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PROPERTIES OF SOLUTIONS FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. Some properties of non-locally bounded solutions for Abel integral equations are given. The case in which there exists two *non-trivial* solutions for such equations is also studied. Besides, some known results about existence, uniqueness and attractiveness of solutions for some Volterra equations are improved.

1. **Introduction.** In this paper we study the behaviour of solutions for nonlinear Volterra integral equations

$$u(x) = \int_0^x k(x-s)g(u(s))ds, \tag{1}$$

where k and g are known functions, called *kernel* and *nonlinearity* respectively, and u is the unknown. This equation shall be referred to as equation (k, g) and the integral operator

$$T_{kg}f(x) = \int_0^x k(x-s)g(f(s))ds$$

as *the associated operator* to the equation (k, g). From now on we suppose that g is a continuous strictly increasing function such that $g(0) = 0$, $g'(x) > 0$ almost everywhere and g transforms null sets into null sets. These assumptions will be denoted by **(B)**.

Abel integral equations are particular cases of Volterra equations with power kernels $k(x) = x^\alpha$ with $\alpha > -1$. In this case the *associated operator* to equation (x^α, g) will be denoted by $T_{\alpha g}$.

Note that, when $g(0) = 0$, the function 0 is a solution of (k, g) known as *trivial* solution. We are interested only in *non-trivial* solutions, i.e. positive solutions u such that $u \not\equiv 0$ *"near zero"*. We shall say that some property holds *near zero* when the property holds on an interval $(0, \delta)$ for some positive δ. Therefore, from now on *solution* means *non-trivial* solutions.

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We shall say that f is *globally attracted by* u when $(T_{kg}^n f(x))_{n\in\mathbb{N}}$ converges to $u(x)$ for any x in the domain of u, where T_{kg}^n denotes the composition of T_{kg} with itself n-times; and we shall say that f is *locally attracted by* u when $(T_{kg}^n f)_{n\in\mathbb{N}}$ converges to u *near zero*.

In this paper, first, we are going to improve some results about existence, uniqueness and attractiveness of solutions for Volterra integral equations with locally bounded kernels. Next, we will introduce some properties about the behaviour of solutions for nonlinear Abel integral equations.

2. **Locally bounded kernels.** Throughout this section we consider equations (k, g) where the nonlinearity g satisfies **(B)** and k is a positive measurable and locally bounded kernel defined on $\mathbb{R}^+$ such that

$$K(x) = \int_0^x k(s)ds$$

is a finite strictly increasing function; these assumptions for kernels together with assumptions **(B)** will be denoted by **(GC)**.

Let (k, g) be an equation satisfying **(GC)**, where kernel k is also a continuous function. It is known that (k, g) has, at most, a unique solution u; and any positive measurable function f, such that $f \not\equiv 0$ *near zero*, is globally attracted by u. These assertions are followed from Proposition 2.1 and Lemma 3.1 in [2]. The aim of this section is to prove the truthfulness of above assertions when the continuity of kernel is removed and the equation (k, g) just satisfies **(GC)**. Note that [2] is strongly based on [1]. Therefore, to attain our aim we need to analyse the use of kernel's continuity in [1, 2].

In [1] are considered equations (k, g) with continuous kernels and satisfying **(GC)**. In that paper, the continuity of kernels was just considered at the beginning of the third section in order to guarantee the continuity of any solution. Note that, even if we remove the continuity of kernel k, any solution of equation (k, g) satisfying **(GC)** is still a continuous function; this is followed from next lemma, where $M(\mathbb{R})$ and $L_1(\mathbb{R})$ denote sets constituted by Lebesgue measurable and Lebesgue integrable functions, respectively, defined on $\mathbb{R}$.

Lemma 1. *Let $f \in L_1(\mathbb{R})$, $h \in M(\mathbb{R})$, and suppose that h is bounded. Then the convolution*

$$f * h(x) = \int_{\mathbb{R}} f(x - s)h(s)ds$$

defines a function that is bounded and uniformly continuous on $\mathbb{R}$.

This lemma appears in [6, pag 357] as exercise 6. To prove the continuity of locally bounded solutions when (k, g) just satisfies **(GC)**, we apply this lemma locally. That is, we fix an interval $[0, b]$, for any positive b, and we define the auxiliary kernel $\tilde{k}$ and the nonlinearity $\tilde{g}$ in the following way

$$\tilde{k}(x) = \begin{cases} k(x) & x \in [0, b], \\ 0 & x \in \mathbb{R}/[0, b] \end{cases} \qquad \text{and} \qquad \tilde{g}(x) = \begin{cases} g(x) & x \in [0, d], \\ 0 & x \in \mathbb{R}/[0, d], \end{cases}$$

where $d = \max_{x\in[0,b]}\{u(x)\}$. Note that $\tilde{k} \in L_1(\mathbb{R})$, $\tilde{g} \circ u$ is a bounded function of $M(\mathbb{R})$ and $u(x) = (k * g \circ u)(x) = (\tilde{k} * \tilde{g} \circ u)(x)$, $x \in [0, b]$. Therefore u is uniformly continuous on $[0, b]$.

