Construction of Structured Incoherent Unit Norm Tight Frames

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Abstract—The exact recovery property of Basis pursuit (BP) and Orthogonal Matching Pursuit (OMP) has a relation with the coherence of the underlying frame. A frame with low coherence provides better guarantees for exact recovery. In particular, Incoherent Unit Norm Tight Frames (IUNTFs) play a significant role in sparse representations. IUNTFs with special structure, in particular those given by a union of several orthonormal bases, are known to satisfy better theoretical guarantees for recovering sparse signals. In the present work, we propose to construct structured IUNTFs consisting of large number of orthonormal bases. For a given r, k, m with k being less than or equal to the smallest prime power factor of m and $r < k$, we construct a CS matrix of size $mk \times (mk \times m^r)$ with coherence at most $\frac{r}{k}$, which consists of m^r number of orthonormal bases and with density $\frac{1}{m}$. We also present numerical results of recovery performance of union of orthonormal bases as against their Gaussian counterparts.

I. INTRODUCTION

Frames are overcomplete spanning systems which are a generalization of bases [17], [20]. A family of vectors $\{\phi_i\}_{i=1}^M$ in $\mathbb{C}^m$ is called a frame for $\mathbb{R}^m$, if there exist constants $0 < A \le B < \infty$ such that

$$A \|z\|^2 \le \sum_{i=1}^M |\langle z, \phi_i \rangle|^2 \le B \|z\|^2, \forall z \in \mathbb{C}^m$$

where A, B are called the lower and upper frame bounds respectively [17]. By taking the frame vectors as columns, a full row rank matrix is obtained. In the rest of the paper, we do not make any distinction between a frame and its associated matrix and the use the two terms interchangeably. The characterization of a few frames is given in the following.

- If $A = B$, then $\{\phi_i\}_{i=1}^M$ is called an $A-$tight frame or simply a tight frame.
- If there exists a constant c such that $\|\phi_i\|_2 = c$ for all $i = 1, 2, \ldots, n$, then $\{\phi_i\}_{i=1}^M$ is an equal norm frame. If $c = 1$, then it is called a unit norm frame.
- If a frame is both unit norm and tight, it is called a unit norm tight frame (UNTF).

UNTFs are known to have good conditioning and provide stable representation. A UNTF exists only for $A = \frac{M}{m}$. It can be noted that a frame which is a concatenation of orthonormal bases is also a UNTF. The coherence of a frame is defined as the maximum absolute value of inner-product between

two distinct normalized frame vectors. A UNTF with small coherence is termed as an incoherent unit norm tight frame (IUNTF).

Compressed Sensing (CS) [11], [18] is a relatively new paradigm in signal processing, which aims at recovering sparse signals from very few linear measurements. Orthogonal Matching Pursuit and Basis Pursuit (BP) are two of the most widely used CS algorithms. The performance of both these algorithms depends on the coherence of the underlying frame.

In [1], [2], [4], [3], it is shown that frames which are a concatenation of several orthonormal bases provide better theoretical recovery guarantees when compared to general frames. In some applications of image/audio processing [8], [9], modeling of data as the superposition of several layers attains importance, which implies the significance of an over-complete representation in terms of union of orthonormal bases. Further, the special structure of underlying frames allows for generating sparse representations through efficient solvers such as block coordinate relaxation (BCR) [7].

However, it is very difficult to construct a frame with small coherence which consists of large number of orthonormal bases in $\mathbb{C}^m$. Most of the existing constructions are dictated by some particular family of numbers (especially primes or their powers). In [5], [6], the authors have constructed $m+1$ number of orthonormal bases for $\mathbb{R}^m$ with coherence $1/\sqrt{m}$, where m is a power of two. Some of the well known structured IUNTFs are mutually unbiased bases (MUBs) ([12], [13], [14], [15], [16]). Two orthonormal bases B and B' of an $m-$dimensional complex inner-product space are called mutually unbiased if and only if $|\langle b, b' \rangle|^2 = \frac{1}{m}$ for all $b \in B$ and $b' \in B'$. At most $m + 1$ mutually unbiased bases of $\mathbb{C}^m$ can exist. If m is a power of a prime, extremal sets containing $m + 1$ mutually unbiased bases are known to exist [12], [13]. However, to the best of our knowledge there exist no constructions of union of orthonormal bases with small coherence for more general sizes.

