PERMUTATIONS ASSIGNED TO SLIM RECTANGULAR LATTICES

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Abstract. Slim rectangular lattices were introduced by G. Grätzer and E. Knapp in Acta Sci. Math. **75**, 29–48, 2009. They are finite semimodular lattices L such that the poset $\mathrm{Ji}\, L$ of join-irreducible elements of L is the cardinal sum of two nontrivial chains. Using deep tools and involved considerations, a 2013 paper by G. Czédli and the present authors proved that a slim semimodular lattice is rectangular iff so is the Jordan–Hölder permutation associated with it. Here, we give an easier and more elementary proof.

1. Introduction

The systematic study of planar semimodular lattices begins with Grätzer and Knapp [10]. By now, there are about two dozen papers devoted to these lattices, and most of these papers are overviewed in a recent book chapter, Czédli and Grätzer [5]. In the class of all planar semimodular lattices, the subclass of slim semimodular lattices plays a distinguished role. The reason is two-fold. First, as it was proved in Grätzer and Knapp [10], each planar semimodular lattice can easily be obtained from a slim semimodular lattice, which is unique by Czédli and Schmidt [8, Lemma 4.1]. Second, slim semimodular lattices play a key role in Czédli and Schmidt [6], which adds a uniqueness part to the classical Jordan–Hölder theorem for groups. By Grätzer and Knapp [11], see also Czédli [2, Lemma 6.4], the class of slim semimodular lattices can easily be obtained from an even more specific class of lattices, which are called *rectangular lattices*. These lattices are the objects studied in this paper.

The *Jordan-Hölder permutation* (briefly, the *permutation*) associated with a slim semimodular lattice was first defined by Stanley [13] and Abels [1]. A systematic treatment for these permutations, with three equivalent definitions, is given in Czédli and Schmidt [9]. The

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importance of a Jordan–Hölder permutation is that it determines the slim semimodular lattice it is associated with. In [4], we determine the number of slim rectangular lattices of length n, both recursively and asymptotically. This makes it more or less necessary to describe the (Jordan–Hölder) permutations associated with these lattices. Although a description is given in [4], its proof is based on heavy tools and only few readers can follow it. The goal of the present paper is to give an *elementary proof*.

2. Basic definitions and the theorem that we prove

Semimodularity means upper semimodularity, that is, $x \prec y$ implies $x \vee z \preceq y \vee z$ for all elements x, y, z in L. A lattice L is *slim*, if it is finite and the poset $\mathrm{Ji}\, L$ of its join-irreducible elements is the union of two chains, say, W_1 and W_2. If, in addition, L is semimodular and $w_1 \wedge w_2 = 0$ holds for all $w_1 \in W_1$ and $w_2 \in W_2$, then L is a *rectangular lattice*. (This convenient definition is due to Czédli and Schmidt [6, Lemma 2.2].) Together with a slim lattice L, we always consider a fixed planar diagram D of L. Besides planarity, our diagrams follow the convention given in Czédli [3]. For example, a planar diagram of a rectangular lattice is given in Figure 1, and also in the rest of figures. We know from Kelly and Rival [12] that D has a *left boundary* (*chain*) and a *right boundary* (*chain*),

$$\begin{aligned} C_\ell(D) &= \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\} \text{ and} \\ C_r(D) &= \{0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1\}\text{, respectively.} \end{aligned}$$

Given a diagram D of a slim semimodular or, in particular, a rectangular lattice L, let $\mathfrak{r}_i = [u_i, v_i]$ be prime intervals, that is, edges, of D, for $i \in \{1, 2\}$. These two edges are *consecutive* if they are opposite sides of a covering square, that is, of a 4-*cell* in the diagram. Following Czédli and Schmidt [6], an equivalence class of the transitive reflexive closure of the "consecutive" relation is called a *trajectory*. Two consecutive edges of a trajectory form a 4-cell *of the trajectory*. For example, in Figure 1, the thick edges like $\mathfrak{p}_\ell(8)$ and $\mathfrak{p}_r(5)$ form a trajectory T_1. The double edges like $\mathfrak{p}_\ell(3)$ and $\mathfrak{p}_r(9)$ form another trajectory T_2. A trajectory begins with an edge on the left boundary chain $C_\ell(D)$, it goes from left to right, it cannot branch out, and it terminates at an edge on the right boundary chain, $C_r(D)$. For example, T_1 begins with $\mathfrak{p}_\ell(8)$ and ends with $\mathfrak{p}_r(5)$. Furthermore,

