensional convective Cahn-Hilliard equation (1.1)-(1.4). We formulate in the spirit of [23] two methods approximating (1.1)-(1.4), namely: implicit multilevel finite volume method and explicit multilevel finite volume method. These new methods are next studied thoroughly and comparison by stability and CPU time with the associated one-level methods discussed in section 3.1 are established. To make this text self-contained for the reader, we recall below the multilevel finite volume approximation as described in Bousquet et. al. [23].

Here N_1 and N_2 are assumed to be divisible by 3. Let N_1^0, N_2^0 and M_0 be integers such that $3N_1^0 = N_1, 3N_2^0 = N_2$ and $\Delta t M_0 = T$. We discretize $\mathcal{M}$ into fine meshes and coarse meshes. The fine mesh consists of $3N_1^0 \times 3N_2^0$ regular cells $(k_{i,j})_{1\leq i\leq 3N_1^0, 1\leq j\leq 3N_2^0}$ of uniform area $\Delta x\Delta y$.
The coarse mesh consists of $N_1^0 N_2^0$ control volumes $(K_{l,m})_{1\leq l\leq N_1^0, 1\leq m\leq N_2^0}$ of uniform area $9\Delta x\Delta y$, where

$$K_{l,m} = (x_{3l-2-1/2}, x_{3m+1/2}) \times (y_{3m-2-1/2}, y_{3m+1/2}).$$

We denote the approximate solutions on the fine grid by $u_{i,j}$, $1 \leq i \leq 3N_1^0, 1 \leq j \leq 3N_2^0$. The approximation on the coarse mesh is given by

$$U_{l,m} = \frac{1}{9}\sum_{\alpha,\beta=0}^{2} u_{3l-\alpha,3m-\beta}, 1 \leq l \leq N_1^0, 1 \leq m \leq N_2^0,$$

and the incremental unknowns are given by the relation

$$Z_{3l-\alpha,3m-\beta} = u_{3l-\alpha,3m-\beta} - U_{l,m}. \tag{4.1}$$

Let $p > 1$ and $q > 1$ be two fixed integers. We discretize (1.1) on the fine mesh by using time step $\Delta t/p$ and on the coarse mesh by using time step Δt. We assume that n is a multiple of $q+1$ and $(u_{i,j}^n)_{1\leq i\leq 3N_1^0,\, 1\leq j\leq 3N_2^0}$ are known, where $u_{i,j}^n$ is an approximation of the average value of u over $k_{i,j}$ at the grid $t = n\Delta t$, for $i = 1, \ldots, 3N_1^0$, $j = 1, \ldots, 3N_2^0$. For $r = 0, 1, \ldots, p$ and $s = 1, 2, \ldots, q+1$, we let $u_{i,j}^{n+r/p}$ be the approximate solution of the mean values over $k_{i,j}$ at time $t_{n+t/p} = n\Delta t + r\Delta t/p$ for $i = 1, \ldots, 3N_1^0, j = 1, \ldots, 3N_2^0$ and $U_{l,m}^{n+s}$ the approximate solution of the mean value on the coarse mesh $K_{l,m}$ at time $t_{n+s} = (n+s)\Delta t$ for $l = 1, \ldots, N_1^0$ and $m = 1, \ldots, N_2^0$.

4.1 Implicit multilevel Finite volume Method

For $0 \leq r \leq p-1$ and $1 \leq s \leq q$, the following multilevel scheme is used to discretize (1.1)-(1.4).

$$\begin{aligned} &\frac{p}{\Delta t}(u_{i,j}^{n+(r+1)/p} - u_{i,j}^{n+r/p}) - (C_h(\mathbf{u}^{n+(r+1)/p}, \tilde{\mathbf{u}}^{n+r/p}))_{i,j} + \varepsilon^2\Delta_h^2 u_{i,j}^{n+(r+1)/p} \\ &\qquad = \nabla_{1,h}^+(\varphi_{i-1/2,j}^{n+r/p}\nabla_{1,h}^- u_{i,j}^{n+(r+1)/p}) + \nabla_{2,h}^+(\varphi_{i,j-1/2}^{n+r/p}\nabla_{2,h}^- u_{i,j}^{n+(r+1)/p}), \end{aligned} \tag{4.2a}$$

$$\begin{aligned} &\frac{U_{l,m}^{n+s+1} - U_{l,m}^{n+s}}{\Delta t} - (C_{3h}(\mathbf{U}^{n+s+1}, \tilde{\mathbf{U}}^{n+s}))_{l,m} + \varepsilon^2\Delta_{3h}^2 U_{l,m}^{n+s+1} \\ &\qquad = \nabla_{1,3h}^+(\Phi_{l-1/2,m}^{n+s}\nabla_{1,3h}^- U_{l,m}^{n+s+1}) + \nabla_{2,3h}^+(\Phi_{l,m-1/2}^{n+s}\nabla_{2,3h}^- U_{l,m}^{n+s+1}), \end{aligned} \tag{4.2b}$$

$$u_{i,j}^{n+(r+1)/p} = u_{i+3N_1^0,j}^{n+(r+1)/p} = u_{i,j+3N_2^0}^{n+(r+1)/p}, \tag{4.2c}$$

$$U_{l,m}^{n+s+1} = U_{l+N_1^0,m}^{n+s+1} = U_{l,m+N_2^0}^{n+s+1}, \tag{4.2d}$$

$$u_{i,j}^0 = \frac{1}{\Delta x\Delta y}, \iint_{k_{i,j}} u^0(x,y)\,dxdy, \tag{4.2e}$$

where $1 \leq i \leq 3N_1^0, 1 \leq j \leq 3N_2^0, 1 \leq l \leq N_1^0$, $1 \leq m \leq N_2^0$ and

$$\varphi_{i-1/2,j}^{n+r/p} = \frac{f'(u_{i,j}^{n+r/p}) + f'(u_{i-1,j}^{n+r/p})}{2}, \quad \varphi_{i,j-1/2}^{n+r/p} = \frac{f'(u_{i,j}^{n}) + f'(u_{i,j-1}^{n+r/p})}{2},$$
$$\Phi_{l-1/2,m}^{n+s} = \frac{f'(U_{l,m}^{n+s}) + f'(U_{l-1,m}^{n+s})}{2}, \quad \Phi_{l,m-1/2}^{n+s} = \frac{f'(U_{l,m}^{n}) + f'(U_{l,m-1}^{n+s})}{2}.$$

The multilevel discretization consists of alternating p steps on (4.2a) with smaller time step $\Delta t/p$, from t_n to t_{n+1} followed by q steps on (4.2b) with time step Δt, the incrementals being frozen at t_{n+1} from t_{n+1} to t_{n+q+1}. Then, using (4.1), we can go back to the fine mesh for p steps from t_{n+q+1} to t_{n+q+2}.

Since (4.2) is a succession of linear equation, the existence and uniqueness of solution follows the existence and uniqueness of solution discussed in section 3, Theorem 3.1.

Theorem 4.1. *The multilevel method defined by the equations* (4.2a)-(4.2e) *is conditionally stable in* $L^\infty(0,T;\mathcal{H}_h)$*, that is, if* $\Delta t \leq \varepsilon^2$ *and* $1 \leq n \leq M$*, then*

$$\|\boldsymbol{u}^n\|_h^2 \leq 2^{\frac{2T}{\varepsilon^2}} \|\boldsymbol{u}^0\|_h^2.$$

Proof. By multiplying (4.2a) by $2\frac{\Delta t}{p}\Delta x \Delta y\, u_{i,j}^{n+(r+1)/p}$ and adding the corresponding equalities for $i = 1, \ldots, 3N_1^0$ and $j = 1, \ldots, 3N_2^0$, after the application of Lemmas 2.2 and 3.1, we obtain

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 - \|\mathbf{u}^{n+r/p}\|_h^2 + \|\mathbf{u}^{n+(r+1)/p} - \mathbf{u}^{n+r/p}\|_h^2 + 2\frac{\Delta t}{p}\varepsilon^2 |\mathbf{u}^{n+(r+1)/p}|_{2,h}^2 \leq \frac{2\Delta t}{p} |\mathbf{u}^{n+(r+1)/p}|_{1,h}^2. \tag{4.3}$$

And then using Young's inequality and Lemma 2.5, (4.3) yields

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 - \|\mathbf{u}^{n+r/p}\|_h^2 + \|\mathbf{u}^{n+(r+1)/p} - \mathbf{u}^{n+r/p}\|_h^2 \leq \frac{\Delta t}{2p\varepsilon^2} \|\mathbf{u}^{n+(r+1)/p}\|_h^2.$$

Thus we have

$$\left[1 - \frac{\Delta t}{2p\varepsilon^2}\right] \|\mathbf{u}^{n+(r+1)/p}\|_h^2 \leq \|\mathbf{u}^{n+r/p}\|_h^2.$$