In this paper, we provide constructions for structured IUNTFs, more specifically, concatenation of orthonormal bases with small coherence, first for sizes governed by primes or their powers and then for composite dimensions using poly-nomials over finite fields and recently introduced composition rule for binary matrices [19].

The paper is organized in several sections. In Section II, we briefly review the basics of compressed sensing. Section III lists the enhanced recovery properties for frames which are a union of orthonormal bases. Section IV discusses our construction for sizes governed by primes or their powers. In Section V, we describe the construction methodology for

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general sizes using a recently proposed composition rule for binary matrices [19].

II. BASICS OF COMPRESSED SENSING

A. Compressed Sensing

Compressed Sensing (CS) aims to recover a sparse signal $x \in \mathbb{R}^M$ from a few of its linear measurements $y \in \mathbb{R}^m$. A vector is called sparse if only a few of its elements are non-zero. Sparsity is measured using the $\| \cdot \|_0$ norm, $\|x\|_0 := |\{j \in \{1, 2, \ldots, M\} : x_j \neq 0\}|$. A signal x is said to be s-sparse if $\|x\|_0 \leq s$. The measurement vector y is obtained from the linear system $y = \Phi x$, where Φ is an $m \times M$ ($m < M$) matrix. Sparse solutions can be obtained by the following minimization problem,

$$P_0(\Phi, y) : \min_x \|x\|_0 \text{ subject to } \Phi x = y.$$

However, $P_0(\Phi, y)$ is combinatorial in nature and is known to be NP-hard [10]. A common way to obtain approximate solutions for P_0 is by using greedy methods [4]. Another approach is to solve a convex relaxation of $P_0(\Phi, y)$ ([11]),

$$P_1(\Phi, y) : \min_x \|x\|_1 \text{ subject to } \Phi x = y.$$

The coherence of the matrix Φ is defined as

$$\mu_\Phi = \max_{1 \leq i,j \leq M, \ i \neq j} \frac{|\phi_i^T \phi_j|}{\|\phi_i\|_2 \|\phi_j\|_2},$$

which gives bounds on the guaranteed recovery of sparse signals via Orthogonal Matching Pursuit (OMP) and Basis Pursuit (BP) [4].

Theorem II.1. *[18] An arbitrary $s-$sparse signal x can be uniquely recovered as a solution to problems $P_0(\Phi, y)$ (using OMP and BP) and $P_1(\Phi, y)$, provided*

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu_\Phi}\right). \tag{1}$$

The density of the frame Φ is key to minimizing the computational complexity associated with the matrix-vector multiplication. Here, by density, one refers to the ratio of number of nonzero entries to the total number of entries of the matrix. The frames constructed in this paper have small density, which aids in faster processing.

III. RECOVERY GUARANTEES FOR CONCATENATION OF ORTHONORMAL BASES

Union of orthonormal bases provides better recovery properties compared to general frames. The enhanced recovery properties for both OMP and BP are given below.

Theorem III.1. *[3] Suppose a frame Φ is a union of Q orthonormal bases such that its coherence is μ_Φ. Let x be a superposition of s_i atoms from the i-th basis, $i = 1, \ldots, Q$. Without loss of generality, assume that $0 < s_1 \leq s_2 \leq \cdots \leq s_Q$. Then OMP and BP recover the signal x provided*

$$\sum_{i=2}^{Q} \frac{\mu_\Phi s_i}{1 + \mu_\Phi s_i} < \frac{1}{2(1 + \mu_\Phi s_1)}.$$

Corollary III.2. *[1] Suppose that Φ is a concatenation of two orthonormal bases with coherence μ_Φ, and let x be a signal consisting of s_1 atoms from the first basis and s_2 atoms from the second basis, where $s_1 \leq s_2$. Then the above condition holds whenever*

$$2\mu_\Phi^2 s_1 s_2 + \mu_\Phi s_2 < 1.$$

Theorem III.3. *[3] If Φ consists of Q orthonormal bases, then OMP and BP recover any $s-$sparse signal provided*

$$s < \left[\sqrt{2} - 1 + \frac{1}{2(Q-1)}\right]\mu_\Phi^{-1}. \tag{2}$$

For small values of Q the bound in (2) is less restrictive than the general bound given in (1).