The results presented in [2] were obtained considering equations (k, g) with continuous kernels and satisfying **(GC)**. In that paper, there are two reasons to consider the continuity of the kernel. The first one is that we were trying to extend the results presented in [1]. As we have seen in previous paragraph, the continuity of the kernel is not necessary to guarantee the truthfulness of those results. The other one is that we needed, at the beginning of the second section, to define a continuous increasing kernel $\bar{k}$ such that $k \leq \bar{k}$. To do it we defined the following $\bar{k}$,

$$\bar{k}(x) = \begin{cases} 0 & x < 0, \\ \sup\{k(s) : s \in [0, x]\} & x \geq 0. \end{cases}$$

Note that the locally boundedness of k is sufficient to find a positive continuous increasing kernel such that k is upper bounded by it.

To sum up, there is no problem to replace continuous kernels in papers [1, 2] by locally bounded ones. Therefore, for equations (k, g) verifying **(GC)**, we have that

Theorem 1. *There exists at most a unique positive solution. Moreover such solution is a uniformly continuous, increasing function and a global attractor of all positive functions f such that $f \not\equiv 0$ near zero.*

3. **Abel kernels.** A particular case of equations considered in previous section, is the nonlinear Abel integral equation

$$u(x) = \int_0^x (x - s)^\alpha g(u(s))\, ds, \qquad \alpha > 0. \tag{2}$$

Therefore, equation (2) has at most one solution which is a continuous function and a global attractor.

For $\alpha \in (-1, 0)$, Abel kernels are non-locally bounded functions. In this case Abel equations may have locally bounded solutions, non-locally bounded solutions or both (see [3]). In this section, our aim is to study the attracting behaviour of non-locally bounded solutions.

First, let us consider Abel integral equations

$$u(x) = \int_0^x (x - s)^\alpha (u(s))^\beta\, ds,$$

where $\alpha \in (-1, 0)$ and $\beta > -1/\alpha > 1$. This kind of equation has a non-locally bounded solution (see [4]) $u(x) = dx^\gamma$, being

$$\gamma = \frac{1+\alpha}{1-\beta} \quad \text{and } d = B(\alpha + 1, \gamma\beta + 1)^{1/(1-\beta)}, \tag{3}$$

where $B(x, y)$ denotes the Euler beta function. Note that $\gamma < 0$ and $d > 0$. We will show in next theorem that u does not attract the functions of the family

$$\mathcal{U} = \{cx^\gamma : c > 0, c \neq d\}.$$

Theorem 2. *If $f \in \mathcal{U}$ then we have that:*

(a) *For $c > d$, $\lim_{n\to+\infty} T^n_{\alpha\beta} f(x) = +\infty$, with $x \in (0, +\infty)$.*
(b) *For $c < d$, $\lim_{n\to+\infty} T^n_{\alpha\beta} f(x) = 0$, with $x \in (0, +\infty)$.*

Proof. For $f \in \mathcal{U}$ we have that

$$T_{\alpha\beta} f(x) = \int_0^x (x - s)^\alpha (cs^\gamma)^\beta\, ds = c^\beta B(\alpha + 1, \gamma\beta + 1) x^\gamma = \varphi(c) x^\gamma,$$

being

$$\varphi(c) = c^\beta B(\alpha + 1, \gamma\beta + 1), \tag{4}$$

so $T^n_{\alpha\beta}f(x) = \varphi^n(c)x^\gamma$, where φ^n denotes the composition of φ with itself n-times. We have to study the sequence $(\varphi^n(c))_{n\in\mathbb{N}}$ in order to calculate the limit of the sequence $\left(T^n_{\alpha\beta}f(x)\right)_{n\in\mathbb{N}}$.