Based on (2.17), for $\dfrac{\Delta t}{2p\,\varepsilon^2} \leq \dfrac{1}{2}$, we have

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 \leq 2^{\frac{2\Delta t}{p\,\varepsilon^2}} \|\mathbf{u}^{n+r/p}\|_h^2.$$

After p iterations, we obtain

$$\|\mathbf{u}^{n+1}\|_h^2 \leq 2^{\frac{2\Delta t}{\varepsilon^2}} \|\mathbf{u}^n\|_h^2. \tag{4.4}$$

We now perform q iterations on the coarse grid, (4.2b), using time step Δt and the relations (4.1). At time $t_{n+s} = (n+s)\Delta t, 2 \leq s \leq q+1$, the incremental unknowns $Z_{i,j}$ are frozen at time $(n+1)\Delta t$. Multiplying (4.2b) by $18\Delta t \Delta x \Delta y\, U_{l,m}^{n+s+1}$ and adding the equalities for $l = 1, \ldots, N_1^0$ and $m = 1, \ldots, N_2^0$, together with Lemmas 2.5 and 3.1 and Young's inequality, we obtain

$$\|\mathbf{U}^{n+s+1}\|_{3h}^2 - \|\mathbf{U}^{n+s}\|_{3h}^2 + \|\mathbf{U}^{n+s+1} - \mathbf{U}^{n+s}\|_{3h}^2 \leq \frac{\Delta t}{2\varepsilon^2} \|\mathbf{U}^{n+s+1}\|_{3h}^2.$$

Thus we have

$$\left[1-\frac{\Delta t}{2\varepsilon^2}\right]\|\mathbf{U}^{n+s+1}\|_{3h}^2 \leq \|\mathbf{U}^{n+s}\|_{3h}^2.$$

Using (2.17), for $\dfrac{\Delta t}{2\varepsilon^2} \leq \dfrac{1}{2}$, we have

$$\|\mathbf{U}^{n+s+1}\|_{3h}^2 \leq 2^{\frac{2\Delta t}{\varepsilon^2}}\|\mathbf{U}^{n+s}\|_{3h}^2. \tag{4.5}$$

From the definition of the increments $Z_{3l-\alpha,3m-\beta}^{n+1}$, we have

$$u_{3l-\alpha,3m-\beta}^{n+s} = U_{l,m}^{n+s} + Z_{3l-\alpha,3m-\beta}^{n+1}, \quad 1 \leq l \leq N_1^0, 1 \leq m \leq N_2^0, \quad \alpha,\beta = 0,1,2.$$

Taking the sum over α and β, we get

$$\sum_{\alpha,\beta=0}^{2}|u_{3l-\alpha,3m-\beta}^{n+s}|^2 = \sum_{\alpha,\beta=0}^{2}|U_{l,m}^{n+s} + Z_{3l-\alpha,3m-\beta}^{n+1}|^2 = 9|U_{l,m}^{n+s}|^2 + \sum_{\alpha,\beta=0}^{2}|Z_{3l-\alpha,3m-\beta}^{n+1}|^2.$$

For $s = 1,\dots,q+1$, the following relation holds

$$\|\mathbf{u}^{n+s}\|_h^2 = \|\mathbf{U}^{n+s}\|_{3h}^2 + \|\mathbf{Z}^{n+1}\|_h^2. \tag{4.6}$$

By adding $\|\mathbf{Z}^{n+1}\|_h^2$ to both sides of inequality (4.5) and using (4.6), we get

$$\|\mathbf{u}^{n+s+1}\|_h^2 \leq 2^{\dfrac{2\Delta t}{\varepsilon^2}}\|\mathbf{u}^{n+s}\|_h^2.$$

After q iterations, and using (4.4), we have

$$\|\mathbf{u}^{n+q+1}\|_h^2 \leq 2^{\dfrac{2\Delta t\ (q+1)}{\varepsilon^2}}\|\mathbf{u}^{n}\|_h^2.$$

By induction over n, we obtain

$$\|\mathbf{u}^{n}\|_h^2 \leq 2^{\dfrac{2n\Delta t}{\varepsilon^2}}\|\mathbf{u}^{0}\|_h^2 \leq 2^{\dfrac{2T}{\varepsilon^2}}\|\mathbf{u}^{0}\|_h^2.$$

This completes the proof. □

Theorem 4.2. *Suppose that the solution $u(x,t)$ of* (1.1)-(1.4) *is sufficiently smooth.*
Assume that $\Delta t < 4\varepsilon^2$, (4.9), *and* (4.14) *are satisfied.*
Assume that $\Delta t, \Delta x$ and Δy satisfy (4.11).
Then, the solution of the finite volume discretization (4.2a)-(4.2e) *converges to the solution of* (1.1) *in the discrete L^2-norm with rate of convergence $\mathcal{O}(\Delta t + (3\Delta x)^2 + (3\Delta y)^2)$.*

Proof. Let n is a multiple of $q+1$. Let

$$v_{i,j}^{n+r/p} = \iint_{k_{i,j}} u(x,y,t_{n+r/p})dxdy,$$

be the cell average of u at time $t_{n+r/p}$ on the cell $k_{i,j}$ for $1 \leq i \leq 3N_1^0, 1 \leq j \leq 3N_2^0, 0 \leq r \leq p$. Denote

$$s = \max_{-L_1\leq x\leq L_1, -L_2\leq y\leq L_2, 0\leq t\leq T}|u(x,y,t)|.$$

Making use of Taylor expansion, we obtain

$$\frac{\upsilon_{i,j}^{n+(r+1)/p}-\upsilon_{i,j}^{n+r/p}}{\Delta t/p}-(C_h(\boldsymbol{\upsilon}^{n+(r+1)/p},\tilde{\boldsymbol{\upsilon}}^{n+r/p}))_{i,j}+\varepsilon^2\Delta_h^2\upsilon_{i,j}^{n+(r+1)/p}=\nabla_{1,h}^{+}\left[\psi_{i-\frac{1}{2},j}^{n+r/p}\nabla_{1,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]$$
$$+\nabla_{2,h}^{+}\left[\psi_{i,j-\frac{1}{2}}^{n+r/p}\nabla_{2,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]+\tau_{i,j}^{n+r/p}, \tag{4.7}$$

where $\tau_{i,j}^{n+r/p}\in\mathcal{H}_h$ is the truncation error of the finite volume discretization (4.2a) for $1\le i\le 3N_1^0, 1\le j\le 3N_2^0$. There exists a positive constant c_1 such that

$$\max_{i,j,n}|\tau_{i,j}^{n+r/p}|\le c_1\left(\frac{\Delta t}{p}+\Delta x^2+\Delta y^2\right), 1\le r\le p \text{ and } \le 1\le i\le 3N_0.$$

Let $\mathbf{e}^{n+r/p}=\boldsymbol{\upsilon}^{n+r/p}-\mathbf{u}^{n+r/p}$, where $u_{i,j}^{n+r/p}$ is the solution of (4.2a). Substituting $u_{i,j}^{n+r/p}=\upsilon_{i,j}^{n+r/p}-e_{i,j}^{n+r/p}$ in (4.2a), and using (4.7), we obtain

$$\frac{e_{i,j}^{n+(r+1)/p}-e_{i,j}^{n+r/p}}{\Delta t/p}-(C_h(\mathbf{e}^{n+(r+1)/p},\tilde{\boldsymbol{\upsilon}}^{n+r/p}-\tilde{\mathbf{e}}^{n+r/p}))_{i,j}+\varepsilon^2\Delta_h^2 e_{i,j}^{n+(r+1)/p}=(C_h(\boldsymbol{\upsilon}^{n+(r+1)/p},\tilde{\mathbf{e}}^{n+r/p}))_{i,j}$$
$$+\nabla_{1,h}^{+}\left[\left(3(\upsilon_{i,j}^{n+r/p}\,e_{i,j}^{n+r/p}+\upsilon_{i-1,j}^{n+r/p}\,e_{i-1,j}^{n+r/p})-\frac{3}{2}[(e_{i,j}^{n+r/p})^2+(e_{i-1,j}^{n+r/p})^2]\right)\nabla_{1,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]$$
$$+\nabla_{2,h}^{+}\left[\left(3(\upsilon_{i,j}^{n+r/p}\,e_{i,j}^{n+r/p}+\upsilon_{i,j-1}^{n+r/p}\,e_{i,j-1}^{n+r/p})-\frac{3}{2}[(e_{i,j}^{n+r/p})^2+(e_{i,j-1}^{n+r/p})^2]\right)\nabla_{2,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]$$
$$+\nabla_{1,h}^{+}\left(\varphi_{i-\frac{1}{2},j}^{n+r/p}\nabla_{1,h}^{-}e_{i,j}^{n+(r+1)/p}\right)+\nabla_{2,h}^{+}\left(\varphi_{i,j-\frac{1}{2}}^{n+r/p}\nabla_{2,h}^{-}e_{i,j}^{n+(r+1)/p}\right)+\tau_{i,j}^{n+r/p}. \tag{4.8}$$