IV. CONSTRUCTION METHOD FOR PRIME AND PRIME POWER SIZES

In this section, we provide our construction method for a union of orthonormal bases for the case when m is a prime or a prime power. Consider the finite field $\mathbb{F}_p = \{f_1, f_2, \ldots, f_p\}$ where p is a prime or a prime power. Let S^p be the collection of polynomials of degree at most r (where $r < p - 1$), which do not contain the constant term. It is easy to check that the cardinality of S^p is $|S^p| = p^r$. For $P \in S^p$, define the set $S_P^p = \{P_j = P + f_j : j = 1, \ldots, p\}$. Fix any ordered $k-$tuple $z \in \mathbb{F}_p^k$ with $r < k \leq p$. For simplicity, we consider $z = (f_1, \ldots, f_k)$. An ordered $k-$tuple is formed after evaluating P_j at each of the points of z i.e., $d_j^P := (P_j(f_1), \cdots, P_j(f_k))$. From the $k-$tuple d_j^P we form a binary vector v_j^P of length pk using

$$v_j^P(p(m-1) + n) = \begin{cases} 1, & \text{if } P_j(f_m) = f_n \\ 0, & \text{otherwise} \end{cases}$$

where $1 \leq m \leq k, 1 \leq n \leq p$. Form a binary matrix V^P of size $pk \times p$ by taking v_j^P, as columns for $j = 1, \ldots, p$.

It can be verified that the matrix V^P satisfies the following properties.

1) V^P has k number of row-blocks with each row-block being of size p. Each column v_j^P of V^P has exactly k number of ones and contains a single 1-valued entry in each block. Also, due to the construction, it is easy to see that every row of V^P contains a single 1-valued entry. Therefore, each row-block is a column (or row) permutation of an identity matrix.
2) The density of V^P is $\frac{1}{p}$.
3) For $i \neq j$, there are no common points between any two distinct $k-$tuples d_i^P and d_j^P. This is true because $P + f_i$ and $P + f_j$ have no common root. As a result there is no overlap (i.e., no two columns contain 1 at the same position) between any two distinct columns of V^P.

We now discuss the construction procedure to produce a unitary matrix from V^P. Let U_P be a $k \times k$ unitary matrix. A new matrix Φ^P is obtained by replacing, in each column of V_P, every 1-valued entry with a distinct row of U_P. The 0-valued entries are replaced by a row of zeros. It is clear that the size of the matrix Φ^P is $pk \times pk$. The orthonormality of the rows of Φ^P follows from the fact that U_P is unitary.

A new matrix Φ is constructed by concatenating Φ^P for $P \in S^p$. The size of Φ is $pk \times (pk \times p^r)$. Let $\alpha = \max_{P,i,j} |u_P(i,j)|$, where $u_{k,P}(i,j)$ denotes the (i,j)−th entry of U_P. The following theorem bounds the coherence of Φ

Theorem IV.1. *The coherence μ_Φ of Φ is at most $\min(r\alpha^2, 1)$.*

Proof: The proof follows from the definition of α and from the fact that any two distinct polynomials P^1 and P^2 belonging to S^p can have at most r number of common roots.

Theorem IV.2. *For $r < k \leq p$, where p is a prime or a prime power, if there is a $k \times k$ unitary matrix such that the largest of the absolute values of its entries (α) satisfies $r\alpha^2 < 1$, then there exists a CS matrix which is a union of p^r number of orthonormal bases, with coherence being at most $r\alpha^2$ and with density $\frac{1}{p}$.*

Remark IV.3. *One can take DCT (Discrete Cosine Transform) and DFT (Discrete Fourier Transform) matrices in the real and complex cases respectively. The DCT matrix is defined for $0 \leq i \leq k-1, 0 \leq j \leq k-1$ as*

$$U(i,j) = \begin{cases} \sqrt{1/k} \cos((\pi/k)(j+0.5)i) & \text{for } i = 0 \\ \sqrt{2/k} \cos((\pi/k)(j+0.5)i) & \text{otherwise} \end{cases}$$

For the DCT matrix $\alpha \leq \sqrt{2/k}$ and therefore the coherence is at most $2r/k$. In the complex case, it can be seen that the coherence is at most r/k, when the DFT matrix is used.