> a trajectory starts going up, possibly in zero steps, then it can turn to the lower right, and it continues down, possibly in zero steps. (1)

It follows from this "traffic rule" that two trajectories have at most one 4-cell in common. For example, T_1 and T_2 have one common 4-cell in Figure 1, the grey 4-cell.

The symmetric group of all $\{1,\dots,n\} \to \{1,\dots,n\}$ permutations, where n is the length of D, is denoted by S_n. With the notation

$$\mathfrak{p}_\ell(i) = [c_{i-1}, c_i], \qquad \mathfrak{p}_r(i) = [d_{i-1}, d_i]$$

for the edges on the boundary chains, the permutation $\pi = \pi_D \in S_n$ associated with D is defined by the following rule:

$$j = \pi(i) \iff \mathfrak{p}_\ell(i) \text{ and } \mathfrak{p}_r(j) \text{ belong to the same trajectory.}$$

For example, the permutation associated with D in Figure 1 is

$$\pi_D = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 3 & 9 & 6 & 2 & 10 & 7 & 5 & 1 & 4 \end{pmatrix};$$

T_1 is responsible for the equality $\pi_D(8) = 5$, while T_2 for $\pi_D(3) = 9$.

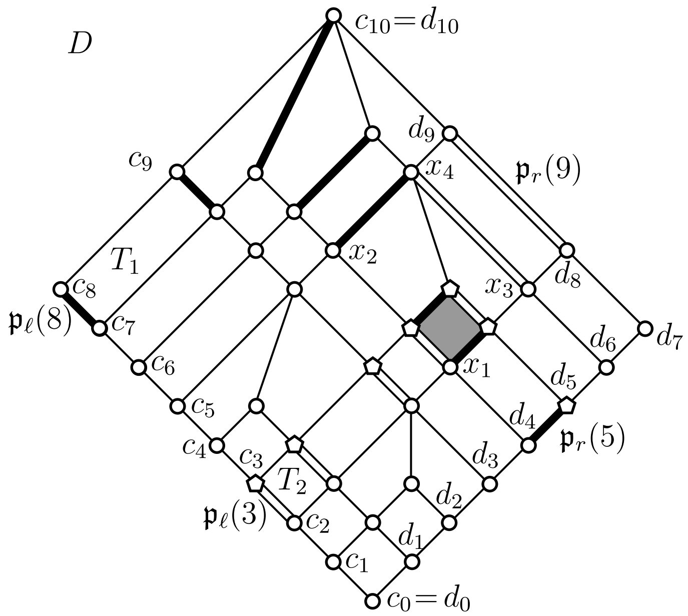


FIGURE 1. Trajectories and forks

According to [4], a permutation $\pi \in S_n$ is called *rectangular* if it satisfies the following three properties.

(i) For all i and j, if $\pi^{-1}(1) < i < j \le n$, then $\pi(i) < \pi(j)$.
(ii) For all i and j, if $\pi(1) < i < j \le n$, then $\pi^{-1}(i) < \pi^{-1}(j)$.
(iii) $\pi(n) < \pi(1)$.

The name "rectangular" is explained by the following statement, which is the theorem we are going to prove.