Multiplying (4.8) by $\frac{2\Delta t\Delta x\Delta y}{p}e_{i,j}^{n+(r+1)/p}$ and summing for $i=1,\ldots,3N_1^0$ and $j=1,\ldots,3N_2^0$, together with (2.15) and Lemmas 3.1 and 2.1, we obtain

$$\|\mathbf{e}^{n+(r+1)/p}\|_h^2-\|\mathbf{e}^{n+r/p}\|_h^2+\|\mathbf{e}^{n+(r+1)/p}-\mathbf{e}^{n+r/p}\|_h^2+\frac{2\Delta t}{p}\varepsilon^2\|\mathbf{e}^{n+(r+1)/p}\|_{2,h}^2\le\frac{2\Delta t}{p}\|\mathbf{e}_{i,j}^{n+(r+1)/p}\|_{1,h}^2$$
$$-\frac{2\Delta t\Delta x\Delta y}{p}\sum_{i=1}^{3N_1^0}\sum_{j=1}^{3N_2^0}\left[\left(3(\upsilon_{i,j}^{n+r/p}\,e_{i,j}^{n+r/p}+\upsilon_{i-1,j}^{n+r/p}\,e_{i-1,j}^{n+r/p})-\frac{3}{2}[(e_{i,j}^{n+r/p})^2+(e_{i-1,j}^{n+r/p})^2]\right)\nabla_{1,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]\nabla_{1,h}^{-}e_{i,j}^{n+(r+1)/p}$$
$$-\frac{2\Delta t\Delta x\Delta y}{p}\sum_{i=1}^{3N_1^0}\sum_{j=1}^{3N_2^0}\left[\left(3(\upsilon_{i,j}^{n+r/p}\,e_{i,j}^{n+r/p}+\upsilon_{i,j-1}^{n+r/p}\,e_{i,j-1}^{n+r/p})-\frac{3}{2}[(e_{i,j}^{n+r/p})^2+(e_{i,j-1}^{n+r/p})^2]\right)\nabla_{2,h}^{-}\upsilon_{i,j}^{n+(r+1)/p}\right]\nabla_{2,h}^{-}e_{i,j}^{n+(r+1)/p}$$
$$+\frac{2\Delta t}{p}(C_h(\boldsymbol{\upsilon}^{n+(r+1)/p},\tilde{\mathbf{e}}^{n+r/p}),\mathbf{e}^{n+(r+1)/p})_h+\frac{2\Delta t}{p}(\boldsymbol{\tau}^{n+r/p},\mathbf{e}^{n+(r+1)/p})_h.$$

Using the approach implemented on the proof of Theorem 3.3, we deduce in the fine mesh taking

$$\Delta tc\le\frac{p}{2}, \tag{4.9}$$

then

$$\|\mathbf{e}^{n+(r+1)/p}\|_h^2+\frac{1}{p}\,\Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+(r+1)/p}|_{1,h}^2\le\frac{3\Delta t\,\delta_2\,4^{\Delta tc}}{p\Delta x\Delta y}\|\mathbf{e}^{n+r/p}\|_h^2\;|\mathbf{e}^{n+(r+1)/p}|_{1,h}^2$$
$$+4^{\Delta tc}\Big[\left(1+\frac{\Delta tc_9}{p}\right)\|\mathbf{e}^{n+r/p}\|_h^2+\frac{\Delta t\,c_8}{p}\|\tilde{\mathbf{e}}^{n+r/p}\|_h^2+\frac{\Delta t}{p}\,c_7\left(\frac{\Delta t}{p}+\Delta x^2+\Delta y^2\right)^2\Big]. \tag{4.10}$$

For

$$\frac{3\delta_2\,c_7}{\Delta x\Delta y}\left(\frac{\Delta t}{p}+\Delta x^2+\Delta y^2\right)^2\le\frac{1}{2T}\varepsilon^2\eta^2 4^{-Tc}\exp\Big(-T\,c_{10}\Big), \tag{4.11}$$

as shown in Theorem 3.3, we obtain

$$\|\mathbf{e}^{n+(r+1)/p}\|_h^2 + \frac{1}{2p}\Delta t\varepsilon^2\eta^2|\mathbf{e}^{n+(r+1)/p}|_{1,h}^2 \\ \leq 4^{\frac{\Delta t\, c}{p}}\left[\left(1+\frac{\Delta tc_9}{p}\right)\|\mathbf{e}^{n+r/p}\|_h^2 + \frac{\Delta tc_8}{p}\|\tilde{\mathbf{e}}^{n+r/p}\|_h^2 + \frac{\Delta t}{p}c_7\left(\frac{\Delta t}{p}+\Delta x^2+\Delta y^2\right)^2\right], \quad (4.12)$$

which after p iterations gives

$$\|\mathbf{e}^{n+1}\|_h^2 \leq 4^{\Delta tc}\exp\left(\Delta t\, c_{10}\right)\left[\|\mathbf{e}^n\|_h^2 + \Delta t\, c_7\left(\frac{\Delta t}{p}+\Delta x^2+\Delta y^2\right)^2\right]. \quad (4.13)$$

In a similar way for $1 \leq s \leq q$, and being on the coarse mesh and for

$$\Delta tc \leq \frac{1}{2}, \quad (4.14)$$

we get

$$\|\mathbf{E}^{n+s+1}\|_{3h}^2 + \frac{1}{2}\Delta t\varepsilon^2\eta^2|\mathbf{E}^{n+s+1}|_{1,3h}^2 \leq 4^{\Delta t\, c}\Big[(1+\Delta tc_9)\|\mathbf{E}^{n+s}\|_{3h}^2 + \Delta tc_8\|\tilde{\mathbf{E}}^{n+s}\|_{3h}^2 + \Delta tc_7(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2\Big], \quad (4.15)$$

where $\mathbf{E}^{n+s} = \mathbf{\Upsilon}^{n+s} - \mathbf{U}^{n+s}$ and $\mathbf{\Upsilon}^{n+s}$ and $\mathbf{U}^{n+s}$ are exact cell average and numerical solutions on the coarse mesh, respectively. For $n+s < m_0 - 1$, we have

$$\|\mathbf{E}^{n+s+1}\|_{3h}^2 \leq 4^{\Delta tc}\exp\left(\Delta t\left(c_9+c_8\right)\right)\Big[\|\mathbf{E}^{n+s}\|_{3h}^2 + \Delta t\, c_7\left(\Delta t+(3\Delta x)^2+(3\Delta y)^2\right)^2\Big], \quad (4.16)$$

As we said at the beginning of this section, the numerical increments $Z_{i,j}$'s are fixed between steps $n+1$ and $n+q+1$ and therefore for $1 \leq s \leq q$, $1 \leq l \leq N_1^0$ and $1 \leq m \leq N_2^0$,

$$Z_{3l-\alpha,3m-\beta}^{n+s+1} = Z_{3l-\alpha,3m-\beta}^{n+1} = u_{3l-\alpha,3m-\beta}^{n+1} - U_{l,m}^{n+1}, \quad \alpha,\beta = 0,1,2.$$

Using (4.1), we have

$$e_{3l-\alpha,3m-\beta}^{n+s+1} = E_{l,m}^{n+s+1} + \zeta_{3l-\alpha,3m-\beta}^{n+s+1},$$

where

$$\zeta_{3l-\alpha,3m-\beta}^{n+s+1} = (\upsilon_{3l-\alpha,3m-\beta}^{n+s+1} - \Upsilon_{l,m}^{n+s+1}) - Z_{3l-\alpha,3m-\beta}^{n+1},$$

is the difference of the numerical increment from exact increment. It is clear from the definition of increments that

$$\sum_{\alpha,\beta=0}^{2}(\upsilon_{3l-\alpha,3m-\beta}^{n+s+1} - \Upsilon_{l,m}^{n+s+1}) = \sum_{\alpha,\beta=0}^{2} Z_{3l-\alpha,3m-\beta}^{n+1} = 0,$$

and hence $\sum_{\alpha,\beta=0}^{2}\zeta_{3l-\alpha,3m-\beta}^{n+s+1} = 0$. As a result for $s = 1,\ldots,q$,

$$\sum_{\alpha,\beta=0}^{2}\left(e_{3l-\alpha,3m-\beta}^{n+s+1}\right)^2 = 9\left(E_{l,m}^{n+s+1}\right)^2 + \sum_{\alpha,\beta=0}^{2}\left(\zeta_{3l-\alpha,3m-\beta}^{n+s+1}\right)^2. \quad (4.17)$$