A. A special case

In this section we discuss a special case with $r = 1, k = p$. Observe that for $r = 1$, any two distinct polynomials, $P^{(1)}$ and $P^{(2)}$ belonging to S^p have exactly one common root (i.e., 0). Therefore, the intersection between, $v_j^{P^{(1)}}$ (j−th column of $V^{P^{(1)}}$) and $v_i^{P^{(2)}}$ (i−th column of $V^{P^{(2)}}$) is exactly one.

Let $H_{p \times p}$ be an orthogonal matrix whose entries are unimodular (i. e., $|h(i,j)| = 1$). In the real case, one can take H as the Hadamard matrix of order p. For $p = 2^i : i \geq 2$ Hadamard matrices of order p are known to exist. In the complex case, H can be chosen as the discrete Fourier transform matrix (DFT). In the construction process, we replace the unitary matrix U_P with $\frac{1}{\sqrt{p}}H$. Then the following hold,

1) The inner-product between two columns of Φ corresponding to the same polynomial is zero.
2) The absolute value of the inner product between two columns of Φ corresponding to two different polynomials is $\frac{1}{p}$.

As a result Φ becomes a union of p mutually unbiased bases with coherence $\mu_\Phi = \frac{1}{p}$.

V. CONSTRUCTION FOR THE COMPOSITE CASE

For the composite case, we use the following composition rule given in [19] for combining binary matrices. The following result has been proved there.

Lemma V.1 (Lemma 4 in [19]). *For $i = 1, 2$, let Ψ_i be a binary (containing $0, 1$) matrix of size $m_i \times M_i$ consisting*

of k_i number of row blocks each having a size n_i so that the intersection between any two columns is at most r_i and assume that $r = \max\{r_1, r_2\} < k \leq \min\{k_1, k_2\} \leq \min\{n_1, n_2\}$. Then, the composition rule, denoted by $*$, produces a matrix $\Psi = \Psi_1 * \Psi_2$ of size $n_1 n_2 k \times M_1 M_2$ containing k number of row blocks each having size $n_1 n_2$ with the intersection between any two columns being at most r and density of Ψ being $\frac{1}{n_1 n_2}$.

Let p and q be two distinct primes or prime powers. With $r < k \leq \min\{p, q\}$, $P \in S^p$ and $Q \in S^q$, we apply composition rule on the matrices V^P and V^Q to obtain a new binary matrix $V^{P,Q} = V^P * V^Q$. It is easy to see that $V^{P,Q}$ satisfies the following properties,
1) The size of $V^{P,Q}$ is $pqk \times pq$.
2) $V^{P,Q}$ has k number of row-blocks and each block is of size pq
3) There is no overlap between any two distinct columns of $V^{P,Q}$.
4) The density of $V^{P,Q}$ is $\frac{1}{pq}$.

Let U be a $k \times k$ unitary matrix. For each column of $V^{P,Q}$ we replace each of its 1-valued entries with a distinct row of U to obtain a new unitary matrix $\Psi^{P,Q}$ of size $pqk \times pqk$.

The matrix Ψ is constructed by concatenating $\Psi^{P,Q}$ for $P \in S^p$ and $Q \in S^q$. Let $\alpha = \max_{i,j} |u_{i,j}|$ where $u_{i,j}$ is the (i,j)th element in U. The following properties of Ψ can be easily established.
1) The size of Ψ is $pqk \times (pqk \times (pq)^r)$.
2) Ψ is a union of $(pq)^r$ number of orthonormal bases.

We show next that the coherence μ_Ψ of Ψ is at most $\min(r\alpha^2, 1)$. For the proof of this result, we first give the concatenation property of the composition rule.