Theorem 2.1. *The permutation $\pi \in S_n$ is rectangular if and only if there exists a slim, semimodular, rectangular, planar diagram, such that $\pi_D = \pi$.*

In Figure 1, D contains seven pentagon-shaped elements like c_3 and d_5. Let D' denote the diagram (of the sublattice) that we obtain by omitting these seven elements from D. We say that D' is obtained from D by *removing a fork*. What is really important is the converse procedure: we say that D is obtained from D' by *adding a fork* to the 4-cell $\{x_1, x_2, x_3, x_4\}$; see Czédli and Schmidt [7]. By a *grid* we mean a direct product of two nontrivial chains. If the chains are of lengths a and b, then the permutation assigned to this grid is

$$\begin{pmatrix} 1 & \dots & a & a+1 & \dots & a+b \\ b+1 & \dots & b+a & 1 & \dots & b \end{pmatrix}.$$

By [7, Theorem 12], see also Czédli and Schmidt [8] for a more explicit statement, every slim rectangular lattice (diagram) can be obtained from a grid by adding forks in a finite number of steps.

3. A lemma and the proof of Theorem 2.1

Let D be a slim, rectangular, planar diagram, and let $\pi = \pi_D \in S_n$ be the permutation assigned to D. Assume that $i < j$. Trajectories starting at $\mathfrak{p}_\ell(i)$ and $\mathfrak{p}_\ell(j)$ meet if and only if (i, j) is an inversion in π, that is $\pi(i) > \pi(j)$. So the number of 4-cells in D is exactly $\operatorname{inv}(\pi)$. Every 4-cell of D contains 4 edges of the diagram so in $4\operatorname{inv}(\pi)$ we count edges twice except for the edges on $C_\ell(D)$ and $C_r(D)$. We conclude that the number of edges in the diagram D is:

$$\frac{4\operatorname{inv}(\pi) + 2\operatorname{len}(\pi)}{2} = 2\operatorname{inv}(\pi) + n.$$

From Euler's polyhedron formula for graphs, we get

$$|D| = (2\operatorname{inv}(\pi) + n) - \operatorname{inv}(\pi) + 1 = n + \operatorname{inv}(\pi) + 1.$$

Let $\pi \in S_n$. For all $1 \le i \le n$ we define π_i^- to be the permutation in S_{n-1} for which

$$\pi_i^-(j) = \begin{cases} \pi_i^*(j), & \text{if } j < i \\ \pi_i^*(j+1), & \text{otherwise,} \end{cases}$$

where

$$\pi_i^*(j) = \begin{cases} \pi(j), & \text{if } \pi(j) < \pi(i) \\ \pi(j) - 1, & \text{otherwise,} \end{cases}$$

for all $1 \le j \le n$.

Now let $\pi \in S_n$ again. For any inversion (i, j) in π let $\pi_{i,j}^+ \in S_{n+1}$ be the permutation such that

$$\pi_{i,j}^+(k) = \begin{cases} \pi_j'(k), & \text{if } k \le i, \\ \pi(j) + 1 & \text{if } k = i+1, \\ \pi_j'(k-1) & \text{otherwise,} \end{cases}$$

where

$$\pi_j'(k) = \begin{cases} \pi(k), & \text{if } \pi(k) \le \pi(j), \\ \pi(k) + 1, & \text{if } \pi(k) > \pi(j). \end{cases}$$

Lemma 3.1. *Let D be a slim, semimodular, rectangular, planar diagram, and $\pi = \pi_D$ the permutation assigned to D. Let D' be the rectangular diagram we get from D by adding a fork to the 4-cell at the meet of trajectories starting at $\mathfrak{p}_\ell(i)$ and $\mathfrak{p}_\ell(j)$. Then we have $\pi_{D'} = \pi^+_{i,j}$.*