Multiplying (4.17) by $\Delta x \Delta y$ and taking the sum for $l = 1, \ldots, N_1^0$ and $m = 1, \ldots, N_2^0$, we obtain

$$\|\mathbf{e}^{n+s+1}\|_h^2 = \|\mathbf{E}^{n+s+1}\|_{3h}^2 + \|\boldsymbol{\zeta}^{n+s+1}\|_h^2. \tag{4.18}$$

We now estimate the term $\boldsymbol{\zeta}^{n+s+1}$. From the definition of increments for $\mathbf{u} \in \mathcal{H}_h$, we have

$$U_{l,m} = u_{3l-\alpha,3m-\beta} + \mathcal{O}(\Delta x + \Delta y), \tag{4.19}$$

from which

$$\Upsilon_{l,m}^{n+s+1} - \upsilon_{3l-\alpha,3m-\beta}^{n+s+1} = \Upsilon_{l,m}^{n+s} - \upsilon_{3l-\alpha,3m-\beta}^{n+s} + \Delta t\, \mathcal{O}(\Delta x + \Delta y).$$

Hence

$$\boldsymbol{\zeta}^{n+s+1} = \boldsymbol{\zeta}^{n+s} + \Delta t\, \mathcal{O}(\Delta x + \Delta y). \tag{4.20}$$

(4.20) gives

$$\|\boldsymbol{\zeta}^{n+s+1}\|_h^2 = \|\boldsymbol{\zeta}^{n+s}\|_h^2 + \Delta t\, \mathcal{O}(\Delta t\, (\Delta x + \Delta y)^2),$$

which with the application of Young's inequality, (2.18) gives

$$\|\boldsymbol{\zeta}^{n+s+1}\|_h^2 \le \|\boldsymbol{\zeta}^{n+s}\|_h^2 + \Delta t\, c_{11}\, (\Delta t + \Delta x^2 + \Delta y^2)^2, \tag{4.21}$$

where c_{11} is a constant independent of Δt, Δx and Δy.
Combining (4.16), (4.18) and (4.21), we obtain

$$\|\mathbf{e}^{n+s+1}\|_h^2 \le 4^{\Delta t c} \exp\left(\Delta t\, (c_9 + c_8)\right) \Big[\|\mathbf{e}^{n+s}\|_h^2 + \Delta t\, c_{12}\, (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2\Big],$$

which after s iterations gives

$$\|\mathbf{e}^{n+s+1}\|_h^2 \le 4^{\Delta t\, s\, c} \exp\left(\Delta t\, s\ (c_9 + c_8)\right) \Big[\|\mathbf{e}^{n+1}\|_h^2 + \Delta t\, s\, c_{12}\, (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2\Big], \tag{4.22}$$

where $c_{12} = \max\{c_7, 4^{-\Delta t c} \exp\left(-\Delta t\, (c_9 + c_8)\right)\, c_{11}\}$. Together with (4.13), (4.22) becomes

$$\begin{aligned}\|\mathbf{e}^{n+s+1}\|_h^2 \le{}& 4^{\Delta t\, (s+1)\, c} \exp\left[\Delta t\, s\ (c_9 + c_8)\right] \Bigg[\exp\left(\Delta t\, c_{10}\right) \Big[\|\mathbf{e}^{n}\|_h^2 + \Delta t c_{12}\, (\Delta t + \Delta x^2 + \Delta y^2)^2\Big] \\ & + \Delta t\, s\, c_{12}\, (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2\Bigg].\end{aligned}$$

Since $m_0 A \ge 1$, it follows from this inequality that

$$\|\mathbf{e}^{n+s+1}\|_h^2 \le 4^{\Delta t\, (s+1)\, c} \exp\left(\Delta t\, (s+1)\, c_{10}\right) \Big[\|\mathbf{e}^{n}\|_h^2 + \Delta t\, (s+1)\, c_{12}\, (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2\Big].$$

Thus, after n iterations, we get

$$\|\mathbf{e}^{n+s+1}\|_h^2 \le 4^{\Delta t\, (n+s+1)\, c} \exp\left(\Delta t\, (n+s+1)\ (c_{10})\right) \Big[\Delta t\, (n+s+1)\, c_{12}\, (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2\Big]. \tag{4.23}$$

For the case $n + s \ge m_0 - 1$, we have

$$\begin{aligned}\|\mathbf{E}^{n+s+1}\|_{3h}^2 \le{}& 4^{\Delta tc}\, (1 + \Delta t\, c_9)\, \|\mathbf{E}^{n+s}\|_{3h}^2 + \Delta t\, A\, c_8\, 4^{\Delta tc} \Big[\|\mathbf{E}^{n+s}\|_{3h}^2 + \|\mathbf{E}^{n+s-1}\|_{3h}^2 + \cdots + \|\mathbf{E}^{n+s-m_0+1}\|_{3h}^2\Big] \\ & + \Delta t\, c_{12}\, 4^{\Delta tc} (\Delta t + (3\Delta x)^2 + (3\Delta y)^2)^2,\end{aligned} \tag{4.24}$$

which implies

$$\max\Big\{\|\mathbf{E}^{n+s+1}\|_{3h}^2,\dots,\|\mathbf{E}^{n+s-m_0+2}\|_{3h}^2,\Big\} \le 4^{\Delta tc}\exp\left(\Delta t\, c_{10}\right)\,\max\Big\{\|\mathbf{E}^{n+s}\|_{3h}^2,\|\mathbf{E}^{n+s-1}\|_{3h}^2,\dots,\|\mathbf{E}^{n+s-m_0+1}\|_{3h}^2\Big\}$$
$$+\,\Delta t\, c_{12}\, 4^{\Delta tc}(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2. \tag{4.25}$$

Using (4.18), (4.21) and (4.25), we obtain

$$\begin{aligned}
&\max\Big\{\|\mathbf{e}^{n+s+1}\|_h^2,\dots,\|\mathbf{e}^{n+s-m_0+2}\|_h^2\Big\}\\
&\quad\le 4^{\Delta tc}\exp\left(c_{10}\right)\ \max\Big\{\|\mathbf{e}^{n+s}\|_h^2,\|\mathbf{e}^{n+s-1}\|_h^2,\dots,\|\mathbf{e}^{n+s-m_0+1}\|_h^2\Big\}+\Delta t\, c_{12}\, 4^{\Delta tc}(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2\\
&\quad\le 4^{\Delta t(n+s-m_0+1)c}\exp\left[\Delta t\, c_{10}(n+s-m_0+1)\right]\Big[\ \max\Big\{\|\mathbf{e}^{m_0-1}\|_h^2,\|\mathbf{e}^{m_0-2}\|_h^2,\dots,\|\mathbf{e}^{0}\|_h^2\Big\}\\
&\qquad+\Delta t\,(n+s-m_0+1)c_{12}\,(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2\Big].
\end{aligned}\tag{4.26}$$

Using (4.23), (4.26) and induction on n, one obtains

$$\begin{aligned}
\|\mathbf{e}^n\|_h^2 &\le 4^{\Delta t(n)c}\exp\left[n\,\Delta t\,(c_{10})\right]\Big[\|\mathbf{e}^0\|_h^2+n\,\Delta t c_{12}\,(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2\Big]\\
&\le 4^{Tc}\exp\left[T\,(c_{10})\right]\Big[Tc_{12}\,(\Delta t+(3\Delta x)^2+(3\Delta y)^2)^2\Big],
\end{aligned}\tag{4.27}$$

for $n=1,\dots,M$. Therefore, we have

$$\|\mathbf{e}^n\|_h \le C(\Delta t+(3\Delta x)^2+(3\Delta y)^2),$$

where C is a constant independent of $\Delta t, \Delta x$ and Δy. This completes the proof. $\square$

The rate of convergence depends on the mesh size of the coarse mesh.

4.2 Explicit Multilevel Finite Volume Method

For $0\le r\le p-1$ and $1\le s\le q$, we discretize (1.1) using explicit multilevel finite volume method.

$$\begin{aligned}
&\frac{p}{\Delta t}(u_{i,j}^{n+(r+1)/p}-u_{i,j}^{n+r/p})-(C_h(\mathbf{u}^{n+r/p},\mathbf{u}^{n+r/p}))_{i,j}+\varepsilon^2\Delta_h^2 u_{i,j}^{n+r/p}\\
&\qquad=\nabla_{1,h}^{+}(\varphi_{i-1/2,j}^{n+r/p}\nabla_{1,h}^{-}u_{i,j}^{n+r+/p})+\nabla_{2,h}^{+}(\varphi_{i,j-1/2}^{n+r/p}\nabla_{2,h}^{-}u_{i,j}^{n+r/p}),
\end{aligned}\tag{4.28a}$$

$$\begin{aligned}
&\frac{U_{l,m}^{n+s+1}-U_{l,m}^{n+s}}{\Delta t}-(C_{3h}(\mathbf{U}^{n+s},\mathbf{U}^{n+s}))_{l,m}+\varepsilon^2\Delta_{3h}^2 U_{l,m}^{n+s}\\
&\qquad=\nabla_{1,3h}^{+}(\varPhi_{l-1/2,m}^{n+s}\nabla_{1,3h}^{-}U_{l,m}^{n+s})+\nabla_{2,3h}^{+}(\varPhi_{l,m-1/2}^{n+s}\nabla_{2,3h}^{-}U_{l,m}^{n+s}).
\end{aligned}\tag{4.28b}$$

$$u_{i,j}^{n+r/p}=u_{i+3N_1^0,j}^{n+r/p}=u_{i,j+3N_2^0}^{n+r/p},\tag{4.28c}$$

$$U_{l,m}^{n+s}=U_{l+N_1^0,m}^{n+s}=U_{l,m+N_2^0}^{n+s},\tag{4.28d}$$

$$u_{i,j}^0=\frac{1}{\Delta x\Delta y}\iint\limits_{k_{i,j}}u^0(x,y)dxdy,\tag{4.28e}$$

where $1\le i\le 3N_1^0$, $1\le l\le N_1^0$, $1\le j\le 3N_2^0$ and $1\le m\le N_2^0$.