Lemma V.2. *Let $V * W$ be the result of composition of the matrices V and W using the rule given in [19]. Then $[V_1, V_2] * [W_1, W_2] = [V_1 * W_1, V_1 * W_2, V_2 * W_1, V_2 * W_2]$ where $[V, W]$ denotes the column-wise concatenation of the two matrices V and W.*

Proof. The composition rule given in [19] is a column-wise operation. For constructing $V * W$ the support of each column of V is combined in an appropriate manner with the support of each column of W. Therefore, the procedure maintains the concatenation property. $\square$

Theorem V.3. *The coherence μ_Ψ of Ψ is at most $\min(r\alpha^2, 1)$.*

Proof. Let $P^{(1)}, P^{(2)} \in S^p$ and $Q^{(1)}, Q^{(2)} \in S^q$, such that $P^{(1)} \neq P^{(2)}$ or $Q^{(1)} \neq Q^{(2)}$. Consider the composition of the column concatenated matrices, $[V^{P^{(1)}}, V^{P^{(1)}}]$ $[V^{Q^{(1)}}, V^{Q^{(2)}}]$. Note that $V^{P^{(1)}}$ and $V^{P^{(2)}}$ have at most r-intersections among any pairs of their columns. Similarly $V^{Q^{(1)}}$ and $V^{Q^{(2)}}$ also have at most r-intersections among their columns. Therefore, from Lemma V.1, the resultant matrix after composition $[V^{P^{(1)}}, V^{P^{(2)}}] * [V^{Q^{(1)}}, V^{Q^{(2)}}]$, has at most r-intersections among its columns. However, from Lemma V.2, we have $[V^{P^{(1)}}, V^{P^{(1)}}] * [V^{Q^{(1)}}, V^{Q^{(2)}}] = [V^{P^{(1)}} * V^{Q^{(1)}}, V^{P^{(1)}} * V^{Q^{(2)}}, V^{P^{(2)}} * V^{Q^{(1)}}, V^{P^{(2)}} * V^{Q^{(2)}}]$. This proves that $V^{P^{(1)}} * V^{Q^{(1)}}$ and $V^{P^{(2)}} * V^{Q^{(2)}}$ have at most r-intersections among any pair of their columns. $\square$

Using the above result recursively, we have the following result for general m.

Theorem V.4. *Let $m = p_1 \ldots p_t$, where $p_1, \ldots, p_t$ are primes or prime powers and $r < k \le \min\{p_1, \ldots, p_t\}$ and U be a $k \times k$ unitary matrix with largest absolute entry α. Then, there exists a CS matrix of size $mk \times (mk \times m^r)$, which is a union of m^r number of orthonormal bases, with coherence being at most $\min(r\alpha^2, 1)$ and density being $\frac{1}{m}$.*

Proof: For $P_1 \in S^{p_1}$, using the construction procedure described in section IV, a binary matrix V^{P_1} is obtained. Now, applying the composition rule in [19] on binary matrices V^{P_i} successively for $i = 1, \ldots, t$, we obtain a new binary matrix $V^{P_1, \ldots, P_t} = V^{P_1} * V^{P_2} \cdots * V^{P_t}$ of size $mk \times m$. By construction, every column of $V^{P_1, \ldots, P_t}$ contains k number of 1-valued entries and there is no intersection between any two columns of $V^{P_1, \ldots, P_t}$. Let $P_i^{(1)}, P_i^{(2)} \in S^{p_i}$ such that $P_i^{(1)} \ne P_i^{(2)}$ for at least one $i \in \{1, \ldots, t\}$, then by composition rule the intersection between any column of $V^{P_1^{(1)}, \ldots, P_t^{(1)}}$ and any column of $V^{P_1^{(2)}, \ldots, P_t^{(2)}}$ is at most r. This can be seen by iteratively applying the concatenation Lemma V.2 to expand $[V^{P_1^{(1)}}, V^{P_1^{(2)}}] * [V^{P_2^{(1)}}, V^{P_2^{(2)}}] * \cdots * [V^{P_t^{(1)}}, V^{P_t^{(2)}}]$ into individual composited matrices. Now, as described previously, embedding a unitary matrix $U_{k \times k}$ into $V^{P_1, \ldots, P_t}$, we can construct a unitary matrix $\Psi_U^{P_1, \ldots, P_t}$ of size $mk \times mk$. The matrix $\Psi_{m,k}^r$ is constructed by concatenating $\Psi_U^{P_1, \ldots, P_t}$ by taking $P_i \in S^{p_i}$. Therefore, $\Psi_{m,k}^r$ is a union of m^r number of orthonormal bases, with coherence being at most $\min(r\alpha^2, 1)$ and density being $\frac{1}{m}$.