Proof. For this proof, consider D as a subdiagram of D'. Intervals on $C_\ell(D')$ and $C_r(D')$ of D' will be denoted by $\mathfrak{p}'_\ell(i)$ and $\mathfrak{p}'_r(j)$, respectively. Let $\operatorname{len}(\pi) = n$. Let H be the 4-cell determined by the meet of trajectories starting at $\mathfrak{p}_\ell(i)$ and $\mathfrak{p}_\ell(j)$. It follows easily from (1) that adding a fork to H splits $\mathfrak{p}_\ell(i)$ into two intervals, namely $\mathfrak{p}'_\ell(i)$ and $\mathfrak{p}'_\ell(i+1)$. The same holds for $\mathfrak{p}_r(\pi(j))$, the resulting intervals are $\mathfrak{p}'_r(\pi(j))$ and $\mathfrak{p}'_r(\pi(j)+1)$. If $k < i$ then $\mathfrak{p}'_\ell(k) = \mathfrak{p}_\ell(k)$, and if $k > i+1$ then $\mathfrak{p}'_\ell(k) = \mathfrak{p}_\ell(k-1)$. If $k < \pi(j)$ then $\mathfrak{p}'_r(k) = \mathfrak{p}_r(k)$, and if $k > \pi(j)+1$ then $\mathfrak{p}'_r(k) = \mathfrak{p}_r(k-1)$. From this, it is clear that trajectories starting at $\mathfrak{p}'_\ell(k)$, with $k \le i$ end at $\mathfrak{p}'_r(\pi'(k))$, where $\pi'(k)$ is $\pi(k)$ if $\pi(k) < \pi(j)$, and $\pi'(k) = \pi(k)+1$ otherwise. The only trajectory in D' that we do not have in D starts at $\mathfrak{p}'_\ell(i+1)$ and ends at $\mathfrak{p}'_r(\pi(j)+1)$. The remaining case is $k > i+1$. In this case the trajectory starting at $\mathfrak{p}'_\ell(k)$ ends at $\mathfrak{p}'_r(\pi'(k-1))$, where $\pi'(k)$ is $\pi(k)$ if $\pi(k) \le \pi(j)$ and $\pi(k)+1$ otherwise. This proves that $\pi_{D'} = \pi^+_{i,j}$. $\square$

Before we turn to prove Theorem 2.1, we want to illustrate the algorithm given in the proof with an example. In this example, we use the notations of the proof without explaining them here.

Let us start with the rectangular permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 3 & 5 & 8 & 2 & 1 & 4 \end{pmatrix}.$$

We will construct the slim, rectangular lattice for which π is assigned to. For π, we have $c = 2$, $h = 6$ and $t = 7$. The algorithm gives us the permutation

$$\pi^-_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 2 & 4 & 7 & 1 & 3 \end{pmatrix}.$$

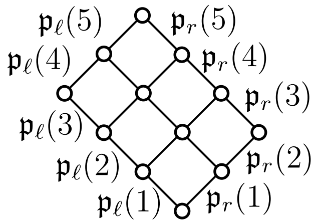
To ease our notations, let $\varrho = \pi^-_6$. For ϱ, we have $c = 2$, $h = 3$, $t = 6$. The next permutation is

$$\varrho^-_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 6 & 1 & 2 \end{pmatrix}.$$

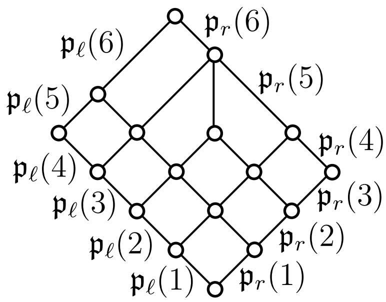
Again, let $\tau = \varrho^-_3$. For τ, we have $c = 3$, $h = 3$, $t = 6$. The final permutation is

$$\tau^-_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

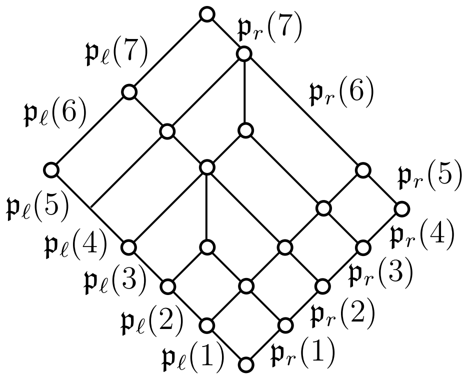
The algorithm stops here, since τ_3^- is a permutation of a grid:



Since $\tau = (\tau_3^-)^+_{2,5}$, we add a fork to the cell at the meet of trajectories starting at $\mathfrak{p}_\ell(2)$ and $\mathfrak{p}_\ell(5)$. The lattice we get is

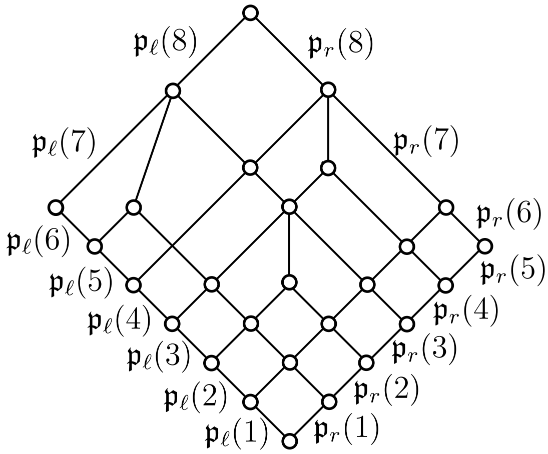


Again, $\tau = \varrho_3^-$ and $\varrho = (\varrho_3^-)^+_{2,5}$, so we add a fork to the cell at the meet of trajectories starting at $\mathfrak{p}_\ell(2)$ and $\mathfrak{p}_\ell(5)$. The lattice we get is



For the last step, remind that $\varrho = \pi_6^-$ and $\pi = (\pi_6^-)^+_{5,6}$. So we add a fork to the cell at the meet of trajectories starting at $\mathfrak{p}_\ell(5)$ and $\mathfrak{p}_\ell(6)$.

The lattice of π is:



Now we can prove our main result, Theorem 2.1.

Proof. We will give an algorithm to find D. The algorithm constructs a smaller permutation from π, until we have a permutation of a grid. From this smaller rectangular diagram we can construct D by adding forks to it, in the sense of Lemma 3.1.

Let π be the given permutation in the theorem, which satisfies the properties (i)—(iii) in the definition of the rectangular permutations. Let $k = \pi^{-1}(1)$. From (iii), we have that $k > 1$. Let

$$c = \min\left(\{1, 2, \dots, n\} \setminus \{\pi(k), \pi(k+1), \dots, \pi(n)\}\right).$$

Finally let $h = \pi^{-1}(c)$. From (i), we have that $\pi(k) = 1, \pi(k+1) = 2, \dots, \pi(t) = c - 1$, where $t = k + c - 2$.

We have two possibilities. First assume, that $h = 1$. By (i) we have that $\pi(k) < \pi(k+1) < \cdots < \pi(n)$. We claim that there is no gap in the strictly increasing sequence $1 = \pi(k), \pi(k+1), \dots, \pi(n)$ of integers. To see this, suppose the contrary. Then the smallest gap is just c, and we obtain that $\pi(1) = \pi(h) = c < \pi(n)$, contradicting (iii). Hence, there is no gap, and we obtain that $\pi(k) = 1, \pi(k+1) = 2, \dots, \pi(n) = n-k+1$. Thus, $\pi(1) = c = n - k + 2$ and $\pi(i) \geq \pi(1)$ for $i \in \{1, \dots, k-1\}$. Hence, (ii) excludes the existence of a pair $\langle i, j\rangle$ with $1 \leq i < j \leq k-1$ but $\pi(i) > \pi(j)$, and we conclude that $\pi(1) = c = n - k + 2, \pi(2) = n - k + 3, \dots, \pi(k-1) = n$. Clearly, π is associated to a $k - 1$ by $n - k + 1$ grid, and the algorithm stops.