It is easy to show that 0 and d are the only fixed points of the φ. We have that

$$\varphi'(c) = \beta c^{\beta-1}B(\alpha+1,\gamma\beta+1) \geq 0,$$

and

$$\varphi''(c) = \beta(\beta-1)c^{\beta-2}B(\alpha+1,\gamma\beta+1) \geq 0,$$

so both, φ and φ' are strictly increasing functions. Note that $\varphi'(d) = \beta > 1$. Then

(a) For $c > d$, $\varphi(c) > \varphi(d) = d$. Moreover, by the Mean Value Theorem we have that there exists a $\xi \in (d,c)$ such that

$$\varphi(c) - d = \varphi(c) - \varphi(d) = \varphi'(\xi)(c-d) > \varphi'(d)(c-d) > c-d.$$

Then we can assure that $\varphi(c) > c$, and therefore, by the monotony of φ, $(\varphi^n(c))_{n\in\mathbb{N}}$ is an increasing sequence. This sequence is not upper bounded since any limit of such sequence, by the continuity of φ, would be a fixed point of φ greater than d, which is absurd, so $\lim_{n\to\infty}\varphi^n(c) = +\infty$.

(b) In a similar way we obtain, for $c < d$, that the sequence $(\varphi^n(c))_{n\in\mathbb{N}}$ is strictly decreasing and converges to 0.

Then we can conclude that

$$\lim_{n\to+\infty} T^n_{\alpha\beta}f(x) = \lim_{n\to+\infty}\varphi^n(c)x^\gamma = \begin{cases} 0, & \text{if } c \in [0,d), \\ +\infty, & \text{if } c > d, \end{cases}$$

for $x \in (0,+\infty)$; which completes the proof. □

For positive β, in particular for $\beta > -\alpha^{-1}$, operators $T_{\alpha\beta}$ are monotone increasing. Therefore, it is immediate to see that last Theorem remains true on

$$\hat{\mathcal{U}} = \{f \in L^1(\mathbb{R}) : f \geq cx^\gamma, c > d \text{ or } f \leq cx^\gamma, c < d\}.$$

In the rest of this section, we are going to see that the behaviour described in Theorem 2 remains true for a wider class of Abel integral equations than those considered in last Theorem. To do it, first, we need to recall briefly the examples, presented in [3], about nonlinear Abel equations (x^α, g) with two solutions, one locally bounded and the other one non-locally bounded. Let us consider an Abel integral equation $(x^\alpha,\tilde{g})$ with a locally bounded solution $\tilde{u}$, where $\tilde{g}$ is a nonlinearity verifying **(B)** and an Abel integral equation (x^α, x^β) with a non-locally bounded solution $u_\beta(x) = dx^\gamma$, where $\alpha \in (-1,0)$, $\beta > -\alpha^{-1}$, and d and γ are defined in (3). Then, for any fixed positive ε, we define the following nonlinearity:

$$g(x) = \begin{cases} C\tilde{g}(x) & \text{if } 0 \leq x < \varepsilon \\ x^\beta & \text{if } x \geq \varepsilon, \end{cases} \tag{5}$$

where C is the unique constant that makes g continuous. The equation (x^α, g) has two solutions, one, u_1, is locally bounded and another one, u_2 is not. Solution u_1 is equal to $\tilde{u}$ *near zero* and $u_2 \equiv u_\beta$ *near zero*.

In order to study the attracting behaviour of the solutions when both kind of solutions exist, we need the following lemma. In the proof of next lemma we need the proposition 2.3 of [3]; this proposition requires that the nonlinearity has an ω-lower estimation, with $\omega \in (0,1)$. A function g has an ω-lower estimation if $g \geq x^\omega$ *near zero*.

Lemma 2. *Let u_1 and u_2 denote the locally bounded solution and the non-locally bounded solution of equation (x^α, g), respectively. Then $u_2 \geq u_1$.*

Proof. Let us suppose that there exists an x_1 such that $u_2(x_1) < u_1(x_1)$, and let us consider the function

$$u_3 = \min\{u_1, u_2\}.$$

Since $u_3 \leq u_1$, proposition 2.3 in [3] assures the convergence of $(T^m_{\alpha g} u_3)_{m\in\mathbb{N}}$ to u_1. On the other hand, since $u_3 \leq u_2$ and the operator $T_{\alpha g}$ is monotone, we have that $T^m_{\alpha g} u_3 \leq u_2$ for every natural m. Therefore,

$$T^m_{\alpha g} u_3(x_1) \leq u_2(x_1) < u_1(x_1), \qquad \forall m \in \mathbb{N},$$

which is impossible since $\lim_{m\to\infty} T^m_{\alpha g} u_3(x_1) = u_1(x_1)$. □

For next theorem we will consider Abel equations (x^α, g), where $\alpha \in (-1, 0)$ and g is defined as in (5), assuming the existence of an ω–lower estimation for $\tilde{g}$, with $\omega \in (0, 1)$, and $\beta > -\alpha^{-1}$; and d and γ are defined in (3).