VI. NUMERICAL SIMULATIONS

This section presents the numerical results for demonstrating the recovery performance of frames constructed via embedding DCT matrix. The column size of the constructed matrix is $mk * m^r$ and the coherence is at most $\frac{2r}{k}$. For obtaining small coherence, it is necessary to consider $r \ll k$. Since k is the smallest prime factor in m, for large values of k, m is also proportionally large. For example, if $m = k = 17$ and $r = 1$, the column size is in the order of 10^3, whereas if $r = 2$, the column size is in the order of 10^4. In the results shown here, as an example, it is assumed that $r = 1$ and $m = k \le 17$ to ease the computational demand. The comparison is performed with respect to Gaussian random matrices. A total of 1000 different signals are considered for each sparsity level and the reconstruction performance is measured. The reconstruction is considered good if the SNR (defined below) is greater than 100dB. If x is the original signal and $\hat{x}$ is the estimated signal, then

$$SNR = 10 \log_{10} \frac{\|x\|}{\|\hat{x} - x\|}.$$

The solutions are computed using the orthogonal matching pursuit (OMP) algorithm. The stopping criterion is considered to be the actual sparsity of the signal. Fig. 1 provides comparison of the success rates of reconstructions between structured frames and their Gaussian counterparts. For a given sparsity level, if 90 percent of the signals are reconstructed accurately
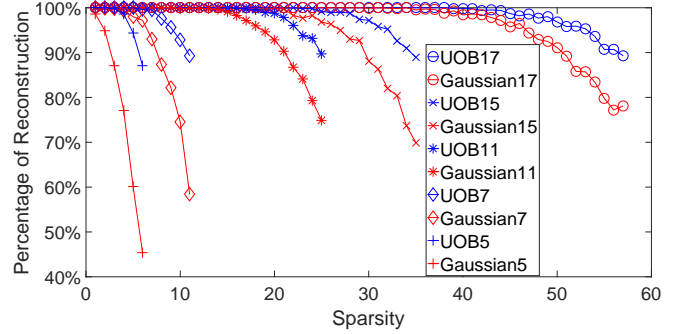

Fig. 1. Comparison of reconstruction performance of unions of orthonormal bases for different m and Gaussian random matrices of the same size. UOB stands for Union of Orthonormal Bases. The sizes of the matrices are given by $mk \times (mk * m^r)$ where $k = m$ and $r = 1$. The results are reported for $m = 5, 7, 11, 15, 17$. UOB17, for e.g., represents the results for the matrix with $m = k = 17$ and $r = 1$. The x and y axes in this plot respectively represent the sparsity (that is, zero norm of solution to be recovered) of solution and success rate. The performance is only shown for sparsity levels for which at least 90% of the signals have been accurately reconstructed.

(i.e., their SNR values are above the threshold of 100 dB) then we consider that the performance is good for that sparsity level. In the above figure, only the performance for sparsity levels satisfying the aforementioned condition is shown. It can be seen from this plot that the constructed structured frames show superior performance compared to Gaussian random matrices.

VII. CONCLUSION

In the present work, we have constructed union of orthonormal bases for general sizes. The matrices for sizes governed by primes and their powers are constructed using polynomials over finite fields. For constructing frames of general sizes, a recently proposed composition rule has been used. Construction of mutually unbiased bases has also been given for the prime power cases. Numerical results show that the constructed structured frames show superior performance when compared to Gaussian Random matrices of the same sizes.

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