Theorem 4.3. *We assume that the following satisfied for some δ, $0 < \delta < 1$:*

$$32\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^2 \le \frac{1-\delta}{2\varepsilon^2}\min\{p,81\}, \tag{4.29}$$

$$16\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) \le \varepsilon^2\delta\eta^2(1-\delta)\min\{p,9\}, \tag{4.30}$$

$$\frac{72\Delta t}{p\Delta x\Delta y}\left((|\alpha_1|+|\alpha_2|)^2+\frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\boldsymbol{u}^0\|_h^2\right)\|\boldsymbol{u}^0\|_h^2 \le \varepsilon^2\delta^2\eta^2\exp\left(\frac{-2T}{\varepsilon^2}\right), \tag{4.31}$$

$$\frac{8\Delta t}{\Delta x\Delta y}\left((|\alpha_1|+|\alpha_2|)^2+\frac{4}{81\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\boldsymbol{u}^0\|_h^2\right)\|\boldsymbol{u}^0\|_h^2 \le \varepsilon^2\delta^2\eta^2\exp\left(\frac{-2T}{\varepsilon^2}\right). \tag{4.32}$$

Then the multilevel method defined by the equations (4.28a) - (4.28e) *is* $L^\infty(0,T;\mathcal{H}_h)$ *stale in the following sense:*

$$\|\boldsymbol{u}^n\|^2 \le \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\boldsymbol{u}^{n-1}\|^2 \le \cdots \le \exp\left(\frac{n\Delta t}{\varepsilon^2}\right)\|\boldsymbol{u}^0\|^2 \le \exp\left(\frac{T}{\varepsilon^2}\right)\|\boldsymbol{u}^0\|^2, \quad n=1,2\ldots,M_0, \tag{4.33}$$

$$\|\boldsymbol{u}^{s(q+1)+r/p}\|^2 \le \exp\left(\frac{r\Delta t}{p\varepsilon^2}\right)\|\boldsymbol{u}^{s(q+1)}\|^2, \quad r=1,2,\ldots,p. \tag{4.34}$$

Proof. To prove this theorem we use the approach discussed in Theorem 3.4. We assume n is a multiple of $q+1$. Multiplying (4.28a) by $2\frac{\Delta t}{p}\Delta x\Delta y u_{i,j}^{n+r/p}$ and taking the sum for $i=1,\ldots,3N_1^0$ and $j=1,\ldots,3N_2^0$ together with (2.16) and Lemma 2.5, we obtain

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 - \|\mathbf{u}^{n+(r+1)/p}-\mathbf{u}^{n+r/p}\|_h^2 + \frac{\Delta t}{p}\varepsilon^2|\mathbf{u}^{n+r/p}|_{2,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2. \tag{4.35}$$

To estimate the term $\|\mathbf{u}^{n+(r+1)/p}-\mathbf{u}^{n+r/p}\|_h^2$, we multiply (4.28a) by $2\frac{\Delta t}{p}\Delta x\Delta y(u_{i,j}^{n+(r+1)/p}-u_{i,j}^{n+r/p})$ and summing from $i=1$ to $i=3N_1^0$ and from $j=1$ to $j=3N_2^0$, we find

$$\begin{aligned}\|\mathbf{u}^{n+(r+1)/p}-\mathbf{u}^{n+r/p}\|_h^2 \le{}& \frac{36\Delta t^2}{p^2\,\Delta x\Delta y}(|\alpha_1|+|\alpha_2|)^2\|\mathbf{u}^{n+r/p}\|_h^2|\mathbf{u}^{n+r/p}|_{1,h}^2 + \frac{64\Delta t^2}{p^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^2\varepsilon^4|\mathbf{u}^{n+r/p}|_{2,h}^2\\ &+\frac{144\Delta t^2}{p^2\,\Delta x^2\Delta y^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^{n+r/p}\|_h^4|\mathbf{u}^n|_{1,h}^2 + \frac{16\Delta t^2}{p^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)|\mathbf{u}^{n+r/p}|_{1,h}^2.\end{aligned}$$

Using (4.29), we obtain

$$\begin{aligned}\|\mathbf{u}^{n+(r+1)/p}-\mathbf{u}^{n+r/p}\|_h^2 \le{}& \frac{36\Delta t^2}{p^2\,\Delta x\Delta y}(|\alpha_1|+|\alpha_2|)^2\|\mathbf{u}^n\|_h^2|\mathbf{u}^{n+r/p}|_{1,h}^2 + \frac{\Delta t}{p}\varepsilon^2(1-\delta)|\mathbf{u}^{n+r/p}|_{2,h}^2\\ &+144\frac{\Delta t^2}{p^2\,\Delta x^2\Delta y^2}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^{n+r/p}\|_h^4|\mathbf{u}^{n+r/p}|_{1,h}^2 + \frac{16\Delta t^2}{p^2h^2}|\mathbf{u}^{n+r/p}|_{1,h}^2.\end{aligned} \tag{4.36}$$

On substitution of (4.36) into (4.35), we get

$$\begin{aligned}\|\mathbf{u}^{n+(r+1)/p}\|_h^2 + \frac{\Delta t}{p}\varepsilon^2\delta|\mathbf{u}^{n+r/p}|_{2,h}^2 \le{}& \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2\\ &+\frac{36\Delta t^2}{p^2\,\Delta x\Delta y}\left[(|\alpha_1|+|\alpha_2|)^2+\frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2\right]\|\mathbf{u}^{n+r/p}\|_h^2|\mathbf{u}^{n+r/p}|_{1,h}^2\end{aligned}$$

Using 2.24 and (4.30), we obtain

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 - \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2 + \frac{\Delta t}{p}\varepsilon^2\delta^2\eta^2|\mathbf{u}^{n+r/p}|_{1,h}^2$$
$$- \frac{36\Delta t^2}{p^2\,\Delta x\Delta y}\left[(|\alpha_1|+|\alpha_2|)^2 + \frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2\right]\|\mathbf{u}^{n+r/p}\|_h^2|\mathbf{u}^{n+r/p}|_{1,h}^2 \le 0. \tag{4.37}$$

In a similar fashion, from (4.28b) together with the assumptions (4.29), (4.30) and (4.32), we obtain

$$\|\mathbf{U}^{n+m+1}\|_{3h}^2 - \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{U}^{n+m}\|_{3h}^2 + \Delta t\varepsilon^2\delta^2\eta^2|\mathbf{U}^{n+m}|_{1,3h}^2$$
$$\le \frac{4\Delta t^2}{\Delta x\Delta y}\left[(|\alpha_1|+|\alpha_2|)^2 + \frac{4}{81\,\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{U}^{n+m}\|_{3h}^2\right]\|\mathbf{U}^{n+m}\|_{3h}^2|\mathbf{U}^{n+m}|_{1,3h}^2. \tag{4.38}$$

Now we need to prove the following by induction on n

$$\|\mathbf{u}^{n+(r+1)/p}\|_h^2 + \frac{\Delta t}{2p}\varepsilon^2\delta^2\eta^2|\mathbf{u}^{n+r/p}|_{1,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{n+r/p}\|_h^2, \quad \text{for } r = 0,1,\ldots,p-1, \tag{4.39}$$

$$\|\mathbf{U}^{n+s+1}\|_{3h}^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{U}^{n+s}|_{1,3h}^2 \le \exp\left(\frac{\Delta t}{\varepsilon^2}\right)\|\mathbf{U}^{n+s}\|_{3h}^2, \quad \text{for } s = 1,2,\ldots,q. \tag{4.40}$$

We first show (4.39) and (4.40) hold by induction on r and s when $n = 0$. We first show

$$\|\mathbf{u}^1\|_h^2 + \frac{\Delta t}{2p}\varepsilon^2\delta^2\eta^2\sum_{r=0}^{p-1}\exp\left(\frac{(p-1-r)\Delta t}{p\varepsilon^2}\right)|\mathbf{u}^{r/p}|_{1,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{r/p}\|_h^2. \tag{4.41}$$