The other case is when $h > 1$. In this case we will show that π_h^- is also rectangular and $(\pi_h^-)^+_{h-1,t-1} = \pi$, which by Lemma 3.1 means that π can be derived from a smaller rectangular diagram by adding a fork to it.

In this case π is of the form:

$$\begin{pmatrix} 1 & \dots & h-1 & h & h+1 & \dots & k & k+1 & \dots & t & \dots & n \\ \pi(1) & \dots & \pi(h-1) & c & \pi(h+1) & \dots & 1 & 2 & \dots & c-1 & \dots & \pi(n) \end{pmatrix}.$$

By definition, π_h^- is of the following form; we can disregard the vertical lines, which serve organizing purposes.

$$\left(\begin{array}{ccc|cc|cccc|cc} 1 & \dots & h-1 & h & \dots & k-1 & k & \dots & t-1 & \dots & n-1 \\ \pi(1)-1 & \dots & \pi(h-1)-1 & \pi(h+1)-1 & \dots & 1 & 2 & \dots & c-1 & \dots & \pi(n)-1 \end{array}\right)$$

Since π satisfies (i)–(iii), it is straightforward to see that so does π_h^-.

Next, we prove that $(\pi_h^-)^+_{h-1,t-1} = \pi$. The idea is the following. In the first row of the permutation π_h^-, we have to increase every number larger than $h-1$ by 1. Similarly, in the second row, we have to increase every number larger than $c-1$ by 1. Also, we have to add a new column to the matrix of π_h^- after the column that corresponds to $\pi_h^-(h-1) = \pi(h-1)-1$. That will split π_h^- into four parts.

Note that, in order to evaluate $(\pi_h^-)^+_{h-1,t-1}(x)$ for $x > h$, we have to look at $\pi_h^-(x-1)$. Hence, cases (1)–(4) below will match the partition of the matrix of π_h^- by vertical lines.

Since $\pi(h-1)-1 > c-1$, $(h-1, t-1)$ is an inversion in π_h^-. So $(\pi_h^-)^+_{h-1,t-1}$ makes sense. By its definition, we see that

(1) if $x < h$ then $(\pi_h^-)^+_{h-1,t-1}(x) = (\pi_h^-)'_{t-1}(x) = \pi_h^-(x)+1 = \pi_h^*(x)+1 = \pi(x)-1+1 = \pi(x)$, since $\pi_h^-(x) > c-1$;
(2) if $h < x < k$ then $(\pi_h^-)^+_{h-1,t-1}(x) = (\pi_h^-)'_{t-1}(x-1) = \pi_h^-(x-1)+1 = \pi_h^*(x)+1 = \pi(x)-1+1 = \pi(x)$, since $\pi_h^-(x-1) > c-1$;
(3) if $k \le x \le t$ then $(\pi_h^-)^+_{h-1,t-1}(x) = (\pi_h^-)'_{t-1}(x-1) = \pi_h^-(x-1) = \pi_h^*(x) = \pi(x)$, since $\pi_h^-(x-1) \le c-1$;
(4) if $x > t$ then $(\pi_h^-)^+_{h-1,t-1}(x) = (\pi_h^-)'_{t-1}(x-1) = \pi_h^-(x-1)+1 = \pi_h^*(x)+1 = \pi(x)-1+1 = \pi(x)$, since $\pi_h^-(x-1) > c-1$.

Also, by definition, $(\pi_h^-)^+_{h-1,t-1}(h) = c = \pi(h)$.

Finally let us prove that if we have a slim, rectangular lattice, the permutation assigned to this lattice is rectangular. By [7, Theorem 12], see also Czédli and Schmidt [8], every slim rectangular lattice (diagram) can be obtained from a grid by adding forks in a finite number of steps. Clearly, every grid has a rectangular permutation. By Lemma 3.1, adding a fork to our lattice is the same as performing a $^+$ operation on the corresponding permutation. A straightforward calculation shows that the $^+$ operation preserves rectangularity; the details are omitted. □

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