Theorem 3. *Let f be a function such that $f(x) = cx^\gamma$ near zero. Then we have that:*

(a) *For $c > d$, $\lim_{m\to\infty} T^m_{\alpha g} f(x) = +\infty$ on $(0, +\infty)$.*
(b) *For $c < d$, $\lim_{m\to\infty} T^m_{\alpha g} f(x) = u_1(x)$ on the domain of definition of u_1.*

Proof.

(a) From the definition of g, this case is followed from Theorem 2 *near zero*, i.e., there exists an interval, denoted by $(0, a]$, where $\lim_{m\to\infty} T^m_{\alpha g} f(x) = +\infty$. For $x > a$, let $\underline{f}(x) = \liminf_{m\to\infty} T^m_{\alpha g} f(x)$; it is just necessary to note that $\underline{f} \equiv +\infty$ on $(0, a]$, so $T_{\alpha g}\underline{f}(x) = +\infty$ for all positive x. Therefore, the limit of $(T^m_{\alpha g} f)_{m\in\mathbb{N}}$ exists and is $+\infty$ on $(a, +\infty)$.
(b) Let $(0, \mu]$ be the biggest interval where $f(x) = cx^\gamma$, and let M be any positive constant greater than ε and $\min\{f(x) : x \in (0, \mu]\}$. Taking into account the definition of M and f on $(0, \mu]$, there exists a positive δ_0 such that $f(x) \geq M$ on $(0, \delta_0]$. In particular, taking $\delta_0 = (M/c)^{1/\gamma}$, we get the biggest interval on which the above assertion is true . Then,

$$T_{\alpha g} f(x) = \int_0^x (x-s)^\alpha g(f(s))\, ds = \int_0^x (x-s)^\alpha (cs^\gamma)^\beta\, ds = \varphi(c) x^\gamma,$$

for all $x \in (0, \delta_0]$, where φ is defined in (4). Since $c < d$, as was seen in part (b) of the proof of Theorem 2, we have that $T_{\alpha g} f(x) < f(x)$ on $(0, \delta_0]$. So, $\left(T^m_{\alpha g} f\right)_{m\in\mathbb{N}}$ is a monotone decreasing sequence; note that $T_{\alpha g}$ is a monotone operator. And from the non-locally boundedness of f it follows that this sequence is lower bounded by the locally bounded solution u_1 on $(0, \delta_0]$. Therefore, $\left(T^m_{\alpha g} f\right)_{m\in\mathbb{N}}$ has a pointwise limit which will be denoted by v.

Now we have to prove that v is locally bounded *near zero*. Let us define the sequence $\delta_m = (M/\varphi^m(c))^{1/\gamma}$, being $\varphi^m = \varphi \circ \varphi \circ \ldots \circ \varphi$, m-times. This sequence is decreasing and converges to zero, as it is followed immediately from part (b) of the proof of Theorem 2.

Now, from the construction of $(\delta_m)_{m\in\mathbb{N}}$ and the definition of M and f, we have, for any natural m, that $T^m_{\alpha g} f \geq M$ on $(0, \delta_m]$ and $T^m_{\alpha g} f(\delta_{m-1}) < M$.

For any $t \in (0, \delta_0]$ let us consider any natural number p such that $\delta_p < t$. Let c_t be a positive constant such that $(M/\varphi^p(c_t))^{1/\gamma} = t$. Since $\delta_p < t$ we have that $c < c_t$. Therefore, $f < f_{c_t}$ holds on $(0, \delta_0]$, where $f_{c_t}(x) = c_t x^\gamma$

and $T^p_{\alpha g} f(t) < T^p_{\alpha g} f_{c_t}(t) = M$. In conclusion, we have that for any $t \in (0, \delta_0]$ there exists a natural p such that $T^p_{\alpha g} f(t) < M$. Since $v(t)$ is the infimum of $\left(T^p_{\alpha g} f(t)\right)_{p \in \mathbb{N}}$, then $v < M$ on $(0, \delta_0]$.

From the locally boundedness of v, by virtue of the Monotone Convergence Theorem, we have that v is a locally bounded solution for equation (x^α, g) and then, since u_1 is the unique locally bounded solution of such equation, $v \equiv u_1$.

□

Remark 1. Note that, since $\beta > 1$, then for any $\delta > 0$, the nonlinearity g defined in (5) verifies the following condition,

$$\int_\delta^{+\infty} \left(\frac{s}{g(s)}\right)^{1/(\alpha+1)} \frac{ds}{s} < +\infty. \tag{6}$$

This condition, given by W. Mydlarczyk [5, Theorem 2.4], is equivalent to the existence of a blow-up for the locally bounded solution. Therefore, from Lemma 2, since the locally bounded solution is upper bounded by the non-locally bounded one, we can conclude that both solutions have a blow-up.

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