For $n = 0$, the relation (4.37) becomes

$$\|\mathbf{u}^{(r+1)/p}\|_h^2 + \frac{\Delta t}{p}\varepsilon^2\delta^2\eta^2|\mathbf{u}^{r/p}|_{1,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{r/p}\|_h^2$$
$$+ 36\frac{\Delta t^2}{p^2\,\Delta x\Delta y}\left[(|\alpha_1|+|\alpha_2|)^2 + \frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^{r/p}\|_h^2\right]\|\mathbf{u}^{r/p}\|_h^2|\mathbf{u}^{r/p}|_{1,h}^2. \tag{4.42}$$

For $r = 0$ using (4.31), we get

$$\|\mathbf{u}^{1/p}\|_h^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2|\mathbf{u}^0|_{1,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2.$$

Let us assume that (4.41) holds up to $r-1$. From the assumption for $s = 1,2,\ldots,r-1$, we have

$$\|\mathbf{u}^{s/p}\|_h^2 \le \|\mathbf{u}^{(s-1)/p}\|_h^2$$

and

$$\|\mathbf{u}^{s/p}\|_h^2 \le \exp\left(\frac{s\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^0\|_h^2 \tag{4.43}$$

The relation (4.42) becomes

$$\|\mathbf{u}^{(r+1)/p}\|_h^2 + \frac{\Delta t}{p}\varepsilon^2\delta^2\eta^2|\mathbf{u}^{r/p}|_{1,h}^2 \le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{r/p}\|_h^2$$
$$+ 36\frac{\Delta t^2}{p^2\,\Delta x\Delta y}\exp\left(\frac{2r\Delta t}{p\varepsilon^2}\right)\left[(|\alpha_1|+|\alpha_2|)^2 + \frac{4}{\Delta x\Delta y}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\|\mathbf{u}^0\|_h^2\right]\|\mathbf{u}^0\|_h^2|\mathbf{u}^{r/p}|_{1,h}^2$$
$$\le \exp\left(\frac{\Delta t}{p\varepsilon^2}\right)\|\mathbf{u}^{r/p}\|_h^2 + \frac{\Delta t}{2p}\varepsilon^2\delta^2\eta^2|\mathbf{u}^{r/p}|_{1,h}^2, \tag{4.44}$$

which shows us that (4.39) is true for $n = 0$. From (4.44), we have

$$\|\mathbf{u}^1\|_h^2 + \frac{\Delta t}{p}\varepsilon^2\delta^2\eta^2 \sum_{r=0}^{p-1} \exp\left(\frac{(p-1-r)\Delta t}{p\varepsilon^2}\right) |\mathbf{u}^{r/p}|_{1,h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{u}^0\|_h^2,$$

which implies

$$\|\mathbf{u}^1\|_h^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{u}^0\|_h^2. \tag{4.45}$$

We then show (4.40) by using induction on s for $n = 0$. From the definition of $\mathbf{U}$, we have

$$\|\mathbf{U}^n\|_{3h}^2 \leq \|\mathbf{u}^n\|_h^2. \tag{4.46}$$

For $s = 1$, from (4.29), we have

$$\begin{aligned} &\|\mathbf{U}^2\|_{3h}^2 - \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{U}^1\|_{3h}^2 + \Delta t\varepsilon^2\delta^2\eta^2 |\mathbf{U}^1|_{1,3h}^2 \\ &\quad - \frac{4\Delta t^2}{\Delta x \Delta y}\left[(|\alpha_1| + |\alpha_2|)^2 + \frac{4}{81\,\Delta x\Delta y}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\|\mathbf{U}^1\|_{3h}^2\right] \|\mathbf{U}^1\|_{3h}^2 |\mathbf{U}^1|_{1,3h}^2 \leq 0. \end{aligned}$$

Then using (4.45) and (4.46), we have

$$\begin{aligned} &\|\mathbf{U}^2\|_{3h}^2 - \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{U}^1\|_{3h}^2 + \Delta t\varepsilon^2\delta^2\eta^2 |\mathbf{U}^1|_{1,3h}^2 \\ &\quad - \frac{4\Delta t^2}{\Delta x \Delta y}\exp\left(\frac{2\Delta t}{\varepsilon^2}\right)\left[(|\alpha_1| + |\alpha_2|)^2 + \frac{4}{81\Delta x\Delta y}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\|\mathbf{u}^0\|_h^2\right] \|\mathbf{u}^0\|_h^2 |\mathbf{U}^1|_{1,3h}^2 \leq 0, \end{aligned} \tag{4.47}$$

and using (4.32), we arrive at

$$\|\mathbf{U}^2\|_{3h}^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2 |\mathbf{U}^1|_{1,3h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{U}^1\|_{3h}^2.$$

We now assume that (4.40) holds true up to the order $q - 1$, that is

$$\|\mathbf{U}^q\|_{3h}^2 + \frac{\Delta t}{2}\varepsilon^2\delta^2\eta^2 |\mathbf{U}^{q-1}|_{1,3h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{U}^{q-1}\|_{3h}^2. \tag{4.48}$$

and we observe that

$$\|\mathbf{U}^{s+1}\|_{3h}^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{U}^s\|_{3h}^2, \text{ for } s = 1, \ldots, q-1. \tag{4.49}$$

From (4.38) and (4.49) together with (4.32) we obtain the result. Thus using (4.1) and (4.6), we find

$$\|\mathbf{u}^{s+1}\|_h^2 \leq \exp\left(\frac{\Delta t}{\varepsilon^2}\right) \|\mathbf{u}^s\|_h^2, \text{ for } s = 0, \ldots, q.$$

Now suppose that (4.39) and (4.40) holds up to the order n. Using the same approach as in the case $n = 0$, it can be easily proved by induction on r and s. Hence, (4.39) and (4.40) hold for any $n = z(q + 1)$, where $z \in \mathbb{N}_+$.

Therefore, the proof is complete. $\square$

Remark 4.1. *By the subscript $3h$, we mean the discrete operators, discrete norms and semi-norms are applied on the coarser discretization.*

Remark 4.2. *To compare the stability regions of the explicit finite volume methods, we use $\frac{\Delta t}{p}$ on the fine mesh and Δt on the coarser mesh as discussed in this section.*

- *When $p \leq 9$, the multilevel method has the same region of stability as the one-level method on the fine mesh but smaller region of stability than the one-level method on the coarse mesh.*
- *When $p \geq 81$, the multilevel method has the same region of stability as the one-level method on the coarse mesh but smaller region of stability than the one-level method on the fine mesh.*
- *When $9 < p < 81$, the multilevel method is less restrictive than the one-level method on the fine mesh and more restrictive than the one-level method on the coarse mesh.*

5 Numerical Simulations

In this section, some numerical simulations of the 2D convective Cahn-Hilliard equation, (1.1), with specified initial condition and periodic boundary conditions at some values of T are presented. All the results are computed in a matlab platform using Windows 10 Intel CORE i3, 6G RAM PC and the parameters are chosen as: $\alpha_1 = \alpha_2 = \frac{1}{6}, p = 5$ and $q = 8$.
For the one-level finite volume methods, we use the following temporal and spatial step sizes

- One-level method on the fine mesh (Fine): time step $\Delta t/p$ and spatial step sizes $\Delta x = \Delta y$.
- One-level method on the coarse mesh (Coarse): time step size Δt and spatial step sizes $3\Delta x = 3\Delta y$.

For the implicit one-level method, $\tilde{\mathbf{u}}^n$ is given by the relation:

$$\tilde{\mathbf{u}}^n = \frac{1}{2}\left(\mathbf{u}^n + \mathbf{u}^{n-1}\right), \text{ for } n = 1, 2, \ldots, M-1.$$

Similarly for the implicit multilevel method, for a non-negative integer m and $n = m(q+1)$, we use the following approximations:

$$\begin{aligned}
\tilde{\mathbf{u}}^{m(q+1)+r/p} &= \frac{1}{2}\left(\mathbf{u}^{m(q+1)+r/p} + \mathbf{u}^{m(q+1)+(r-1)/p}\right), \text{ for } r = 1, \ldots, p-1, \\
\tilde{\mathbf{u}}^{m(q+1)} &= \mathbf{u}^{m(q+1)}, \\
\tilde{\mathbf{U}}^{m(q+1)+s} &= \frac{1}{2}\left(\mathbf{U}^{m(q+1)+s} + \mathbf{U}^{m(q+1)+s-1}\right), \text{ for } s = 1, \ldots, q,
\end{aligned}$$

and for both implicit methods $\tilde{\mathbf{u}}^0 = \mathbf{u}^0$.
To test the numerical methods, we consider the exact solution

$$u(x, y, t) = \sin\left(\frac{2\pi\, x}{L}\right) \sin\left(\frac{2\pi\, y}{L}\right) \cos(2\pi\, t),$$

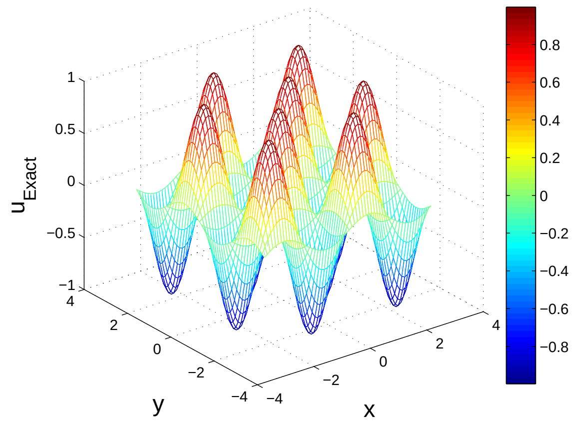
where $L_1 = L_2 = L = 3$, from which the source term is obtained on substitution of (1.1). As shown by Fig. 2, it is observed that the numerical results obtained using the multilevel finite volume methods are close to the results obtained from one-level methods on the fine mesh as compared to the one-level on the coarse mesh. There is no need to plot u versus y because of the similarities with u versus x.
Tables 1-2 show that we can save more time using the multilevel method as compared to the one-level methods on the fine mesh. From the numerical simulations, it is observed that all methods are second order accurate in space and the solutions obtained from the multilevel methods are intermediate between the ones obtained from one-level methods on the fine mesh and on the coarse mesh.

Method	$\Delta x (= \Delta y)$	Δt	L_2-error	CPU time	L_2 Rate
Fine	0.2	0.01	0.0518	1.032	
	0.1	0.0025	0.0131	15.134	1.9594
	0.05	0.000625	0.0033	1232.888	2
Coarse	0.2	0.01	0.3947	0.150	
	0.1	0.0025	0.1128	0.272	1.5661
	0.05	0.000625	0.0291	3.822	1.8806
	0.025	0.00015625	0.0073	159.934	1.9607
Multilevel	0.2	0.01	0.0518	1.341	
	0.1	0.0025	0.0279	4.643	0.8927
	0.05	0.000625	0.0098	271.597	1.5094
	0.025	0.00015625	0.0027	15701.626	1.8598

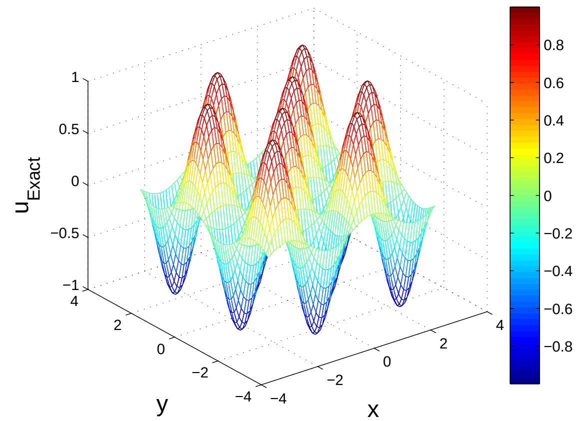
Table 1: Convergence rate, CPU time and L_2-error for some values of spatial step sizes and Δt for the implicit methods at $T = 0.01$.

Method	$\Delta x (= \Delta y)$	Δt	L_2-error	CPU time	L_2 Rate
Fine	0.2	0.0002	0.0441	1.765	
	0.1	0.0000125	0.0112	73.639	1.9773
	0.05	0.00000078125	0.0031	10267.945	1.8532
Coarse	0.2	0.0002	0.3749	0.571	
	0.1	0.0000125	0.0983	2.079	1.9312
	0.05	0.00000078125	0.0249	117.542	1.9810
Multilevel	0.2	0.0002	0.0449	0.469	
	0.1	0.0000125	0.0143	10.725	1.6507
	0.05	0.00000078125	0.0043	723.252	1.7336

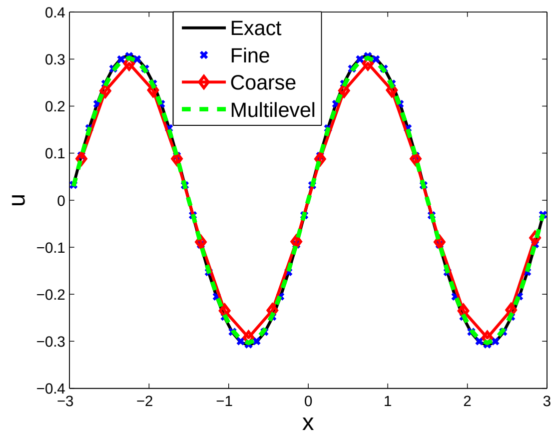
Table 2: Convergence rate, CPU time and L_2-error for some values of spatial step sizes and Δt for the explicit methods at $T = 0.001$.



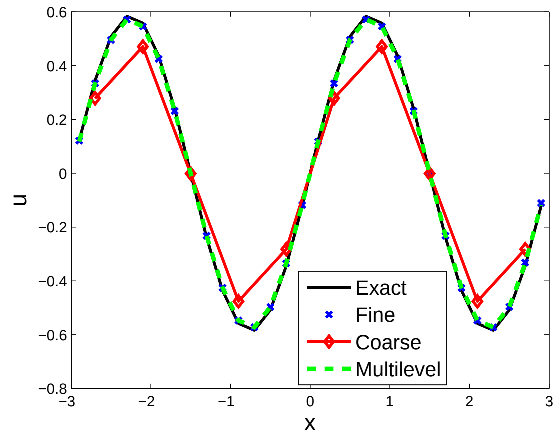
(a) Exact when $\Delta x = \Delta y = 0.1$ and $\Delta t = 0.0025$ at $T = 0.01$.



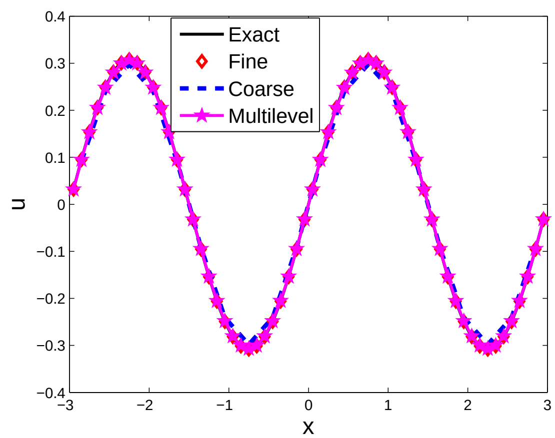
(b) Exact when $\Delta x = \Delta y = 0.1$ and $\Delta t = 0.0000125$ at $T = 0.001$.



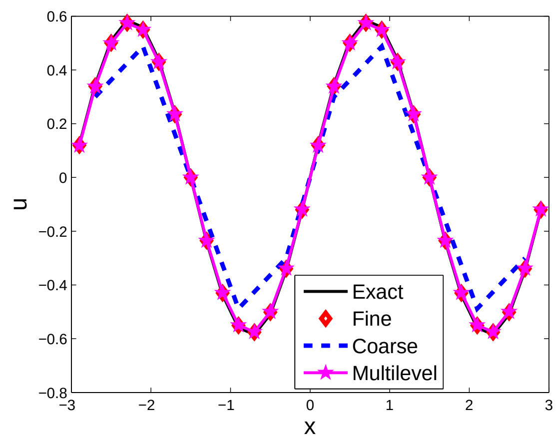
(c) u versus x obtained from implicit methods when $\Delta x = \Delta y = 0.1, \Delta t = 0.0025$ at the cells with centre $y = 0.15$ and $T = 0.01$.



(d) u versus x obtained from implicit methods when $\Delta x = \Delta y = 0.2, \Delta t = 0.01$ at the cells with centre $y = 0.3$ and $T = 0.01$.



(e) u versus x obtained from explicit methods when $\Delta x = \Delta y = 0.1, \Delta t = 0.0000125$ at the cells with centre $y = 0.15$ and $T = 0.001$.



(f) u versus x obtained from explicit methods when $\Delta x = \Delta y = 0.2, \Delta t = 0.0002$ at the cells with centre $y = 0.3$ and $T = 0.001$.

Figure 2: Numerical results for some values of spatial step sizes and Δt.

6 Conclusion

We have extended the work of [22, 23] in two directions: first, we have considered a nonlinear equation in which the nonlinear term has been linearized following Mickens' rules. Secondly, we have shown that the method can be adapted to higher order partial differential equations. In this paper, four numerical methods have been presented and analyzed. The implicit methods discussed here are linear and easy to implement. Existence, uniqueness of solutions for the schemes formulated are discussed and detailed convergence analysis of implicit schemes is furnished. We compare the multilevel methods with the one-level methods by means of stability, convergence and CPU time. It is shown that the multilevel methods are faster than the one-level methods on the fine mesh. We also study the stability of these schemes which allow us to make a classification based on region of stability. But as the numerical experiments reveal, comparing these schemes only with the stability is misleading, hence the CPU time is good indicator for a classification. From the convergence analysis, it is proven that all the methods are second order accurate in space and it is validated by numerical experiments. Our future plan is to extend this work to Navier Stokes equations and its variants.

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Appendix A Taylor's expansion about $\upsilon_{i,j}^{n}$

In this section, we prove the Taylor's expansion given by (3.17), (3.18) and (3.19). Note for simplicity that we omit $|_{i,j}^{n}$ on the expanded terms (right hand sides of each equations).

Proof of (3.17).
To find the Taylor's expansion of the approximation of the fourth order derivative, we use the relation

$$\Delta_h^2 \upsilon_{i,j}^{n+1} = \Delta_{1,h}^2 \upsilon_{i,j}^{n+1} + \Delta_{2,h}\Delta_{1,h}\upsilon_{i,j}^{n+1} + \Delta_{1,h}\Delta_{2,h}\upsilon_{i,j}^{n+1} + \Delta_{2,h}^2 \upsilon_{i,j}^{n+1}. \tag{A.1}$$

It is clear that

$$\begin{aligned}\Delta_{1,h}^2 \upsilon_{i,j}^{n+1} &= u_{xxxx} + \mathcal{O}(\Delta t + \Delta x^2).\\ \Delta_{2,h}^2 \upsilon_{i,j}^{n+1} &= u_{yyyy} + \mathcal{O}(\Delta t + \Delta y^2).\end{aligned}$$

We only find the Taylor's expansion of the second term of the right hand side of (A.1) and hence the expansion of the third term can be obtained accordingly. Using central difference approximation, we have

$$\Delta_{2,h}\Delta_{1,h}\upsilon_{i,j}^{n+1} = \frac{(u_{i+1,j+1}^{n+1} - 2u_{i,j+1}^{n+1} + u_{i-1,j+1}^{n+1}) - 2(u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}) + (u_{i+1,j-1}^{n+1} - 2u_{i,j-1}^{n+1} + u_{i-1,j-1}^{n+1})}{\Delta x^2\, \Delta y^2}. \tag{A.2}$$

Then applying Taylor's expansion, one can verify that

$$\Delta_{2,h}\Delta_{1,h}\upsilon_{i,j}^{n+1} = u_{xxyy} + \mathcal{O}(\Delta t + \Delta x^2 + \Delta x \Delta y + \Delta y^2), \tag{A.3}$$

and

$$\Delta_{1,h}\Delta_{2,h}\upsilon_{i,j}^{n+1} = u_{yyxx} + \mathcal{O}(\Delta t + \Delta x^2 + \Delta x \Delta y + \Delta y^2). \tag{A.4}$$

Therefore, we conclude (3.17).

Proof of (3.18).

$$\nabla_{1,h}^{+}(\psi_{i-\frac{1}{2},j}^{n}\nabla_{1,h}^{-}\upsilon_{i,j}^{n+1}) = \frac{3}{2\Delta x}\left((\upsilon_{i+1,j}^{n})^2 - (\upsilon_{i-1,j}^{n})^2\right)\frac{\left(\upsilon_{i+1,j}^{n+1} - \upsilon_{i,j}^{n+1}\right)}{\Delta x} + \frac{3}{2}\left((\upsilon_{i,j}^{n})^2 + (\upsilon_{i-1,j}^{n})^2\right)\Delta_{1,h}\upsilon_{i,j}^{n+1} - \Delta_{1,h}\upsilon_{i,j}^{n+1}. \tag{A.5}$$

Clearly $\Delta_{1,h}\upsilon_{i,j}^{n+1} = u_{xx} + \mathcal{O}(\Delta t + \Delta x^2)$. Making use of Taylor's expansion, we obtain

$$\frac{3}{2\Delta x^2}\left((\upsilon_{i+1,j}^{n})^2 - (\upsilon_{i-1,j}^{n})^2\right)\left(\upsilon_{i+1,j}^{n+1} - \upsilon_{i,j}^{n+1}\right) = 6uu_x^2 + 3\Delta x u u_x\, u_{xx} + \mathcal{O}(\Delta t + \Delta x^2),$$

and

$$\frac{3}{2}\left((\upsilon_{i,j}^n)^2 + (\upsilon_{i-1,j}^n)^2\right)\Delta_{1,h}\upsilon_{i,j}^{n+1} = 3u^2 u_{xx} - 3uu_x u_{xx} + \mathcal{O}(\Delta t + \Delta x^2),$$

which give

$$\nabla_{1,h}^+(\psi_{i-\frac{1}{2},j}^n \nabla_{1,h}^- \upsilon_{i,j}^{n+1}) = 6uu_x^2 + 3u^2 u_{xx} - u_{xx} + +\mathcal{O}(\Delta t + \Delta x^2). \tag{A.6}$$

In a similar way, we obtain

$$\nabla_{2,h}^+(\psi_{i,j-\frac{1}{2}}^n \nabla_{2,h}^- \upsilon_{i,j}^{n+1}) = 6uu_y^2 + 3u^2 u_{yy} - u_{yy} + +\mathcal{O}(\Delta t + \Delta y^2). \tag{A.7}$$

Combining (A.6) and (A.7), we obtain (3.18).

Proof of (3.19).
We recall that

$$\begin{aligned} C_h(\boldsymbol{\upsilon}^{n+1}, \tilde{\boldsymbol{\upsilon}}^n)_{i,j} =& \frac{\alpha_1}{\Delta x}\left[\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i+1,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i,j}^n\right]\Big] + \frac{\alpha_2}{\Delta x}\left[\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i-1,j}^n\right] \\ &+ \frac{\alpha_1}{\Delta y}\Big[\upsilon_{i,j+1}^{n+1}\tilde{\upsilon}_{i,j+1}^n - \upsilon_{i,j-1}^{n+1}\tilde{\upsilon}_{i,j}^n + \frac{\alpha_2}{\Delta y}\left[\upsilon_{i,j+1}^{n+1}\tilde{\upsilon}_{i,j}^n - \upsilon_{i,j-1}^{n+1}\tilde{\upsilon}_{i,j-1}^n\right]. \end{aligned} \tag{A.8}$$

Then applying Taylor's expansion, we obtain

$$\frac{\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i+1,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i,j}^n}{\Delta x} = c_0\left[3uu_x + \frac{\Delta x}{2}u\, u_{xx} + \Delta x\, u_x^2\right] + \mathcal{O}(\Delta t + \Delta x^2),$$

and

$$\frac{\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i-1,j}^n}{\Delta x} = c_0\left[3uu_x - \frac{\Delta x}{2}u\, u_{xx} - \Delta x\, u_x^2\right] + \mathcal{O}(\Delta t + \Delta x^2),$$

where

$$c_0 = a_1 + a_2 + \cdots + a_{m_0}.$$

Hence

$$\begin{aligned} \frac{\alpha_1}{\Delta x}\left[\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i+1,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i,j}^n\right] + \frac{\alpha_2}{\Delta x}\left[\upsilon_{i+1,j}^{n+1}\tilde{\upsilon}_{i,j}^n - \upsilon_{i-1,j}^{n+1}\tilde{\upsilon}_{i-1,j}^n\right] &= 3c_0(\alpha_1+\alpha_2)\, uu_x \\ + \frac{\Delta x\, c_0}{2}(\alpha_1 - \alpha_2)\, u\, u_{xx} + \Delta x c_0(\alpha_1 - \alpha_2)\, u_x^2 + \mathcal{O}(\Delta t + \Delta x^2).& \end{aligned} \tag{A.9}$$

In a similar way, one obtains

$$\begin{aligned} \frac{\alpha_1}{\Delta y}\left[\upsilon_{i,j+1}^{n+1}\tilde{\upsilon}_{i,j+1}^n - \upsilon_{i,j-1}^{n+1}\tilde{\upsilon}_{i,j}^n\right] + \frac{\alpha_2}{\Delta y}\left[\upsilon_{i,j+1}^{n+1}\tilde{\upsilon}_{i,j}^n - \upsilon_{i,j-1}^{n+1}\tilde{\upsilon}_{i,j-1}^n\right] &= 3c_0\,(\alpha_1+\alpha_2)\, uu_y \\ + \frac{\Delta y\, c_0}{2}(\alpha_1 - \alpha_2)\, u\, u_{yy} + \Delta y\, c_0(\alpha_1 - \alpha_2)\, u_y^2 + \mathcal{O}(\Delta t + \Delta y^2).& \end{aligned} \tag{A.10}$$

Combining (A.9) and (A.10), we obtain (3.19).